Department of Physics
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Doctoral Dissertation

# Gravitational Theories with Torsion 

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## Abstract

We give a complete formulation of Poincaré gauge theory, starting from the fibre bundle formulation to the resultant Riemann-Cartan spacetime. We also introduce several diverse gravity theories descendent from the Poincaré gauge theory. Especially, the cosmological effect of the simple scalar-torsion $\left(0^{+}\right)$mode in Poincaré gauge theory of gravity is studied. In the theory, we treat the geometric effect of torsion as an effective quantity, which behaves like dark energy, and study the effective equation of state (EoS) of the model.

We concentrate on the two cases of the constant curvature solution and positive kinetic energy. In the former, we find that the torsion EoS has different values corresponding to the stages of the universe. For example, it behaves like the radiation (matter) EoS of $w_{r}=1 / 3\left(w_{m}=0\right)$ in the radiation (matter) dominant epoch, while in the late time the torsion density is supportive for the accelerating universe. In the latter case of positive kinetic energy, we find the (affine) curvature is not constant in general and hence requires numerical solution. Our numerical analysis shows that the EoS in general has an asymptotic behavior in the high redshift regime, while it could cross the phantom divide line in the low redshift regime. By further analysis of the Laurent series expansion, we find that the early evolution of the torsion density $\rho_{T}$ has a radiation-like asymptotic behavior of $O\left(a^{-4}\right)$ where $a(t)$ denotes the scale factor, along with a stable point of the torsion pressure $\left(P_{T}\right)$ and a density ratio $P_{T} / \rho_{T} \rightarrow 1 / 3$ in the high redshift regime ( $z \gg 0$ ), this is different from the previous result in the literature. Some numerical illustrations are also demonstrated.

We construct the extra dimension theory of teleparallel gravity by using differential forms. In particular, we discuss the Kaluza-Klein and braneworld scenarios by direct dimensional reduction and specifying the shape of fibre. The FLRW cosmological scenario of the braneworld theory in teleparallel gravity demonstrates its equivalence to general relativity (GR) in the field equations, namely they possess the same Friedmann equation.

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## 1

## Introduction

The recent cosmological observations, such as those from type Ia supernovae [1, 2], cosmic microwave background radiation [3, 4], large scale structure [5, 6] and weak lensing [7], reveal that our universe is subject to a period of accelerated expansion.

Although general relativity (GR) developed in the last century has been successful in many ways of explaining various experimental results in gravity, the nature of the accelerating universe now rises as a small cloud shrouding it. We thereby look for a more general theory that comprises GR yet being able to explain the accelerating problem referred to as dark energy [8].

In general, there are two ways to resolve the phenomenon of the late-time accelerated universe [8] either by modified gravity or by modified matter theories. Modified gravity asserts that considering alternative geometry may be responsible for forces that we are not able to explain, usually by modifying geometric Lagrangians or changing the geometric framework of spacetime. Modified matter theories include some negative pressure matter that could result in the expanding effect. In this thesis, we adopt the viewpoint of an alternative gravity theory by considering the so-called Poincaré gauge theory (PGT) [9, 10, 11, which integrates the gauge covariant idea into spacetime.

PGT starts with the consideration of gauging the Poincaré group $\mathcal{P}=\mathbb{R}^{1,3} \rtimes$ $S O(1,3)$, where $\mathbb{R}^{1,3}$ denotes the Minkowski spacetime $\left(\mathbb{R}^{4},\langle\cdot, \cdot\rangle_{\mathbb{R}^{1,3}}\right)$ and $\left\langle e_{\mu}, e_{\nu}\right\rangle_{\mathbb{R}^{1,3}}=$ $\operatorname{diag}(-1,+1,+1,+1)$, into gravity and ends up in effect as a Riemann-Cartan spacetime $(M, g, \nabla)$, where $M$ is the spacetime manifold, $g$ is a metric and $\nabla$ is a general metric-compatible connection on $M$. Such a general connection can be decomposed into $\nabla=\bar{\nabla}-K$ such that $\bar{\nabla}$ is the Riemannian one and $K$ is the contortion tensor related
to torsion tensor $T$ of $\nabla$. As a result, PGT is in general a gravitational theory with torsion [9, 10] that couples to the spin of the matter field. Gauge theory with the Poincaré group can be considered as a natural extension of GR, in the sense that it contains GR as a degenerate case. In fact, it comprises a large class including GR, the Einstein-Cartan theory [12], teleparallel gravity, and quadratic Poincaré gauge theory.

Historically, Einstein effectively assumed vanishing torsion ad hoc in 1915, later in 1928 he attempted to utilize the teleparallel (purely torsional) theory to unify gravitation and electromagnetism [13]. Around the same time, Élie Cartan, as a mathematician who constructed the idea of torsion in 1922, communicated with Einstein about his work. In their sequence of communications [14], they set up a large portion of the foundation for the teleparallel gravity theory of nowadays.

The birth of gauge theory was innovated in the hand of Hermann Weyl in 1918 [15], while in 1929 he achieved the concept of $U(1)$-gauge theory [16] we know nowadays and introduced the vierbein (orthonormal basis, tetrad) into general relativity. The success of local gauge theory in 1950s brought new life into the gravity with torsion. Utiyama gave a first attempt in gauging $S O(1,3)$ into spacetime without success [17], mainly due to the Riemannian connection used. On this track, Sciama then introduced torsion and related it to spin [18], and later Kibble showed how to describe gravity with torsion as a local gauge theory of the Poincaré group [19]. Later in 1976 Hehl et al formulated a complete gravitation theory that demonstrates Poincaré gauge invariance and eventually results in gravity with torsion [9]. The great success of gauge theory in fundamental physics leads us to believe that gravity should also belong to the roll of gauge theories, since all the other fundamental interactions like the electroweak and the strong are beautifully formulated by such rules. In this sense, this provides a best guiding principle to follow in searching for an alternative gravity theory.

The framework of PGT is based on the gauge principles of Yang-Mill's theory of non-Abelian group. Through the use of principal fibre bundles in mathematics, one derives a more clear vision of gauge structures and its essence which is eventually beneficial for the transition between different Lie groups.

We set out from the formulation of PGT in fibre bundle language [20], 21] (Chapter 2) which is also general for all gauge theories, and try to address the story of PGT as complete as possible in a united and compact way with the minimal offering of bundle
materials for essential study. Such construction will then provide an integrated and clear view for the Poincaré gauge gravity and Riemann-Cartan spacetime, which in the end leads to several diverse alternative gravity theories, such as the Einstein-Cartan theory, GR, teleparallel gravity, and quadratic PGT.

In particular, we shall investigate a specific quadratic theory of the scalar-torsion mode in PGT [22], [23], [23](Chapter 3), that possesses dynamical torsion field. This particular mode is also called simple $0^{+}$mode or $S N Y$-model, [25], which is one of the six modes: $0^{ \pm}, 1^{ \pm}$and $2^{ \pm}$labeled by spin and parity, based on the linearlized theory of PGT [26, 27]. In the theory, the $0^{+}$mode is known to have no interaction with any fundamental source [28] and thus it could have a significant magnitude without being much noticed within the current universe. In particular, it has been studied by Shie, Nester and Yo (SNY) that the spin- $0^{+}$mode is divided into two classes: one with negative energy density but exhibiting late time de-Sitter universe served as dark energy; the other with normal positive energy condition that is also responsible for the late-time acceleration but demonstrates the early oscillation in various physical quantities. In this thesis, we concentrate on these two classes and present the numerical solutions of the late-time acceleration behavior and their corresponding equation of state (EoS), defined by $w=p / \rho$, where $\rho$ and $p$ are the energy density and pressure of the relevant component of the universe respectively. This type of cosmology provides some realistic and interesting features so that it has been explored in numerous discussions [29, 30, 31, 32, 33, 34, 35, 36, 37]. Consequently, this mode naturally becomes a subject to study [38.

In PGT, there is another interesting degenerate case called teleparallelism (in contrast to GR of zero torsion), where curvature vanishes identically on the spacetime and torsion is the only responsiblility for the gravitational force. The name is so dubbed simply because without curvature every vector field is parallel. Such construction is possible if we adopt the so-called Weitzenböck connection $\nabla^{W}$ on a Riemann-Cartan space. As indicated early, such a spacetime was considered by Einstein [13, who had unified theory concerns, and later it was developed into a type called teleparallel equivalent to general relativity (TEGR). The Lagrangian is in a special form such that it is almost equivalent to GR in every aspect. TEGR is equivalent to GR in the field equations and the matter evolution so that they cannot be told from the dynamics. One distinction between TEGR and GR is the local Lorentz violation at the Lagrangian level [39], i.e, TEGR does not respect local Lorentz
transformation. However such local Lorentz violation terms appear in the form of an exact differential such that it can be regarded as a boundary term were we in a closed manifold. In any case this term does not affect the field equations such that it gives the same action as GR, which is what we meant by "almost equivalent". We also give an account from bundle formalism why such violation occurs. In Chapter 4, we shall provide more discussions.

As is known, there exists another type of modified gravity theory called extra dimension theory that consists of a higher dimensional spacetime (called bulk, generally higher than four) and a 4-dimensional submanifold as our living space(-time). The five dimension gravity generally induces gravity on the four-dimensional spacetime along with a type of force. First extra dimension theory that unifies electromagnetism and gravitation was initiated by Nordström [40] around 1914 as well as Kaluza 41 ] and Klein [42], known as the KK theory. In KK theory, electromagnetic field is from the projection of five-dimensional spacetime whose fibre is a small circle $S^{1} \cong U(1)$. It is usually used to explain the hierarchy problems with the effective Planck scale in 4-dimension by dimensional reduction. There is another type of extra dimension theory called large extra dimension or ADD model, proposed by Arkani-Hamed, Dimopoulos and Dvali 43] in 1998. It was proposed to explain why gravity is so weak compared to other forces. The ADD theory assumes that the fields of the Standard Model are confined on the 4 -dimensional membrane, with only gravity being able to propagate through the large extra dimension that is spatial. Thus it is also referred to as the braneworld theory.

We construct the extra dimension theory for TEGR gravity [44, [45]. In order to build such theory, it is necessary to search for the torsion relations between the brane and the bulk mimicking the Gauss-Codacci equation. In general, such relations could be complicated in component form. Thus we adopt differential forms to reduce the large amount of computation and to serve as a rigorous tool. In the construction, we keep our geometric setting as general as possible to contain branworld theory and KK theory under the same geometric framework. Some of the aspects have been explored in the literature [46, 47, 48, 49, 50], which can be compared with our results. In the end, we utilize our extra dimension theory for TEGR for FLRW cosmology as an application and derive a result consistent with GR.

## 2

## Poincaré Gauge Gravity Theory

...It is possible that ECT will prove to be a better classical limit of a future quantum theory of gravitation than the theory without torsion.

- A. Trautman

Poincaré Gauge Theory for gravity (PGT) is a theory that incorporates gravity as a gauge theory of the Poincaré group $\mathcal{P}=\mathbb{R}^{1,3} \rtimes S O(1,3)$. To develop such a theory, one needs to find an ambient space where both theories cooperate.

Recall that to describe (local) gauge symmetry, one requires gauge invariance of an internal group $G$, that is a Lie group, and the covariant transition induced by an external group of diffeomorphisms $\operatorname{Diff}(M)$ between two observers of spacetime, and hence such a theory must be based on a 4-dimensional Lorentzian manifold. For a gauge theory modelled on a spacetime, a principal fibre bundle is then a natural candidate. For example, electromagnetism of $U(1)$ symmetry can be formulated on a $U(1)$-principal bundle.

Therefore one finds that the best suited mathematical theory that depicts PGT is the principal fibre bundle theory. Below we mainly follow the treatment of [21], [51, [52], [53], [54], and [55] in principal fibre bundle theory to provide essential bundle material to clearly address PGT.

### 2.1 Preliminaries and Notations

### 2.1.1 Geometric construction of PGT

Definition 1. (Principal G-bundle)

Let $G$ be a Lie group. A principal $G$-bundle consists of a pair of differentiable manifolds $P$ called the total space and $M$ called base manifold with a differentiable (surjective) projection $\pi: P \rightarrow M$ and an action of $G$ on $P$ such that

1. For every $g \in G$, there exists a diffeomorphism $R_{g}: P \rightarrow P$ such that $R_{g_{1} g_{2}}(p)=$ $R_{g_{2}} \circ R_{g_{1}}(p)$ for all $g_{1}, g_{2} \in G$ and $p \in P$. And if $e \in G$ is the identity element, then $R_{e}(p)=p$ for all $p \in P$. We also require the group action of $G$ acts freely on $P,(p, g) \in P \times G \mapsto R_{g}(P) \in P$ such that $R_{g}(p) \neq p$ for all $g \neq e$. We also write $R_{g}(p)=p g$.

The action of $G$ on $P$ then defines an equivalence relation. Define $p \sim q$ for $p, q \in P$ $\Leftrightarrow$ if there exists $g \in G$ such that $p=q g$.
2. $M$ is the quotient space of the equivalence relation $\sim$ induced by $G, M=P / G$. Hence $\pi^{-1}(\pi(p))=\{p g \mid g \in G\}$ (the orbit of $G$ through $p$ ). If $x \in M$, then $\pi^{-1}(x)$ is called the fibre above $x$.
3. $P$ is locally trivial. For each $x \in M$, there exists an open set $U$ containing $x$ and a diffeomorphism $T_{U}: \pi^{-1}(U) \rightarrow U \times G$ such that $T_{U}(p)=\left(\pi(p), \psi_{U}(p)\right)$ satisfying $\psi(p g)=\psi(p) g$ for all $g \in G$. The map $T_{U}$ is called a a local trivialization or a choice of gauge (in physics language).

In practice the base manifold $M$ corresponds to the 4-dimensional spacetime in consideration, while for all $p \in \pi^{-1}(x)$ there exists a map $G \rightarrow \pi^{-1}(x)$ by $g \mapsto p g$, which is a diffeomorphism depending on $p$. Thus all fibres $\pi^{-1}(x)$ for $x \in M$ are isomorphic to $G$ called the internal (symmetry) group or gauge group.

A principal $G$-bundle provides a space for phase factors, see [56], while a connection on $P$ yields a gauge potential in physics language, which we denote

Definition 2. Let $\mathfrak{g}$ be the Lie algebra of $G$. A connection is a $\mathfrak{g}$-valued 1-form $\omega$ on $P$, denoted by $\Lambda^{1}(P ; \mathfrak{g}):=\Lambda^{1}(P) \otimes \mathfrak{g}$, such that

1. Let $A \in \mathfrak{g}$, define the fundamental vector field corresponding to $A$ on $P$ by

$$
\begin{equation*}
A_{p}^{*}:=\left.\frac{d}{d t}(p \exp (t A))\right|_{t=0} \tag{2.1}
\end{equation*}
$$

then we require $\omega_{p}\left(A_{p}^{*}\right)=A$.
2. For $g \in G$, let $\mathfrak{a d _ { g }}: \mathfrak{g} \rightarrow \mathfrak{g}$ be the associated adjoint mart. We require $\omega_{p g}\left(R_{g *} X_{p}\right)=$ $\mathfrak{a d}_{g^{-1}} \omega_{p}\left(X_{p}\right)$ for all $g \in G, p \in P$ and $X \in T P$ a vector field on $P$. i.e, $R_{g}^{*} \omega=\mathfrak{a d}_{g^{-1}} \omega$

On a principal fibre bundle $P$, the notion of vertical is already distinguished by the projection $\pi: P \rightarrow M$, specifically if $w \in T_{p} P$, a vector on $P$, is vertical if $\pi_{*}(w)=0$. In contrast, for the notion of horizontal we need extra structure to specify it. A connection $\omega$ on $P$ essentially tells us what vector is horizontal, defined by $\omega(v)=0$ if $v \in T_{p} P$. Therefore given a connection 1-form $\omega$ on $(P, \pi, M, G)$, we may decompose a vector field $w \in T_{p} P$ as $w=w^{V}+w^{H}$ such that $w^{V}$ is vertical $\pi_{*}\left(w^{V}\right)=0$ and $w^{H}$ is horizontal $\omega\left(w^{H}\right)=0$.

With the technical definitions above, we can then describe gravitational gauge theories. We recall that Einstein's general relativity can be reformulated on the following principal bundle whose structure group is $S O(1,3)$,

Definition 3. (Linear frame bundle)
Let $M$ be a 4-dim Lorentz manifold (spacetime). Define a generalized frame (a generalized observer) at $x \in M$ to be a linear isomorphism $u: \mathbb{R}^{1,3} \rightarrow T_{x} M$, and thus $\left\{u\left(e_{0}\right), \ldots, u\left(e_{3}\right)\right\} \in T_{x} M$ is a basis for $T_{x} M$ (not necessarily orthonormal), where $e_{0}, \ldots, e_{3}$ is the canonical basis of the Minkowski spacetime $\mathbb{R}^{1,3}$. Let $L_{x}(M):=\left\{u_{x}\right.$ : $\mathbb{R}^{1,3} \rightarrow T_{x} M$ a isomorphism $\}$ be the set of all generalized observers at $x \in M$. Define the set

$$
\begin{equation*}
L(M):=\bigcup_{x \in M} L_{x}(M) \tag{2.2}
\end{equation*}
$$

with $\pi\left(u_{x}\right):=x$ for $u_{x} \in L_{x}(M)$,, and the action of $G L\left(\mathbb{R}^{1,3}\right)$ on $L(M)$ by $R_{A}: L(M) \rightarrow$ $L(M)$ with $R_{A}\left(u_{x}\right):=u_{x} \circ A$ for $A \in G L\left(\mathbb{R}^{1,3}\right)$. Equipped with a suitable differentiable structure on $L(M)$, one verifies that $\left(L(M), \pi, M, G L\left(\mathbb{R}^{1,3}\right)\right.$ is a principal fibre bundle called the linear frame bundle.

One sees that if $u_{x}$ is an observer at $x$, then the right action $R_{A}, A \in G L\left(\mathbb{R}^{1,3}\right)$, brings it from one frame to another. Furthermore, on a semi-Riemannian ${ }^{[2]}$ spacetime $(M, g)$ we

[^0]can consider an orthonormal frame (or tetrad) $u_{x} \in L_{x}(M)$ at $x \in M$ such that the $g\left(u_{x}(v), u_{x}(w)\right)=\langle v, w\rangle_{\mathbb{R}^{1,3}}$ for all $v, w \in \mathbb{R}^{1,3}$. An orthonormal frame then corresponds to an observer in the usual physical sense. We can further define an orthonormal frame bundle.

Definition 4. (Orthonormal frame bundle)
Let $F_{x}(M):=\left\{u_{x} \in L_{x}(M) \mid g \circ u_{x}=\langle\cdot, \cdot\rangle_{\mathbb{R}^{1,3}}\right\}$ be the set of all orthonormal frames. Define the set

$$
\begin{equation*}
F(M):=\bigcup_{x \in M} F_{x}(M) \tag{2.3}
\end{equation*}
$$

with $\pi: F(M) \rightarrow M$ by $\pi\left(u_{x}\right)=x$ for $u_{x} \in F_{x}(M)$, and the right action of $G=O(1,3)$ by $R_{A}\left(u_{x}\right)=u_{x} \circ A$ for $A \in O(1,3)$. With the differential structure induced from $L(M)$, one can verify that $F(M)$ is also a principal fibre bundle called the orthonormal frame bundle.

Since we know that the hyper-rotation group

$$
O(1,3):=\left\{A \in G L\left(\mathbb{R}^{1,3}\right) \mid\langle A v, A w\rangle_{\mathbb{R}^{1,3}}=\langle v, w\rangle_{\mathbb{R}^{1,3}}\right\}
$$

where $\langle\cdot, \cdot\rangle_{\mathbb{R}^{1,3}}$ is the Lorentzian inner product of Minkowski spacetime, has 4 components:

$$
\begin{align*}
& L_{+}^{\uparrow}:=\left\{B \in O(1,3) \mid \operatorname{det} B=1, B_{0}^{0} \geq 1\right\}, \\
& L_{-}^{\uparrow}:=\left\{B \in O(1,3) \mid \operatorname{det} B=-1, B_{0}^{0} \geq 1\right\}  \tag{2.4}\\
& L_{+}^{\downarrow}:=\left\{B \in O(1,3) \mid \operatorname{det} B=1, B_{0}^{0} \leq-1\right\}, \\
& L_{-}^{\downarrow}:=\left\{B \in O(1,3) \mid \operatorname{det} B=-1, B_{0}^{0} \leq-1\right\}
\end{align*}
$$

where the connected component $L_{+}^{\uparrow}$ is usually referred as the Lorentz group $S O(1,3)$. It follows that $F(M)$ can contain up to 4 components in general. If we assume $F(M)$ has 4 components for simplicity, then the base manifold (spacetime) $M$ is then called space and time orientable. A choice of one component for $F(M)$ corresponds to a space and time orientation. Let $F_{0}(M)$ be such a choice, then the principal fibre bundle $\left(F_{0}(M), \pi, M, S O(1,3)\right)$ describes a spacetime with Lorentz gauge covariance. Since the fibre bundle $\left(F_{0}(M), \pi, M, S O(1,3)\right)$ construction is canonical whenever $M$ exists, we conclude that Einstein's general relativity has local Lorentz gauge freedom.

So far, we have established the living space for gauge gravitational theories, yet we do not have notion of curvature or shape for $P$ at this moment. Curvature is the core
of general relativity in the sense that it is responsible for the gravitational force one perceives but one should emphasize that curvature is not a property pertaining to a spacetime. In the theory of General Relativity in 1915, Einstein adopted the unique Levi-Civita (Riemannian) connection for defining curvature of a spacetime. The following reformulation of the Levi-Civita connection on a linear frame bundle equivalent to the one in general GR textbooks is helpful for PGT generalization later. We begin with defining curvature

Definition 5. If $\psi \in \Lambda^{k}(P, V)$, then we define $\psi^{H} \in \Lambda^{k}(P, V)$ by $\psi^{H}\left(X_{1}, \ldots, X_{k}\right):=$ $\psi\left(X_{1}^{H}, \ldots, X_{k}^{H}\right)$, where $X_{i} \in T P$.

Definition 6. (Exterior covariant derivative)
If $\psi \in \Lambda^{k}(P, V)$, then we define $D^{\omega} \psi:=(d \psi)^{H} \in \Lambda^{k+1}(P, V)$ with respect to the connection $\omega$.

With the exterior covariant derivative in particular for $V=\mathfrak{g}$ of $G$, we can define the notion of curvature on a principal fibre bundle.

Definition 7. (Curvature of a connection)
If $\omega$ is a connection 1-form on $P$, the curvature of the connection is defined as the 2-form $\Omega^{\omega}:=D^{\omega} \omega \in \Lambda^{2}(P, \mathfrak{g})$.

In terms of physics context, $\omega$ is referred to as the gauge potential and $\Omega^{\omega}$ is the field strength (corresponding to $\omega$ ). The following form is usually more familiar in the physics literature than Definition (7).

Theorem 1. If $G$ is a matrix group, then the curvature 2-form can be expressed by

$$
\begin{equation*}
\Omega^{\omega}=d \omega+\omega \wedge \omega \tag{2.5}
\end{equation*}
$$

where $\omega$ is regarded as a matrix in $\mathfrak{g}$ with each entry a real-valued 1-form $\omega_{\mu}{ }^{\nu} \in \Lambda^{1}(P, \mathbb{R})$,
write $\omega=\left(\begin{array}{cccc}\omega_{1}{ }^{1} & \omega_{1}{ }^{2} & \cdots & \omega_{1}{ }^{n} \\ \omega_{2}{ }^{1} & \omega_{2}{ }^{2} & \cdots & \omega_{2}{ }^{n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n}{ }^{1} & \omega_{n}{ }^{2} & \cdots & \omega_{n}{ }^{n}\end{array}\right)$ and the wedge $\omega \wedge \omega$ is defined as

$$
\left(\begin{array}{cccc}
\omega_{1}{ }^{1} & \omega_{1}{ }^{2} & \cdots & \omega_{1}{ }^{n}  \tag{2.6}\\
\omega_{2}{ }^{1} & \omega_{2}{ }^{2} & \cdots & \omega_{2}{ }^{n} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{n}{ }^{1} & \omega_{n}{ }^{2} & \cdots & \omega_{n}{ }^{n}
\end{array}\right) \wedge\left(\begin{array}{cccc}
\omega_{1}{ }^{1} & \omega_{1}{ }^{2} & \cdots & \omega_{1}{ }^{n} \\
\omega_{2}{ }^{1} & \omega_{2}{ }^{2} & \cdots & \omega_{2}{ }^{n} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{n}{ }^{1} & \omega_{n}{ }^{2} & \cdots & \omega_{n}{ }^{n}
\end{array}\right)=\left(\begin{array}{cccc}
\omega_{1}^{k} \wedge \omega_{k}{ }^{1} & \omega_{1}{ }^{k} \wedge \omega_{k}{ }^{2} & \cdots & \omega_{1}{ }^{k} \wedge \omega_{k}{ }^{n} \\
\omega_{2}^{k} \wedge \omega_{k}{ }^{1} & \omega_{2}^{k} \wedge \omega_{k}{ }^{2} & \cdots & \omega_{2}^{k} \wedge \omega_{k}{ }^{n} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{n}{ }^{k} \wedge \omega_{k}{ }^{1} & \omega_{n}{ }^{k} \wedge \omega_{k}{ }^{2} & \cdots & \omega_{n}{ }^{k} \wedge \omega_{k}{ }^{n}
\end{array}\right)
$$

In the case of a linear frame bundle $L(M)$, 2.5 leads to the usual definition of the curvature tensor on a Riemannian manifold ( $M, g, \nabla$ ) by pullback of local sections $\sigma_{U_{i}}: U_{i} \subseteq M \rightarrow L(M)$, given by

$$
\begin{equation*}
R(X, Y) Z:=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z \tag{2.7}
\end{equation*}
$$

for $X, Y, Z \in \mathfrak{X}(M)$, the collection of all $C^{\infty}$-vector fields. In particular, the curvature 2-from (2.5) reduces to the Faraday tensor $F_{\mu \nu}=A_{\mu, \nu}-A_{\mu, \nu}$ for $G=U(1)$ an Abelian group,

$$
\Omega_{U}^{\omega}=d \omega_{U}:=\frac{1}{i} d\left(A_{\mu} d x^{\mu}\right)=\frac{1}{i} d A_{\mu} \wedge d x^{\mu}=\frac{1}{i}\left(A_{\nu, \mu}-A_{\mu, \nu}\right) d x^{\mu} \wedge d x^{\nu}
$$

where $\omega_{U}:=\sigma_{U}^{*} \omega:=\frac{1}{i} A_{\mu} d x^{\mu} \in \Lambda(U, \mathfrak{u}(1) \cong i \mathbb{R})$ is a local 1-form on $U \subseteq M$ pulled back by a local section $\sigma_{U}: U \subset M \rightarrow L(M)$. As for the particle force where $G=S U(2)$ is non-Abelian, the field strength (2.5) applies to see non-linear dependence on the gauge potential $\omega_{\mu}$ as in the case of PGT. We are ready for introducing torsion, which appears as another field strength in the PGT as we explain later.

Definition 8. (The canonical 1-form, the soldering form)
For a generalized observer $u \in L(M)$, define the canonical 1-form $\varphi \in \Lambda^{1}\left(L(M), \mathbb{R}^{1,3}\right)$ by $\varphi\left(X_{u}\right):=u^{-1}\left(\pi_{*}\left(X_{u}\right)\right)$, where $X \in T L(M)$ is a vector field on $L(M)$ and $u^{-1}: T_{x} M \rightarrow$ $\mathbb{R}^{1,3}$ is the inverse map of the linear isomorphism $u$. The restricted canonical 1-form on $F(M)$ is defined similarly on $F(M)$, which is denoted by the same notation $\varphi$.

In order to describe the interaction between the external symmetry of the spacetime and the internal symmetry of the gauge, a representation of $G$ is needed. Recall that a
representation of a group $G$ on a vector space $V$ is a group homomorphism $\rho: G \rightarrow G L(V)$ such that $\rho(g h)=\rho(g) \cdot \rho(h)$ for all $g, h \in G$. With a representation of a Lie group $G$, a fundamental object is defined:

Definition 9. (Basic differential forms)
Let $V$ be a vector space and $\bar{\Lambda}^{k}(P, V)$ be a space of basic differential $k$-forms on $P$ such that, for a given representation $\rho: G \rightarrow G L(V)$, it is both $G$-invariant and horizontal:

1. $\left(G\right.$-invariant) For $\alpha \in \bar{\Lambda}^{k}(P, V), X_{1}, \ldots, X_{k} \in T_{p} P, p \in P$, we have,

$$
\begin{equation*}
\alpha\left(R_{g_{*}} X_{1}, \ldots, R_{g_{*}} X_{k}\right)=\rho\left(g^{-1}\right) \cdot \alpha\left(X_{1}, \ldots, X_{k}\right) \quad \text { or } \quad R_{g}^{*} \alpha=\rho\left(g^{-1}\right) \cdot \alpha \tag{2.8}
\end{equation*}
$$

2. (horizontal) If one of $X_{1}, \ldots, X_{k}$ is vertical, then $\alpha\left(X_{1}, \ldots, X_{k}\right)=0$.
with the definitions above, immediately one finds a fact that the canonical 1-form $\varphi$ is a basic differential 1-form with respect to the representation $\rho: O(1,3) \rightarrow G L\left(\mathbb{R}^{1,3}\right)$. The exterior covariant derivative of basic differential forms are defined similarly:

Definition 10. (Exterior covariant derivative of $\bar{\Lambda}^{k}(P, V)$ )
Given a connection $\omega$ on $P$, with respect to the adjoint representation $V=\mathfrak{g}, \rho: G \rightarrow$ $G L(V)$ by $g \mapsto \rho(g)=\mathfrak{a d}_{g}$. We define

$$
\begin{equation*}
D^{\omega}: \bar{\Lambda}^{k}(P, V) \rightarrow \bar{\Lambda}^{k+1}(P, V), \text { by } \quad D^{\omega} \psi:=(d \psi)^{H} \tag{2.9}
\end{equation*}
$$

one can easily see that $D^{\omega} \psi$ satisfies (2.8) since

$$
R_{g}^{*} D^{\omega} \psi=R_{g}^{*}(d \psi)^{H}=\left(R_{g}^{*} d \psi\right)^{H}=\left(d R_{g}^{*} \psi\right)^{H}=\mathfrak{a d}_{g^{-1}} \cdot(d \psi)^{H}=\mathfrak{a d}_{g^{-1}} \cdot D^{\omega} \psi
$$

Similarly, the exterior covariant derivative of $\bar{\Lambda}^{k}(P, V)$ has another form.
Theorem 2. For $\psi \in \bar{\Lambda}^{k}(P, V)$, we have $D^{\omega} \psi=d \psi+\omega \dot{\wedge} \psi$.
where the notation $\psi \in \bar{\Lambda}^{k}(P, V)$ is defined as
$\alpha \dot{\wedge} \psi\left(X_{1}, \ldots, X_{j}, X_{j+1}, \ldots, X_{j+k}\right):=\frac{1}{j!k!} \sum_{\sigma} \rho_{*}\left(\alpha\left(X_{\sigma(1)}, \ldots, X_{\sigma(j)}\right)\right) \cdot \alpha\left(X_{\sigma(j+1)}, \ldots, X_{\sigma(j+k)}\right)$
for $\alpha \in \bar{\Lambda}^{j}(P, \mathfrak{g})$ and $\sigma$ is a permutation of $\{1, \ldots, j+k\}$. Finally, we may present the notion of torsion stemming from the canonical 1-form $\varphi$ of the frame bundle $F(M)$.

