The $Spin(4)$ gauge theory of space, time, gravitation, matter and dark matter

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A novel gauge-theoretic framework for spacetime and gravitation is proposed, in which a *Cartan khronon* field dynamically breaks the symmetry between space and time, enabling the emergence of temporality within a fundamentally Euclidean setting. Based on a $Spin(4)$ gauge structure, the theory provides a real-valued formulation of chiral spacetime, wherein the effects typically attributed to dark matter could instead be explained by the dynamics of gravitation. New results are presented with implications for a broad range of phenomena, including cosmology, large-scale structure, gravitational waves, black holes, and potential signatures accessible to laboratory experiments. By avoiding the pitfalls of complexification and reinterpreting chiral spacetime geometry through a real, dynamical, and gauge-theoretic lens, the *Cartan khronon* theory offers a fresh and compelling framework for revisiting the foundations of spacetime and gravitation.

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I. INTRODUCTION

The Euclidean path integral is a powerful and widely used tool in modern theoretical physics, particularly in the study of quantum field theories. The formulation and development of gauge theories, such as quantum electrodynamics and quantum chromodynamics have relied heavily on this method. A more rigorous and mathematically well-defined formulation of the path integral for the fields in the standard model of particle physics, indispensable in the study of non-perturbative phenomena, is often only possible in Euclidean space. The bridge between the results obtained in Euclidean space and the observables in Minkowski spacetime is established via a Wick rotation, an analytic continuation that relates Schwinger functions in the Euclidean regime to Wightman functions in the Lorentzian regime [1].

We may also recall that the quantum fields that describe particles in the standard model are most naturally associated with unitary representations not of the non-compact Lorentz group directly, but of the compact Euclidean rotation group, via the so-called unitarian trick. This mathematical structure further underscores the central role that Euclidean space plays in the formulation of quantum field theory.

In light of this foundational importance, it is natural to entertain the possibility that the Euclidean formulation of physics offers not merely a convenient mathematical tool, but perhaps a more fundamental description of reality itself. An expression of this profound perspective appears in a 1978 lecture, reprinted in [2]:

> In fact, one could take the attitude that quantum theory and indeed the whole of physics is really defined in the Euclidean region and that it is simply a consequence of our perception that we interpret it in the Lorentzian regime.

Yet, integrating gravitation into this Euclidean framework introduces significant challenges. Unlike the gauge theories of particle physics, which are formulated on fixed spacetime backgrounds, gravity governs the dynamics of spacetime itself. The principle of general covariance in general relativity implies that there is no preferred time coordinate, complicating the straightforward application of Euclidean techniques to gravitational theories. Stephen Hawking immediately acknowledged this difficulty [2]:

> I feel that one should adopt a similar Euclidean approach in quantum gravity and supergravity. Of course one cannot simply replace the time coordinates by imaginary quantities because there is no preferred set of time coordinates in general relativity.

The “problem of time” is a generic structural issue in quantum gravity [3, 4], not unique to the Euclidean approach. However, the Euclidean context makes it particularly acute, as the formalism lacks any time direction, distinguished or otherwise, from the outset. While the Euclidean path integral remains a powerful tool, its application to gravity forces a confrontation with the fact that time in general relativity is not a background parameter but rather a subtle concept whose operational definition remains challenging and tied to the choice of coordinates.

It is natural to seek a pregeometric foundation for a Euclidean description of spacetime, particularly in formulations where metric geometry is not fundamental but composed from a primordial spinor [5, 6]. In this framework, the spacetime manifold and its geometric properties, such as distance and angle, emerge from bilinear combinations of a fundamental spinor field, without presupposing a metric structure [7, 8].

This is consistently realised in the new kind of gauge theory that addresses the longstanding problem of time by allowing the spinor bilinear to spontaneously select a direction in an abstract 4-dimensional space, thereby giving rise to the arrow of time as an emergent feature [9]. We propose that in such a theory, the distinction between Euclidean and Lorentzian signatures can be understood as a matter of perception arising from how the spinorial substrate is interpreted. The new Lorentz gauge theory appears to be a rather unique setting in which such an approach is possible, or at the very least, it stands out as minimal in terms of assumptions. Conventional gauge theories of gravity often require a proliferation of auxiliary fields. The common framework of Poincaré gravity models [10, 11] can also be formulated in terms of Yang-Mills -like gauge fields and Higgs-like scalar field [12, 13], though it is perhaps best motivated as a contraction limit of the framework of anti/de Sitter gravity models which are at least based on a semi-simple symmetry group [14, 15]. However, on top of the spin connection, this requires 5+4×4 extra field components for the symmetry breaking field + the translation/extra-dimensional rotation gauge fields, and moreover, in order to describe the chiral aspect of spacetime, complexification is required in the conventional frameworks which thereby moreover doubles the total number of real field components. In contrast, the Lorentz gauge theory introduces, in addition to the spin connection, only the 4 real components of the *Cartan khronon* scalar field ϕ^I and grounds spacetime and gravity on the semi-simple and compact gauge group[1]. The khronon ϕ^I with the $SO(4)$ index I can be regarded as a bilinear formed from the primordial spinor and the symmetry group taken as the double-cover $Spin(4)$.