Definition 11. (Torsion 2-form)
The torsion 2-form of $\omega$ is defined by $\Theta^{\omega}:=D^{\omega} \varphi \in \bar{\Lambda}^{2}\left(F(M), \mathbb{R}^{1,3}\right)$, where $\varphi$ is the canonical 1-form on $F(M)$ and $\omega$ is a connection $\omega$ given on $F(M)$.

Therefore by Theorem. (2) one has

$$
\begin{equation*}
\Theta^{\omega}:=D^{\omega} \varphi=d \varphi+\omega \dot{\lambda} \varphi \tag{2.10}
\end{equation*}
$$

In particular, Riemannian geometry describing GR is of one special case
Theorem 3. (Levi-Civita connection)
On the orthonormal frame bundle $F(M)$, there exists a unique connection $\widetilde{\omega} \in$ $\Lambda^{1}(F(M), \mathfrak{s o}(1,3))$ such that $\widetilde{\omega}$ has vanishing torsion, $D^{\widetilde{\omega}} \varphi=0$, which is called the

## Levi-Civita connection.

The above bundle tools contain minimal geometric conception for us to understand PGT theory as a whole. From this viewpoint, GR can be reformulated on the frame bundle $\pi: F(M) \rightarrow M$. Below is a dictionary of terminology between general principal fibre bundles and gauge theories, see [53], [56], and 57]:

| Principal fibre bundle theory |  | Gauge theory |
| :---: | :---: | :---: |
| total space | $P$ | space of phase factors |
| base space | $M$ | spacetime |
| structure group | $G$ | gauge group |
| local section | $\sigma$ | local gauge |
| connection 1-form | $\omega$ | gauge potential |
| curvature 2-form | $\Omega$ | field strength |

With the bundle tools above, we can start to explore the Poincaré gauge theory of gravitation.

### 2.2 Poincaré gauge theory on the affine frame bundle $\mathbb{A}(M)$

PGT is, by definition, a gauge gravity theory of the Poincaré group $\mathcal{P}=\mathbb{R}^{1,3} \rtimes S O(1,3)$. Since we have seen that GR can be formulated by the frame bundle ( $F(M), \pi, M, S O(1,3), \widetilde{\omega})$,
one may ask what is the suitable bundle description for PGT? That is we intend to find a principal bundle $(P, \pi, M, G)$ with $G=\mathbb{R}^{1,3} \rtimes S O(1,3)$.

Of course, we may just assume that such a fibre bundle exists. However, we want to search for a natural construction like the frame bundle $L(M)$ in Definition (3). Recall that the Poincaré group $\mathcal{P}$ is the semidirect product of translations in Minkowski $\mathbb{R}^{1,3}$ and hyper-rotations $S O(1,3)$, defined by all rigid motion of the Minkowski spacetime $\mathbb{R}^{1,3}$. Analyzing the Poincaré group helps us to construct the PGT, [58].

Definition 12. (Affine space)
If $E$ is an affine space with a vector space $V$ as the space of translations. We require that $V$ acts freely and transitively on $E$. By this we mean:

- (freely) for $p \in E$ and $v \in V$, if $v+p=p \Leftrightarrow v=0$,
- (transitively) for all $p, q \in E$, there exists $v \in V$ such that $p+v=q$.

If $E_{1}, E_{2}$ are two affine spaces, $V_{1}$ and $V_{2}$ are their groups of translations respectively, then the map $f: E_{1} \rightarrow E_{2}$ is called an affine map if there exists a linear map $\beta: V_{1} \rightarrow V_{2}$ such that $f(v+p)=\beta_{f}(v)+f(p)$ for all $v \in V, p \in E$. In particular, if $E_{1}=E_{2}=E$ we define the affine group $G A(E)$ of $E$ by

$$
G A(E):=\{f: E \rightarrow E \mid f \text { is a bijective affine map }\}
$$

then we have an exact sequence from the above,

$$
\begin{equation*}
0 \longrightarrow V \xrightarrow{\alpha} G A(E) \xrightarrow{\beta} G L(V) \longrightarrow 1 \tag{2.11}
\end{equation*}
$$

in the sense that $\operatorname{Im} \alpha=\operatorname{Ker} \beta$, which indicates that $G A(E)$ splits into a semi-direct product $G A(E)=V \rtimes G L(V)$. Thus if we take $V=\mathbb{R}^{1,3}$, one obtains $G A\left(\mathbb{R}^{1,3}\right)=$ $\mathbb{R}^{1,3} \rtimes G L\left(\mathbb{R}^{1,3}\right) \supset \mathcal{P}$, all rigid motion of the Minkowski space $\mathbb{R}^{1,3}$, including Lorentz group $S O(1,3)$, parity transformation, time reversal and translation.

Let $M$ be the spacetime. If we regard $\mathbb{R}^{1,3}$ as an affine space $\mathbb{A}^{1,3}$ and also the tangent space $T_{x} M$ at $x \in M$ as another affine space $A_{x} M$ (tangent affine space). Then elements in $A_{x} M$ are of the form $\bar{u}=\left(p, u\left(e_{0}\right), \ldots, u\left(e_{3}\right)\right) \in A_{x} M$, where $p \in A_{x} M$, $\left\{e_{0}, \ldots, e_{3}\right\}$ is the standard basis for $\mathbb{R}^{1,3}$, with $u: \mathbb{R}^{1,3} \rightarrow T_{x} M$ a linear isomorphism such that $\left\{u\left(e_{0}\right), \ldots, u\left(e_{3}\right)\right\}$ forms a vector basis for $T_{x} M$. One can then identify an element $\bar{u} \in A_{x} M$ as an affine transformation $\widetilde{u}: \mathbb{A}^{1,3} \rightarrow A_{x} M$ by

$$
\widetilde{u}\left(0 ; e_{0}, \ldots, e_{3}\right):=\left(p, u\left(e_{0}\right), \ldots, u\left(e_{3}\right)\right) \in A_{x} M, \quad\left(\left(0 ; e_{0}, \ldots, e_{3}\right) \in \mathbb{A}^{1,3}\right)
$$

Denote $\mathbb{A}_{x} M:=\left\{\widetilde{u}=(p, u) \mid \widetilde{u}: \mathbb{A}^{1,3} \rightarrow A_{x} M\right.$, an affine transformation $\}$. Then there is a $1-1$ correspondence of $\bar{u} \in A_{x} M \stackrel{1-1}{\longleftrightarrow} \widetilde{u} \in \mathbb{A}_{x} M$ such that we can identity $A_{x} M \cong \mathbb{A}_{x} M$. As a comparison,

$$
\begin{aligned}
& \text { a linear frame } u \in F_{x} M \Leftrightarrow u: \mathbb{R}^{1,3} \rightarrow T_{x} M \quad(u \text { linear isomorphism }) \\
& \text { an affine frame } \widetilde{u} \in \mathbb{A}_{x} M \quad \Leftrightarrow \quad \widetilde{u}: \mathbb{A}^{1,3} \rightarrow A_{x} M \quad(\widetilde{u} \text { affine transformation })
\end{aligned}
$$

If we define the set $\mathbb{A}(M):=\bigcup_{x \in M} \mathbb{A}_{x} M$ and $R_{g}: \mathbb{A}(M) \rightarrow \mathbb{A}(M)$ the right action of the gauge group $G A\left(\mathbb{R}^{1,3}\right)=\left\{g=(A, \xi) \mid A \in G L\left(\mathbb{R}^{1,3}\right), \xi \in \mathbb{R}^{1,3}\right\}$ on $\mathbb{A}(M)$ by $R_{(A, \xi)}(p, u):=(p+u \cdot \xi, u \circ A)$, where $(p, u) \in \mathbb{A}_{x} M$ and $(A, \xi) \in G A\left(\mathbb{R}^{1,3}\right)$, and define the projection map $\widetilde{\pi}: \mathbb{A}(M) \rightarrow M$ by $\widetilde{\pi}(p, u):=x$ for all affine frames $(p, u)$ at $x \in M$. Then with proper differential structure on $\mathbb{A}(M)$, one can prove that $\left(\mathbb{A}(M), \tilde{\pi}, M, G A\left(\mathbb{R}^{1,3}\right)\right)$ forms a principal fibre bundle called the affine frame bundle, [20], 52]. Such is the living space of the PGT. Since we want to study the relationship between the PGT and GR, we should investigate the connection between $\mathbb{A}(M)$ and $L(M)$. Recall that we have the exact sequence (2.11) where

$$
\begin{gathered}
\alpha: \mathbb{R}^{1,3} \rightarrow G A\left(\mathbb{R}^{1,3}\right) \quad \text { by } \quad \alpha(\xi):=\left(\begin{array}{cc}
I_{4 \times 4} & \xi \\
0 & 1
\end{array}\right) \\
\beta: G A\left(\mathbb{R}^{1,3}\right) \rightarrow G L\left(\mathbb{R}^{1,3}\right) \quad \text { by } \quad \beta\left(\left(\begin{array}{cc}
A & \xi \\
0 & 1
\end{array}\right)\right):=A \\
\gamma: G L\left(\mathbb{R}^{1,3}\right) \rightarrow G A\left(\mathbb{R}^{1,3}\right) \quad \text { by } \quad \gamma(A):=\left(\begin{array}{cc}
A & 0 \\
0 & 1
\end{array}\right)
\end{gathered}
$$

and thus $\beta \circ \gamma=I d$ on $G L\left(\mathbb{R}^{1,3}\right)$. Corresponding to the homomorphisms $\alpha, \beta$, $\gamma$, we define a natural projection $\beta: \mathbb{A}(M) \rightarrow L(M)$ by $\beta(p, u):=u$ and $\gamma: L(M) \rightarrow \mathbb{A}(M)$ by $\gamma(u):=(0, u)$ so that $\beta \circ \gamma=I d$ on $L(M)$. A connection $\widetilde{\omega}$ defined on the affine frame bundle $\tilde{\pi}: \mathbb{A}(M) \rightarrow M$ is called a generalized affine connection. Taking $V=\mathbb{R}^{1,3}$ in the exact sequence 2.11) results in the splitting exact sequence of Lie algebras:

$$
\begin{equation*}
0 \longrightarrow \mathbb{R}^{1,3} \longrightarrow \mathfrak{g a}\left(\mathbb{R}^{1,3}\right) \longrightarrow \mathfrak{g l}\left(\mathbb{R}^{1,3}\right) \longrightarrow 1 \tag{2.12}
\end{equation*}
$$

so that we obtain a decomposition of the Lie algebra $\mathfrak{g a}\left(\mathbb{R}^{1,3}\right)=\mathfrak{g l}\left(\mathbb{R}^{1,3}\right) \oplus \mathbb{R}^{1,3}$.
Thus for a generalized affine connection $\widetilde{\omega}$, the pull-back $\gamma^{*} \widetilde{\omega} \in \Lambda^{1}\left(L(M), \mathfrak{g a}\left(\mathbb{R}^{1,3}\right)\right)$ can be decomposed into two 1 -forms

$$
\begin{equation*}
\gamma^{*} \widetilde{\omega}=\omega \oplus \psi \tag{2.13}
\end{equation*}
$$

according to their Lie-algebra value such that $\omega \in \Lambda^{1}\left(L(M), \mathfrak{g l}\left(\mathbb{R}^{1,3}\right)\right)$ and $\psi \in \bar{\Lambda}^{1}\left(L(M), \mathbb{R}^{1,3}\right)$. In fact, by a theorem in [52], it turns out that $\omega$ defines a connection on $L(M)$. While from another theorem of McInnes, there is a $1: 1$ correspondence between an affine connection and a linear connection

$$
\begin{equation*}
\widetilde{\omega} \stackrel{1-1}{\longrightarrow}(\omega, \psi) . \tag{2.14}
\end{equation*}
$$

In summary, one has
Theorem 4. (Kobayashiళ Nomizu, [52])
Let $\widetilde{\omega}$ be a generalized affine connection on $\left(\mathbb{A}(M), \widetilde{\pi}, M, G A\left(\mathbb{R}^{1,3}\right)\right)$ with the decomposition (2.13), then the affine curvature 2-form $\widetilde{\Omega}^{\omega} \in \bar{\Lambda}\left(\mathbb{A}(M), \mathfrak{g a}\left(\mathbb{R}^{1,3}\right)\right)$ has the decomposition

$$
\begin{equation*}
\gamma^{*} \widetilde{\Omega}^{\omega}=\Omega^{\omega}+D^{\omega} \psi \tag{2.15}
\end{equation*}
$$

where the covariant derivative $D^{\omega}$ is defined by $\omega$ on $L(M)$.

In particular, if a generalized affine connection $\widetilde{\omega}$ coincidentally has the form

$$
\begin{equation*}
\gamma^{*} \widetilde{\omega}=\omega \oplus \varphi \tag{2.16}
\end{equation*}
$$

such that the $\mathbb{R}^{1,3}$ Lie-algebra valued component $\varphi$ is the canonical 1-form in Definition. 8 , then we call such a connection $\widetilde{\omega}$ an affine connection (c.f. generalized affine connection). Thus by (2.15), we see that an affine connection $\widetilde{\omega}$ on $\mathbb{A}(M)$ yields

$$
\begin{equation*}
\gamma^{*} \widetilde{\Omega}^{\omega}=\Omega^{\omega}+\Theta^{\omega} \tag{2.17}
\end{equation*}
$$

$$
\text { affine curvature }=\text { spacetime curvature }+ \text { torsion }
$$

where $\Theta^{\omega}$ is the torsion form on $L(M)$ as given by $\Theta^{\omega}:=D^{\omega} \varphi$ in 2.10. This shows that on the affine frame bundle $\left(\mathbb{A}(M), \widetilde{\pi}, M, G A\left(\mathbb{R}^{1,3}\right), \widetilde{\omega}\right)$ where $\widetilde{\omega}$ is an affine connection, the curvature and torsion of spacetime on $M$ are united as one affine curvature $\widetilde{\Omega}^{\omega}$. In other words, the affine curvature $\widetilde{\Omega}^{\omega}$ splits into curvature $\Omega$ and torsion $\Theta^{\omega}$ on the (base) spacetime $M$. This is reminiscent of a quotation in [20]:
"This removes the objection that torsion is a feature of space-time structure which has no analogue in gauge theory."

To understand the above statement more clearly, we remark that, for example, one can define the curvature 2-form $\Omega^{\omega}$ of the $U(1)$-bundle ( $P, \pi, M, U(1), \omega=\frac{1}{i} A_{\mu} d x^{\mu}$ ) for Maxwell's electrodynamics, called the electromagnetic field strength $F=d \omega$. But one
cannot define the torsion of the electromagnetic potential $A_{\mu}$. Similarly, one cannot speak of the torsion of the $S U(n)$-gauge field. What makes gravity special is exactly the existence of the canonical 1-form $\varphi$ in $L(M)$, Definition 8. However, from the viewpoint of bundles torsion is nothing more than a byproduct of the specific group $\mathcal{P}$ used.

Here, one can raise a naïve question as a summary for this section:

## Why is Poincaré gauge gravity theory (PGT) naturally related to gravity with torsion?

The answer is the following:
Since the gauging of the Poincaré group $\mathcal{P}$ into gravity theory necessarily leads to the affine frame bundle $\left(\mathbb{A}(M), \widetilde{\pi}, M, G A\left(\mathbb{R}^{1,3}\right), \widetilde{\omega}\right)$, and an affine connection $\widetilde{\omega}$ on $\mathbb{A}(M)$ is pulled back naturally on the frame bundle $L(M)$ by $\gamma$ such that it results in the splitting (2.16) according to the Lie-algebra, and thus the affine curvature splits accordingly by (2.17), hence the torsion and the curvature on the spacetime.

In fact, if we require the PGT to be the bundle defined by $(\mathcal{P}(M), \widetilde{\pi}, M, \mathcal{P}, \widetilde{\omega})$ with structure group, called the Poincaré bundle as $\mathcal{P}$, then we will need to reduce from $\left(\mathbb{A}(M), \widetilde{\pi}, M, G A\left(\mathbb{R}^{1,3}\right)\right)$ to the Poincaré bundle. However such a reduction is a delicate issue, which is beyond our scope, see [20]. Thus in the following discussion, we assume the Poincaré bundle as $\mathcal{P}$ exists as a sub-bundle of $\left(\mathbb{A}(M), \widetilde{\pi}, M, G A\left(\mathbb{R}^{1,3}\right)\right)$

The above bundle construction clearly describes the gauge scenario for PGT. However, when we project from the total bundle space $\mathcal{P}(M)$ to the spacetime $M$, the resultant field strength we observe on $M$ is simply curvature and torsion. Thus for most PGT theorists, their actual computation does not involve the bundle formulation, except that it provides a clear account of gauging gravity. Rather, one can only regard the projected version of the spacetime with torsion and curvature, namely $(M, g, \nabla)$ with $\nabla$ a general connection compatible to $g$ on $M$, and such a local version on $M$ is what we introduce next.

### 2.3 Riemann-Cartan geometry

Given a semi-Riemannian spacetime $(M, g), \mathrm{GR}$ is formulated on the tuple $(M, g, \widetilde{\nabla})$ where $\widetilde{\nabla}$ is the Levi-Civita connection. If one considers a general connection $\nabla$, then $(M, g, \nabla)$ is called a metric-affine spacetime, which can be regarded as the most general spacetime under the framework of differentiable geometry. Here a general connection
$\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ on $M$ is an operator such that [54],

$$
\begin{align*}
\nabla_{f X+g Y} Z & =f \nabla_{X} Z+g \nabla_{Y} Z \\
\nabla_{X}(Y+Z) & =\nabla_{X} Y+\nabla_{X} Z  \tag{2.18}\\
\nabla_{X}(f Y) & =f \nabla_{X} Y+X(f) Y
\end{align*}
$$

where $X, Y, Z \in \mathfrak{X}(M), f, g \in C^{\infty}(M)$. With a connection, one can then define the torsion tensor and the curvature tensor by [54,

Definition 13. (Curvature tensor and torsion tensor)
The Riemann curvature tensor $R(\cdot, \cdot): \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ and the torsion tensor $T(\cdot, \cdot): \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ of $\nabla$ are defined by

$$
\begin{align*}
& R(X, Y) Z=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z \\
& T(X, Y) Z=\nabla_{X} Y-\nabla_{Y} X-[X, Y] \tag{2.19}
\end{align*}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$.
Notice that on a metric-affine spacetime $(M, g, \nabla)$ one does not have to place the condition of metric compatibility (or metricity) constrained by

$$
\begin{equation*}
d(g(X, Y))=g(\nabla X, Y)+g(X, \nabla Y) \tag{2.20}
\end{equation*}
$$

which is equivalent to a usual coordinate form $g_{\alpha \beta, \mu}=\Gamma_{\mu \alpha}{ }^{\nu} g_{\nu \beta}+\Gamma_{\mu \beta}{ }^{\nu} g_{\nu \alpha}$ by taking coordinate vector fields $X=\frac{\partial}{\partial x^{\alpha}}, Y=\frac{\partial}{\partial x^{\beta}}$, and $Z=\frac{\partial}{\partial x^{\mu}}$ in 2.20.

Definition 14. (Riemann-Cartan spacetime, underlying space of PGT)
The tuple $(M, g, \nabla)$ with a metric compatible (2.20) connection $\nabla$ on $M$ is called the Riemann-Cartan spacetime, which is the underlying space of PGT, [12].

The Definition (14) of PGT is equivalent to the version of bundle formulation. That is $(M, g, \nabla) \Leftrightarrow(\mathcal{P}(M), \widetilde{\pi}, M, \mathcal{P}, \widetilde{\omega})$. However, this requires some steps which we now explain.

Starting from the Poincaré bundle $(\mathcal{P}(M), \widetilde{\pi}, M, \mathcal{P}, \widetilde{\omega})$, to obtain curvature and torsion defined on $M$, one requires a local section.

Let $U \subseteq M$ be an open neighborhood in $M$ and $\sigma_{U}: U \rightarrow F(M)$ be a local (smooth) section on $U$, a $C^{\infty}$-map such that $\pi \circ \sigma_{U}=I d_{U}$. With the given local section $\sigma_{U}$ on $U \subseteq M$, equivalent to choosing an observer, we derive a locally defined curvature 2-form on $U$ from the Poincaré bundle $(\mathcal{P}(M), \widetilde{\pi}, M, \mathcal{P}, \widetilde{\omega})$ via

$$
\begin{align*}
& \sigma_{U}^{*} \Omega^{\omega} \stackrel{(2.5)}{=} \sigma_{U}^{*}(d \omega+\omega \dot{\wedge} \omega)=d\left(\sigma_{U}^{*} \omega\right)+\left(\sigma_{U}^{*} \omega\right) \dot{\wedge}\left(\sigma_{U}^{*} \omega\right) \\
& \sigma_{U}^{*} \Theta^{\omega} \stackrel{(2.10)}{=} \sigma_{U}^{*}(d \varphi+\omega \dot{\wedge} \varphi)=d\left(\sigma_{U}^{*} \varphi\right)+\left(\sigma_{U}^{*} \omega\right) \dot{\wedge}\left(\sigma_{U}^{*} \varphi\right) \tag{2.21}
\end{align*}
$$

if we denote $\Omega_{U}:=\sigma_{U}^{*} \Omega^{\omega} \in \Lambda^{2}(U, \mathfrak{s o}(1,3)), T_{U}:=\sigma_{U}^{*} \Theta^{\omega} \in \Lambda^{2}\left(U, \mathbb{R}^{1,3}\right), \omega_{U}:=\sigma_{U}^{*} \omega \in$ $\Lambda^{1}(U, \mathfrak{s o}(1,3))$, and $\vartheta_{U}:=\sigma_{U}^{*} \varphi \in \Lambda^{1}\left(U, \mathbb{R}^{1,3}\right)$, then we can rewrite the above as

$$
\begin{align*}
\Omega_{U} & =d \omega_{U}+\omega_{U} \wedge \omega_{U} \in \Lambda^{2}(U, \mathfrak{s o}(1,3))  \tag{2.22}\\
T_{U} & =d \vartheta_{U}+\omega_{U} \wedge \vartheta_{U} \in \Lambda^{2}\left(U, \mathbb{R}^{1,3}\right)
\end{align*}
$$

so that we return to the case in (2.6), where the Lie group is a matrix group. If we omit the sub-index $U$ for simplicity without confusion and write

$$
\begin{array}{ll}
\Omega=\Omega^{\beta \alpha} E_{\alpha \beta}, & T=T^{\alpha} \widetilde{e}_{\alpha},  \tag{2.23}\\
\omega=\omega^{\alpha \beta} E_{\alpha \beta}, & \vartheta=\vartheta^{\alpha} \widetilde{e}_{\alpha},
\end{array}
$$

where $\left\{\tilde{e}_{\alpha} \in \mathbb{R}^{1,3}\right\}$ are the standard basis of the Minkowski spacetime and $\left\{E_{\alpha \beta} \in \mathfrak{s o}(1,3)\right\}$ is a basis of the Lie algebra $\mathfrak{s o}(1,3)$ with $\alpha, \beta, \ldots=0,1,2,3$. Notice that since the Lie-algebra basis $E_{\alpha \beta}=-E_{\beta \alpha}$ is anti-symmetric so that $\Omega^{\beta \alpha}=-\Omega^{\alpha \beta}$ and $\omega^{\beta \alpha}=-\omega^{\alpha \beta}$ is anti-symmetric in the indices $\alpha$ and $\beta$. One also remarks that the indices of $\Omega_{\alpha \beta}, \omega_{\alpha}{ }^{\beta}, T^{\beta}$, $\ldots$. etc. have nothing to do with the spacetime index or coordinate. i.e, one cannot say $T^{\alpha}$ is a 1-tensor or $\omega_{\alpha}{ }^{\beta}$ is a 2 -tensor simply by judging from the number of indices.

With the basis expansion (2.23), one may further rewrite (2.22) with (2.6) in a usual form recognizable in several references [11], 12]

$$
\begin{align*}
\Omega^{\beta}{ }_{\alpha} & =\nabla \omega_{\alpha}{ }^{\beta}=d \omega_{\alpha}{ }^{\beta}+\omega_{\gamma}{ }^{\beta} \wedge \omega_{\alpha}{ }^{\gamma}=\frac{1}{2} R^{\beta}{ }_{\alpha \mu \nu} \vartheta^{\mu} \wedge \vartheta^{\nu},  \tag{2.24}\\
T^{\alpha} & =\nabla \vartheta^{\alpha}=d \vartheta^{\alpha}+\omega_{\beta}{ }^{\alpha} \wedge \vartheta^{\beta}=\frac{1}{2} T_{\mu \nu}{ }^{\alpha} \vartheta^{\mu} \wedge \vartheta^{\nu}
\end{align*}
$$

both the last equalities may be verified via the relationships:

$$
\begin{array}{lc}
\Omega_{\alpha}^{\beta}(X, Y)=\vartheta^{\beta}\left(R(X, Y) e_{\alpha}\right), & T^{\alpha}(X, Y)=\vartheta^{\alpha}(T(X, Y)), \quad \text { or }  \tag{2.25}\\
\Omega_{\beta \alpha}(X, Y)=g\left(R(X, Y) e_{\alpha}, e_{\beta}\right), & T_{\alpha}(X, Y)=g\left(\left(T(X, Y), e_{\alpha}\right)\right.
\end{array}
$$

where $e_{\alpha}=\sigma(x)\left(\widetilde{e}_{\alpha}\right) \in \mathfrak{X}(M)$ is an orthonormal basis (tetrad) induced by the section $\sigma$, $X, Y \in \mathfrak{X}(M)$ and $R\left(e_{\mu}, e_{\nu}\right) e_{\gamma}:=R^{\sigma}{ }_{\gamma \mu \nu} e_{\sigma}, T\left(e_{\mu}, e_{\nu}\right):=T_{\mu \nu}{ }^{\sigma} e_{\sigma}$ are defined by 2.19). One can also verify the following useful identities by direct computations, [52]:

$$
\begin{equation*}
\nabla T^{\alpha}=\Omega^{\alpha}{ }_{\beta} \wedge \vartheta^{\beta}, \quad\left(1^{s t} \text { Bianchi }\right), \quad \nabla \Omega^{\alpha \beta}=0, \quad\left(2^{\text {nd }} \text { Bianchi }\right) \tag{2.26}
\end{equation*}
$$

Also the requirement of the metricity condition 2.20 in a tetrad $e_{\alpha}$ is written as

$$
\begin{equation*}
0=\omega_{\alpha}{ }^{\mu} \eta_{\mu \beta}+\omega_{\beta}{ }^{\mu} \eta_{\mu \alpha}, \quad \omega_{\alpha \beta}=-\omega_{\beta \alpha} \tag{2.27}
\end{equation*}
$$

where $\eta_{\alpha \beta}:=g\left(e_{\alpha}, e_{\beta}\right)$ and in the last line we denote $\omega_{\alpha \beta}:=\omega_{\alpha}{ }^{\mu} \eta_{\mu \beta}$ for simplicity. Here we remark that the connection 1 -form $\omega_{\alpha}{ }^{\beta} \in \Lambda^{1}(M, \mathbb{R})$ is as equivalently defined by $\nabla e_{\alpha}:=\omega_{\alpha}{ }^{\beta} e_{\beta}$.

In summary, if we pullback from $\left(\mathbb{A}(M), \widetilde{\pi}, G A\left(\mathbb{R}^{1,3}\right), \widetilde{\omega}\right)$, the underlying space of PGT, to the spacetime $M$, the resultant elements are $\left(M, g, \nabla, \vartheta^{\alpha}\right)$ with $\varphi \Leftrightarrow \vartheta^{\alpha}$ a tetrad as the $\mathbb{R}^{1,3}$-gauge potential for translation and $\nabla \Leftrightarrow \omega_{\alpha}{ }^{\beta}$ as the $\mathfrak{s o}(1,3)$-gauge potential for rotations satisfying (2.27), called Riemann-Cartan spacetime, 9]. The two field strength torsion and curvature $\left(\Omega^{\alpha}{ }_{\beta}, T^{\beta}\right)$ of $\nabla$ are given by $(2.24)$. Below we write the geometric correspondence of the PGT in gauge theoretic language (c.f. [11], [12]):

| PGT | translational $\mathbb{R}^{1,3}$ | rotational $S O(1,3)$ |
| :---: | :---: | :---: |
| gauge potentials (on $L(M)$ ) | $\varphi$ | $\omega$ |
| gauge potentials (on $M$ ) | $\vartheta^{\alpha}$ | $\omega_{\alpha}{ }^{\beta}$ |
| field strengths (on $L(M)$ ) | $\Theta^{\omega}=D^{\omega} \varphi$ | $\Omega^{\omega}=D^{\omega} \omega$ |
| field strengths (on $M$ ) | $T^{\alpha}=\nabla \vartheta^{\alpha}$ | $\Omega^{\beta}{ }_{\alpha}=\nabla \omega_{\alpha}{ }^{\beta}$ |

### 2.3.1 Changes of frames

It is important to study changes of frames under different observers in spacetime. Some gauge quantities may change correspondingly under the transition and here we wish to give a clear view.

In the definition of a principal fibre bundle $P$, it is defined to have a local trivialization on $T_{U}: \pi^{-1}(U) \subseteq P \rightarrow U \times G$ such that $T_{U}(p)=\left(\pi(p), \psi_{U}(p)\right)$ with $\psi_{U}: \pi^{-1}(U) \rightarrow G$ satisfying $\psi_{U}(p g)=\psi_{U}(p) \cdot g$ for all $p \in \pi^{-1}(U), g \in G$.

One can then define the transition function on the overlap
Definition 15. (Transition function)
Let $U, V$ be two open sets in $M$ such that $U \cap V \neq \phi$, and let $T_{U}: \pi^{-1}(U) \rightarrow U \times G$ and $T_{V}: \pi^{-1}(V) \rightarrow V \times G$ be two local trivializations of a principal bundle $(P, \pi, M, G)$. Then the transition function from $T_{U}$ to $T_{V}$ is defined as the map $\Psi_{U V}: U \cap V \rightarrow P$ by

$$
\begin{equation*}
\Psi_{U V}(x):=\psi_{U}(p) \psi_{V}(p)^{-1} \tag{2.28}
\end{equation*}
$$

where $p \in P, x \in M$ such that $\pi(p)=x$ and $\psi_{V}(p)^{-1}$ denotes the group inverse, not the functional inverse.

The definition of the transition function is well-defined since if $p, q \in \pi^{-1}(U)(p=q g$ for some $g \in G$ ), then $\psi_{U}(q g) \psi_{V}(q g)^{-1}=\psi_{U}(q) \psi_{V}(q)^{-1}$. For the transition function it is easy to verify that it possesses the following properties

Proposition 1. The transition function $\Psi_{U V}$ satisfies,

$$
\begin{align*}
\Psi_{U U}(y) & =e, & & (\forall y \in U) \\
\Psi_{U V}(y) & =\Psi_{V U}^{-1}(y), & & (\forall y \in U \cap V)  \tag{2.29}\\
\Psi_{U V}(y) \cdot \Psi_{V W}(y) \cdot \Psi_{W U}(y) & =e & & (\forall y \in U \cap V \cap W)
\end{align*}
$$

The change of the two frames is equivalent to the transition between two sections.
Proposition 2. Let $T_{U}: \pi^{-1}(U) \rightarrow U \times G$ and $T_{V}: \pi^{-1}(V) \rightarrow V \times G$ be two local trivializations of $P$ as given above, one can define a section $\sigma_{U}: U \rightarrow P$ associated to the trivialization $T_{U}$ given by $\sigma_{U}(x):=T_{U}^{-1}(x, e)$, along with $\sigma_{V}: V \rightarrow P$ similarly defined. For $Y \in T_{x} M$, we have

$$
\begin{equation*}
\sigma_{V *}\left(Y_{x}\right)=\left[L_{\Psi_{U V}^{-1}(x) *}\left(\Psi_{U V *}\left(Y_{x}\right)\right)\right]_{\sigma_{V}(x)}^{*}+R_{\Psi_{U V}(x) *} \circ \sigma_{U *}\left(Y_{x}\right) \tag{2.30}
\end{equation*}
$$

Proof. First we notice that since $\sigma_{U}(x)=T_{U}^{-1}(x, e)$ we have $T_{U}\left(\sigma_{U}(x) g\right)=(x, g)$, in particular $T_{U}\left(\sigma_{U}(x)\right)=(x, e)=\left(\pi\left(\sigma_{U}(x)\right), \psi_{U}\left(\sigma_{U}(x)\right)\right)$. Thus we find $\psi_{U}\left(\sigma_{U}(x)\right)=e$ for all $x \in \pi^{-1}(U)$. Now let $\gamma: \mathbb{R} \rightarrow M$ be a curve such that $\gamma(0)=x$ and $\gamma^{\prime}(0)=Y$, then

$$
\begin{align*}
\sigma_{V *}\left(Y_{x}\right) & :=\left.\frac{d}{d t}\right|_{t=0} \sigma_{V}(\gamma(t))=\frac{d}{d t}\left[\sigma_{U}(\gamma(t)) \Psi_{U V}(\gamma(t))\right] \\
& =\left.\frac{d}{d t}\right|_{t=0}\left[\sigma_{U}(x) \Psi_{U V}(\gamma(t))\right]+\left.\frac{d}{d t}\right|_{t=0}\left[\sigma_{U}(\gamma(t)) \Psi_{U V}(x)\right]  \tag{2.31}\\
& =\left.\frac{d}{d t}\right|_{t=0}\left[\sigma_{V}(x) \Psi_{U V}^{-1}(x) \Psi_{U V}(\gamma(t))\right]+R_{\Psi_{U V}(x) *} \sigma_{U x}\left(Y_{x}\right)
\end{align*}
$$

where we have used $\Psi_{V U}(x)=\Psi_{U V}^{-1}(x)=\psi_{V}(x) \cdot \psi_{U}^{-1}(x)$. If we define a curve $\alpha$ in $G$ by $\alpha(t):=\Psi_{U V}^{-1}(x) \Psi_{U V}(\gamma(t))$, we find $\alpha(0)=e$ and

$$
\alpha^{\prime}(0):=\left.\frac{d}{d t}\right|_{t=0}\left(\Psi_{U V}^{-1}(x) \Psi_{U V}(\gamma(t))\right)=L_{\Psi_{U V}^{-1}(x) *}\left(\Psi_{U V *}\left(Y_{x}\right)\right) \in \mathfrak{g}
$$

Then we may write $\alpha(t)=e^{t A}$ with $A:=L_{\Psi_{U V}^{-1}(x) *}\left(\Psi_{U V *}\left(Y_{x}\right)\right) \in \mathfrak{g}$ and the first term in the last equality of 2.30 becomes

$$
\left.\frac{d}{d t}\right|_{t=0}\left(\sigma_{V}(x) \Psi_{U V}^{-1}(x) \Psi_{U V}(\gamma(t))\right)=\left.\frac{d}{d t}\right|_{t=0}\left(\sigma_{V}(x) \cdot e^{t A}\right):=\left[L_{\Psi_{U V}^{-1}(x) *}\left(\Psi_{U V *}\left(Y_{x}\right)\right)\right]_{\sigma_{V}(x)}^{*}
$$

where the last equality is due to (2.1).

With the proposition above, we have

Corollary 1. On the frame bundle $\left(L(M), \pi, M, G L\left(\mathbb{R}^{1,3}\right), \omega\right)$, let $\sigma_{U}: U \rightarrow L(M)$, $\sigma_{V}: V \rightarrow L(M)$ be two local sections corresponding to the local trivializations $T_{U}$ and $T_{V}$ respectively. For the canonical 1-form $\varphi \in \bar{\Lambda}^{1}\left(L(M), \mathbb{R}^{1,3}\right)$ and torsion 2-form $\Theta^{\omega} \in$ $\bar{\Lambda}^{2}\left(L(M), \mathbb{R}^{1,3}\right)$ with canonical representation $\rho: G L\left(\mathbb{R}^{1,3}\right) \rightarrow G L(4, \mathbb{R})$ and $Y_{x} \in T_{x} M$ one has following local forms $\sigma_{U}^{*} \varphi \in \Lambda^{1}\left(U, \mathbb{R}^{1,3}\right), \sigma_{U}^{*} \Theta^{\omega} \in \Lambda^{2}\left(U, \mathbb{R}^{1,3}\right), \sigma_{V}^{*} \varphi \in \Lambda^{1}\left(V, \mathbb{R}^{1,3}\right)$, and $\sigma_{V}^{*} \Theta^{\omega} \in \Lambda^{2}\left(V, \mathbb{R}^{1,3}\right)$ and they are related by

$$
\begin{align*}
&\left(\sigma_{V}^{*} \varphi\right)\left(Y_{x}\right)=\varphi\left(\sigma_{V *} Y_{x}\right)  \tag{2.32}\\
&\left(\sigma_{V}^{*} \Theta^{\omega}\right)\left(Y_{x}\right)=\varphi\left(\Psi_{U V}(x)\right)^{-1} \cdot\left(\sigma_{U}^{*} \varphi\right)\left(Y_{x}\right) \\
&=\left(\Psi_{U V}(x)\right)^{-1} \cdot\left(\sigma_{U}^{*} \Theta^{\omega}\right)\left(Y_{x}\right)
\end{align*}
$$

Proof. By applying the canonical 1-form $\varphi$ directly on 2.30

$$
\left(\sigma_{V}^{*} \varphi\right)\left(Y_{x}\right)=\varphi\left(\sigma_{V *} Y_{x}\right)=\varphi\left(R_{\Psi_{U V}(x) *} \sigma_{U x *}\left(Y_{x}\right)\right)=\left(\Psi_{U V}(x)\right)^{-1} \cdot\left(\sigma_{U}^{*} \varphi\right)\left(Y_{x}\right)
$$

where the two equalities are true because $\varphi$ is a basic differential 1-form in Def. (9) and similarly for the torsion 2-form.

We explain why the local form $\sigma^{*} \varphi \in \Lambda^{1}\left(U, \mathbb{R}^{1,3}\right)$ is important, simply because it gives a coframe $\vartheta^{\alpha} \in T M$ on $M$.

Remark 1. (Coframe on M)
Let $\sigma: U \subseteq M \rightarrow L(M)$ be a local section on $U$ and $\left(E_{\alpha}\right) \in \mathbb{R}^{1,3}, \alpha=0,1,2,3$, be the standard basis. Since $\sigma(x) \in L(M)$ for $x \in U$, denoting $\sigma(x)\left(E_{\alpha}\right):=e_{\alpha}(x) \in T_{x} U$, then $\left(e_{\alpha}\right) \in \mathfrak{X}(U)$ form a basis on $U$ and let the coframe $\left(\vartheta^{\alpha} \in T^{*} U\right)$ dual to $\left(e_{\alpha}\right)$ be such that $\vartheta^{\alpha}\left(e_{\beta}\right):=\delta_{\beta}^{\alpha}$. Therefore

$$
\begin{equation*}
\left(\sigma^{*} \varphi\right)\left(e_{\alpha}\right):=\varphi\left(\sigma_{*} e_{\alpha}\right):=(\sigma(x))^{-1} \cdot \pi_{*}\left(\sigma_{*} e_{\alpha}\right)=(\sigma(x))^{-1}\left(e_{\alpha}\right)=E_{\alpha}=\left(\vartheta^{\beta} \otimes E_{\beta}\right)\left(e_{\alpha}\right) \tag{2.33}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\sigma^{*} \varphi=\vartheta^{\beta} \otimes E_{\beta}, \quad\left(\sigma^{*} \varphi\right)^{\beta}=\vartheta^{\beta} \tag{2.34}
\end{equation*}
$$

This indicates that the pull back of the canonical 1-form $\varphi$ gives a coframe $\vartheta^{\alpha}$ corresponding to the given section $\sigma: U \rightarrow L(M)$.

By the remark, Corollary 1 then implies the transition of the two frames on $M$.

Remark 2. (Local transition)
Let $\left(e_{\alpha}\right) \in \mathfrak{X}(U)$ and $\left(\vartheta^{\alpha}:=\left(\sigma_{U}^{*} \varphi\right)^{\alpha}\right) \in \Lambda^{1}(U)$ be a set of frame and coframe on $U \subset M$ as defined above by a local section $\sigma_{U}: U \rightarrow L(M)$ and $\left(\widetilde{e}_{\alpha}\right) \in \mathfrak{X}(V)$ and $\left(\widetilde{\vartheta}^{\alpha}:=\left(\sigma_{V}^{*} \varphi\right)^{\alpha}\right) \in \Lambda^{1}(V)$ be another set of frame and coframe on $V \subset M$ similarly defined by a section $\sigma_{V}: V \rightarrow L(M)$. Also denote the transition function $\Psi_{U V}: U \cap V \rightarrow G L(1,3)$ in a matrix form $A_{\alpha}^{\beta}(x):=\left(\Psi_{U V}(x)\right)_{\alpha}^{\beta}$ for $x \in U \cap V$, then (2.32) states that if $\widetilde{e}_{\alpha}(x)=A_{\alpha}^{\beta}(x) e_{\beta}$, then the local tetrad and torsion 2-form follow the transformation

$$
\begin{equation*}
\widetilde{\vartheta}^{\beta}(x)=\left(A^{-1}\right)_{\alpha}^{\beta}(x) \vartheta^{\alpha}(x), \quad \widetilde{T}^{\beta}(x)=\left(A^{-1}\right)_{\alpha}^{\beta}(x) T^{\alpha}(x), \tag{2.35}
\end{equation*}
$$

where $T^{\alpha} \in \Lambda^{2}(U, \mathbb{R})$ are as defined in (2.21) and (2.22).
In fact, by (2.30) we also obtain a transformation law for a connection 1-form $\omega$ and curvature 2-form $\Omega^{\omega}$ on a general principal bundle ( $P, \pi, M, G, \omega$ ).

Proposition 3. With the same notations above and define $\omega_{U}:=\sigma_{U}^{*} \omega \in \Lambda^{1}(U, \mathfrak{g}), \omega_{V}:=$ $\sigma_{V}^{*} \omega \in \Lambda^{1}(V, \mathfrak{g})$, we have the transformation behavior between two local trivializations

$$
\begin{equation*}
\omega_{V}\left(Y_{x}\right)=L_{\Psi_{U V}(x) *}^{-1}\left(\Psi_{U V *}\left(Y_{x}\right)\right)+\mathfrak{a d}_{\Psi_{U V}^{-1}(x)}\left(\omega_{U}\left(Y_{x}\right)\right), \quad \forall Y_{x} \in T_{x} M \tag{2.36}
\end{equation*}
$$

in particular, when the Lie group $G$ is a matrix group we obtain a familiar form

$$
\begin{equation*}
\omega_{V}=\Psi_{U V}^{-1} \cdot d \Psi_{U V}+\Psi_{U V}^{-1} \cdot \omega_{U} \cdot \Psi_{U V} \tag{2.37}
\end{equation*}
$$

therefore one says that "a connection does not transform like a tensor" due to the extra piece $\Psi_{U V}^{-1} \cdot d \Psi_{U V}$.

The transformation of the local curvature 2-forms $\Omega_{U}^{\omega}:=\sigma_{U}^{*} \Omega^{\omega} \in \bar{\Lambda}^{2}(U, \mathfrak{g})$ and $\Omega_{V}^{\omega}:=$ $\sigma_{V}^{*} \Omega^{\omega} \in \bar{\Lambda}^{2}(V, \mathfrak{g})$ is

$$
\begin{equation*}
\Omega_{V}^{\omega}(x)=\mathfrak{a d}_{\Psi_{U V}^{-1}(x)}\left(\Omega_{U}^{\omega}(x)\right), \Omega_{V}^{\omega}=\Psi_{U V}^{-1} \cdot \Omega_{U}^{\omega} \cdot \Psi_{U V}, \quad(\text { if } G \text { is a matrix group }) \tag{2.38}
\end{equation*}
$$

One remarks that Proposition (3) is true for all principal bundles, while (2) is only defined on the frame bundle $L(M)$ over $M$.

### 2.3.2 Computational Aspects

So far we only setup the essential equipment for the PGT, but we have not yet specified the gravitational dynamics to follow, a gravitational Lagrangian. Thus a specific gravity

Lagrangian shall give us the unique evolution of the PGT spacetime. Before we turn to imposing meaningful Lagrangians on ( $M, g, \nabla$ ), first we demonstrate some computational skills.

Given a RC-spacetime $(M, g, \nabla)$ with a coframe $\vartheta^{\alpha}$ such that the connection 1-form $\nabla e_{\alpha}:=\omega_{\alpha}{ }^{\beta} e_{\beta}$. Then we can define the volume 4 -form on $M$

$$
\begin{equation*}
\eta:=\star 1=\vartheta^{0} \wedge \vartheta^{1} \wedge \vartheta^{2} \wedge \vartheta^{3}=\frac{1}{4!} \epsilon_{\alpha \beta \mu \nu} \vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \vartheta^{\mu} \wedge \vartheta^{\nu} \tag{2.39}
\end{equation*}
$$

by the star operator (Hodge dual) $\star: \Lambda^{k}(M) \rightarrow \Lambda^{4-k}(M)$. Using the Hodge dual, we define a convenient basis called $\eta$-basis for $\Lambda(M):=\oplus_{k} \Lambda^{k}(M)$, see [59]-62], [11], which is a vector space.

$$
\begin{align*}
\eta_{\alpha} & :=\star \vartheta_{\alpha}, & & (3 \text {-form }) \\
\eta_{\alpha \beta} & :=\star\left(\vartheta_{\alpha} \wedge \vartheta_{\beta}\right), & & (1 \text {-form })  \tag{2.40}\\
\eta_{\alpha \beta \gamma} & :=\star\left(\vartheta_{\alpha} \wedge \vartheta_{\beta} \wedge \vartheta_{\gamma}\right), & & (0 \text {-form }) \\
\eta_{\alpha \beta \gamma \delta} & :=\star\left(\vartheta_{\alpha} \wedge \vartheta_{\beta} \wedge \vartheta_{\gamma} \wedge \vartheta_{\delta}\right), & & (0)
\end{align*}
$$

where $\vartheta_{\alpha}:=g_{\alpha \beta} \vartheta^{\beta}$, and each of which has $\operatorname{dim}\{\eta\}=\binom{4}{0}, \operatorname{dim}\left\{\eta_{\alpha}\right\}=\binom{4}{1}, \operatorname{dim}\left\{\eta_{\alpha \beta}\right\}=\binom{4}{2}$, $\operatorname{dim}\left\{\eta_{\alpha \beta \gamma}\right\}=\binom{4}{3}, \operatorname{dim}\left\{\eta_{\alpha \beta \gamma \delta}\right\}=\binom{4}{4}$. This is to be compared to the $\vartheta$-basis, that also forms a basis for each $\Lambda^{k}(M)$.

$$
\underbrace{1}_{0 \text {-form }}, \quad \underbrace{\vartheta^{\alpha}}_{\text {1-form }}, \underbrace{\vartheta^{\alpha} \wedge \vartheta^{\beta}}_{\text {2-form }}, \quad \underbrace{\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \vartheta^{\gamma}}_{3 \text {-form }}, \quad \underbrace{\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \vartheta^{\gamma} \wedge \vartheta^{\delta}}_{\text {4-form }}
$$

In fact, one can verify that the $\eta$-basis can be written in another form,

$$
\begin{align*}
\eta_{\alpha} & :=i_{e_{\alpha}} \eta, \quad \eta_{\alpha \beta}:=i_{e_{\beta}} \eta_{\alpha},  \tag{2.41}\\
\eta_{\alpha \beta \gamma} & :=i_{e_{\gamma}} \eta_{\alpha \beta}, \quad \eta_{\alpha \beta \gamma \delta}:=i_{e_{\delta}} \eta_{\alpha \beta \gamma}
\end{align*}
$$

where $i_{V}: \Lambda^{k}(M) \rightarrow \Lambda^{k-1}(M)$ denotes the interior product by $\left(i_{V} \alpha\right)\left(X_{1}, \cdots, X_{k-1}\right):=$ $\alpha\left(V, X_{1}, \ldots, X_{k-1}\right)$ for a given vector (field) $V \in T_{M}$ with $\alpha \in \Lambda^{k}(M)$, and $X_{1}, \ldots, X_{k-1} \in$ $\mathfrak{X}(M)$. In some of the literature, it is also denoted by symbol $V\rfloor \alpha:=i_{V} \alpha$, where we take both interchangeably.

The $\eta$-basis is useful because of the following identities that help to reduce computation.

## Proposition 4.

$$
\begin{align*}
\vartheta^{\alpha} \wedge \eta_{\beta} & =\delta_{\beta}^{\alpha} \eta \\
\vartheta^{\alpha} \wedge \eta_{\beta \gamma} & =\delta_{\gamma}^{\alpha} \eta_{\beta}-\delta_{\beta}^{\alpha} \eta_{\gamma}  \tag{2.42}\\
\vartheta^{\alpha} \wedge \eta_{\beta \gamma \sigma} & =\delta_{\beta}^{\alpha} \eta_{\gamma \sigma}+\delta_{\gamma}^{\alpha} \eta_{\sigma \beta}+\delta_{\sigma}^{\alpha} \eta_{\beta \gamma} \\
\vartheta^{\alpha} \wedge \eta_{\beta \gamma \mu \nu} & =\delta_{\nu}^{\alpha} \eta_{\beta \gamma \mu}-\delta_{\mu}^{\alpha} \eta_{\nu \beta \gamma}+\delta_{\gamma}^{\alpha} \eta_{\mu \nu \beta}-\delta_{\beta}^{\alpha} \eta_{\gamma \mu \nu}
\end{align*}
$$

Proof. There are at least three ways to compute the first identity: the first is to utilize
Lemma 1. For $\psi, \phi \in \Lambda^{k}(M)$, we have $\langle\psi, \phi\rangle=-\star(\phi \wedge \star \psi)$.
A second way is compute it directly

$$
\vartheta^{\alpha} \wedge \star \vartheta_{\beta}=\vartheta^{\alpha} \wedge\left(\frac{1}{3!} \varepsilon_{\beta \mu \nu \gamma} \vartheta^{\mu} \wedge \vartheta^{\nu} \wedge \vartheta^{\gamma}\right)=\frac{1}{3!} \varepsilon_{\beta \mu \nu \gamma} \varepsilon^{\alpha \mu \nu \gamma} \eta=\delta_{\beta}^{\alpha} \eta
$$

or a third way by considering

$$
0=i_{e_{\beta}}\left(\vartheta^{\alpha} \wedge \eta\right)=i_{e_{\beta}}\left(\vartheta^{\alpha}\right) \wedge \eta-\vartheta^{\alpha} \wedge i_{e_{\beta}}(\eta)
$$

due to $\vartheta^{\alpha} \wedge \eta$ being a 5 -form, which vanishes identically on 4 -dimensional $M$. The second identity can be proved by considering

$$
i_{e_{\gamma}}\left(\vartheta^{\alpha} \wedge \eta_{\beta}\right)=\delta_{\gamma}^{\alpha} \eta_{\beta}-\vartheta^{\alpha} \wedge\left(i_{r_{\gamma}}\left(\eta_{\beta}\right)\right)
$$

where $i_{e_{\gamma}}\left(\eta_{\beta}\right)=\eta_{\beta \gamma}$ by using (2.41). The third then follows a similar iteration trick.
The following covariant derivatives of the $\eta$-basis are also useful in computation.

## Proposition 5.

$$
\begin{equation*}
D \eta_{\alpha}=\eta_{\alpha \gamma} \wedge T^{\gamma}, \quad D \eta_{\alpha \beta}=\eta_{\alpha \beta \gamma} \wedge T^{\gamma}, \quad D \eta_{\alpha \beta \mu}=\eta_{\alpha \beta \mu \gamma} T^{\gamma} \tag{2.43}
\end{equation*}
$$

with $D \eta=0$ and $D \eta_{\alpha \beta \mu \nu}=0$.
Proof. We only prove the second, the rest are similar. Since

$$
D \eta_{\alpha \beta}=\frac{1}{2!} D\left(\varepsilon_{\alpha \beta \mu \nu} \vartheta^{\mu} \wedge \vartheta^{\nu}\right)=\varepsilon_{\alpha \beta \mu \nu} T^{\mu} \wedge \vartheta^{\nu}=T^{\mu} \wedge \star\left(\vartheta_{\alpha} \wedge \vartheta_{\beta} \wedge \vartheta_{\mu}\right)
$$

With the convenient notations above, next we introduce the decomposition of the field strengths.

Under the local Lorentz group $S O(1,3)$, the torsion and the curvature are decomposed into 3 and 6 irreducible pieces respectively, see [11], 63]. The torsion tensor of 24 independent components $T_{\mu \nu}{ }^{\alpha}$ is decomposed into

$$
\begin{align*}
T^{\alpha} & ={ }^{(1)} T^{\alpha}+{ }^{(2)} T^{\alpha}+{ }^{(3)} T^{\alpha} \quad \in \Lambda^{2}(M)  \tag{2.44}\\
(24) & =(16)+(4)+(4)
\end{align*}
$$

where

$$
\begin{align*}
& { }^{(1)} T^{\alpha}:=T^{\alpha}-{ }^{(2)} T^{\alpha}-{ }^{(3)} T^{\alpha}, \quad \text { (tensor) } \\
& { }^{(2)} T^{\alpha}:=\frac{1}{3} \vartheta^{\alpha} \wedge i_{e \beta}\left(T^{\beta}\right), \quad \text { (vector) }  \tag{2.45}\\
& { }^{(3)} T^{\alpha}:=\frac{1}{3} \star\left(\star\left(\vartheta_{\alpha} \wedge T^{\alpha}\right) \wedge \vartheta^{\alpha}\right), \quad \text { (axial vector) }
\end{align*}
$$

and the curvature of 36 independent components $R^{\gamma}{ }_{\alpha \beta \mu}$ is decomposed according to

$$
\begin{equation*}
\Omega^{\alpha \beta}=\sum_{I=1}^{6}{ }^{(I)} \Omega^{\alpha \beta} \quad \in \Lambda^{2}(M) \tag{2.46}
\end{equation*}
$$

where

$$
\begin{align*}
{ }^{(1)} \Omega_{\alpha \beta} & :=\Omega_{\alpha \beta}-\sum_{I=2}^{6}{ }^{(I)} \Omega_{\alpha \beta} \\
{ }^{(2)} \Omega_{\alpha \beta} & :=(-1) \star\left(\vartheta_{[\alpha} \wedge \Psi_{\beta]}\right) \\
{ }^{(3)} \Omega_{\alpha \beta} & :=-\frac{1}{12} \star\left(X \wedge \vartheta_{\alpha} \wedge \vartheta_{\beta}\right)  \tag{2.47}\\
{ }^{(4)} \Omega_{\alpha \beta} & :=(-1) \vartheta_{[\alpha} \wedge \Phi_{\beta]} \\
{ }^{(5)} \Omega_{\alpha \beta} & \left.:=-\frac{1}{2} \vartheta_{[\alpha} \wedge e_{\beta]}\right\rfloor\left(\vartheta^{\gamma} \wedge R i c_{\gamma}\right) \\
{ }^{(6)} \Omega_{\alpha \beta} & :=-\frac{1}{12} R \vartheta_{\alpha} \wedge \vartheta_{\beta}
\end{align*}
$$

with

$$
\begin{align*}
\operatorname{Ric}_{\alpha} & \left.\left.\left.:=e_{\beta}\right\rfloor \Omega^{\beta}{ }_{\alpha}, \quad R:=e_{\alpha}\right\rfloor \operatorname{Ric}^{\alpha}, \quad X^{\alpha}:=\star\left(\Omega^{\alpha \beta} \wedge \vartheta_{\beta}\right), \quad X:=e_{\alpha}\right\rfloor X^{\alpha}, \\
\Psi_{\alpha} & \left.\left.:=X_{\alpha}-\frac{1}{4} \vartheta_{\alpha} \wedge X-\frac{1}{2} e_{\alpha}\right\rfloor\left(\vartheta^{\beta} \wedge X_{\beta}\right), \quad \Phi_{\alpha}:=\operatorname{Ric}_{\alpha}-\frac{1}{4} R \vartheta_{\alpha}-\frac{1}{2} e_{\alpha}\right\rfloor\left(\vartheta^{\beta} \wedge \operatorname{Ric}_{\beta}\right) \tag{2.48}
\end{align*}
$$

where the first term, called the Ricci 1-form is recognized as $\left.\operatorname{Ric}_{\alpha}:=e_{\beta}\right\rfloor \Omega^{\beta}{ }_{\alpha}=R_{\alpha \mu} \vartheta^{\mu}$ with Ricci tensor coefficients. In general $R_{\alpha \beta} \neq R_{\beta \alpha}$ in the PGT and more generally in RiemannCartan geometry (independent of any physics) due to the presence of torsion, and thus it has 16 independent components. The rest of the 5 irreducible pieces have their own properties, see [11]. Thus we see from above that if $T^{\alpha}=0$, then ${ }^{(2)} \Omega^{\alpha \beta}={ }^{(3)} \Omega^{\alpha \beta}={ }^{(5)} \Omega^{\alpha \beta}=0$, which recovers the Riemannian space $V_{4}$ describing GR.

We also define a convenient quantity to compare the connection of PGT $\omega_{\alpha}{ }^{\beta}$ to the Riemannian one $\widetilde{\omega}_{\alpha}{ }^{\beta}$, the contortion 1-form $K_{\alpha}{ }^{\beta}:=\widetilde{\omega}_{\alpha}{ }^{\beta}-\omega_{\alpha}{ }^{\beta}$. Then we can write $T^{\alpha}=K^{\alpha}{ }_{\beta} \wedge \vartheta^{\beta}$. In fact, the metricity condition (2.27) imposes that $K_{(\alpha \beta)} \equiv 0$. It turns out that one can solve the contortion 1-form in terms of torsion

$$
\begin{equation*}
K_{\alpha \beta}=i_{e_{[\alpha}} T_{\beta]}-\frac{1}{2} i_{e_{\alpha}}\left(i_{e_{\beta}} T_{\gamma}\right) \vartheta^{\gamma} \tag{2.49}
\end{equation*}
$$

The differential form formalism, compared to the components formalism, usually can be avoid much of the complicated computation. We conclude this section with an important example:

## Example 1. (Gravity Lagrangian)

Let the gravity Lagrangian be of the 4 -form $\mathcal{L}_{G}:=\frac{1}{2 \kappa} \Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}$, we want to translate it into a component form. Expand $\Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}=\left(\frac{1}{2} R^{\alpha \beta}{ }_{\mu \nu} \vartheta^{\mu} \wedge \vartheta^{\nu}\right) \wedge \eta_{\alpha \beta}$ and using (2.42) repeatedly one obtains $\frac{1}{2} R^{\alpha \beta}{ }_{\mu \nu}\left(\delta_{\beta}^{\nu} \delta_{\alpha}^{\mu}-\delta_{\alpha}^{\nu} \delta_{\beta}^{\mu}\right) \eta=R \eta$. Hence we conclude that the HilbertEinstein Lagrangian of $G R$ can be written into the 4-form on $M$, given by [59].

$$
\begin{equation*}
\mathcal{L}_{G}:=\frac{1}{2 \kappa} \Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}=\frac{1}{2 \kappa} R \sqrt{-g} d^{4} x \tag{2.50}
\end{equation*}
$$

where $\kappa=8 \pi G / c^{4}$.

## 2.4 the Einstein-Cartan theory

The Einstein-Cartan theory is a special case of Poincaré gauge gravity theory, and can be considered as a generalization of GR with torsion.

Definition 16. (Einstein-Cartan Theory)
The Einstein-Cartan theory is defined by the tuple $(M, g, \nabla)$ with the gravity action $\mathcal{L}_{E C}:=\frac{1}{2 \kappa} \Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}$, where $\nabla$ is a metric-compatible connection.

In fact, the connection $\omega_{\alpha}{ }^{\beta}$ in 2.50 need not be restricted to that of Levi-Civita. Hence although the Einstein-Cartan theory and GR share the same form of the action, they have a different geometry in general, especially when spin matter appears. Recall that in GR the metric is coupled to the energy-momentum tensor $\mathcal{T}_{i j}$. In the Einstin-Cartan theory the metric (or coframe) is coupled to the canonical energy-momentum tensor (of the Noether type), i.e,

$$
\begin{equation*}
\mathcal{T}_{i j}:=g_{i j} \mathcal{L}_{\mathrm{M}}-\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial\left(\phi^{a ; i}\right)} \phi_{; j}^{a} \tag{2.51}
\end{equation*}
$$

where $\mathcal{L}_{\mathrm{M}}=\mathcal{L}_{\mathrm{M}}\left(\phi^{a}, D_{i} \phi^{a}\right)$ is the matter Lagrangian of the matter field $\phi^{a}$, the index $a=1, \ldots, k$ denotes the vector components of the matter field, and the indices $i, j, \ldots=$ $0,1,2,3$ denote the coordinate indices of the spacetime. Note that in general $\mathcal{T}_{i j} \neq \mathcal{T}_{j i}$. The reason for such an asymmetry is due to the presence of torsion, which shall be explained later.

The torsion tensor in the Einstein-Cartan theory is coupled to the spin current tensor of a matter field $\phi$ defined by [11]

$$
\begin{equation*}
\mathcal{S}_{i j}{ }^{k}=\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}}\right) \rho_{[i j] b}{ }^{a} \cdot \phi^{b}=-\mathcal{S}_{j i}{ }^{k} \tag{2.52}
\end{equation*}
$$

where $\rho_{[i j] b}{ }^{a}:=\left(\rho_{*} E_{i j}\right)_{b}^{a}$ is a Lie-algebra representation of $\rho: S O(1,3) \rightarrow G L(V)$ and $\left\{E_{i j} \in \mathfrak{s o}(1,3)\right\}$ is a basis of $\mathfrak{s o}(1,3)$ (for details see A.10). The spin current 2.52) comes from the conserved current of the internal $S O(1,3)$ symmetry, see Appendix. A and couples to torsion. Thus in PGT the spin does play a role in the spacetime evolution, unlike GR. However, in terms of differential forms the canonical energy-momentum tensor and the spin-current of matter are found to be simpler expressed by the 3 -forms, see [11], [59], [64]:

$$
\begin{equation*}
\mathcal{T}_{\alpha}=\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \vartheta^{\alpha}}, \quad \mathcal{S}_{\alpha \beta}=\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \omega^{\alpha \beta}} \tag{2.53}
\end{equation*}
$$

where more specific explanation will be given later.
The field equations for the Einstein-Cartan theory are given in component form by [18], 19,

$$
\begin{align*}
R_{i j}-\frac{1}{2} g_{i j} R & =\kappa \mathcal{T}_{i j}  \tag{2.54}\\
T_{i j}^{k}+\delta_{i}^{k} T_{j l}^{l}-\delta_{j}^{k} T_{i l}^{l} & =\kappa \mathcal{S}_{i j}{ }^{k}
\end{align*}
$$

One can also write the (Einstein-Cartan)-Sciama-Kibble equation in a simpler form. Before the derivation, we first observe a property that

Theorem 5. If the spin current of matter vanishes, $\mathcal{S}_{i j}{ }^{k} \equiv 0$, then the Einstein-Cartan theory reduces to GR. Hence $G R$ is a degenerate case of the Einstein-Cartan.

Proof. If $\mathcal{S}_{i j}{ }^{k} \equiv 0$, then from $2.54{ }_{2}$ contracting the indices $j$ and $k$, one has $T_{i l}{ }^{l} \equiv 0$, which indicates that in fact the whole torsion tensor $T_{i j}{ }^{k} \equiv 0$ again by $(2.54)_{2}$, and thus recovers GR.

Thus in the Einstein-Cartan theory without spin matter the second equation of (2.54) is simply null. Now we derive (2.54) by using the differential forms formalism.

Theorem 6. (Sciama-Kibble equation)
In differential forms, the Einstein-Cartan field equations are written as

$$
\begin{align*}
& \frac{1}{2} \eta_{\alpha \beta \gamma} \wedge \Omega^{\gamma \beta}=\kappa \mathcal{T}_{\alpha} \\
& \frac{1}{2} \eta_{a \beta \gamma} \wedge T^{\gamma}=\kappa \mathcal{S}_{\alpha \beta} \tag{2.55}
\end{align*}
$$

Proof. Since the Einstein-Cartan theory is a special case of PGT, the two gauge potentials $\left(\vartheta^{a}, \omega_{\alpha}^{\beta}\right)$ are considered as independent variables. Gravitational variation is of the form

$$
\begin{equation*}
\delta \mathcal{L}_{G}=\delta \vartheta^{\alpha} \wedge \frac{\delta \mathcal{L}_{G}}{\delta \vartheta^{\alpha}}+\delta \omega^{\alpha \beta} \wedge \frac{\delta \mathcal{L}_{G}}{\delta \omega^{\alpha \beta}}+d(\cdots) \tag{2.56}
\end{equation*}
$$

where $d(\cdots)$ is some exact differential term which vanishes upon the integration over a closed 4-manifold by Stoke's theorem. Similarly,

$$
\begin{equation*}
\delta \mathcal{L}_{\mathrm{M}}=\delta \vartheta^{\alpha} \wedge \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \vartheta^{\alpha}}+\delta \omega^{\alpha \beta} \wedge \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \omega^{\alpha \beta}}+d(\cdots) \tag{2.57}
\end{equation*}
$$

Consider the total variation

$$
\begin{equation*}
\delta\left(\eta_{\alpha \beta} \wedge \Omega^{\alpha \beta}\right)=\delta \eta_{\alpha \beta} \wedge \Omega^{\alpha \beta}+\eta_{\alpha \beta} \wedge \delta \Omega^{\alpha \beta} \tag{2.58}
\end{equation*}
$$

where the variational formula $\delta(\psi \wedge \phi)=\delta \psi \wedge \phi+\psi \wedge \delta \phi$ is used for all $\psi \in \Lambda^{p}(M)$, $\phi \in \Lambda^{q}(M)$ and since

$$
\begin{align*}
\delta \eta_{\alpha \beta} & =\frac{1}{2!} \delta\left(\varepsilon_{\alpha \beta \mu \nu} \vartheta^{\mu} \wedge \vartheta^{\nu}\right)=\delta \vartheta^{\mu} \wedge \eta_{\alpha \beta \mu} \\
\delta \Omega^{\alpha \beta} & =d \delta \omega^{\alpha \beta}+\delta \omega^{\gamma \beta} \wedge \omega^{\alpha}{ }_{\gamma}+\omega^{\gamma \beta} \wedge \delta \omega^{\alpha}{ }_{\gamma}  \tag{2.59}\\
& =d \delta \omega^{\alpha \beta}+\omega_{\gamma}{ }^{\alpha} \wedge \delta \omega^{\gamma \beta}+\omega_{\gamma}{ }^{\beta} \wedge \delta \omega^{\alpha \gamma}=D \delta \omega^{\alpha \beta} .
\end{align*}
$$

then (2.58) reads

$$
\begin{align*}
\delta\left(\eta_{\alpha \beta} \wedge \Omega^{\alpha \beta}\right) & =\delta \vartheta^{\mu} \wedge\left(\eta_{\alpha \beta \mu} \wedge \Omega^{\alpha \beta}\right)+\eta_{\alpha \beta} \wedge D \delta \omega^{\alpha \beta} \\
& =\delta \vartheta^{\mu} \wedge\left(\eta_{\alpha \beta \mu} \wedge \Omega^{\alpha \beta}\right)+D\left(\eta_{\alpha \beta} \wedge \delta \omega^{\alpha \beta}\right)-\left(D \eta^{\alpha \beta}\right) \wedge\left(\delta \eta_{\alpha \beta}\right)  \tag{2.60}\\
& =\delta \vartheta^{\mu} \wedge\left(\eta_{\alpha \beta \mu} \wedge \Omega^{\alpha \beta}\right)+\delta \omega^{\alpha \beta} \wedge\left(T^{\gamma} \wedge \eta_{\alpha \beta \gamma}\right)+D\left(\eta_{\alpha \beta} \wedge \delta \omega^{\alpha \beta}\right)
\end{align*}
$$

where we have used $D\left(\eta_{\alpha \beta} \wedge \delta \omega^{\alpha \beta}\right)=D \eta_{\alpha \beta} \wedge \delta \omega^{\alpha \beta}+\eta_{\alpha \beta} \wedge D \delta \omega^{\alpha \beta}$ in the second equality and the identity $D \eta_{\alpha \beta}=T^{\gamma} \wedge \eta_{\alpha \beta \gamma}$ from (2.43) is used in the last equality. Since the last term $D\left(\eta_{\alpha \beta} \wedge \delta \omega^{\alpha \beta}\right)=d\left(\eta_{\alpha \beta} \wedge \delta \omega^{\alpha \beta}\right)$ in 2.60 vanishes upon integration, by comparison to 2.56) one derives the field equations:

$$
\begin{align*}
& \frac{\delta \mathcal{L}_{E C}}{\delta \vartheta^{\mu}}=\frac{1}{2 \kappa} \eta_{\mu \alpha \beta} \wedge \Omega^{\alpha \beta}=-\mathcal{T}_{\mu} \\
& \frac{\delta \mathcal{L}_{E C}}{\delta \omega^{\alpha \beta}}=\frac{1}{2 \kappa} \eta_{a \beta \gamma} \wedge T^{\gamma}=\mathcal{S}_{\alpha \beta} \tag{2.61}
\end{align*}
$$

Remark 3. Here we remark that the variational formula $2.59{ }_{1}$ can be derived rigorously from the formula in Eq.(33) of [65], namely

Theorem 7. For all $\phi \in \Lambda^{p}(M)$, and an arbitrary frame $\left(e_{\alpha}\right) \in T M$, one has

$$
\begin{equation*}
(\delta \cdot \star-\star \cdot \delta) \phi=\delta \vartheta^{\alpha} \wedge\left(i_{e_{\alpha}}(\star \phi)\right)-\star\left[\delta \vartheta^{\gamma} \wedge i_{e_{\gamma}} \phi\right]+\delta g_{\alpha \beta}\left[\vartheta^{(\alpha} \wedge\left(i_{e^{\beta)}}(\star \phi)\right)-\frac{1}{2} g^{\alpha \beta} \star \phi\right] \tag{2.62}
\end{equation*}
$$

which indicates that the Hodge dual operator does not commute with the variational operator $\delta$ in general. To obtain 2.59 1, one applies $\phi=\vartheta^{\alpha} \wedge \vartheta^{\beta}$ with the orthonormal frame condition such that $\delta g_{\alpha \beta} \equiv 0$.

Here we remark that the translation from (2.55) to (2.54) is direct as the following: for 2.551

$$
\begin{equation*}
\frac{1}{2} \eta_{\alpha \beta \gamma} \wedge \Omega^{\gamma \beta}=\frac{1}{4} \eta_{\alpha \beta \gamma} \wedge\left(R_{\mu \nu}^{\gamma \beta} \vartheta^{\mu} \wedge \vartheta^{\nu}\right)=\frac{1}{2}\left(R_{\beta \alpha}-\frac{1}{2} g_{\alpha \beta} R\right) \eta^{\beta}=\kappa \mathcal{T}_{\alpha \beta} \eta^{\beta} \tag{2.63}
\end{equation*}
$$

where we first expand $\Omega^{\gamma \beta}$ with 2.24 and apply 2.42 repeatedly. Notice in the last equality we have defined the coefficients of the 3 -form expansion $\mathcal{T}_{\alpha}$ in terms of the 3 -form basis $\eta^{\alpha}$ as the energy-momentum tensor in the usual sense. The coefficient $\mathcal{S}_{\alpha \beta \gamma}$ called spin current tensor is similarly defined i.e,

$$
\begin{equation*}
\mathcal{T}_{\alpha}:=\mathcal{T}_{\alpha \beta} \eta^{\beta}, \quad \mathcal{S}_{\alpha \beta}=\mathcal{S}_{\alpha \beta \gamma} \eta^{\gamma} \tag{2.64}
\end{equation*}
$$

Then we see from this expansion that in general $\mathcal{T}_{\alpha \beta} \neq \mathcal{T}_{\beta \alpha}$ and $\mathcal{S}_{\alpha \beta \gamma} \neq \mathcal{S}_{\alpha \gamma \beta}$. The translation for $(2.55)_{2}$ is similar.

As we have seen from the proof of Theorem (5) in $(2.54)_{2}$ that the relationship between torsion and spin currents is algebraic ${ }^{3}$ Basically, the Einstein-Cartan field equations (2.55) are of $1^{\text {st }}$ order partial differential equations (PDEs) in $\left(\vartheta^{\alpha}, \omega_{\alpha}{ }^{\beta}\right)$ if it is supported by a spin fluid source, [9], [66]. With a detailed examination of the field equations, in the end one finds that the torsion field $T^{\alpha}=D \vartheta^{\alpha}$ is not a dynamical field, and hence not propagating. This is unlike the typical Yang-Mills gauge theory, $\mathcal{L}_{\mathrm{YM}} \sim F \wedge F$, whose field equations are generally of $2^{\text {nd }}$ order PDEs in field variables. Thus from the gauge theoretical point of view, EC is a degenerate theory. Following the essence of Yang-Mills, Hehl, Nitsch, and Von der Heyde [67] constructed a more general framework for PGT that in addition to the linear scalar curvature $R$ in the gravity action (Einstein-Cartan), quadratic terms of $\left(\Omega^{\beta}{ }_{\alpha}, T^{\alpha}\right)$ in the Lagrangian should be also in consideration, hence the introduction of quadratic PGT, [26], 67].

[^1]
### 2.5 Quadratic PGT (qPG)

### 2.5.1 Field equations for general PGT

On a Riemann-Cartan spacetime $(M, g, \nabla)$, we consider a general Lagrangian of the form

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}\left(g_{\alpha \beta}, \vartheta^{\alpha}, d \vartheta^{\alpha}, \omega_{\alpha}^{\beta}, d \omega_{\alpha}^{\beta}, \phi^{a}, d \phi^{a}\right) \tag{2.65}
\end{equation*}
$$

which is a total Lagrangian containing both gravity and matter field of $\phi \in \Lambda^{p}(U \subseteq M, V)$ and the index $a$ denotes the component in the vector space $V$. However, if we require $\mathcal{L}$ to have the Lorentz symmetry, i.e, under the transformation of two frames at $x \in M$

$$
\begin{equation*}
\tilde{e}_{\alpha}(x)=(A(x))_{\alpha}^{\beta} \vartheta^{\beta}(x), \quad \widetilde{\vartheta}^{\alpha}=\left(A^{-1}(x)\right)_{\beta}^{\alpha} \vartheta^{\beta}(x) \tag{2.66}
\end{equation*}
$$

where $A=\left(A_{\alpha}^{\beta}\right): U \subseteq M \rightarrow S O(1,3)$, the Lagrangian $\mathcal{L}$ has an invariant value. Then $d$ and $\omega_{\alpha}{ }^{\beta}$ in $\mathcal{L}$ can only be involved with certain combination such that under (2.66) the Lorentz symmetry is preserved. In fact, it can only be the combination of the exterior covariant derivative given in Definition (10) or Theorem (2)

$$
D \phi=d \phi+\omega \dot{\wedge} \phi, \quad \Leftrightarrow \quad D=d \phi+\rho_{\beta}^{\alpha} \omega_{\alpha}^{\beta} \wedge \phi
$$

where $\rho: S O(1,3) \rightarrow G L(V)$. Therefore
Proposition 6. ( $S O(1,3)$-invariant Lagrangian)
By imposing the Lorentz symmetry, $G=S O(1,3)$, the general Lagrangian (2.65) is reduced down to the form

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}\left(g_{\alpha \beta}, \vartheta^{\alpha}, \phi^{a}, D \phi^{a}, T^{\alpha}, \Omega^{\beta}{ }_{\alpha}\right) \tag{2.67}
\end{equation*}
$$

One defines the variation of the independent arguments

$$
\begin{equation*}
\delta \mathcal{L}=\delta \vartheta^{\alpha} \wedge \frac{\partial \mathcal{L}}{\partial \vartheta^{\alpha}}+\delta T^{\alpha} \wedge \frac{\partial \mathcal{L}}{\partial T^{\alpha}}+\delta \Omega^{\beta}{ }_{\alpha} \wedge \frac{\partial \mathcal{L}}{\partial \Omega^{\beta}{ }_{\alpha}}+\delta \phi^{a} \wedge \frac{\partial \mathcal{L}}{\partial \phi^{a}}+\delta\left(D \phi^{a}\right) \wedge \frac{\partial \mathcal{L}}{\partial\left(D \phi^{a}\right)} \tag{2.68}
\end{equation*}
$$

Moreover, if we replace the variations of $\delta T^{\alpha}$ and $\delta \Omega^{\beta}{ }_{\alpha}$ by

$$
\begin{equation*}
\delta T^{\alpha}=D\left(\delta \vartheta^{\alpha}\right)+\delta \omega_{\beta}{ }^{\alpha} \wedge \vartheta^{\beta}, \quad \delta \Omega^{\beta}{ }_{\alpha}=D \delta \omega_{\alpha}{ }^{\beta} \tag{2.69}
\end{equation*}
$$

where the identities are derived by applying the relation $[\delta, d]=0$. As a consequence, one has

$$
\begin{align*}
\delta \mathcal{L}=\delta \vartheta^{\alpha} \wedge\left[\frac{\partial \mathcal{L}}{\partial \vartheta^{\alpha}}\right. & \left.+D\left(\frac{\partial \mathcal{L}}{\partial T^{\alpha}}\right)\right]+\delta \omega^{\alpha \beta} \wedge\left[\rho_{\alpha \beta}{ }^{a}{ }_{b} \phi^{b} \wedge \frac{\partial \mathcal{L}}{\partial\left(D \phi^{a}\right)}+\vartheta_{[\alpha} \wedge \frac{\partial \mathcal{L}}{\partial T^{\beta]}}+D\left(\frac{\partial \mathcal{L}}{\partial \Omega^{\beta \alpha}}\right)\right] \\
& +\delta \phi^{a} \wedge \frac{\delta \mathcal{L}}{\delta \phi^{a}}+d\left(\delta \vartheta^{\alpha} \wedge \frac{\partial \mathcal{L}}{\partial T^{\alpha}}+\delta \omega_{\alpha}{ }^{\beta} \wedge \frac{\partial \mathcal{L}}{\partial \Omega^{\beta}{ }_{\alpha}}+\delta \phi^{a} \wedge \frac{\partial \mathcal{L}}{\partial D \phi^{a}}\right) \tag{2.70}
\end{align*}
$$

where $\rho_{\alpha \beta}{ }^{a}{ }_{b}$ is defined in A.10 and $D \delta \vartheta^{\alpha} \wedge \frac{\partial \mathcal{L}}{\partial T^{\alpha}}=D\left(\delta \vartheta^{\alpha} \wedge \frac{\partial \mathcal{L}}{\partial T^{\alpha}}\right)+\delta \vartheta^{\alpha} \wedge D\left(\frac{\partial \mathcal{L}}{\partial T^{\alpha}}\right)$ is used. By the comparison of (2.68) to (2.70), we obtain [11]

$$
\begin{align*}
\frac{\delta \mathcal{L}}{\delta \vartheta^{\alpha}} & =\frac{\partial \mathcal{L}}{\partial \vartheta^{\alpha}}+D\left(\frac{\partial \mathcal{L}}{\partial T^{\alpha}}\right) \\
\frac{\delta \mathcal{L}}{\delta \omega^{\beta \alpha}} & =\rho_{\alpha \beta}{ }^{a}{ }_{b} \phi^{b} \wedge \frac{\partial \mathcal{L}}{\partial\left(D \phi^{a}\right)}+\vartheta_{[\alpha} \wedge \frac{\partial \mathcal{L}}{\partial T^{\beta]}}+D\left(\frac{\partial \mathcal{L}}{\partial \Omega^{\beta \alpha}}\right) \tag{2.71}
\end{align*}
$$

In particular, if the general Lagrangian (2.65) is decomposable into the form

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}_{G}\left(\vartheta^{\alpha}, T^{\alpha}, \Omega_{\alpha}{ }^{\beta}\right)+\mathcal{L}_{M}\left(\vartheta^{\alpha}, \phi^{a}, D \phi^{a}\right) \tag{2.72}
\end{equation*}
$$

where $\mathcal{L}_{G}$ denotes the gravitational Lagrangian in PGT and $\mathcal{L}_{M}$ is the matter Lagrangian. (2.70) simply tells us

Theorem 8. (Field equations for PGT)
The field equations for the Lagrangian (2.72) is given by

$$
\begin{align*}
\frac{\partial \mathcal{L}_{G}}{\partial \vartheta^{\alpha}}+D\left(\frac{\partial \mathcal{L}_{G}}{\partial T^{\alpha}}\right) & =\mathcal{T}_{\alpha} \\
\vartheta_{[\alpha} \wedge \frac{\partial \mathcal{L}_{G}}{\partial T^{\beta]}}+D\left(\frac{\partial \mathcal{L}_{G}}{\partial \Omega^{\beta \alpha}}\right) & =\mathcal{S}_{\alpha \beta} \tag{2.73}
\end{align*}
$$

where $\mathcal{T}_{\alpha} \in \Lambda^{3}(M)$ and $\mathcal{S}_{\alpha \beta} \in \Lambda^{3}(M)$ are the (canonical) energy-momentum current and spin-current of matter defined by [11]

$$
\begin{align*}
\mathcal{T}_{\alpha} & :=\frac{\delta \mathcal{L}_{M}}{\delta \vartheta^{\alpha}}=\frac{\partial \mathcal{L}_{M}}{\partial \vartheta^{\alpha}}+D\left(\frac{\partial \mathcal{L}_{M}}{\partial T^{\alpha}}\right) \\
\mathcal{S}_{\alpha \beta} & :=\frac{\delta \mathcal{L}_{M}}{\delta \omega^{[\alpha \beta]}}=\rho_{\alpha \beta}{ }^{a}{ }_{b} \phi^{b} \wedge \frac{\partial \mathcal{L}}{\partial\left(D \phi^{a}\right)}+\vartheta_{[\alpha} \wedge \frac{\partial \mathcal{L}_{M}}{\partial T^{\beta]}}+D\left(\frac{\partial \mathcal{L}_{M}}{\partial \Omega^{\beta \alpha}}\right), \tag{2.74}
\end{align*}
$$

with the evolution of the matter field given by

$$
\begin{equation*}
\frac{\delta \mathcal{L}_{M}}{\delta \phi}=\frac{\partial \mathcal{L}_{M}}{\partial \phi}-D\left(\frac{\partial \mathcal{L}_{M}}{\partial(D \phi)}\right)=0 . \tag{2.75}
\end{equation*}
$$

Together (2.73) and (2.75) constitute the complete evolution of PGT in the presence of the matter field $\phi$.

In fact, computing terms like $\frac{\partial \mathcal{L}_{G}}{\partial \vartheta^{\alpha}}, \frac{\partial \mathcal{L}_{G}}{\partial T^{\alpha}}, \ldots$ in 2.73 require more work. The following theorem can make computation easier, see [11]

Theorem 9. (Field equations for PGT)
The field equations for the Lagrangian (2.72) are given by

$$
\begin{equation*}
D H_{\alpha}-t_{\alpha}=\mathcal{T}_{\alpha}, \quad D H_{\alpha \beta}-s_{\alpha \beta}=\mathcal{S}_{\alpha \beta} \tag{2.76}
\end{equation*}
$$

where $H_{\alpha} \in \Lambda^{2}(M)$ and $H_{\alpha \beta} \in \Lambda^{2}(M)$ are called the translational excitation and the Lorentz excitation (field momenta), defined by

$$
\begin{equation*}
H_{\alpha}:=-\frac{\partial \mathcal{L}_{G}}{\partial T^{\alpha}}, \quad H_{\alpha \beta}:=-\frac{\partial \mathcal{L}_{G}}{\partial \Omega^{\alpha \beta}} \tag{2.77}
\end{equation*}
$$

with $t_{\alpha} \in \Lambda^{3}(M)$ and $s_{\alpha \beta} \in \Lambda^{3}(M)$ called the gravitational energy-momentum and the gravitational spin current, respectively, given defined by

$$
\begin{align*}
t_{\alpha} & :=\frac{\partial \mathcal{L}_{G}}{\partial \vartheta^{\alpha}}=i_{e_{\alpha}} \mathcal{L}_{G}+\left(i_{e_{\alpha}} T^{\beta}\right) \wedge H_{\beta}+\left(i_{e_{\alpha}} \Omega^{\gamma \beta}\right) \wedge H_{\beta \gamma},  \tag{2.78}\\
s_{\alpha \beta} & :=\frac{\partial \mathcal{L}_{G}}{\partial \omega^{\alpha \beta}}=-\vartheta_{[\alpha} \wedge H_{\beta]}
\end{align*}
$$

Both the last equalities in (2.78) require some non-trivial procedures concerning the local diffeomorphisms and local Lorentz invariance, see [11]. With this theorem, one easily derives the Einstein-Cartan theory (2.55) alternatively.

Indeed, with the Einstein-Cartan Lagrangian (16), one computes that the gravitational excitations from 2.77) and (2.78)

$$
\begin{equation*}
H_{\alpha}:=-\frac{\partial \mathcal{L}_{G}}{\partial T^{\alpha}}=0, \quad H_{\alpha \beta}:=-\frac{\partial \mathcal{L}_{G}}{\partial \Omega^{\alpha \beta}}=\eta_{\alpha \beta} \tag{2.79}
\end{equation*}
$$

and the gravitational currents

$$
\begin{equation*}
t_{\alpha}=\Omega^{\beta \gamma} \wedge \eta_{\beta \gamma \alpha}, \quad s_{\alpha \beta}=0 \tag{2.80}
\end{equation*}
$$

which recover 2.55.

### 2.5.2 Quadratic PGT Lagrangians

In the last two sections, ones sees that the Einstein-Cartan theory as a Poincaré gauge theory is in many ways degenerate. Unlike Yang-Mills, there is a self-interaction of the form $F \wedge \star F$, quadratic in the field strength. Naturally, one would expect that gravity as a gauge theory may have analogous terms like $T \wedge \star T$ and $\Omega \wedge \star \Omega$. Using the irreducible decomposition of field strengths for gravity in PGT, (2.44)-(2.48) one can consider a general Lagrangian quadratic in PGT field strengths,

$$
\begin{align*}
& \mathcal{L}_{q P G}=\frac{1}{2 \kappa}\left[a_{0} \Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}-2 \Lambda \eta+T^{\alpha} \wedge\left(\sum_{I=1}^{3} a_{I} \star^{(I)} T_{\alpha}\right)\right] \\
&-\frac{1}{2 \varrho} \Omega^{\alpha \beta} \wedge\left(\sum_{I=1}^{6} b_{I} \star^{(I)} \Omega_{\alpha \beta}\right) \tag{2.81}
\end{align*}
$$

where the self-coupling constants $a_{0}, \ldots, a_{3}, b_{1}, \ldots, b_{6}$ are defined to be dimensionless, $[\kappa]=T^{2} / M L$, and $[\varrho]=T / M L^{2}$. In [35], the above quadratic combinations in (2.81) are classified into 2 types,:

$$
\begin{align*}
\mathcal{L}_{\text {weak }}^{+} & =\frac{1}{2 \kappa}\left[a_{0} \Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}-2 \Lambda \eta+T^{\alpha} \wedge\left(\sum_{I=1}^{3} a_{I} \star^{(I)} T_{\alpha}\right)\right] \\
\mathcal{L}_{\text {strong }}^{+} & =-\frac{1}{2 \varrho} \Omega^{\alpha \beta} \wedge\left(\sum_{I=1}^{6} b_{I} \star^{(I)} \Omega_{\alpha \beta}\right) \tag{2.82}
\end{align*}
$$

where both $\mathcal{L}_{\text {weak }}^{+}$and $\mathcal{L}_{\text {strong }}^{+}$belong to the parity even pieces, which are easily observed by the presence of Hodge dual operator. In fact, in [35] they considered a wider class of quadratic combinations including terms like

$$
\begin{align*}
\mathcal{L}_{\text {weak }}^{-} & =\frac{b_{0}}{2 \kappa}{ }^{(3)} \Omega_{\alpha \beta} \wedge \vartheta^{\alpha} \wedge \vartheta^{\beta}+\frac{1}{\kappa}\left(\sigma_{1}{ }^{(1)} T^{\alpha} \wedge{ }^{(1)} T_{\alpha}+\sigma_{2}{ }^{(2)} T^{\alpha} \wedge{ }^{(3)} T_{\alpha}\right) \\
\mathcal{L}_{\text {strong }}^{-} & =-\frac{1}{2 \varrho}\left(\mu_{1}{ }^{(1)} \Omega^{\alpha \beta} \wedge{ }^{(1)} \Omega_{\alpha \beta}+\mu_{2}{ }^{(2)} \Omega^{\alpha \beta} \wedge{ }^{(4)} \Omega_{\alpha \beta}+\mu_{3}{ }^{(3)} \Omega^{\alpha \beta} \wedge{ }^{(6)} \Omega_{\alpha \beta}+\mu_{4}{ }^{(5)} \Omega^{\alpha \beta} \wedge{ }^{(5)} \Omega_{\alpha \beta}\right) \tag{2.83}
\end{align*}
$$

which are considered as parity odd terms. Also it is shown in 35] that the most general quadratic combinations can be written as

$$
\begin{align*}
\mathcal{L}_{\mathrm{qPG}}= & \frac{1}{2 \kappa}\left[\left(a_{0} R-2 \Lambda+b_{0} X\right) \eta\right. \\
& \left.+\frac{a_{2}}{3} \mathcal{V} \wedge \star \mathcal{V}-\frac{a_{3}}{3} \mathcal{A} \wedge \star \mathcal{A}-\frac{2 \sigma_{2}}{3} \mathcal{V} \wedge \star \mathcal{A}+a_{1}{ }^{(1)} T^{\alpha} \wedge \star^{(1)} T_{\alpha}\right] \\
& -\frac{1}{2 \varrho}\left[\left(\frac{b_{6}}{12} R^{2}-\frac{b_{3}}{12} X^{2}+\frac{\mu_{3}}{12} R X\right) \eta+b_{4}{ }^{(4)} \Omega^{\alpha \beta} \wedge \star^{(4)} \Omega_{\alpha \beta}\right. \\
& \left.+{ }^{(2)} \Omega^{\alpha \beta} \wedge\left(b_{2} \star{ }^{(2)} \Omega_{\alpha \beta}+\mu_{2}{ }^{(4)} \Omega_{\alpha \beta}\right)+{ }^{(5)} \Omega^{\alpha \beta} \wedge\left(b_{5} \star^{(5)} \Omega_{\alpha \beta}+\mu_{4}{ }^{(5)} \Omega_{\alpha \beta}\right)\right] \tag{2.84}
\end{align*}
$$

where $\mathcal{V}:=i_{e_{\gamma}}\left(T^{\gamma}\right)$ and $\mathcal{A}:=\star\left(\vartheta_{\alpha} \wedge T^{\alpha}\right)$ denote the vector and axial parts of the torsion tensor, respectively. It has up to 8 independent variables in the Lagrangian, but where do the physics of these terms lie? This is a question to be thoroughly explored. However, recently, Shie, Nester, Yo (SNY, [25]) have chosen a small subclass out of (2.84) by the Hamiltonian analysis of Poincaré gauge gravity [68], 69]. The SNY-model is simple but physically interesting, which we explore in the next chapter.

In this chapter we have gone through the basic formulation of PGT in principal fibre bundle language, which is suitable for the general gauge theory as well. Such a formulation helps the transition from the typical Yang-Mills theory to gravity gauge theory and shows their many similarities. Later, we project the PGT gauge potentials and strengths $\left(\varphi, \omega, \Omega^{\omega}, \Theta^{\omega}\right)$ onto the spacetime $\left(\vartheta^{\alpha}, \omega_{\alpha}{ }^{\beta}, \Omega^{\alpha \beta}, T^{\alpha}\right)$ via a local section $\sigma: U \subset M \rightarrow P$.

Such reduction addresses that the Poincaré gauge group invariance results in the field strength of torsion and curvature one observes on the spacetime $M$, and clearly address why we obtain torsion and curvature simultaneously when gauging the Poincaré group into the spacetime. The resultant spacetime is called the Riemann-Cartan spacetime, described by $(M, g, \nabla)$. Thus this is the version that appears most in the physics literature.

We also studied a simplest Lagrangian, $\mathcal{L}_{G}=\frac{1}{2 \kappa} \Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}$ 2.50 on a Riemann-Cartan spacetime, known as the Einstein-Cartan theory developed around 1961 by Sciama and Kibble. We also sketched the quadratic PGT as a natural extension mimicking the YangMills theory. It can be seen that PGT has a more general structure than GR by its nature, and thus contains more variety and possibility.

Below we show a diagram, Fig. (2.5.2), that may best represent the relationship between different geometries to conclude this chapter.


Figure 2.1: Riemann-Cartan (RC) space and its subcases.

## 3

## Scalar-torsion Mode of PGT

### 3.1 Lagrangian for the scalar-torsion mode

As shown in Chater 2, qPG gravity consists of a Riemann-Cartan spacetime ( $M, g, \nabla$ ) and a Lagrangian 4-form

$$
\begin{equation*}
\mathcal{L}(g, \vartheta, \Gamma)=\mathcal{L}_{G}+\mathcal{L}_{M}, \tag{3.1}
\end{equation*}
$$

where $\vartheta^{\alpha}$ is a set of tetrad, $\omega_{\alpha}{ }^{\beta} \in \Lambda^{1}(M)$ is the connection 1-form with respect to $\vartheta^{\alpha}, \mathcal{L}_{M}$ is the matter Lagrangian, and $\mathcal{L}_{G}$ is the gravitational Lagrangian that can be made up by certain combinations. In [25], SNY studied the spin $0^{+}$mode, given by [25, 38]

$$
\begin{equation*}
\mathcal{L}_{G}=\frac{a_{0}}{2} R \eta+\frac{b}{24} R^{2} \eta+\frac{a_{1}}{8} T^{\alpha} \wedge \star\left({ }^{(1)} T_{\alpha}-2^{(2)} T_{\alpha}-\frac{1}{2}{ }^{(3)} T_{\alpha}\right), \tag{3.2}
\end{equation*}
$$

where ${ }^{(I)} T^{\alpha}$ are irreducible torsion pieces in (5.19), and the coefficients of $\mathcal{L}_{G}$ in (3.2) are constrained by the positivity argument [25] such that

$$
\begin{equation*}
a_{1}>0, \quad b>0 . \tag{3.3}
\end{equation*}
$$

### 3.2 Cosmology in Scalar-torsion Mode of PGT

To study the cosmology of the scalar-torsion gravity in PGT, we consider the homogeneous and isotropic FLRW universe, given by the metric

$$
\begin{equation*}
d s^{2}=-d t^{2}+a^{2}(t)\left(\frac{d r^{2}}{1-k r^{2}}+r^{2} d \Omega^{2}\right) \tag{3.4}
\end{equation*}
$$

where $k$ is the sectional curvature of the spatial homogeneous universe and we set $k=0$ for simplicity.

For (3.2) of the SNY model in the FRLW cosmology with no spin source $S_{i j k} \equiv 0$, defined in (2.52), the field equations of PGT (2.76) lead to [25]

$$
\begin{align*}
\dot{H} & =\frac{\mu}{6 a_{1}} R+\frac{1}{6 a_{1}} \mathcal{T}-2 H^{2}  \tag{3.5}\\
\dot{\Phi}(t) & =\frac{a_{0}}{2 a_{1}} R+\frac{\mathcal{T}}{2 a_{1}}-3 H \Phi+\frac{1}{3} \Phi^{2}  \tag{3.6}\\
\dot{R} & =-\frac{2}{3}\left(R+\frac{6 \mu}{b}\right) \Phi \tag{3.7}
\end{align*}
$$

where $\mu=a_{1}+a_{0}, H=\dot{a}(t) / a(t)$, and $\Phi(t)=T_{t}$, which is the time component of the torsion trace, defined by $T_{i}:=T_{i j}{ }^{j}$, and the coordinate indices $i, j, k, \ldots$ run from $0, \ldots 3$. Here, $R$ in (3.5)-(3.7) denotes the affine curvature in (2.48). In addition, we have the relation

$$
\begin{equation*}
R=\bar{R}+2 T_{; j}^{j}-\frac{2}{3} T_{k} T^{k} \tag{3.8}
\end{equation*}
$$

where $\bar{R}=6\left(\dot{H}+2 H^{2}\right)$ represents the curvature of the Levi-Civita connection induced by (3.4). The energy-momentum tensor $\mathcal{T}_{i j}$ is defined as (2.64) and $\mathcal{T}$ stands for the trace $\mathcal{T}_{i}{ }^{i}$. Explicitly, one has

$$
\begin{align*}
\mathcal{T}_{t t} & =\rho_{M}=\frac{b}{18}\left(R+\frac{6 \mu}{b}\right)(3 H-\Phi)^{2}-\frac{b}{24} R^{2}-3 a_{1} H^{2}  \tag{3.9}\\
\mathcal{T} & =3 p_{M}-\rho_{M}
\end{align*}
$$

with the subscript $M$ representing the ordinary matter including both dust and radiation. To see the geometric effect of torsion, we can write down the Friedmann equations as

$$
\begin{array}{rlrl}
H^{2} & =\frac{\rho_{c}}{3 a_{0}}, & \rho_{c}=\rho_{M}+\rho_{T} \\
\dot{H} & =-\frac{\rho_{c}+p_{t o t}}{3 a_{0}}, & & p_{t o t}=p_{M}+p_{T} \tag{3.10}
\end{array}
$$

with $a_{0}=(8 \pi G)^{-1}$ in GR, where $\rho_{c}$ and $p_{\text {tot }}$ denote the critical energy density and total pressure of the universe, while $\rho_{T}$ and $p_{T}$ correspond to the energy density and pressure of some effective field, respectively. By comparing the equation of motion of the scalar-torsion mode in PGT (3.44) to the Friedmann equations (3.10), one obtains

$$
\begin{align*}
\rho_{T} & =3 \mu H^{2}-\frac{b}{18}\left(R+\frac{6 \mu}{b}\right)(3 H-\Phi)^{2}+\frac{b}{24} R^{2} \\
p_{T} & =\frac{1}{3}\left(\mu(R-\bar{R})+\rho_{T}\right) \tag{3.11}
\end{align*}
$$

which will be regarded as the torsion dark energy density and pressure, respectively.

$$
\begin{equation*}
\dot{\rho}_{c}+3 H\left(\rho_{c}+p_{t o t}\right)=0, \tag{3.12}
\end{equation*}
$$

which can also be derived by applying the identity

$$
\bar{\nabla}_{j} \bar{G}^{i j}=\bar{\nabla}_{j}\left(\bar{R}^{i j}-\frac{1}{2} \bar{R} g^{i j}\right)=\bar{\nabla}_{j}\left(\mathcal{T}^{i j}+\mathcal{T}_{T}^{i j}\right)=0
$$

where $\bar{\nabla}$ is the covariant derivative with respect to the Levi-Civita connection and $\mathcal{T}_{T}{ }^{i}{ }_{j}=\operatorname{diag}\left(-\rho_{T}, p_{T}, p_{T}, p_{T}\right)$ is the effective energy-momentum tensor of the torsion dark energy.

In addition, from (3.5) - 3.7), one can check that the continuity equation for the torsion field is also valid, i.e.

$$
\begin{equation*}
\dot{\rho}_{T}+3 H\left(\rho_{T}+p_{T}\right)=0 \tag{3.13}
\end{equation*}
$$

Consequently, we obtain the continuity equation for the ordinary matter to be

$$
\begin{equation*}
\dot{\rho}_{M}+3 H\left(\rho_{M}+p_{M}\right)=0 . \tag{3.14}
\end{equation*}
$$

By assuming no coupling between radiation and dust, the matter densities of radiation $\left(w_{r}=1 / 3\right)$ and dust $\left(w_{m}=0\right)$ in scalar-torsion cosmology share the same evolution behaviors as in GR, i.e. $\rho_{r} \propto a^{-4}$ and $\rho_{m} \propto a^{-3}$, respectively. In order to investigate the cosmological evolution, it is natural to define the total EoS by [8]

$$
\begin{equation*}
w_{t o t}=-1-\frac{2 \dot{H}}{3 H^{2}}=\frac{p_{t o t}}{\rho_{c}} \tag{3.15}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
w_{t o t}=\Omega_{M} w_{M}+\Omega_{T} w_{T}, \tag{3.16}
\end{equation*}
$$

where $\Omega_{\alpha}=\rho_{\alpha} / \rho_{c}$ and $w_{\alpha}=p_{\alpha} / \rho_{\alpha}$ with $\alpha=M, T$, representing the energy density ratios and EoSs of matter and torsion, respectively. Note that the EoS in (3.16), which is commonly used in the literature, e.g. [8], can be determined from the cosmological observations in [1, 2, 3, 6, 70]. In particular, it can be used to distinguish the modified gravity theories from the $\Lambda$ CDM [8].

Consequently, the evolution of the torsion dark energy can be described solely in terms of $w_{T}$ by

$$
\begin{equation*}
\rho_{T}(z)=\rho_{T}^{(0)} \exp \left\{3 \int_{0}^{z} d z^{\prime} \frac{1+w_{T}\left(z^{\prime}\right)}{1+z^{\prime}}\right\} . \tag{3.17}
\end{equation*}
$$

In the following sections, we focus on this important quantity.

### 3.3 Numerical Results of Torsion Cosmology

The evolution of torsion cosmology is determined by (3.5) - 3.7). In general, one needs to solve the dynamics of $R, \Phi$ and $H$ by the system of ordinary differential equations. However, one easily sees that in (3.7) there exists a special case: the constant scalar affine curvature solution, $R=-6 \mu / b$ [25]. Recall that in order to conform with the positive kinetic energy argument, the condition (3.3) is required. Such a condition leads to the negative curvature $R=-6 \mu / b<0$ in this case with a negative matter density $\rho<0$, the condition of $a_{1}<-a_{0}<0$ is required [25].

We concentrate on the EoS of the scalar-torsion mode in both special and normal cases which we define later. We also present the cosmological evolution of the density ratio, defined by $\Omega=\rho / \rho_{c}$, from a high redshift to the current stage.

### 3.3.1 Special Case: $R=$ const.

In this special case, we take the assumption of $a_{1}<0, \mu<0$ and $a_{0}>0$ in [25]. The evolution equations (3.5) - (3.7) reduce to

$$
\begin{align*}
\rho_{M} & =-3 a_{1} H^{2}-\frac{3}{2} \frac{\mu^{2}}{b}  \tag{3.18}\\
\rho_{T} & =\frac{3}{2} \frac{\mu^{2}}{b}+3 \mu H^{2},  \tag{3.19}\\
\dot{H} & =-\left(1+w_{M}\right)\left(\frac{3}{4} \frac{\mu^{2}}{a_{1} b}+\frac{3}{2} H^{2}\right) . \tag{3.20}
\end{align*}
$$

To employ numerical calculation, we rescale the parameters as below:

$$
\begin{align*}
m^{2} & =\rho_{m}^{(0)} / 3 a_{0}, \quad \widetilde{a}_{0}=a_{0} / m^{2} b, \quad \widetilde{a}_{1}=-a_{1} / m^{2} b \\
\widetilde{t} & =m \cdot t, \quad \widetilde{\mu}=\widetilde{a}_{1}-\widetilde{a}_{0}, \quad \widetilde{H}^{2}=H^{2} / m^{2}, \quad \widetilde{R}=R / m^{2} \tag{3.21}
\end{align*}
$$

where $\rho_{m}^{(0)}$ is the matter density at $z=0$ and the scalar affine curvature is $\widetilde{R}=6 \widetilde{\mu}>0$. From (3.18), (3.19) and (3.20), we obtain the following dimensionless equations,

$$
\begin{align*}
& \widetilde{H}^{2}=\frac{\widetilde{a}_{0}}{\widetilde{a}_{1}}\left(a^{-3}+\chi a^{-4}\right)+\frac{\widetilde{\mu}^{2}}{2 \widetilde{a}_{1}},  \tag{3.22}\\
& \frac{\rho_{T}}{\rho_{m}^{(0)}}=\frac{\widetilde{\mu}^{2}}{2 \widetilde{a}_{0}}-\frac{\widetilde{\mu}}{\widetilde{a}_{0}} \widetilde{H}^{2}  \tag{3.23}\\
& \widetilde{H} \widetilde{H}^{\prime}=\left(1+w_{M}\right)\left(\frac{3}{4} \widetilde{\mu}^{2} \widetilde{a}_{1}-\frac{3}{2} \widetilde{H}^{2}\right), \tag{3.24}
\end{align*}
$$



Figure 3.1: Evolutions of (a) the energy density ratio $\Omega$ and (b) the torsion $\operatorname{EoS} w_{T}$ with $\Omega_{m}^{(0)}=27.5 \%$, where the solid (black), dashed (blue), and dotted-dashed (red) lines stand for torsion, matter and radiation, respectively.
where the prime "".stands for $d / d \ln a$ and $\chi=\rho_{r}^{(0)} / \rho_{m}^{(0)}$. Using (3.13), 3.16) and (3.23), we find that

$$
\begin{equation*}
w_{T}=-1-\frac{\dot{\rho}_{T}}{3 H \rho_{T}}=-1-\frac{4}{3} \frac{\dot{\bar{H}}}{2 \widetilde{H}^{2}-\widetilde{\mu}} . \tag{3.25}
\end{equation*}
$$

From (3.22)-3.25), it is easy to see that the evolution of $\rho_{T}$ is automatically determined without solving any differential equation for given values of $\widetilde{a}_{0}$ and $\widetilde{a}_{1}$. The numerical results of this special case are shown in Fig. 3.3, where we have chosen $\widetilde{a}_{0}=76, \widetilde{a}_{1}=100$ and $\chi=3.07 \times 10^{-4}$ corresponding to $\Omega_{m}^{(0)}=\widetilde{H}_{z=0}^{-2} \simeq 27.5 \%$. In Fig. 3.3a, we plot the energy density ratios of torsion, matter and radiation, $\Omega_{T}, \Omega_{m}$ and $\Omega_{r}$, respectively. Notice that $\rho_{T}$ depends on the parameters $\widetilde{a}_{0}$ and $\widetilde{a}_{1}$, and there exists a late-time de-Sitter solution when $\widetilde{H}^{2}=\widetilde{\mu}^{2} / 2 \widetilde{a}_{1}$. In the high redshift regime, in which $\widetilde{H}^{2} \gg \widetilde{\mu}, \widetilde{\mu}^{2} / \widetilde{a}_{1}$, we observe that the torsion density ratio $\Omega_{T}$ is a constant which can also be estimated from (3.18) and (3.19), namely

$$
\begin{equation*}
\frac{\rho_{M}}{\rho_{T}}=\frac{3 \widetilde{a}_{1} \widetilde{H}^{2}-3 \widetilde{\mu}^{2} / 2}{3 \widetilde{\mu}^{2} / 2-3 \widetilde{\mu} \widetilde{H}^{2}} \simeq-\frac{\widetilde{a}_{1}}{\widetilde{\mu}} \tag{3.26}
\end{equation*}
$$

which manifests itself as a negative constant. In Fig. 3.3b, we show that the torsion EoS $w_{T}$ acts as matter $w_{m}=0$ and radiation $w_{r}=1 / 3$ in the matter-dominant ( $\rho_{m} \gg \rho_{r}$ ) and radiation-dominant $\left(\rho_{r} \gg \rho_{m}\right)$ stages, respectively, which are interesting asymptotic behaviors. We also observe that in the low redshift regime of $\log a \simeq 0, w_{T}$ is smaller than unity, indicating the existence of a late-time acceleration epoch.

### 3.3.2 The Normal Case

The normal case corresponds to both the kinetic energy and the matter density being positive, i.e, the parameters $a_{0}, a_{1}$ and $b$ are subject to the condition (3.3). It is also convenient to rescale the parameters in the form

$$
\begin{align*}
\widetilde{a}_{0} & =a_{0} / m^{2} b, \quad \widetilde{a}_{1}=a_{1} / m^{2} b, \quad \tilde{t}=t \cdot m, \quad \widetilde{\mu}=\widetilde{a}_{0}+\widetilde{a}_{1}, \\
\widetilde{H}^{2} & =H^{2} / m^{2}, \quad \widetilde{\Phi}=\Phi / m, \quad \widetilde{R}=R / m^{2}, \tag{3.27}
\end{align*}
$$

where $m^{2}=\rho_{m}^{(0)} / 3 a_{0}$. Using the above rescaling parameters, (3.5) - 3.7) and (3.44) are then rewritten as

$$
\begin{align*}
& \widetilde{H} \widetilde{H}^{\prime}=\frac{\widetilde{\mu}}{6 \widetilde{a}_{1}} \widetilde{R}-\frac{\widetilde{a}_{0}}{2 \widetilde{a}_{1}} a^{-3}-2 \widetilde{H}^{2}  \tag{3.28}\\
& \widetilde{H} \widetilde{\Phi}^{\prime}=\frac{\widetilde{a}_{0}}{2 \widetilde{a}_{1}}\left(\widetilde{R}-3 a^{-3}\right)-3 \widetilde{H} \widetilde{\Phi}+\frac{1}{3} \widetilde{\Phi}^{2},  \tag{3.29}\\
& \widetilde{H} \widetilde{R}^{\prime}=-\frac{2}{3}(\widetilde{R}+6 \widetilde{\mu}) \widetilde{\Phi}  \tag{3.30}\\
& \frac{1}{18}(\widetilde{R}+6 \widetilde{\mu})(3 \widetilde{H}-\widetilde{\Phi})-\frac{\widetilde{R}^{2}}{24}-3 \widetilde{a}_{1} \widetilde{H}^{2}=3 \widetilde{a}_{0}\left(a^{-3}+\chi a^{-4}\right), \tag{3.31}
\end{align*}
$$

where we have used $\stackrel{e m}{T}=3 P_{M}-\rho_{M}=-\rho_{m}=-3 a_{0} m^{2} a^{-3}$ due to $w_{r}=p_{r} / \rho_{r}=1 / 3$ and $w_{m}=p_{m} / \rho_{m}=0$. From (3.16) and (3.28) (3.31), we have

$$
\begin{equation*}
w_{T}=\frac{1}{3} \frac{\widetilde{\mu}\left(\widetilde{R}-\bar{R} / m^{2}\right)}{3 \widetilde{\mu} \widetilde{H}^{2}-(\widetilde{R}+6 \widetilde{\mu})(3 \widetilde{H}-\widetilde{\Phi})^{2} / 18+\widetilde{R}^{2} / 24}+\frac{1}{3} . \tag{3.32}
\end{equation*}
$$

To perform the numerical computations, we need to specify two parameters: $\widetilde{a}_{0}$ and $\widetilde{a}_{1}$, along with two initial conditions: $\widetilde{R}$ and $\widetilde{H}$. Thus, the initial condition for $\widetilde{\Phi}$ is automatically determined by (3.31). The numerical results are shown in Fig. 3.4, where the initial conditions at $z=0$ are set as $\left(\widetilde{a}_{0}, \widetilde{a}_{1}, \widetilde{R}_{0}, \widetilde{H}_{0}\right)=(2,1,14,2),(2,1,13,2),(3,1,8,2)$ for solid, dot-dashed, and dashed lines, respectively. Note that $\chi=3.07 \times 10^{-4}$ originates from the WMAP-5 data, and $\widetilde{H}=2$ corresponds to $\Omega_{m}^{(0)}=\widetilde{H}_{0}^{-2}=0.25$.

In Fig. 3.4 , we show the evolution of the density ratio, $\Omega_{T}=\rho_{T} / \rho_{c}$, as a function of the redshift $z$. The figure demonstrates that the torsion density $\rho_{T}$ dominates the universe in the high redshift regime $(z \gg 1)$ with the general parameter and initial condition selection, while the matter-dominated regime is reached only within a very short time interval. In Fig. 3.4 b , we show that $w_{T}$ has an asymptotic behavior at the high redshift regime, i.e.


Figure 3.2: Evolutions of (a) the energy density ratio $\Omega_{T}$ and (b) the torsion EoS $w_{T}$ in the universe as functions of the redshift $z$ with $\Omega_{m}^{(0)}=25 \%$ and $\chi=3.07 \times$ $10^{-4}$, where the solid, dotted-dashed and dashed lines correspond to ( $\left.\widetilde{a}_{0}, \widetilde{a}_{1}, \widetilde{R}_{0}, \widetilde{H}_{0}\right)=$ $(2,1,14,2),(2,1,13,2),(3,1,8,2)$, respectively.
$w_{z \gg 0} \rightarrow 1 / 3$. Moreover, in the low redshift regime, it may even have a phantom crossing behavior, i.e., the torsion EoS could cross the phantom divide line of $w_{T}=-1$. As a result, the scalar-torsion mode is able to account for the late-time accelerating universe. We also notice that from the numerical illustrations of the normal case one observes asymptotic behavior in the high red shift regime. Compared to Refs. [25, 34] only oscillating behavior are indicated. However, we find such asymptotic behavior in the normal case is not only generic, but rather a property that can be proved universally. Below we give a further account.

### 3.4 Asymptotic Behavior of High Redshift

### 3.4.1 Semi-analytical solution in high redshift

In the following, we shall provide the semi-analytical solution for the positive energy case of the scalar-torsion mode in the large scalar affine curvature limit $R \gg 6 \mu / b$ which is commonly achieved in the high redshift regime ( $a \ll 1$ ). In such circumstances, we may
write the energy density of ordinary matter and torsion in series expansions of $a(t)$ as

$$
\begin{align*}
\rho_{M} & =\frac{\rho_{m}^{(0)}}{a^{3}}+\frac{\rho_{r}^{(0)}}{a^{4}} \\
\frac{\rho_{T}}{\rho_{m}^{(0)}} & =\sum_{k=-c}^{\infty} A_{-k} a^{k}, \tag{3.33}
\end{align*}
$$

respectively. Before the analysis, we adopt the rescaling $(3.27)$ and write $(3.28)$ as functions of $\tilde{t}$ as

$$
\begin{align*}
& \frac{d \widetilde{H}}{d \widetilde{t}}=\frac{\widetilde{\mu}}{6 \widetilde{a}_{1}} \widetilde{R}-\frac{\widetilde{a}_{0}}{2 \widetilde{a}_{1} a^{3}}-2 \widetilde{H}^{2}  \tag{3.34}\\
& \frac{d \widetilde{\Phi}}{d \widetilde{t}}=\frac{\widetilde{a}_{0}}{2 \widetilde{a}_{1}}\left(\widetilde{R}-\frac{3}{a^{3}}\right)-3 \widetilde{H} \widetilde{\Phi}+\frac{1}{3} \widetilde{\Phi}^{2}  \tag{3.35}\\
& \frac{d \widetilde{R}}{d \widetilde{t}} \simeq-\frac{2}{3} \widetilde{R} \widetilde{\Phi}  \tag{3.36}\\
& \frac{\widetilde{R}}{18}(3 \widetilde{H}-\widetilde{\Phi})-\frac{\widetilde{R}^{2}}{24}-3 \widetilde{a}_{1} \widetilde{H}^{2}=3 \widetilde{a}_{0}\left(\frac{1}{a^{3}}+\frac{\chi}{a^{4}}\right), \tag{3.37}
\end{align*}
$$

In (3.36), we have taken the approximation of $R \gg 6 \mu / b$ for the high redshift regime. With the above rescaling, we shall argue that the lowest order of $\rho_{T}$ does not exceed $a^{-4}$ in the following discussion. We formulate the statement as a theorem.

Theorem 10. In the high redshift regime $(a \ll 1)$, $\rho_{T}=O\left(a^{-4}\right)$.
Proof. First we expand

$$
\begin{equation*}
\widetilde{H}^{2}(t)=\sum_{k=-c}^{\infty} r_{k} a^{k-4}, \quad\left(r_{k}<\infty\right) \tag{3.38}
\end{equation*}
$$

where $c$ is some integer, so that we have

$$
\begin{equation*}
\frac{d \widetilde{H}}{d \widetilde{t}}=\sum_{k=-c}^{\infty}\left(\frac{k-4}{2}\right) r_{k} a^{k-4} . \tag{3.39}
\end{equation*}
$$

Using (3.34), (3.36), (3.38) and (3.39), we obtain

$$
\begin{align*}
\widetilde{R} & =\frac{3 \widetilde{a}_{1}}{\widetilde{\mu}}\left(\sum_{k=-c}^{\infty} k \cdot r_{k} a^{k-4}\right)+\frac{3 \widetilde{a}_{0}}{\widetilde{\mu} a^{3}},  \tag{3.40}\\
\widetilde{\Phi} & =-\frac{3}{2} \widetilde{H} \cdot \frac{\widetilde{a}_{1}\left(\sum_{k=-c}^{\infty} k(k-4) r_{k} a^{k-4}\right)-\frac{3 \widetilde{a}_{0}}{a^{3}}}{\widetilde{a}_{1}\left(\sum_{k=-c}^{\infty} k r_{k} a^{k-4}\right)+\frac{\widetilde{a}_{0}^{3}}{a^{3}}}, \tag{3.41}
\end{align*}
$$

Substituting (3.38), (3.39), (3.40) and (3.41) into (3.35), and comparing the lowest power (requiring $c>-1$, otherwise losing its leading position) of $a$ in the high redshift, $a \ll 1$, we derive the following relation

$$
\begin{equation*}
\left(\frac{\widetilde{a}_{1}}{\widetilde{\mu}}\right)^{2} c^{2} \cdot r_{-c}^{3}\left[c^{2}+\left(5-\frac{\widetilde{a}_{0}}{\widetilde{\mu}}\right) c+4\right]=0 \tag{3.42}
\end{equation*}
$$

which leads to $r_{-c}=0$ if $c \geq 1$ as $0<\widetilde{a}_{0} / \widetilde{\mu}<1$ and $c^{2}+\left(5-\frac{\widetilde{a}_{0}}{\widetilde{\mu}}\right) c+4 \neq 0$. This is equivalent to saying that (3.38) has the form

$$
\begin{equation*}
\widetilde{H}^{2}=\frac{r_{0}}{a^{4}}+\frac{r_{1}}{a^{3}}+\frac{r_{2}}{a^{2}}+\frac{r_{3}}{a}+r_{4}+\cdots \tag{3.43}
\end{equation*}
$$

Finally, we achieve our claim from (3.10) that

$$
\begin{align*}
\frac{\rho_{T}}{\rho_{m}^{(0)}}= & -\left(\frac{\chi}{a^{4}}+\frac{1}{a^{3}}\right)+\widetilde{H}^{2} \\
& =-\left(\frac{\chi}{a^{4}}+\frac{1}{a^{3}}\right)+\left(\frac{r_{0}}{a^{4}}+\frac{r_{1}}{a^{3}}+\frac{r_{2}}{a^{2}}+\frac{r_{3}}{a}+r_{4}+\cdots\right)=O\left(\frac{1}{a^{4}}\right) . \tag{3.44}
\end{align*}
$$

Note that the last equality follows since $r_{0} \neq \chi$, which will be explained later.
We now write the expansion in (3.44), by the theorem above, simply as

$$
\begin{equation*}
\frac{\rho_{T}}{\rho_{m}^{(0)}}=\sum_{k=-4}^{\infty} A_{-k} a^{k} \tag{3.45}
\end{equation*}
$$

We shall only take first few dominating terms for a sufficient demonstration. By the procedure in the proof of the theorem, we can as well compare terms of various orders to yield the following relations:

$$
\begin{align*}
O\left(a^{-10}\right): & 3\left(A_{4}+\chi\right)\left(1+\frac{\widetilde{a}_{1}}{\widetilde{\mu}} A_{3}\right)^{2}=0,  \tag{3.46}\\
O\left(a^{-9}\right): & 2\left(1+\frac{\widetilde{a}_{1}}{\widetilde{\mu}} A_{3}\right)\left[\frac{\widetilde{a}_{0}}{\widetilde{\mu}}\left(1+\frac{\widetilde{a}_{1}}{\widetilde{\mu}} A_{3}\right) A_{3}+\frac{4 \widetilde{a}_{1}}{\widetilde{\mu}}\left(A_{4}+\chi\right) A_{2}\right]=0,  \tag{3.47}\\
O\left(a^{-8}\right): & \left(1+\frac{\widetilde{a}_{1}}{\widetilde{\mu}} A_{3}\right)\left[4 \frac{\widetilde{a}_{0}}{\widetilde{\mu}}\left(1+3 \frac{\widetilde{a}_{1}}{\widetilde{\mu}} A_{3}\right) A_{2}\right. \\
& \left.-\left(3 A_{2}+\frac{\widetilde{a}_{1}}{\widetilde{\mu}}\left(A_{2}\left(2+5 A_{3}\right)-18 A_{1}\left(A_{4}+\chi\right)\right)\right)\right]=0 . \tag{3.48}
\end{align*}
$$

From (3.46), (3.47) and (3.48), one concludes a relation,

$$
\begin{equation*}
A_{3}=-\frac{\widetilde{\mu}}{\widetilde{a}_{1}}=-\frac{\left(\widetilde{a}_{0}+\widetilde{a}_{1}\right)}{\widetilde{a}_{1}}<-1 \tag{3.49}
\end{equation*}
$$

with $A_{1}, A_{2}$ and $A_{4}$ left as arbitrary constants to be determined by initial conditions and (3.37). Note that (3.49) implies $r_{1}=-\widetilde{a}_{0} / \widetilde{a}_{1}<0$ in (3.38). However, due to the observational data that $a=1$ at the current stage, the radiation density is much smaller than the dust density ( $\chi \ll 1$ ), whereas the torsion density is the same order as the dust density, as seen from (3.44),

$$
\begin{equation*}
\frac{\rho_{T}^{(0)}}{\rho_{m}^{(0)}}=\left[\left(r_{0}-\chi\right)+r_{2}+\cdots\right]-\left(1+\left|r_{1}\right|\right) \simeq O(1) \tag{3.50}
\end{equation*}
$$




Figure 3.3: Evolutions of (a) $w_{T}$ and (b) $\left|w_{T}-1 / 3\right|$ as function of the redshift $z$ and the scale parameter $a$, respectively, where the parameters and initial conditions are chosen as $\widetilde{a}_{0}=2, \widetilde{a}_{1}=1, \widetilde{H}_{0}=2, \widetilde{R}_{0}=14$ and $\chi=\rho_{r}^{(0)} / \rho_{m}^{(0)}=3.1 \times 10^{-4}$.

Subsequently, we have that $\left[\left(r_{0}-\chi\right)+r_{2}+\cdots\right] \leq \max \left\{O(1), O\left(\left|r_{1}\right|\right)\right\}$, along with the assumption $r_{k}<\infty$ for each $k$. As a result, we conclude that $r_{k}$, for all $k \neq 1$, should not be too large, which forbids the possibility $r_{0}=\chi$. This argument shows the validity of the last equality in (3.44) with the non-vanishing $O\left(1 / a^{4}\right)$ coefficient.

From (3.16), via the continuity equation [24], we obtain

$$
\begin{equation*}
w_{T}=-1-\frac{\rho_{T}^{\prime}}{3 \rho_{T}} \simeq-1+\frac{1}{3}\left(\frac{4 A_{4} a^{-4}+3 A_{3} a^{-3}}{A_{4} a^{-4}+A_{3} a^{-3}}\right) \simeq \frac{1}{3}\left(1-\frac{A_{3}}{A_{4}} a\right), \tag{3.51}
\end{equation*}
$$

where the prime " " stands for $d / d \ln a$ and we have used 3.45 for $a \ll 1$.

### 3.4.2 Numerical computations

In this subsection, we perform numerical computations to support the analysis above. As an illustration, we take the parameters $\widetilde{a}_{0}=2$ and $\widetilde{a}_{1}=1$ and initial conditions

$$
\widetilde{H}(z=0)=\widetilde{H}_{0}=2, \quad \widetilde{R}(z=0)=\widetilde{R}_{0}=14
$$

and show the evolutions of $w_{T}, \widetilde{\Phi}$, and $\widetilde{R}$ in Figs. 3.3, 3.4a and 3.4b, respectively.

In Fig. 3.3a, we demonstrate the EoS of torsion as a function of the redshift $z$. As seen from the figure, in the high redshift regime $w_{T}$ approaches $1 / 3$, which indeed shows an asymptotic behavior. Fig. 3.3b indicates that $\left|w_{T}-1 / 3\right|$ approximates a straight line in the scale factor $a$ in the log-scaled coordinate since the slope in the log-scaled coordinates


Figure 3.4: Evolutions of (a) the rescaled affine curvature $\widetilde{R}$ and (b) the torsion $\widetilde{\Phi}$ as functions of the scale parameter $a$ in the log scale with the parameters and initial conditions taken to be the same as Fig. 3.3.
is nearly 1 . The singularity in the interval $[0.1,1]$ corresponds to the crossing $1 / 3$ of $w_{T}$. Thus, the numerical results concur with our semi-analytical approximation in (3.51). In Fig. 3.4 we observe that the behaviors of $\widetilde{R}$ and $\widetilde{\Phi} \propto 1 / a^{2}$ in the high redshift regime are consistent with the results in (3.40) and (3.41), given by

$$
\begin{align*}
& \widetilde{R} \simeq \frac{2 \widetilde{a}_{1}}{\widetilde{\mu}} A_{2} a^{-2}  \tag{3.52}\\
& \widetilde{\Phi} \simeq 3 \widetilde{H} \propto a^{-2} \tag{3.53}
\end{align*}
$$

respectively, where $\widetilde{H}^{2} \simeq\left(\chi+A_{4}\right) a^{-4}$ from 3.38). Note that from 3.52), the behavior of the affine curvature $\widetilde{R}$ is highly different from that of the Riemannian scalar curvature $\bar{R}=-\mathcal{T} / a_{0}=\rho_{m} / a_{0}$, which is proportional to $1 / a^{3}$ in both the matter (dust) and radiation dominated eras.

## 4

## Teleparallel Gravity

> ... Torsion is not just a tensor, but rather a very specific tensor that is intrinsically related to the translation group, as was shown by Elie Cartan in 1923-24.
> - Friedrich W. Hehl

### 4.1 A Degenerate Theory of PGT

From Chapter 2, we have seen that from the affine frame bundle theory where PGT resides, the gauge group $G=\mathbb{R}^{1,3} \rtimes S O(1,3)$ results in the decomposition of the affine connection $\gamma^{*} \widetilde{\omega}=\omega+\varphi$ according to the Lie algebra $\mathfrak{g}=\mathbb{R}^{1,3} \oplus \mathfrak{s o}(1,3)$, and from (2.17) we see

$$
\begin{equation*}
D^{\omega} \gamma^{*} \widetilde{\omega}=\underbrace{D^{\omega} \omega}_{\mathfrak{s o}(1,3)}+\underbrace{D^{\omega} \varphi}_{\mathbb{R}^{1,3}} \tag{4.1}
\end{equation*}
$$

It is then natural to ponder the question whether there could be some connection $\omega$ such that either $D^{\omega} \omega=0$ or $D^{\omega} \varphi=0$ which takes only a 1 -sided Lie-algebra value.

This naïve question leads to two different gravitational theories. Of course the case $D^{\omega} \varphi=T^{\omega}=0$ (vanishing torsion) is known as Riemannian geometry of GR; the other one $D^{\omega} \omega=\Omega^{\omega}=0$ (vanishing curvature), is called the teleparallel geometry (also known as the absolute parallelism). It is also clear from the Lie algebra (4.1) (only the tail of the Lie-algebra value $\mathbb{R}^{1,3}$ is left) to see the dubbed name translational gauge theory. Thus GR and teleparallel gravity can be considered as two extreme yet complementary degenerate cases. For clarity, we summarize as follows:

Definition 17. (Teleparallel geometry: bundle, vanishing curvature)

On the orthonormal frame bundle $(F(M), \pi, M, S O(1,3))$, if there exists a connection 1-form $\omega \in \Lambda^{1}(F(M), \mathfrak{s o}(1,3))$ such that $D^{\omega} \omega \equiv 0$, then we call the tuple $(M, g, \nabla)$ teleparallel geometry, where the connection $\nabla$ on $M$ is induced by $D^{\omega}$ on $F(M)$.

Gravitational theorists, who do not necessarily have to know bundle theories, usually take an alternative version on the (projected) base manifold.

Definition 18. (Teleparallel geometry: base manifold, Weitzenböck connection)
Given a semi-Riemannian spacetime $(M, g)$ with a set of (local, not necessarily) frame $\left\{e_{\alpha} \in T M\right\}, \alpha=0,1,2,3$, and a connection $\nabla^{W}$ on $M$ satisfying

$$
\begin{equation*}
\nabla_{e_{\alpha}}^{W} e_{\beta}=0, \quad(\text { for all } \alpha, \beta) \tag{4.2}
\end{equation*}
$$

the tuple $\left(M, g, \nabla^{W},\left\{e_{\alpha}\right\}\right)$ is defined as teleparallel geometry with respect to $\left\{e_{\alpha}\right\}$, also called a Weitzenböck spacetime along with the connection $\nabla^{W}$ called the Weitzenböck connection

The definition of a Weitzenböck connection automatically indicates all the connection 1-forms vanish, $\omega_{\alpha}{ }^{\beta}:=\vartheta^{\beta}\left(\nabla^{W} e_{\alpha}\right) \equiv 0$, and hence the curvature vanishes

$$
\begin{equation*}
\Omega^{\mu}{ }_{\nu}=d \omega_{\mu}{ }^{\nu}+\omega_{\nu}{ }^{\gamma} \wedge \omega_{\gamma}{ }^{\mu} \equiv 0, \quad \Leftrightarrow \quad R^{\mu}{ }_{\nu \alpha \beta} \equiv 0 . \tag{4.3}
\end{equation*}
$$

We emphasize that two facts that are neglected sometimes:
Remark 4. 1. The curvedness or flatness is not an absolute property of a spcetime manifold, rather it changes with the connection assigned.
2. Initially the flatness of teleparallel gravity is with respect to one chosen frame, say $e_{\alpha}$, but since $R^{\mu}{ }_{\nu \alpha \beta} \equiv 0$ is a tensor equation, we have $R(X, Y) Z=0$ for all $X, Y, Z \in T M$. In particular, if we choose another frame $\tilde{e}_{\beta}$ for computation, we still have curvature zero. Hence the parallelism is well-defined and independent of any chosen frame as long as there exists one frame $e_{\alpha}$ such that (4.2) holds.

However, one natural question may arise: whether the two definitions are equivalent or not? A closer look of the question reveals its essence:

$$
\begin{array}{rll}
D \omega_{\alpha}{ }^{\beta}:=d \omega_{\alpha}{ }^{\beta}+\omega_{\gamma}{ }^{\beta} \wedge \omega_{\alpha}{ }^{\gamma} \equiv 0 & \stackrel{?}{\Leftrightarrow} & \omega_{\alpha}{ }^{\beta} \equiv 0  \tag{4.4}\\
\text { (vanishing curvature, Def 17) } & \stackrel{?}{\Leftrightarrow} & \text { (Weitzenböck connection, Def 18) }
\end{array}
$$

Obviously, the direction $(\Leftarrow)$ is simply by (4.3). The other direction requires some effort to see and involves partial differential equations in general. First we consider that there are two frames $\left\{\widetilde{e}_{\alpha}\right\}$ and $\left\{e_{\alpha}\right\}$, where the latter satisfies the Weitzenböck condition (4.2). Since there exists a linear isomorphism $A_{\alpha}^{\beta}(x)$ for all $x \in M$ such that $\tilde{e}_{\alpha}(x)=A_{\alpha}^{\beta}(x) e_{\beta}(x)$, then we may calculate the Weitzenböck connection 1-form expressed in the frame $\left\{\widetilde{e}_{\alpha}\right\}$. Let $\nabla^{W} \widetilde{e}_{\beta}:=\widetilde{\omega}_{\beta}^{\alpha} \widetilde{e}_{\alpha}$, then we have $\widetilde{\omega}_{\beta}^{\alpha}(X)=\widetilde{\vartheta}^{\alpha}\left(\nabla_{X}^{W} \widetilde{e}_{\beta}\right)$ where $\widetilde{\vartheta}^{\alpha}$ is the dual basis of $\widetilde{e}_{\alpha}$, and thus

$$
\begin{equation*}
\widetilde{\omega}_{\beta}^{\alpha}\left(\widetilde{e}_{\mu}\right)=\widetilde{\vartheta}^{\alpha}\left(\nabla_{\widetilde{e}_{\mu}}^{W} \widetilde{e}_{\beta}\right)=\widetilde{e}_{\mu}\left(A_{\beta}^{\nu}\right)\left(A^{-1}\right)_{\nu}^{\alpha}, \quad \Leftrightarrow \quad \widetilde{\omega}_{\beta}^{\alpha}=\left(d A_{\beta}^{\nu}\right) \cdot\left(A^{-1}\right)_{\nu}^{\alpha} \neq 0 . \tag{4.5}
\end{equation*}
$$

We see that the Weitzenböck connection 1-form $\widetilde{\omega}_{\beta}^{\alpha}$ in the frame $\left\{\widetilde{e}_{\alpha}\right\}$ is in general nonzero although $\omega_{\beta}{ }^{\alpha} \equiv 0$ in the frame $\left\{e_{\alpha}\right\}$. The computation of changing frames helps answering the question (4.4). Since if we start from any frame $\left\{e_{\alpha}\right\}$ in Def. (17), to acquire the Weitzenböck condition defined by 4.2 we need to solve $A_{\alpha}^{\beta}(x)$ from the coupled partial differential equations, namely

$$
\begin{equation*}
D \widetilde{\omega}_{\alpha}{ }^{\beta}=d \widetilde{\omega}_{\alpha}{ }^{\beta}+\widetilde{\omega}_{\gamma}{ }^{\beta} \wedge \widetilde{\omega}_{\alpha}{ }^{\gamma} \equiv 0 \tag{4.6}
\end{equation*}
$$

then the existence of $A_{\alpha}^{\beta}(x)$ at all $x \in M$ guarantees the equivalence of the other direction.

### 4.1.1 Another construction from PGT

There is another construction that leads to the same teleparallel gravity directly from PGT. This intuitive derivation can be found in [71]. To require vanishing curvature $\Omega^{\beta}{ }_{\alpha} \equiv 0$ for a gravitational theory, one can put manually a Lagrangian multiplier two-form $\Lambda_{\alpha \beta} \in \Lambda^{2}(M)$ to attain the desired constraint from the Lagrangian such that

$$
\begin{equation*}
L_{\mathrm{tot}}=L_{\mathrm{G}}+L_{\mathrm{mat}}-\frac{1}{2 \varrho} \Lambda_{\alpha \beta} \wedge \Omega^{\alpha \beta} \tag{4.7}
\end{equation*}
$$

which leads to the PGT field equations (2.76),

$$
\begin{align*}
\left.\left.D H_{\alpha}-e_{\alpha}\right\rfloor L_{\mathrm{G}}-\left(e_{\alpha}\right\rfloor T^{\beta}\right) \wedge H_{\beta} & =\mathcal{T}_{\alpha} \\
\frac{1}{2 \varrho} D \Lambda_{\alpha \beta}+\vartheta_{[\alpha} \wedge H_{\beta]} & =\mathcal{S}_{\alpha \beta} \tag{4.8}
\end{align*}
$$

where one observes that 4.8$)_{2}$ helps solve the Lagrangian multiplier $\Lambda_{\alpha \beta}$ while 4.8$)_{1}$ is free of $\Lambda_{\alpha \beta}$ and thus defines the field equation for teleparallel gravity, see [72].

So far in this section we introduced three approaches that define teleparallel geometry. However, we have not yet specified a geometric Lagrangian to indicate the evolution of Weitzenböck spacetime. Below we introduce an interesting Lagrangian equivalent to GR.

### 4.2 The Teleparallel Equivalent to General Relativity (TEGR)

The teleparallel equivalent to general relativity (TEGR, or $\mathrm{GR}_{\|}$) is a special type of teleparallel gravity that manifests equivalence to GR in spacetime and matter evolutions. First we give a formal definition.

Definition 19. (TEGR)
$\boldsymbol{T E G R}$ is defined by a Weitzenböck spacetime $\left(M, g, \nabla^{W},\left\{e_{\alpha}\right\}\right)$ with the gravitational Lagrangian (4-form)

$$
\begin{equation*}
L_{G R_{\| \mid}}=-\frac{1}{2 \kappa} T^{\alpha} \wedge \star\left(-{ }^{(1)} T_{\alpha}+2^{(2)} T_{\alpha}+\frac{1}{2}{ }^{(3)} T_{\alpha}\right) \tag{4.9}
\end{equation*}
$$

and vanishing spin source of matter, $\mathcal{S}_{\alpha \beta} \equiv 0$. Here ${ }^{(I)} T_{\alpha}$ are the torsion irreducible pieces defined in (2.44).

With this special TEGR Lagrangian (4.9), the translational and rotational excitations (2.77) and gravitational currents (2.78) can be computed as

$$
\begin{align*}
H_{\alpha} & =\frac{1}{\kappa} \star\left(-{ }^{(1)} T_{\alpha}+2^{(2)} T_{\alpha}+\frac{1}{2}{ }^{(3)} T_{\alpha}\right), & H_{\alpha \beta}=-\frac{\partial \mathcal{L}_{G}}{\partial \Omega^{\alpha \beta}}=0, \\
t_{\alpha} & =i_{e_{\alpha}}\left(\mathcal{L}_{G}\right)+\left(i_{e_{\alpha}}\left(T^{\beta}\right)\right) \wedge H_{\beta}, & s_{\alpha \beta}=\frac{\partial \mathcal{L}_{G}}{\partial \omega^{\alpha \beta}}=0 \tag{4.10}
\end{align*}
$$

As a consequence, only field equation (2.76) 1 survives, given by (4.8)

$$
\begin{equation*}
d H_{\alpha}-t_{\alpha}=\kappa \mathcal{T}_{\alpha} \tag{4.11}
\end{equation*}
$$

where the second field equation degenerates much as in the Einstein-Cartan theory. In fact, the equivalence of TEGR to GR is reflected by two facts: the first one is given by as follows.

Theorem 11. The motion of matter (assumed spinless) in TEGR is equivalent to $G R$ (the proof demonstrated here is provided by Hehl).

Proof. Recall the contortion 1-form

$$
\begin{equation*}
K_{\alpha}{ }^{\beta}:=\widetilde{\omega}_{\alpha}{ }^{\beta}-\omega_{\alpha}{ }^{\beta} \tag{4.12}
\end{equation*}
$$

where $\widetilde{\omega}_{\alpha}{ }^{\beta}$ is the Levi-Civita connection and $\omega_{\alpha}{ }^{\beta}$ is the Weitzenböck connection. From the fact that

$$
\begin{equation*}
T^{\beta}=d \vartheta^{\beta}+\omega_{\alpha}{ }^{\beta} \wedge \vartheta^{\alpha}, \quad 0=d \vartheta^{\beta}+\widetilde{\omega}_{\alpha}{ }^{\beta} \wedge \vartheta^{\alpha} \tag{4.13}
\end{equation*}
$$

we can rewrite torsion 2-form in terms of the contortion

$$
\begin{equation*}
T^{\beta}=K^{\beta}{ }_{\gamma} \wedge \vartheta^{\gamma} \tag{4.14}
\end{equation*}
$$

We also notice the fact that $K^{\alpha \beta}=-K^{\beta \alpha}$ if both $\widetilde{\omega}_{\alpha}{ }^{\beta}$ and $\omega_{\alpha}{ }^{\beta}$ are metric compatible. Now from the energy-momentum conservation law in $U_{4}$ for PGT, we have

$$
\begin{equation*}
\left.\left.\left.D \mathfrak{T}_{\alpha}=\left(e_{\alpha}\right\rfloor T^{\beta}\right) \wedge \mathfrak{T}_{\beta}+\left(e_{\alpha}\right\rfloor \Omega^{\beta \gamma}\right) \wedge \mathfrak{S}_{\beta \gamma}=\left(e_{\alpha}\right\rfloor K^{\beta}{ }_{\gamma}\right) \wedge \vartheta^{\gamma} \wedge \mathfrak{T}_{\beta}-K^{\beta}{ }_{\alpha} \wedge \mathfrak{T}_{\beta} \tag{4.15}
\end{equation*}
$$

where we have used the Weitzenböck connection $\Omega^{\beta \gamma}=0$ and (4.14). On the other hand, the LHS of 4.15) follows from the definition that

$$
\begin{equation*}
D \mathfrak{T}_{\alpha}:=d \mathfrak{T}_{\alpha}-\omega_{\alpha}{ }^{\beta} \wedge \mathfrak{T}_{\beta}:=d \mathfrak{T}_{\alpha}-\left(\widetilde{\omega}_{\alpha}{ }^{\beta}-K_{\alpha}{ }^{\beta}\right) \wedge \mathfrak{T}_{\beta}=\widetilde{D} \mathfrak{T}_{\alpha}+K_{\alpha}{ }^{\beta} \wedge \mathfrak{T}_{\beta}, \tag{4.16}
\end{equation*}
$$

where $\widetilde{D}$ denotes the Levi-Civita connection. When we equate 4.15 and 4.16), we have

$$
\begin{equation*}
\left.\widetilde{D} \mathfrak{T}_{\alpha}+\underline{K}_{\alpha}{ }^{\beta} \wedge \mathfrak{T}_{\beta}=\left(e_{\alpha}\right\rfloor K^{\beta}{ }_{\gamma}\right) \wedge \vartheta^{\gamma} \wedge \mathfrak{T}_{\beta}-\underline{K}^{\beta}{ }_{\alpha} \wedge \mathfrak{T}_{\beta} \tag{4.17}
\end{equation*}
$$

where the two terms cancel due to the fact $K_{\alpha \beta}=-K_{\beta \alpha}$, so that we obtain

$$
\begin{equation*}
\left.\widetilde{D} \mathfrak{T}_{\alpha}=\left(e_{\alpha}\right\rfloor K^{\beta \gamma}\right) \wedge \vartheta_{[\gamma} \wedge \mathfrak{T}_{\beta]} . \tag{4.18}
\end{equation*}
$$

Recall that in TEGR, we assume only spinless matter $\mathfrak{S}_{\gamma \beta}=0$ which has the identity $\vartheta_{[\gamma} \wedge \mathfrak{T}_{\beta]}=0$. Then we derive $\widetilde{D} \mathfrak{T}_{\alpha}=0$, which is the conservation law of GR, indicating that the motion is same as GR. This completes the proof.

Next, we show the equivalence of the gravitational field evolution to GR. There are two levels of the proof to certify such an equivalence. One sets out from the level of the Lagrangian and converts $L_{\mathrm{GR}_{\|}}$in 4.9 into the Hilbert-Einstein Lagrangian of GR $L_{\mathrm{EH}}=\frac{1}{2 \kappa} \widetilde{\Omega}^{\alpha \beta} \wedge \eta_{\alpha \beta}$ via a remarkable identity (see [65])

$$
\begin{equation*}
L_{\mathrm{GR}_{\|}}=-\frac{1}{2 \kappa} \widetilde{\Omega}^{\alpha \beta} \wedge \eta_{\alpha \beta}+d\left(\vartheta^{\alpha} \wedge \star d \vartheta_{\alpha}\right) \tag{4.19}
\end{equation*}
$$

where $\widetilde{\Omega}_{\alpha \beta}$ is the curvature 2-form of Levi-Civita connection $\widetilde{\omega}_{\alpha}{ }^{\beta}$. The other assertion for the equivalence is shown at the level of the gravitational field equations as we present below. First we prove some useful identities.

## Lemma 2.

$$
\begin{gather*}
\frac{1}{2} K^{\mu \nu} \wedge \eta_{\alpha \mu \nu}=\star\left(-{ }^{(1)} T_{\alpha}+2^{(2)} T_{\alpha}+\frac{1}{2}{ }^{(3)} T_{\alpha}\right)  \tag{4.20}\\
\Omega_{\alpha}^{\beta}=\widetilde{\Omega}_{\alpha}{ }^{\beta}-\widetilde{D} K_{\alpha}{ }^{\beta}-K_{\alpha}{ }^{\mu} \wedge K_{\mu}{ }^{\beta} \tag{4.21}
\end{gather*}
$$

Proof. The proof for the first identity can be found in the Appendix C [73]. Here we only prove the second. By definition,

$$
\Omega_{\alpha}^{\beta}=d \omega_{\alpha}{ }^{\beta}-\omega_{\alpha}{ }^{\gamma} \wedge \omega_{\gamma}{ }^{\beta}
$$

with the contortion 1-form $K_{\alpha}{ }^{\beta}=\widetilde{\omega}_{\alpha}{ }^{\beta}-\omega_{\alpha}{ }^{\beta}$. It can be rewritten as

$$
\begin{aligned}
\Omega_{\alpha}^{\beta} & =d\left(\widetilde{\omega}_{\alpha}{ }^{\beta}-K_{\alpha}{ }^{\beta}\right)-\left(\widetilde{\omega}_{\alpha}{ }^{\gamma}-K_{\alpha}{ }^{\gamma}\right) \wedge\left(\widetilde{\omega}_{\gamma}{ }^{\beta}-K_{\gamma}{ }^{\beta}\right) \\
& =\left(d \widetilde{\omega}_{\alpha}{ }^{\beta}-\widetilde{\omega}_{\alpha}{ }^{\gamma} \wedge \widetilde{\omega}_{\gamma}{ }^{\beta}\right)-\left(d K_{\alpha}{ }^{\beta}-K_{\alpha}{ }^{\gamma} \wedge \widetilde{\omega}_{\gamma}{ }^{\beta}-\widetilde{\omega}_{\alpha}{ }^{\gamma} \wedge K_{\gamma}{ }^{\beta}\right)-K_{\alpha}{ }^{\gamma} \wedge K_{\gamma}{ }^{\beta} \\
& =\widetilde{\Omega}^{\beta}{ }_{\alpha}-\widetilde{D} K_{\alpha}{ }^{\beta}-K_{\alpha}{ }^{\gamma} \wedge K_{\gamma}{ }^{\beta}
\end{aligned}
$$

Theorem 12. The $G R_{\|}$field equation (4.8) 1 is equivalent to Einstein's equation (GR).
Proof. Recall that the $\mathrm{GR}_{\|}$choice $H_{\beta}$ is given by (4.10). Via (4.20) it reads

$$
\begin{equation*}
H_{\beta}=\frac{1}{2 \kappa} K^{\mu \nu} \wedge \eta_{\beta \mu \nu} \tag{4.22}
\end{equation*}
$$

Thus the $\mathrm{GR}_{| |}$Lagrangian 4.9) can be rewritten as:

$$
\begin{equation*}
L_{\mathrm{GR}_{| |}}=-\frac{1}{2} T^{\beta} \wedge H_{\beta}=-\frac{1}{4 \kappa}\left(K^{\beta \gamma} \wedge \vartheta_{\gamma}\right) \wedge\left(K^{\mu \nu} \wedge \eta_{\beta \mu \nu}\right)=\frac{1}{2 \kappa}\left(K^{\beta}{ }_{\mu} \wedge K^{\mu \nu} \wedge \eta_{\nu \beta}\right) \tag{4.23}
\end{equation*}
$$

where we have used (4.14), (4.22) and identity (2.42). Then the $\mathrm{GR}_{\|}$field equation 4.8$)_{1}$ reads:

$$
\begin{align*}
&\left.\left.D H_{\alpha}-e_{\alpha}\right\rfloor L_{\mathrm{GR}_{\| \mid}}-\left(e_{\alpha}\right\rfloor T^{\beta}\right) \wedge H_{\beta}=(\frac{1}{2 \kappa} \underbrace{\widetilde{D} K^{\mu \nu} \wedge \eta_{\alpha \mu \nu}}_{(\star)}+K_{\alpha}{ }^{\beta} \wedge H_{\beta}) \\
&-\frac{1}{2 \kappa}\left[e_{\alpha}\right\rfloor\left(K^{\beta}{ }_{\mu} \wedge K^{\mu \nu}\right) \wedge \eta_{\nu \beta}+\underbrace{K^{\beta}{ }_{\mu} \wedge K^{\mu \nu} \wedge \eta_{\nu \beta \alpha}}_{(\star)}]-\left(e_{\alpha}\right\rfloor T^{\beta}) \wedge H_{\beta} \tag{4.24}
\end{align*}
$$

where the term

$$
\begin{equation*}
D H_{\alpha}=\widetilde{D} H_{\alpha}+K_{\alpha}{ }^{\beta} \wedge H_{\beta}=\frac{1}{2 \kappa} \widetilde{D} K^{\mu \nu} \wedge \eta_{\alpha \mu \nu}+K_{\alpha}{ }^{\beta} \wedge H_{\beta} \tag{4.25}
\end{equation*}
$$

utilizes (4.16), 4.22 and the fact $\widetilde{D} \eta_{\alpha_{1} \cdots \alpha_{p}} \equiv 0$ from (2.43), and

$$
\begin{equation*}
\left.\left.e_{\alpha}\right\rfloor L_{\mathrm{GR}_{\|}}=\frac{1}{2 \kappa}\left[e_{\alpha}\right\rfloor\left(K^{\beta}{ }_{\mu} \wedge K^{\mu \nu}\right) \wedge \eta_{\nu \beta}+K^{\beta}{ }_{\mu} \wedge K^{\mu \nu} \wedge \eta_{\nu \beta \alpha}\right] \tag{4.26}
\end{equation*}
$$

One observes that using the important identity 4.21 with the $\mathrm{GR}_{\|}$requirement $R^{\beta \nu} \equiv 0$, the $(\boldsymbol{\star})$ terms collect as

$$
\begin{equation*}
\frac{1}{2 \kappa}\left(\widetilde{D} K^{\beta \nu}+K^{\beta}{ }_{\mu} \wedge K^{\mu \nu}\right) \wedge \eta_{\alpha \beta \nu}=\frac{1}{2 \kappa} \widetilde{R}^{\beta \nu} \wedge \eta_{\alpha \beta \nu}=-\frac{1}{\kappa}\left(\widetilde{R}_{\mu \alpha}-\frac{1}{2} \widetilde{R} g_{\mu \alpha}\right) \eta^{\mu} \tag{4.27}
\end{equation*}
$$

which is immediately recognized as the Einstein tensor. The appearance of this term almost claims the completion of the proof; it remains to prove that all the rest of the terms in (4.24) cancel, which only takes some more steps to see it. Indeed, since the last term in (4.24) can be rewritten as

$$
\begin{align*}
\left.\left(e_{\alpha}\right\rfloor T^{\beta}\right) \wedge H_{\beta} & \left.=\left(e_{\alpha}\right\rfloor\left(K^{\beta \gamma} \wedge \vartheta_{\gamma}\right)\right) \wedge H_{\beta}  \tag{4.28}\\
& \left.\left.=\left(\left(e_{\alpha}\right\rfloor K^{\beta \gamma}\right) \wedge \vartheta_{\gamma}-K^{\beta}{ }_{\alpha}\right) \wedge H_{\beta}=-2\left(e_{\alpha}\right\rfloor K^{\beta}{ }_{\mu}\right) \wedge K^{\mu \nu} \wedge \eta_{\nu \beta}+K_{\alpha}{ }^{\beta} \wedge H_{\beta} \tag{4.29}
\end{align*}
$$

together, the 3 terms remaining vanish,

$$
\begin{equation*}
\left.\left.K_{\alpha}{ }^{\beta} \wedge H_{\beta}-\frac{1}{2 \kappa}\left(e_{\alpha}\right\rfloor\left(K_{\mu}^{\beta}{ }_{\mu} \wedge K^{\mu \nu}\right) \wedge \eta_{\nu \beta}\right)-\left(e_{\alpha}\right\rfloor T^{\beta}\right) \wedge H_{\beta}=0 \tag{4.30}
\end{equation*}
$$

where we have used $\left.\left.e_{\alpha}\right\rfloor\left(K^{\beta}{ }_{\mu} \wedge K^{\mu \nu}\right) \wedge \eta_{\nu \beta}=2\left(e_{\alpha}\right\rfloor K^{\beta}{ }_{\mu}\right) \wedge K^{\mu \nu} \wedge \eta_{\nu \beta}$.

The above two theorems then conclude that TEGR is equivalent to GR in the matter and geometric evolutions. One observes that the use of computational tools developed in Sec.(2.3.2) largely reduces the complicated calculation, while it causes more abstraction and indirectness. For most teleparallel theorists, the component forms of TEGR are then used more often (see [74]). Since the conversion in between is rarely seen, we demonstrate some sketch below.

## Theorem 13.

$$
\begin{align*}
L_{G R_{\|}} & :=-\frac{1}{2} T^{\alpha} \wedge H_{\alpha} \\
& =\frac{1}{2 \kappa}\left(\frac{1}{4} T_{\mu \nu \alpha} T^{\mu \nu \alpha}+\frac{1}{2} T_{\mu \nu \alpha} T^{\alpha \nu \mu}-T_{\mu \sigma}{ }^{\sigma} T^{\mu \delta_{\delta}}\right) \eta  \tag{4.31}\\
& :=-T_{\text {scalar }} \cdot \eta
\end{align*}
$$

where the term $T_{\text {scalar }}:=\frac{1}{2 \kappa}\left(\frac{1}{4} T_{\mu \nu \alpha} T^{\mu \nu \alpha}+\frac{1}{2} T_{\mu \nu \alpha} T^{\alpha \nu \mu}-T_{\mu \sigma}{ }^{\sigma} T^{\mu \delta}{ }_{\delta}\right) \in C^{\infty}(M)$ is referred to as the torsion scalar.

Proof. From (2.45) we simplify (4.9) as

$$
\begin{equation*}
L_{\mathrm{GR}_{\|}}=\frac{1}{2 \kappa} T_{\alpha} \wedge \star\left(T^{\alpha}-\vartheta^{\alpha} \wedge i_{e_{\beta}}\left(T^{\beta}\right)-\frac{1}{2} i_{e^{\alpha}}\left(\vartheta_{\gamma} \wedge T^{\gamma}\right)\right) \tag{4.32}
\end{equation*}
$$

and then we compute the terms above individually

$$
\begin{align*}
\vartheta^{\alpha} \wedge i_{e_{\beta}}\left(T^{\beta}\right) & =-T_{\mu} \vartheta^{\alpha} \wedge \vartheta^{\mu}  \tag{4.33}\\
i_{e^{\alpha}}\left(\vartheta_{\gamma} \wedge T^{\gamma}\right) & =T_{\alpha}-T_{\alpha[\mu \nu]} \vartheta^{\mu} \wedge \vartheta^{\nu}
\end{align*}
$$

where (2.24) and (2.42) are repeatedly used and $T_{\mu}:=T_{\mu \nu}{ }^{\nu}$ is the torsion trace we defined early. Note that here a single notation $T_{\mu}$ can either denote torsion trace or torsion 2-form. However the confusion shall not rise since one can tell from the correct degree of differential forms. After rearrangement, one is left to compute

$$
\begin{align*}
T_{\alpha} \wedge \star T^{\alpha} & =-\frac{1}{2} T_{\mu \nu \alpha} T^{\mu \nu \alpha} \eta, \\
T_{\alpha} \wedge \star\left(T_{\mu} \vartheta^{\alpha} \wedge \vartheta^{\mu}\right) & =T_{\mu} T^{\mu} \eta,  \tag{4.34}\\
T_{\alpha} \wedge \star\left(-\frac{1}{2} i_{e^{\alpha}}\left(\vartheta_{\gamma} \wedge T^{\gamma}\right)\right) & =-\frac{1}{2} T_{\alpha} \wedge \star T^{\alpha}-\frac{1}{2} T^{\mu \nu \alpha} T_{\alpha[\mu \nu]} \eta .
\end{align*}
$$

if we put everything together, we obtain the component form in 4.31).

Remark 5. Most of the literatures that studies teleparallel gravity, the component formulation is usually adopted. One of the merits is that the computation made direct. Since both a tetrad and local coordinate vectors are a basis, one can expand a tetrad in terms of the local coordinate vectors, hence

$$
\begin{equation*}
e_{\alpha}=e_{\alpha}^{i}(x) \frac{\partial}{\partial x^{i}}, \quad \vartheta^{\alpha}=e_{i}^{\alpha}(x) d x^{i} \tag{4.35}
\end{equation*}
$$

where $e_{\alpha}^{i}: U \subseteq M \rightarrow \mathbb{R}$ is the coefficient matrix. Along with the dual condition

$$
\vartheta^{\beta}\left(e_{\alpha}\right)=\left(e_{i}^{\beta}(x) d x^{i}\right)\left(e_{\alpha}^{j}(x) \frac{\partial}{\partial x^{j}}\right)=e_{i}^{\beta}(x) e_{\alpha}^{i}(x)=\delta_{\alpha}^{\beta}, \quad\left(\left.d x^{i}\left(\frac{\partial}{\partial x^{j}}\right)\right|_{x}=\delta_{j}^{i}\right)
$$

for all $x \in U$, one obtains the relation $e_{i}^{\beta}(x) e_{\alpha}^{i}(x)=\delta_{\alpha}^{\beta}$. Thus the entire teleparallel field equation (4.11) in terms of $e_{i}^{\beta}$ and $e_{\alpha}^{i}$ is only the equation of $e_{i}^{\beta}$ alone. More often, teleparallelists tend to call the coefficients $e_{\alpha}^{i}$ and $e_{i}^{\beta}$ as a tetrad and co-tetrad.

### 4.3 Local Lorentz violation in TEGR

Although we have proved that TEGR is equivalent to GR in many aspects, there is one peculiar property that does not share with GR: local Lorentz violation. It has been pointed out in [39], [75] that the TEGR Lagrangian $L_{\mathrm{GR}_{\| \mid}}$does not respect local Lorentz symmetry, where the local violation term is in the form of an exact differential. Hence in effect, as a boundary term, it does not affect the action and field equations. However, the story will be different in the $f(T)$ theory, in which violation terms cannot be eliminated, see [39].

In fact, we can give an account why the TEGR Lagrangian fails to be Lorentz invariant from the viewpoint of the fibre bundle theory. To depict the local Lorentz violation first we define it operationally

## Definition 20. (Local Lorentz Invariance and Violation)

Let $\left\{U_{i} \subseteq M\right\}$ be an open covering of the spacetime $M$, and a family of local Lagrangian 4-forms $\left\{L_{i} \in \Lambda^{4}\left(U_{i}, \mathbb{R}\right)\right\}$ defined on $U_{i}$. If on nonempty overlaps $U_{i} \cap U_{j}$ with transition map $\Psi_{i j}: U_{i} \cap U_{j} \rightarrow S O(1,3)$, the two Lagrangians agree $L_{i}(x)=L_{j}(x)$ (in the sense of up to a constant), $\forall x \in U_{i} \cap U_{j}$, then we say $L_{i}$ or $L_{j}$ is locally invariant on $U_{i} \cap U_{j}$. Moreover, if $L:=\cup_{i} L_{i}$ is a locally invariant on any nonempty overlaps, then $L$ is a locally invariant Lagrangian on the whole $M$. Thus if a Lagrangian is not locally invariant on some nonempty overlaps, we call it Lorentz violation otherwise.

From Section. 2.3.1, we see that if we have two orthonormal frames $\left(e_{\alpha}\right) \in \mathfrak{X}\left(U_{i}\right)$ and $\left(\widetilde{e}_{\beta}\right) \in \mathfrak{X}\left(U_{j}\right)$ defined on $U_{i}$ and $U_{j}$ respectively with $U_{i} \cap U_{j} \neq \phi$ and $\alpha, \beta=0,1,2,3$ with $\left(\vartheta^{\alpha}\right) \in \Lambda^{1}\left(U_{i}\right)$ and $\left(\widetilde{\vartheta}^{\beta}\right) \in \Lambda^{1}\left(U_{j}\right)$ denoting their dual coframes, then by 2.35 the frame changing between two sections $\sigma_{U_{i}}: U_{i} \rightarrow M$ and $\sigma_{U_{j}}: U_{j} \rightarrow M$ with the transition function $\Psi_{U_{j} U_{i}}: U_{i} \cap U_{j} \rightarrow S O(1,3)$ one has the transforms

$$
\begin{equation*}
e_{\beta}(x)=(A(x))_{\beta}^{\alpha} \widetilde{e}_{\alpha}(x) \Leftrightarrow \widetilde{\vartheta}^{\beta}(x)=\left(A^{-1}(x)\right)_{\alpha}^{\beta} \vartheta^{\alpha}(x) \tag{4.36}
\end{equation*}
$$

where $A_{\beta}^{\alpha}:=\left(\Psi_{U_{j} U_{i}}\right)_{\beta}^{\alpha}$ denotes the matrix form of $S O(1,3)$. According to such transformation rule, the TEGR action (4.9) and the corresponding field momenta (4.10) transforms as

$$
\begin{align*}
& \widetilde{\mathcal{L}}_{G}(x)=\mathcal{L}_{G}(x)-\frac{1}{2 \kappa} d\left(\left(A^{-1}(x)\right)_{\beta}^{\alpha} d(A(x))_{\gamma}^{\beta} \wedge \eta^{\gamma}{ }_{\alpha}\right) \\
& \widetilde{H}_{\alpha}(x)=\left(A^{-1}(x)\right)_{\alpha}^{\beta} H_{\beta}(x)-\frac{1}{2 \kappa}\left(A^{-1}(x)\right)_{\alpha}^{\beta}\left(A^{-1}(x)\right)_{\gamma}^{\nu} d(A(x))_{\mu}^{\gamma} \wedge \eta_{\beta}{ }^{\mu}{ }_{\nu}  \tag{4.37}\\
& \widetilde{t}_{\alpha}(x)=\left(A^{-1}(x)\right)_{\alpha}^{\beta} t_{\beta}(x) \\
&+d\left(A^{-1}(x)\right)_{\alpha}^{\beta} \wedge H_{\beta} \\
& \quad-\frac{1}{2 \kappa} d\left(\left(A^{-1}(x)\right)_{\alpha}^{\beta}\left(A^{-1}(x)\right)_{\gamma}^{\nu} d(A(x))_{\mu}^{\gamma} \wedge \eta_{\beta}{ }^{\mu}{ }_{\nu}\right)(x)
\end{align*}
$$

where $x \in U_{i} \cap U_{j}$ and the tilde-symbols refer to quantities in the $U_{j}$ system.
After moments of thought, one recognizes that the violation of 4.37) follows from the fact that they are not basic differential forms (see Def. (9)) from a principal bundle. One sees that if $\alpha \in \bar{\Lambda}^{k}(P, V)$ is a basic differential form, by applying the change of sections (2.30) its local form transforms as

$$
\begin{equation*}
\sigma_{U_{j}}^{*} \alpha=\rho\left(\Psi_{U_{i} U_{j}}^{-1}(x)\right) \cdot \sigma_{U_{i}}^{*} \alpha \tag{4.38}
\end{equation*}
$$

where $\sigma_{U_{i}}: U_{i} \rightarrow P$ is a local section. Some well-behaved examples are: the torsion 1-form $T^{\alpha}$ by (2.32) and (2.35), and the curvature 2-form $\Omega^{\omega}$ by (2.38). An ill-behaved example is the connection 1 -form $\omega$ by (2.36) and (2.37). In fact, we have 51]

Theorem 14. Basic differential forms on a principal bundle $\pi: P \rightarrow M$ are in 1-1 correspondence to global-defined differential forms on $M$ with values in the associated bundle $P \times{ }_{G} V$

$$
\begin{equation*}
\bar{\Lambda}^{k}(P, V) \cong \Lambda^{k}\left(M, P \times_{G} V\right), \tag{4.39}
\end{equation*}
$$

where the associated bundle $P \times_{G} V$ is defined as $(P \times V) / G:=\{[p, v] \mid p \in P, v \in V\}$ by the quotient of the $G$-action $(p, v) \cdot g:=\left(p \cdot g, \rho\left(g^{-1}\right) \cdot v\right)$. Yet forms on $M$ of value in $P \times{ }_{G} V$ are equivalently represented by a family of $\mu_{i} \in \Lambda^{k}\left(U_{i}, V\right)$ with the gluing condition

$$
\begin{equation*}
\mu_{i}=\rho\left(\Psi_{j i}^{-1}\right) \cdot \mu_{j} \tag{4.40}
\end{equation*}
$$

where $\Psi_{j i}: U_{i} \cap U_{j} \rightarrow G$ is the transition function.
Thus using the theorem, we conclude that
Corollary 2. (Local Lorentz violation)
The reason for the local Lorentz violation (4.37) is due to the fact that the TEGR Lagrangian $L_{G R_{\|}}$is not a basic differential form, i.e, not a global-defined scalar on $M$.

In contrast, we know that the Hilbert-Einstein action has a nice transformation behavior ${ }^{1}$ under the change of frames. As one now looks back to the Einstein-Hilbert action, or even the more general Einstein-Cartan theory,

$$
\mathcal{L}_{G}:=\frac{1}{2 \kappa} \Omega^{\alpha \beta} \wedge \eta_{\alpha \beta}=\frac{1}{2 \kappa} R \sqrt{-g} d^{4} x \in \Lambda^{4}(M)
$$

it can be checked that it belongs to $\bar{\Lambda}^{0}(P, V)$. Therefore it is well-glued on each overlap $U_{\alpha} \cap U_{\beta}$ and hence globally defined showing no local Lorentz violation.

Remark 6. There is usually a misleading concept that a scalar without index, e.g. a Lagrangian, is always an invariant scalar under change of frames, since it may only be locally defined. One has to check the gluing condition carefully. It should be now clear

[^2]from the theorem that a locally invariant scalar must be originated from $\bar{\Lambda}^{k}(P, V) \cong$ $\Lambda^{k}\left(M, P \times{ }_{G} V\right)$, and hence the word "tensor" does not mean too much unless it is specified globally or locally.

However it is generally believed that physics law should remain the same form in every frame which is an essence of a gauge theory. This is reminiscent of the Yang-Mill's theory of the non-Abelian gauge. Local (gauge) violation occurs if we consider the field strength defined like $F=d A$ for some gauge potential $A \in \Lambda^{1}(M, \mathfrak{g})$ transforming as

$$
A=\varphi^{-1} d \varphi+\varphi^{-1} \tilde{A} \varphi
$$

by 2.37). To cure the local violation of $F$, the remedy is to add the term $A \wedge A$ such that $\widetilde{F}:=d A+A \wedge A$, which was the Yang-Mills original idea. Thus $\widetilde{F}$ will then transforms as ("a tensor" by an abuse of language) $\widetilde{F}=\Psi^{-1} \widetilde{F} \Psi$, which is now globally defined.

### 4.3.1 Generalized teleparallel theories

Despite the local Lorentz violation term, the interesting property of TEGR equivalence with GR makes some people to conceive an extension of teleparallel gravity to $f(T)$ theory (here $T=T_{\text {scalar }}$ ). This proposal mainly replaces the Lagrangian $L_{\mathrm{GR}_{\|}}$with $L=f(T)$ by mimicking the $f(R)$ phenomenological gravity models, while the underlying geometry remain unchanged. Thus one can see that $f(T)$ theory contains nontrivial local Lorentz violation terms that cannot be treated as boundary terms, see [39]. In effect, the field equations of $f(T)$ deviate from those of GR. However [76] finds that if one considers the weak field limit of the tetrad field on the solar system scale, the geodesic equation of $f(T)$ coincides with that of GR up to the Newtonian limit, which indicates that the current solar system observations hardly distinguish these two theories, regardless of the actual form of $f(T)$. However, $f(T)$ theory does demonstrate a different evolution history on the cosmological scale. Some recent studies show that $f(T)$ has a certain effect in on the sub-horizon scale [77], which modifies the effective Newtonian constant and causes different formation history for the large scale structure. In this background, $f(T)$ serves as an alternative theory to compare with $\Lambda$ CDM. Several papers are devoted in the subsequent investigations of $f(T)$ models.

In addition to $f(T)$ theory, another minimal extension of teleparallel gravity mimicking the scalar-tensor theory in GR was proposed, called teleparallel dark energy. Such a
theory was shown to provide a contrast to that of GR, see [78], [79]. Some other problems of $f(T)$ were also found by Ong, Izumi, Nester, and Chen 80] that in a generic $f(T)$ theory there could be super-luminal propagating modes due to the effects of nonlinear constraints via the analysis of the corresponding characteristic equations and the Hamiltonian structure. Nevertheless, $f(T)$ theory still provides useful phenomenological explanations in cosmology problems [81, [82].

## 5

## Five Dimensional Theories

In this chapter we study the five dimensional theories of teleparallel gravity. Kaluza [41] and Klein [42] attempted to apply extra dimension to unify electromagnetism and gravity ${ }^{11}$

Kaluza's ansatz in 1921 was to split the 5 -dimensional spacetime $\bar{M}$ into a 4 -dimensional spacetime $M$ and the Maxwell's electrodynamics we perceive. However, Kaluza's attempt was to consider the fibre as $\mathbb{R}^{1}$ while Oskar Klein in 1926 modified his theory with curled fibre $S^{1} \cong U(1)$, a compact Abelian Lie group. Nowadays we can interpret the Kaluza-Klein theory as a principal fibre bundle $\pi: P=\bar{M} \rightarrow M$ with gauge group $G=U(1)$, where $M$ denotes our 4-dimensional spacetime. The splitting of $\bar{M}$ into $M$ with electromagnetic force is then due to the nature of the local trivialization of the principal fibre bundle in Def.(1]) with an assigned metric $\bar{g}=\pi^{*} g+\bar{A} \otimes \bar{A}$, where $g$ is the metric on $M$ and $A$ is a connection form on $M$ such that $\bar{A}=\pi^{*} A$, see [57].

In the following we are motivated by the Kaluza-Klein theory to construct fivedimensional theories in teleparallel gravity.

### 5.1 Five-dimensional teleparallel gravity construction

As before, we keep our setting as general as possible. Let $f: M \rightarrow V$ be an isometric embedding of a 4-dimensional Lorentzian manifold $M$ (hypersurface) into a 5-dimensional Lorentzian manifold (the bulk) $(V, \bar{g})$.

Consider a tetrad $\left\{e_{0}, \ldots, e_{3}\right\}$ on $M$ and its natural extension as a tetrad $\left\{\bar{e}_{0}, \ldots, \bar{e}_{3}, \bar{e}_{5}\right\}$

[^3]on $V$, where $\bar{e}_{a}:=f_{*}\left(e_{a}\right)$ and $\bar{e}_{5}$ is the unit normal vector field to $M$. The corresponding dual coframe to $e_{a}$ is $\left\{\vartheta^{0}, \ldots, \vartheta^{3}\right\}$ and to $\bar{e}_{A}$ is $\left\{\bar{\vartheta}^{0}, \ldots, \bar{\vartheta}^{3}, \bar{\vartheta}^{5}\right\}$, and thus $f^{*}\left(\bar{\vartheta}^{a}\right)=\vartheta^{a}$. We shall identify $M$ with $\bar{M}:=f(M) \subset V, X \in \mathfrak{X}(M)$ with $\bar{X} \in \mathfrak{X}(\bar{M})$, and $\vartheta^{a} \in T^{*} M$ with $\bar{\vartheta}^{a} \in T^{*} \bar{M}$...,etc interchangeably.

Indices of the tetrad on $M$ are labeled by small Latin letters $a, b, c, \ldots,=0,1,2,3$. For the local coordinate on $M$ the index is denoted by Greek letters $\mu, \nu, \ldots=0,1,2,3$. Both the spatial components of the local coordinates and tetrad on $M$ share the labelling by middle Latin letters $i, j, k, \ldots=1,2,3$; indices of the tetrad in $V$ are labelled by capital Latin letters $A, B, C, \ldots,=0,1,2,3,5$; indices of the local coordinate, e.g, $d x^{M}=$ $\left(d x^{\mu}, d x^{5}\right)$, in $V$ are denoted by capital Latin letters $M, N=0,1,2,3,5$, where 5 denotes the extra dimension ( $5^{t h}$ dimension). Quantities with a bar, e.g, $\bar{e}_{A}$, are used to mean objects viewed in $V$.


Figure 5.1: The isometric embedding $f: M \rightarrow V$.

The 4-dimensional teleparallel gravity on $M$, as introduced in Chapter 4, is formulated by a tetrad $\vartheta^{a}$, and the Weitzenböck connection defined by (4.2) with respect to $e_{a}$, denoting $\nabla^{W} e_{b}:=\omega_{b}{ }^{a} e_{a}$. The metric $g$ of $M$ is written as

$$
\begin{equation*}
d s^{2}=g_{\mu \nu} d x^{\mu} \otimes d x^{\nu}=\eta_{a b} \vartheta^{a} \otimes \vartheta^{b} \tag{5.1}
\end{equation*}
$$

where $\eta_{a b}$ denotes the Minkowski metric. The metric signature is fixed as $(-,+,+,+, \varepsilon)$ where $\varepsilon:=\bar{g}\left(\bar{e}_{5}, \bar{e}_{5}\right)= \pm 1$ is the sign of the $5^{\text {th }}$ dimension.

With the TEGR Lagrangian (4.9)

$$
\begin{equation*}
\mathcal{T}:=T_{a} \wedge \star\left[{ }^{(1)} T^{a}-2^{(2)} T^{a}-\frac{1}{2}{ }^{(3)} T^{a}\right] \tag{5.2}
\end{equation*}
$$

or the component form (4.31)

$$
\begin{align*}
T & =\frac{1}{4} T_{a b c} T^{a b c}+\frac{1}{2} T_{a b c} T^{c b a}-T_{b a}^{b} T^{c}{ }_{c}^{a}  \tag{5.3a}\\
& =\frac{1}{4} T_{\mu \nu \sigma} T^{\mu \nu \sigma}+\frac{1}{2} T_{\mu \nu \sigma} T^{\sigma \nu \mu}-T^{\nu}{ }_{\nu \mu} T^{\sigma}{ }_{\sigma}{ }^{\mu} . \tag{5.3b}
\end{align*}
$$

so that $\mathcal{T}=T \star 1$. Teleparallel gravity on the bulk $V$ is similarly defined with the Weitzenböck connection 1-form $\left\{\bar{\omega}_{A}^{B} \in \Lambda^{1}(V)\right\}$ on $V$ corresponding to $\left\{\bar{e}_{0}, \ldots, \bar{e}_{3}, \bar{e}_{5}\right\}$, and the 5-form TEGR Lagrangian $\overline{\mathcal{T}}$ for $V$

$$
\begin{equation*}
\overline{\mathcal{T}}=\bar{T}_{A} \wedge \bar{\star}\left[{ }^{(1)} \bar{T}^{A}-2^{(2)} \bar{T}^{A}-\frac{1}{2}{ }^{(3)} \bar{T}^{A}\right] \tag{5.4}
\end{equation*}
$$

where $\bar{天}$ is the Hodge dual operator in $(V, \bar{g})$ and $\bar{T}^{A}:=\nabla^{W} \bar{\vartheta}^{A}=\bar{d} \bar{\vartheta}^{A}+\bar{\omega}_{B}^{A} \wedge \bar{\vartheta}^{B}=\bar{d} \bar{\vartheta}^{A}$ is the torsion 2-form on $V$. Here there are two Cartan differentials $d: \Lambda^{k}(M) \rightarrow \Lambda^{k+1}(M)$ and $\bar{d}: \Lambda^{k}(V) \rightarrow \Lambda^{k+1}(V)$ to be carefully distinguished, along with the requirement $\left.\bar{d}\right|_{\Lambda^{k}(M)}=\left.d\right|_{\Lambda^{k}(M)}$. The gravitational action on $V$ is given by

$$
\begin{equation*}
{ }^{(5)} S=-\frac{1}{2 \kappa_{5}} \int \overline{\mathcal{T}}=-\frac{1}{2 \kappa_{5}} \int \bar{T} \bar{\star} 1, \tag{5.5}
\end{equation*}
$$

where $\kappa_{5}=8 \pi G^{(5)}$ represents the 5-dimensional gravitational coupling, $\bar{T}$ stands for the torsion scalar of $V$, and $\mp 1={ }^{(5)} e d^{5} x=\operatorname{det}\left(e_{M}^{A}\right) d^{5} x$ is the volume form of $V$.

In order to understand 5-dimensional teleparallel gravity, it is necessary to look back to the analysis in that of GR.

### 5.2 Five-dimensional gravity of GR

Here we provide an approach, which is rarely found in the literature, using Cartan's moving frame to derive the Gauss-Codazzi Theorem that provides a useful basis for the later construction. The Gauss-Codazzi equation has the importance of connecting relations between the 5 -dimensions and the 4 -dimensions, and hence gives the projected information of the five-dimensional spacetime down to the 4-dimensional spacetime.

Keeping the notations for the setting from the last section, and suppose now that $(V, \bar{\nabla}, \bar{g})$ and $(M, g, \nabla)$ are both semi-Riemannian manifolds. From Cartan's structure equation (vanishing torsion),

$$
\begin{align*}
& \bar{d} \bar{\vartheta}^{A}+\bar{\omega}_{B}{ }^{A} \wedge \bar{\vartheta}^{B}=0, \quad(\text { on } \bar{M})  \tag{5.6}\\
& d \vartheta^{a}+\omega_{b}{ }^{a} \wedge \vartheta^{b}=0, \quad(\text { on } M)
\end{align*}
$$

in particular, from $(5.6)_{1}$ with $A=5$ and $A=a$ we have

$$
\begin{align*}
& \bar{d} \bar{\vartheta}^{5}+\bar{\omega}_{b}^{5} \wedge \bar{\vartheta}^{b}=0  \tag{5.7}\\
& \bar{d} \bar{\vartheta}^{a}+\bar{\omega}_{5}^{a} \wedge \bar{\vartheta}^{5}+\bar{\omega}_{b}^{a} \wedge \bar{\vartheta}^{b}=0
\end{align*}
$$

also from $\sqrt{5.6})_{2}$, we may replace $\bar{d} \bar{\vartheta} \bar{\vartheta}^{a}=-\omega_{b}{ }^{a} \wedge \vartheta^{b}$ since we require $\left.\bar{d}\right|_{\Lambda^{k}(M)}=\left.d\right|_{\Lambda^{k}(M)}$. Then we derive

$$
\begin{equation*}
\left(\bar{\omega}_{b}{ }^{a}-\omega_{b}^{a}\right) \wedge \vartheta^{b}+\bar{\omega}_{5}^{a} \wedge \bar{\vartheta}^{5}=0 . \tag{5.8}
\end{equation*}
$$

Recall that the extrinsic curvature (or the second fundamental form) is defined by

$$
\begin{equation*}
\nabla_{\bar{X}} \bar{Y}-\nabla_{X} Y=K(X, Y) \bar{e}_{5} \tag{5.9}
\end{equation*}
$$

or alternatively

$$
\begin{equation*}
K(X, Y):=\varepsilon \bar{g}\left(\nabla_{\bar{X}} \bar{Y}-\nabla_{X} Y, \bar{e}_{5}\right)=\varepsilon \bar{g}\left(\nabla_{\bar{X}} \bar{Y}, \bar{e}_{5}\right) \tag{5.10}
\end{equation*}
$$

Taking $X=e_{a}, Y=e_{b}$, one obtains

$$
\begin{equation*}
\bar{\omega}_{a}^{5}\left(e_{b}\right):=\bar{\omega}_{b a}^{5}=K\left(e_{a}, e_{b}\right):=K_{a b} . \tag{5.11}
\end{equation*}
$$

Note that here $K_{a b}$ denotes the extrinsic curvature 2-tensor, not the contortion 1-form (4.12). On the other hand, the metric compatible condition

$$
\begin{equation*}
0=\bar{d} \bar{g}\left(\bar{e}_{a}, \bar{e}_{5}\right)=\bar{g}\left(\bar{\nabla} \bar{e}_{a}, \bar{e}_{5}\right)+\bar{g}\left(\bar{e}_{a}, \bar{\nabla} \bar{e}_{5}\right) \tag{5.12}
\end{equation*}
$$

helps us to derive

$$
\begin{equation*}
\bar{\omega}_{5}^{b}\left(e_{a}\right):=-\eta^{b c} \varepsilon \bar{\omega}_{c}{ }^{5}\left(e_{a}\right)=-\eta^{b c} \varepsilon \bar{\omega}_{a c}{ }^{5}=-\varepsilon K_{a}^{b} \tag{5.13}
\end{equation*}
$$

where (5.11) is used. Finally combined with (5.8) we obtain the relation of connection 1-forms on $V$ and on $M$, given by

$$
\begin{equation*}
\bar{\omega}_{b}^{a}=\omega_{b}^{a}-\varepsilon K_{b}^{a} \bar{\vartheta}^{5} . \tag{5.14}
\end{equation*}
$$

Therefore only with some simple computation, one has attained
Theorem 15. ([83])

$$
\begin{align*}
\bar{R}_{b c d}^{a} & =R^{a}{ }_{b c d}-\varepsilon K^{a}{ }_{c} K_{d b}+\varepsilon K^{a}{ }_{d} K_{c b}, \quad \text { (Gauss) }  \tag{5.15}\\
\bar{R} & =R+2\left(\varepsilon \mathcal{L}_{\bar{e}_{5}} t r K-K^{a b} K_{a b}\right), \quad \text { (Mainardi) }
\end{align*}
$$

Proof. Simply use the relation (5.14) on the curvature 2-form

$$
\begin{equation*}
\bar{\Omega}_{B}^{A}:=\bar{d} \bar{\omega}_{B}^{A}+\bar{\omega}_{C}^{A} \wedge \bar{\omega}_{B}^{C}=\frac{1}{2} \bar{R}_{B C D}^{A} \bar{\vartheta}^{C} \wedge \bar{\vartheta}^{D} . \tag{5.16}
\end{equation*}
$$

With $A=a$ and $B=b$, one can derive the first equation; with $A=a, B=5$ one can derive

$$
\begin{equation*}
\bar{R}^{a}{ }_{5 b 5}=\varepsilon \mathcal{L}_{\bar{e}_{5}} K^{a}{ }_{b}-K^{a c} K_{c b}, \quad \bar{R}^{a}{ }_{5 a 5}=\varepsilon \mathcal{L}_{\bar{e}_{5}} t r K-K^{a b} K_{a b}, \tag{5.17}
\end{equation*}
$$

where $\mathcal{L}_{\bar{e}_{5}}$ is the Lie derivative with respect to the normal direction. The second identity of the curvature scalar is obtained from

$$
\begin{equation*}
\bar{R}=\bar{\Omega}^{A B} \wedge \bar{\eta}_{A B}=\bar{\Omega}^{a b} \wedge \bar{\eta}_{a b}+2 \bar{\Omega}^{a 5} \wedge \bar{\eta}_{a 5} \tag{5.18}
\end{equation*}
$$

One notices that if we put the so-called cylindrical condition (see (5.25), namely $\mathcal{L}_{\bar{e}_{5}} K^{a}{ }_{b} \equiv 0$, then 5.15$)_{2}$ reads $\bar{R}=R-2 K^{a b} K_{a b}$, where the last term is reminiscent of the Maxwell's electrodynamics action $-\frac{1}{4} F_{a b} F^{a b}$ with a suitable reparametrization [84]. If we interpret the extrinsic curvature of $M$ in $V$ as the electrodynamics we perceive, then the five-dimensional gravity shall induce four-dimensional gravity plus Maxwell's electrodynamics. Also notice that there is no extra coupling term between gravity and electromagnetism in this theory.

With these understanding in GR, next we turn to teleparallel gravity.

### 5.3 Effective gravitational action on $M$

To derive the effective teleparallel gravity from $V$ onto $M$, a crucial ingredient is to find a relation analogous to the Gauss equation 5.15). However one notices in TEGR that the Weitzenböck connection leads to vanishing extrinsic curvature 5.10) of $M$ in $V, K_{a b} \equiv 0$, since $\bar{\nabla}^{W} \bar{e}_{5}=\bar{\omega}_{5}{ }^{a} e_{a} \equiv 0$. Thus the Gauss equation 5.15

$$
\bar{R}_{b c d}^{a}=R_{b c d}^{a}-\varepsilon K_{c}^{a}{ }_{c} K_{d b}+\varepsilon K^{a}{ }_{d} K_{c b}
$$

simply becomes a zero identity due to $\bar{R}^{a}{ }_{b c d}=0$ and $R^{a}{ }_{b c d}=0$ in teleparallel gravity, which indicates that the brane is like a flat-paper of $\mathbb{R}^{2}$ put into an Euclidean space $\mathbb{R}^{3}$.

The extra degree of freedom in TEGR actually lies in the torsion 2-form. From 2.24) we may decompose the torsion $\bar{T}^{a}$ of $V$ into normal and parallel components with respect
to $M$

$$
\begin{equation*}
\bar{T}^{a}=T^{a}+\bar{T}^{a}{ }_{b 5} \bar{\vartheta}^{b} \wedge \bar{\vartheta}^{5} \tag{5.19}
\end{equation*}
$$

where $T^{a}=\frac{1}{2} T^{a}{ }_{b c} \vartheta^{b} \wedge \vartheta^{c}$ is the torsion 2-form on $M$. Therefore the second term in (5.19) plays a role like the extrinsic curvature $K_{a b}$ in GR.

Since $f: M \rightarrow V$ is an isometric embedding, for any chart $\left(\chi, U=\left(x^{\mu}\right) \subseteq \mathbb{R}^{4}\right)$ of $M$ we can always find a local normal form $\left(\Phi, W=\left(x^{\mu}, y\right) \subseteq \mathbb{R}^{5}\right)$ such that $F=\Phi^{-1} \circ f \circ \chi$ : $\chi(U) \rightarrow \Phi(W)$ is given by $F\left(x^{\mu}\right)=\left(x^{\mu}, y=0\right)$ and the 5 D metric $\bar{g}$ is in the form

$$
\bar{g}_{M N}=\left(\begin{array}{cc}
g_{\mu \nu}\left(x^{\mu}, y\right) & 0  \tag{5.20}\\
0 & \varepsilon \phi^{2}\left(x^{\mu}, y\right)
\end{array}\right)
$$

where $y=x^{5}$. The choice of the notation $\phi$ is in order for the extra dimensional effect to mimic a scalar field. Within such coordinates, we obtain a preferred frame for $V$ with

$$
\begin{equation*}
\bar{e}_{A}=\left(e_{a}, \frac{1}{\phi} \frac{\partial}{\partial y}\right), \quad \bar{\vartheta}^{A}=\left(\bar{\vartheta}^{a}, \phi d y\right) \tag{5.21}
\end{equation*}
$$

such that $\bar{g}=\eta_{a b} \vartheta^{a} \otimes \vartheta^{b}+\varepsilon \bar{\vartheta}^{5} \otimes \bar{\vartheta}^{5}$. In this case,

$$
\begin{equation*}
\bar{T}^{n}=\bar{d} \bar{\vartheta}^{n}+\bar{\omega}_{A}^{n} \wedge \bar{\vartheta}^{A}=\bar{d} \bar{\vartheta}^{n}=\frac{e_{a}(\phi)}{\phi} \bar{\vartheta}^{a} \wedge \bar{\vartheta}^{n} \tag{5.22}
\end{equation*}
$$

one reads $\bar{T}^{5}{ }_{b 5}=\frac{1}{\phi} e_{b}(\phi)$ where $e_{b}(\phi):=(d \phi)\left(e_{b}\right)$. Thus we conclude that in the frame (5.21), the nonvanishing torsion components of $V$ are $T^{a}{ }_{b c}, \bar{T}^{a}{ }_{5 b}$ and $\bar{T}^{5}{ }_{b 5}=\frac{1}{\phi} e_{b}(\phi)$.

So far we have not yet specified the type of our five-dimensional spacetime. If we now let the ambient space $V$ be a local product of $U \times W$, where $U \subseteq M$ is open in $M$ and $W$ corresponds to the extra spatial dimension. Utilize (5.4) and the local product structure of $V$, we may compute the integration of the five-dimensional action over the base space $U$ of $M$,

$$
\begin{equation*}
S_{\mathrm{bulk}}=\frac{-1}{2 \kappa_{5}} \int_{U} \int_{W}\left(T+\frac{1}{2}\left(T_{a b 5} T^{a b 5}+T_{a 5 b} T^{b 5 a}\right)+\frac{2}{\phi} e_{a}(\phi) t^{a}-t_{5} \cdot t^{5}\right) \phi d y d v o l^{4} \tag{5.23}
\end{equation*}
$$

where $T$ is the (induced) 4-dimensional torsion scalar defined in (5.3) and let $t^{a}:=T_{a b}{ }^{b}$ denote the torsion trace instead of $T_{a}$ to avoid confusion. The equation (5.23) then provides us with a general effective action on the hypersurface $M$ in TEGR theory. Next, we concentrate on two specific theories of braneworld and Kaluza-Klein scenarios.

### 5.3.1 Braneworld Scenario

In the braneworld scenario, we set the hypersurface $M$ located at $y=0$ as a brane and specify the fibre $W=\mathbb{R}$ such that $V=M \times \mathbb{R}$. From (5.23), the general action on the


Figure 5.2: The space $M \times G$ is compactified over the compact Lie group $G$, and after the Kaluza-Klein decomposition we have an effective field theory over $M$ (Image courtesy of Wikipedia).
bulk reads

$$
\begin{equation*}
S_{\mathrm{bulk}}=\frac{-1}{2 \kappa_{5}} \int_{M} \int_{\mathbb{R}}\left\{\phi T+\phi\left(\frac{1}{2}\left(T_{a b 5} T^{a b 5}+T_{a 5 b} T^{b 5 a}\right)+\frac{2}{\phi} e_{a}(\phi) t^{a}-t_{5} t^{5}\right)\right\} d y d v o l^{4} \tag{5.24}
\end{equation*}
$$

The first term of the parentheses $\int_{M} \int_{\mathbb{R}} \phi T \sqrt{-g} d y d^{4} x$ in (5.24) is recognized as the usual TEGR Lagrangian with a non-minimally coupled scalar field $\phi$ on the brane localized in the fifth-dimension, which is analogous to the non-minimally coupled Hilbert action $\int_{M} \int_{\mathbb{R}} \phi R \sqrt{-g} d y d^{4} x$ of 4-dimension in GR. The second term arises from the fifth-dimensional component.

According to the induced-matter theory, the fifth-dimensional component and the flow along the 5th-dimension of the second term in (5.24) can be regarded as the induced-matter from the geometry. It is the projected effect due to the extra spatial dimension. We note that the mathematically equivalent formulation between the induced-matter and braneworld theories has been demontsrated by Ponce de Leon in [85].

### 5.3.2 Kaluza-Klein Theory

If we identity the space $V$ locally as $U \times S^{1}$ where topologically $S^{1} \cong U(1)$ and consider the 4-dimensional effective low-energy theory to require the Kaluza-Klein ansatz in TEGR,

$$
\begin{equation*}
e_{5}\left(g_{\mu \nu}\right)=0 \quad \text { or } \quad \frac{\partial}{\partial y} g_{\mu \nu}=0, \quad(\text { Kaluza-Klein ansatz }) \tag{5.25}
\end{equation*}
$$

which is also called the cylindrical condition to indicate that the function is independent of the fibre level. With this condition the theory allows only the massless Fourier mode
[87. The metric is then reduced to

$$
\bar{g}_{M N}=\left(\begin{array}{cc}
g_{\mu \nu}\left(x^{\mu}\right) & 0  \tag{5.26}\\
0 & \phi^{2}\left(x^{\mu}\right)
\end{array}\right)
$$

with $\varepsilon=+1$. By the $K K$ ansatz, we see $T^{a}{ }_{b 5}=0$ and $t^{5}=0$ so that the extra-dimensional integration is trivial, i.e, $\int_{S^{1}} \phi\left(x^{\mu}\right) d y=2 \pi r \phi\left(x^{\mu}\right)$, where $r$ is the radius of the fifthdimension. As a result, we obtain

$$
\begin{equation*}
S_{\mathrm{KK}}=\frac{-1}{2 \kappa_{4}} \int_{U}\left(\phi T+2 \partial_{\mu} \phi t^{\mu}\right) e d^{4} x \tag{5.27}
\end{equation*}
$$

where $\kappa_{4}:=\kappa_{5} / 2 \pi r$ is the effective 4 -dimensional gravitation coupling constant. We point out that our result of (5.27) disagrees with that given in [49]. One can adopt a simple case with $F(T)=T$ in Eq. (5) of [49] to check that the resultant equation differs from ours in (5.27).

In the next section, we examine our five-dimensional theory of TEGR in a cosmological background.

### 5.4 Friedmann Equation of Braneworld Scenario in TEGR

Before the computation for the TEGR theory, it is necessary to reformulate the FLRW cosmology of GR in differential forms.

### 5.4.1 FLRW cosmology in GR

In GR, the FLRW universe of sectional curvature $k=0$ is described by

$$
\begin{equation*}
g_{\mu \nu}=\operatorname{diag}\left(-1, a^{2}(t), a^{2}(t), a^{2}(t)\right) \tag{5.28}
\end{equation*}
$$

a canonical choice of coframe field is given by

$$
\begin{equation*}
\vartheta^{0}=d t, \quad \vartheta^{i}=a(t) d x^{i} \tag{5.29}
\end{equation*}
$$

First we need to compute the connection 1-form of the Levi-Civita connection $\widetilde{\nabla} e_{a}=$ $\widetilde{\omega}_{a}{ }^{b} \vartheta^{b}$. Start with Cartan's structure equation $(5.6)_{2}$

$$
\begin{equation*}
0=T^{0}=d \vartheta^{\theta}+\widetilde{\omega}_{i}^{0} \wedge \vartheta^{i} \tag{5.30}
\end{equation*}
$$

on the isotropic condition, one can choose $\widetilde{\omega}_{i}{ }^{0}=B(t, \mathbf{x}) d x^{i}$ for some temporal function $B(t, \mathbf{x})$, and the spatial components yield

$$
\begin{equation*}
0=T^{i}=d \vartheta^{i}+\widetilde{\omega}_{j}{ }^{i} \wedge \vartheta^{j}+\widetilde{\omega}_{0}{ }^{i} \wedge \vartheta^{0} \tag{5.31}
\end{equation*}
$$

Again, on the isotropic consideration, we may assume $\widetilde{\omega}_{j}{ }^{i}=C_{k j}{ }^{i}(t, \mathbf{x}) d x^{k}$, then 5.31) tells us that

$$
0=(\dot{a}(t)-B(t, \mathbf{x})) d t \wedge d x^{i}+C_{k j}^{i}(t, \mathbf{x}) d x^{k} \wedge a(t) d x^{j}
$$

By the linear independence of $\left\{d t, d x^{i}\right\}$, one arrives at $B(t, \mathbf{x})=\dot{a}$ and $C_{k j}{ }^{i}(t, \mathbf{x})=0$, which is

$$
\begin{equation*}
\widetilde{\omega}_{i}^{0}=\eta_{i j} \widetilde{\omega}_{0}^{j}=\frac{\dot{a}}{a} \vartheta^{i}, \quad \text { and } \quad \widetilde{\omega}_{j}^{i} \equiv 0 . \tag{5.32}
\end{equation*}
$$

where we have used the relation $\widetilde{\omega}_{0}{ }^{i}=-\eta_{00} \eta^{i j} \widetilde{\omega}_{j}{ }^{0}$ from the metric compatibility. With (5.32), the curvature 2-forms (2.24) are

$$
\begin{align*}
& \Omega^{j k}=\eta^{k m} \Omega^{j}{ }_{m}=\frac{\dot{a}^{2}}{a^{2}} \vartheta^{j} \wedge \vartheta^{k}  \tag{5.33}\\
& \Omega^{0 j}=\eta^{j m} \Omega^{0}{ }_{m}=\frac{\ddot{a}}{a} \vartheta^{0} \wedge \vartheta^{j}
\end{align*}
$$

one can readily verify the Friedmann equation from 2.55)

$$
\begin{aligned}
& \frac{1}{2} \Omega^{\beta \gamma} \wedge \eta_{0 \beta \gamma}=\kappa t_{0}, \quad \Leftrightarrow \quad\left(\frac{\dot{a}}{a}\right)^{2}=\kappa \rho \\
& \frac{1}{2} \Omega^{\beta \gamma} \wedge \eta_{i \beta \gamma}=\kappa t_{i}, \quad \Leftrightarrow \quad 2 \frac{\ddot{a}}{a}+4\left(\frac{\dot{a}}{a}\right)^{2}=\kappa(\rho-p)
\end{aligned}
$$

where the energy-momentum 3 -form is defined by $t_{0}=\rho \eta_{0}$ and $t_{i}=p \delta_{i j} \eta^{j}$. Again, one observes that the use of differential forms reduces tedious Christoffel symbol and curvature components computation. With these helpful computations in the Riemannian case, we may return to TEGR.

### 5.4.2 FLRW Brane Universe

Now we apply the teleparallel braneworld effect in cosmology. Assume that the brane $M$ at $y=0$ is a homogeneous and isotropic universe. The bulk metric $\bar{g}$ is further assumed to be a maximally symmetric 3 -space with spatially flat $(k=0)$ by

$$
\begin{equation*}
\bar{g}_{M N}=\operatorname{diag}\left(-1, a^{2}(t, y), a^{2}(t, y), a^{2}(t, y), \varepsilon \phi^{2}(t, y)\right) \tag{5.34}
\end{equation*}
$$

by choosing the canonical coframe field

$$
\begin{equation*}
\bar{\vartheta}^{0}=d t, \quad \bar{\vartheta}^{i}=a(t, y) d x^{i}, \quad \text { and } \quad \bar{\vartheta}^{5}=\phi(t, y) d y \tag{5.35}
\end{equation*}
$$

Subsequently, the torsion 2-forms are

$$
\begin{equation*}
\bar{T}^{0}=\bar{d} \bar{\vartheta}^{0}=0, \quad \bar{T}^{i}=\bar{d} \bar{\vartheta}^{i}=\frac{\dot{a}}{a} \bar{\vartheta}^{0} \wedge \bar{\vartheta}^{i}+\frac{a^{\prime}}{a \phi} \bar{\vartheta}^{5} \wedge \bar{\vartheta}^{i}, \quad \bar{T}^{5}=\frac{\dot{\phi}}{\phi} \bar{\vartheta}^{0} \wedge \bar{\vartheta}^{5} \tag{5.36}
\end{equation*}
$$

where the dot and prime stand for the partial derivatives respect to $t$ and $y$, respectively. Thus the non-vanishing torsion components of five-dimension are read off

$$
\begin{equation*}
T_{0 i}{ }^{i}=\frac{\dot{a}}{a}, \quad T_{n i}{ }^{i}=\frac{1}{\phi} \frac{a^{\prime}}{a}, \quad T_{n a}^{n}=\frac{e_{a}(\phi)}{\phi} \tag{5.37}
\end{equation*}
$$

With these data, the bulk Lagrangian (5.24) in the FRLW cosmology background has the form

$$
\begin{equation*}
\overline{\mathcal{T}}=\left[T+\left(\frac{3-9 \varepsilon}{\phi^{2}} \frac{a^{\prime 2}}{a^{2}}+6 \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi}\right)\right] d v o l^{5} \tag{5.38}
\end{equation*}
$$

where $\varepsilon=+1$ and $T=6 \dot{a}^{2} / a^{2}$ is the usual 4-dimensional scalar torsion.

### 5.4.3 Equations of Motion

The gravitational field equations on the bulk can be derived from the formulation given in [88, 89]. The equations of motion on $V$ are 4 -forms

$$
\begin{equation*}
\bar{D} \bar{H}_{A}-\bar{E}_{A}=-2 \kappa_{5}{ }^{(5)} \bar{\Sigma}_{A} \tag{5.39}
\end{equation*}
$$

with

$$
\begin{align*}
\bar{H}_{A} & =(-2) \bar{\star}\left({ }^{(1)} \bar{T}_{A}-2^{(2)} \bar{T}_{A}-\frac{1}{2}^{(3)} \bar{T}_{A}\right) \\
\bar{E}_{A} & :=i_{\bar{e}_{A}}(\overline{\mathcal{T}})+i_{\bar{e}_{A}}\left(\bar{T}^{B}\right) \wedge \bar{H}_{B}, \\
\bar{\Sigma}_{A} & :=\frac{\delta \bar{L}_{m a t}}{\delta \bar{\vartheta}^{A}} \tag{5.40}
\end{align*}
$$

where $\bar{\Sigma}_{A}$ is the canonical energy-momentum 4 -form of matter fields, and $\bar{H}_{A}$ can be simplified as 89]

$$
\begin{equation*}
\bar{H}_{A}=\left(\bar{g}^{B C} \bar{K}_{C}^{D}\right) \wedge \bar{\star}\left(\bar{\vartheta}_{A} \wedge \bar{\vartheta}_{B} \wedge \bar{\vartheta}_{D}\right) \tag{5.41}
\end{equation*}
$$

with $\bar{K}_{C}^{D}:=\bar{\omega}_{C}^{D}-\widetilde{\omega}_{C}^{D}$ being the contortion 1-form.
Following the same procedure as in (5.32), we are able to derive the unique Levi-Civita connection 1-form $\widetilde{\omega}_{C}^{D}$ with respect to the coframe in $V$, given by

$$
\begin{align*}
& \widetilde{\omega}_{i}^{0}=\frac{\dot{a}}{a} \bar{\vartheta}^{i}, \quad \widetilde{\omega}_{0}^{i}=\widetilde{\omega}_{i}^{0}, \quad \widetilde{\omega}_{5}^{0}=\varepsilon \frac{\dot{\phi}}{\phi} \bar{\vartheta}^{n}, \quad \widetilde{\omega}_{0}^{5}=\varepsilon \widetilde{\omega}_{5}^{0}  \tag{5.42}\\
& \widetilde{\omega}_{j}^{5}=-\varepsilon \frac{a^{\prime}}{\phi a} \bar{\vartheta}^{j}, \quad \widetilde{\omega}_{5}^{j}=-\varepsilon \widetilde{\omega}_{j}^{i}, \quad \widetilde{\omega}_{j}^{i} \equiv 0 .
\end{align*}
$$

From (5.41) and (5.42), some computation yields the equations of motion for the bulk:

$$
\begin{align*}
& \bar{D} \bar{H}_{0}-\bar{E}_{0}=3\left[\left(\frac{\dot{a}^{2}}{a^{2}}+\frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi}\right)-\frac{\varepsilon}{\phi^{2}}\left(\frac{a^{\prime \prime}}{a}-\frac{a^{\prime}}{a} \frac{\phi^{\prime}}{\phi}\right)-\left(\frac{1+\varepsilon}{2 \phi^{2}}\right) \frac{a^{\prime 2}}{a^{2}}\right] 天 \bar{\vartheta}_{0} \\
& +\frac{3 \varepsilon}{\phi}\left(\frac{\dot{a}^{\prime}}{a}-\frac{a^{\prime}}{a} \frac{\dot{\phi}}{\phi}\right) \mp \bar{\vartheta}_{5}=-\kappa_{5} \bar{\Sigma}_{0},  \tag{5.43}\\
& \bar{D} \bar{H}_{5}-\bar{E}_{5}=\frac{3}{\phi}\left(\frac{a^{\prime}}{a} \frac{\dot{\phi}}{\phi}-\frac{\dot{a}^{\prime}}{a}\right) \mp \bar{\vartheta}_{0}+3\left[\left(\frac{\ddot{a}}{a}+\frac{2 \dot{a}^{2}}{a^{2}}\right)-\left(\frac{1+\varepsilon}{2 \phi^{2}}\right) \frac{a^{\prime 2}}{a^{2}}\right] \mp \bar{\vartheta}_{5} \\
& =-\kappa_{5} \bar{\Sigma}_{5} \text {. }
\end{align*}
$$

Thus (5.43) ${ }_{1}$ is the Friedmann equation of the bulk. Moreover, if we let $\varepsilon=+1$ and expand the energy-momentum 4-form $\bar{\Sigma}_{A}=\bar{T}_{A}^{B} \bar{\star} \bar{\vartheta}_{B}$, we obtain

$$
\begin{equation*}
\left(\frac{\dot{a}^{2}}{a^{2}}+\frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi}\right)-\frac{1}{\phi^{2}}\left(\frac{a^{\prime \prime}}{a}-\frac{a^{\prime}}{a} \frac{\phi^{\prime}}{\phi}\right)-\frac{1}{\phi^{2}} \frac{a^{\prime 2}}{a^{2}}=\frac{\kappa_{5}}{3} \bar{T}_{00} \tag{5.44}
\end{equation*}
$$

Furthermore, if we consider our matter field as a perfect fluid, one can decompose the energy-momentum tensor into bulk and brane parts as 90

$$
\begin{align*}
\bar{T}_{A}^{B}(t, y) & =\left(\bar{T}_{A}^{B}\right)_{\mathrm{bulk}}+\left(\bar{T}_{A}^{B}\right)_{\mathrm{brane}} \\
\left(\bar{T}_{A}^{B}\right)_{\mathrm{brane}} & =\frac{\delta(y)}{\phi} \operatorname{diag}(-\rho(t), P(t), P(t), P(t), 0) \tag{5.45}
\end{align*}
$$

where $\left(\bar{T}_{A}^{B}\right)_{\text {bulk }}$ represents the vacuum energy-momentum tensor or the cosmological constant $\left(\Lambda_{5} / \kappa_{5}\right) \eta_{A}^{B}$ in the bulk, and $\rho(t)$ and $P(t)$ are the energy density and the pressure of the normal matter localized on the brane, respectively.

If the first discontinuity appears in the first derivative of the bulk metric $\bar{g}$, or $\bar{g} \in$ $C^{0}(M) \backslash \bigcup_{k=1} C^{k}(M)$ to be precise, the Dirac delta function would appear in the second derivative of the bulk metric. The FLRW metric leads to the equation of the scale factor with the form at $y=0$

$$
\begin{equation*}
a^{\prime \prime}(t, y)=\delta(y)\left[a^{\prime}\right](t, 0)+\widetilde{a}^{\prime \prime}(t, y) \tag{5.46}
\end{equation*}
$$

where $\widetilde{a}^{\prime \prime}$ denotes the non-distributional part of $a^{\prime \prime}$ and the definition of the jump is

$$
\begin{equation*}
[f](0):=\lim _{\delta \rightarrow 0^{+}} f(\delta)-f(-\delta) \quad(f: M \rightarrow \mathbb{R}) \tag{5.47}
\end{equation*}
$$

which measures the discontinuity of a real-valued function $f$ across the brane. With the form of the scale factor, (5.44) yields the junction condition

$$
\begin{equation*}
\left[a^{\prime}\right](t, 0)=\frac{\kappa_{5}}{3 \varepsilon} \rho a_{0}(t) \phi_{0}(t) \tag{5.48}
\end{equation*}
$$

where $a_{0}(t):=a(t, 0)$ and $\phi_{0}(t):=\phi(t, 0)$ are considered as the scalar factor and a scalar field on the brane, respectively. Furthermore, if we impose the so-called $\mathbb{Z}_{2}$ symmetry 91 ] for the scale factor in Eq. (5.48) on the bulk as a real-valued quantity $f$ must be an odd function $f(x)=-f(-x)$ across the brane, we obtain the Friedmann equation on the brane to be

$$
\begin{equation*}
\frac{\dot{a}_{0}^{2}(t)}{a_{0}^{2}(t)}+\frac{\ddot{a}_{0}(t)}{a_{0}(t)}=-\frac{\kappa_{5}^{2}}{36} \rho(t)(\rho(t)+3 P(t))-\frac{k_{5}}{3 \phi_{0}^{2}(t)}\left(\bar{T}_{55}\right)_{\text {bulk }} \tag{5.49}
\end{equation*}
$$

which is, unexpectedly, found to be the same as the braneworld theory of GR shown in [90]. Hence, this indicates that the cosmological braneworld scenario in TEGR coincides with that of GR, i.e,, there is no distinction between TEGR and GR in the braneworld FLRW cosmology, which again justifies the name of TEGR.

The physical consequence of the cosmological brane scenario here then follows from the discussions in [90]. In particular, if the extra 5th-dimension is compact, one can check if the solutions of $a(t, y)$ and $\phi(t, y)$ derived from (5.43) are well-defined ones, as given in [90]. In effect, we established the diagram


4 D TEGR $\Longleftrightarrow 4{ }^{2}{ }^{2}$ GR
Although the result may appear to be obvious, in fact there exists some non-triviality within the reduction. The 5 -dimensional reduction to the 4 -dimensional of the TEGR and GR involves different connections, the Levi-Civita and the Weitzenböck connection, and different geometric projections. Thus after some moment of thoughts, one realizes it is not that transparent as one though it was.

Finally, we remark that the Friedmann equation (5.43) in the bulk can be identified as $G_{00}=-\kappa_{5} \bar{T}_{00}$ and $G_{05}=0$, which are the same as those in 90 . This result implies that a radiating contribution of the universe can be generated in TEGR due to the extra spatial dimension. It can be viewed as a generic property that there exists a component of dark radiation in the braneworld scenario. We have to mention that there is no extrinsic curvature in TEGR since the projected effects of the dark radiation and discontinuity property of the brane come from torsion itself, which is clearly beyond the expectations of GR [85, 86] as already pointed out in 50.

## 6

## Conclusion

In this thesis, we have given a complete view of Poincaré gauge gravity starting from the affine frame bundle $\left(\mathbb{A}(M), \widetilde{\pi}, M, G L\left(\mathbb{R}^{1,3}\right), \widetilde{\omega}\right)$ to Riemann-Cartan spacetime $(M, g, \nabla)$. It can be seen from such a viewpoint that the torsion is a natural byproduct of gauging the Poincaré group $\mathcal{P}=\mathbb{R}^{1,3} \rtimes S O(1,3)$ into gravity, simply due to the existence of the canonical 1-form $\varphi \in \bar{\Lambda}^{1}\left(L(M), \mathbb{R}^{1,3}\right)$ of the frame bundle $L(M)$ and the decomposition of the Lie algebra $\mathfrak{P}=\mathbb{R}^{1,3} \oplus \mathfrak{s o}(1,3)$.

We have also discussed one of the interesting models in PGT, the torsion-scalar mode that does support the late-time universe acceleration, and investigated its features in a cosmological model. In particular, we have studied that the energy density ratio $\Omega_{T}$ and the torsion EoS $w_{T}$ in two main cases of the scalar-torsion mode in PGT.

For the first case of the negative energy matter density with negative constant affine curvature $R<0$, the torsion EoS $w_{T}$ demonstrates the same behavior as the background fluid in the high redshift regime: in the case of the radiation-dominated era we have $w_{T}=1 / 3$, in the case $w_{T}=0$ of the matter-dominated era it is $w_{T}=0$ and in a later stage of the de-Sitter point we also have $w_{T}=-1$. We also observe that the torsion density ratio of $\Omega_{T}$ in the high redshift regime becomes a negative constant.

In the second case of the scalar-torsion mode where the positive kinetic energy condition holds, the numerical solution of the field equations shows that in general $w_{T}$ has an asymptote to $1 / 3$ in the high redshift regime, while it could cross the phantom divide line in the low redshift regime. With a further analysis we find that such asymptotic behavior of $w_{T}$ and $\Omega_{T}$ in fact can be resolved by a semi-analytical solution. In principle, we apply a Laurent series expansion in the scale factor $a(t)$ for the torsion density $\rho_{T}$ and find
that the series in fact has a cut-off as a the lower bound at $O\left(a^{-4}\right)$ in the high redshift, corresponding to a radiation-like behavior. By a comparison of the next leading-order term of $a^{-3}$ in the field equations, we are able to extract the coefficient $A_{3}=-\mu / a_{1}$, which results in the vanishing of the $a^{-3}$ term in the affine curvature $R$, such that $R$ is only proportional to $a^{-2}$, and this result is consistent with the numerical demonstration.

In the last two chapters, we regard another special geometry descendant from PGT, teleparallel gravity. A clear definition for the Weitzenböck spacetime and its parallelism are provided. In particular, using the basic differential forms on the principal fibre bundle, we are able to understand the reason of local Lorentz violation in teleparallel gravity that generally occurs.

We also construct the extra dimension theory for TEGR in five-dimensional spacetime. With the use of Cartan's moving frame by means of differential forms, we can find the torsion relations between the brane and the bulk, analogous to the Gauss-Codazzi equation in GR. In particular, from the extra dimension theory rigorously constructed, we can show that the Kaluza-Klein theory in teleparallel gravity does not generate a Brans-Dicke type of the effective 4-dimensional Lagrangian as in GR. This result disagrees with the one given in [49].

We further apply our theory as the braneworld theory of teleparallel gravity and investigate its FLRW cosmology solution. To our surprise, it provides equivalent field equations and hence the same solutions as Einstein's general relativity. We thus conclude that the additional radiation of the universe can arise from the extra dimension, which is a generic feature of the branworld theory.

## Appendix A

## The Spin Current

Let $\phi \in \Gamma(U) \otimes V$ be a particle field, a vector-valued local section on spacetime $U \subseteq M$, where $V$ is a vector space under consideration (in general $\mathbb{R}^{n}$ or $\mathbb{C}^{n}$ ). Consider the infinitesimal variation of $\phi$, denoted by $\delta \phi$

$$
\begin{align*}
\delta \int \mathcal{L}_{\mathrm{M}}\left(x, \vartheta^{\alpha}(x), \phi^{a}(x), \phi_{; k}^{a}(x)\right) d v o l^{4} & =\int\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^{a}} \delta \phi^{a}+\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}} \delta \phi_{; k}^{a}\right) d v o l^{4},  \tag{A.1}\\
& =\int\left[\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^{a}} \delta \phi^{a}+D_{\partial_{k}}\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}} \delta \phi^{a}\right)-D_{\partial_{k}}\left(\frac{\partial L}{\partial \phi_{; k}^{a}}\right) \delta \phi^{a}\right] d v o l^{4} \\
& =\int\left[\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^{a}}-D_{\partial_{k}}\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}}\right)\right) \delta \phi^{a}+D_{\partial_{k}}\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}} \delta \phi^{a}\right)\right] d v o l^{4} \tag{A.2}
\end{align*}
$$

where $d v o l^{4}=\sqrt{-g} d^{4} x, D$ is the exterior covariant derivative defined in (5), $\delta \phi_{; k}^{a}:=$ $\delta\left(D \phi^{a}\right)\left(\partial_{k}\right)=D\left(\delta \phi^{a}\right)\left(\partial_{k}\right)$ due to the fixed connection, and we have applied the integration by parts in the second equality. If we define a vector field $J^{k}:=\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{k}^{a}} \delta \phi^{a}$ (general conserved current) and assume $\phi$ satisfies the Euler-Lagrange equation (on-shell)

$$
\begin{equation*}
\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi^{a}}-D_{\partial_{k}}\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}}\right)=0 \tag{A.4}
\end{equation*}
$$

with the infinitesimal invariant (under a Lie group $G$ ) of the Lagrangian, we obtain,

$$
\begin{equation*}
\delta \int \mathcal{L}_{\mathrm{M}}\left(x, \phi^{a}(x), \phi_{; k}^{a}(x)\right) d v o l^{4}=\int D_{\partial_{k}}\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}} \delta \phi^{a}\right) d v o l^{4} \tag{A.5}
\end{equation*}
$$

which will lead to Noether's first theorem for the $G$-symmetry. Now we carefully define the meaning of infinitesimal variation $\delta \phi$. Let $\mathfrak{g}$ be the Lie algebra of $G, \rho: G \rightarrow G L(V)$ be a Lie group representation, and $E \in \mathfrak{g}$ is an element. Let $\phi(t):=\rho\left(e^{t E}\right) \cdot \phi$ and define
$\delta \phi$ as the variation of $\phi(t)$ in a 1-parameter subgroup of $G$,

$$
\begin{equation*}
\delta \phi:=\left.\frac{d}{d t} \phi(t)\right|_{t=0}=\left(\rho_{*} E\right) \cdot \phi \tag{A.6}
\end{equation*}
$$

where in components we write

$$
\begin{equation*}
\delta \phi^{a}=\left(\rho_{*} E\right)_{b}^{a} \phi^{b} \tag{A.7}
\end{equation*}
$$

where $\rho_{*}: \mathfrak{g} \rightarrow \mathfrak{g l}(V)$ is the induced Lie algebra representation, $\left(\xi_{a}\right) \in V$ is a set of chosen basis for $V$, and we expand $\phi:=\phi^{a} \xi_{a}$ so that $\left(\rho_{*} E\right)_{b}^{a}$ denotes the matrix representation of $\rho_{*} E$ under the basis $\xi_{a}$. Thus A.5) in view of the invariance of the Lagrangian under $G$ leads (on shell) to

$$
\begin{equation*}
\operatorname{Div}(J)=D_{\partial_{k}} J^{k} \equiv 0 \tag{A.8}
\end{equation*}
$$

In particular, if we take the Lorentz symmetry with $G=S O(1,3)$, we obtain a conserved current corresponding to each $E_{i j}$ generator in $\mathfrak{s o}(1,3)$, i.e,

$$
\begin{equation*}
J^{k}=\frac{\partial \mathcal{L}}{\partial \phi_{; k}^{a}}\left(\rho_{*} E_{i j}\right)_{b}^{a} \phi^{b} \tag{A.9}
\end{equation*}
$$

which is called the canonical spin angular momentum or (simply spin current).

$$
\begin{equation*}
S_{i j}^{k}=\left(\frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \phi_{; k}^{a}}\right) \rho_{[i j] b}^{a} \cdot \phi^{b}=-S_{j i}^{k}, \quad\left(\text { where } \rho_{[i j] b}^{a}:=\left(\rho_{*} E_{i j}\right)_{b}^{a}\right) \tag{A.10}
\end{equation*}
$$

It follows that we have $\rho_{[i j] b}{ }^{a}:=\left(\rho_{*} E_{i j}\right)_{b}^{a}$ due to the property $E_{i j}=-E_{j i}(\forall i, j)$ for generators of $\mathfrak{s o}(1,3)$. Notice that the indices $a, b$ and $i, j$ are indicating different objects, which should be distinguished. Indices $a, b$ correspond to the vector basis $\left(e_{a}\right) \in V$, hence the matrix indices, while $i, j$ specify which Lie algebra basis of $\mathfrak{g}$ we are referring to, not the spacetime coordinate.

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[^0]:    ${ }^{1}$ The associated adjoint map $\mathfrak{a d _ { g }}: \mathfrak{g} \rightarrow \mathfrak{g}$ is defined as the differential map of $A d_{g}: G \rightarrow G$ at $e \in G$, where $A d_{g}(h):=g h g^{-1}$. Specifically, let $a \in \mathfrak{g}$ find a local curve $t \mapsto c(t)$ on $G$ such that $c(0)=e$, $c^{\prime}(0)=a$, then define $\mathfrak{a} \mathfrak{d}_{g}(a)=\left.\frac{d}{d t}\left(g c(t) g^{-1}\right)\right|_{t=0}$
    ${ }^{2}$ The Semi-Riemannian geometry denotes a differentiable manifold $M$ and a Lorentz metric $g$ that has one negative signature.

[^1]:    $3 "$ Algebraic" means only operations of,,$+- \times, \div$ and $\sqrt[n]{\cdot}$ are involved, especially not differential or integral operators.

[^2]:    ${ }^{1}$ In fact, before Hilbert gave the Lagrangian (2.50) Einstein took one different from the scalar curvature $R$ by a total differential and was not invariant under changes of the coordinate frame. Nevertheless, the corresponding field equations under the variation can still possess Lorentz invariance property.

[^3]:    ${ }^{1}$ Around 1914, Nordström proposed a similar 5D unified field theory attempt of gravitation and Maxwell's electromagnetism before Kaluza and Klein, see 40.