Despite its technical elegance, the main advantage of this formulation is its robust and unambiguous treatment of the flow of time, achieved with a minimal field content. A further conceptual clarification lies in recognising that the units used to measure time and energy are fundamentally arbitrary, and their operational relation to the underlying fields - such as the dimensionless ϕ^I - can be, in a precise technical sense, "imaginary". While the ϕ^I itself is a real scalar, assigning physical units to its evolution involves a transformation familiar from Wick rotation. This does not imply that the coordinates themselves are imaginary, but that the relation between unit conventions and field dynamics is encoded in an abstract, complex-valued structure. The recipe is as follows:

- Introduce the symmetry breaking as $\phi^I = \phi\delta^I_4$.
- Solve the equations of motion for the spin connection.
- Introduce the Euclidean time coordinate $\tau = \sqrt{\kappa}\phi$.
- Introduce the Lorentzian time $t \leftarrow -i\tau$, so that $\phi = m_P t$.

The $\leftarrow$ denotes, not an equality but *substitution*. In particular, both τ and t are real, and similarly, both m_P and $\kappa^{-\frac{1}{2}}$ are real units of energy (in our convention the speed of light $c \equiv 1$, and as the notation suggests, m_P will be identified with the Planck mass). We could generalise the final step of the recipe by allowing rotations by an arbitrary angle in the complex plane, but we omit this as it is unnecessary for the purposes of the present article. Moreover, the choice of ϕ being linear in time(s) in the last two steps was made solely for clarity; the formulation remains manifestly coordinate-invariant throughout, and we will in fact implement the recipe with $\phi(x)$ taken as a generic function of the coordinates in the course of this work. Readers who prefer a visual roadmap may wish to glance at the schematic diagram in section VIII, which summarises the structure of the theory and the implementation of the above recipe.

For concreteness, consider the example of a flat radiation-dominated Friedmann-Lemaître-Robertson-Walker (FLRW) universe, characterised by a scale factor that grows like the square root of the time coordinate. The square of the dimensionless scale factor can then be taken to be ϕ, choosing an appropriate normalisation. So the line element in this case reads $\mathrm{d}s^2 = \mathrm{d}\tau\mathrm{d}\tau + \phi\mathrm{d}x_i\mathrm{d}x^i = -\mathrm{d}t\mathrm{d}t + \phi\mathrm{d}x_i\mathrm{d}x^i$ in terms of the Euclidean time τ and the Lorentzian time t, respectively. The line element is manifestly real in both cases. If we insisted on using the dimensionless field ϕ as the time coordinate, we would write the line element as $\mathrm{d}s^2 = \kappa\mathrm{d}\phi\mathrm{d}\phi + \phi\mathrm{d}x_i\mathrm{d}x^i = -m_P^{-2}\mathrm{d}\phi\mathrm{d}\phi + \phi\mathrm{d}x_i\mathrm{d}x^i$. This illustrates that the Wick-rotated picture can be interpreted as the Lorentzian observers living in imaginary space dimension or, equivalently, as measuring distances in imaginary energy units from the Euclidean perspective (or mutatis mutandis from the opposite perspective). A yet third expression of the same element in the two "frames" is $\mathrm{d}s^2 = \mathrm{d}\tau\mathrm{d}\tau + \kappa^{-\frac{1}{2}}\tau\mathrm{d}x_i\mathrm{d}x^i = -\mathrm{d}t\mathrm{d}t + m_P t\mathrm{d}x_i\mathrm{d}x^i$. Both "frames" are dynamically equivalent descriptions of the

[1] For a recent different take on compact gravity structure group, see [16].

same field configurations, and as long as one doesn't confuse the units in one "frame" with the units in the other, both descriptions are real. Whereas the radiation-dominated example serves to illustrate the simple idea, across this article we'll construct much more nontrivial examples from first principles, including an exact rotating solution and the perturbed FLRW universe without any symmetries, and we'll see in all these examples that the physics can be consistently described in both the real Euclidean and the real Lorentzian "frames". Therefore we will be confident to conclude that the above recipe is universally applicable.

Significant progress and many interesting results have been obtained within the Euclidean quantum gravity programme, see e.g. [2, 17, 18]. However, these advances often rely on complexified coordinates and/or metrics, coordinate-dependent (or tetrad-frame-dependent) constructions, or other auxiliary prescriptions introduced to recover the desired aspects of Lorentzian dynamics [19–27]. Such strategies, while often effective within specific contexts, reflect the fact that a fully coherent and compelling formulation of Euclidean quantum gravity remains elusive at the foundational level. One of the approaches considered is "the real way" wherein one introduces an extended action parameterised by two real numbers providing a handle to interpolate between Euclidean and Lorentzian solutions [28]. Ref.[29] defines a mapping W on phase space spanned by E, K such that[2] for functions $f(E, K)$ on phase space we have $W(f(E, K)) = f(iE, -iK)$. A dynamical approach has been considered in the literature as well, wherein the gravitational metric is Euclidean but has a disformal relation, via a vector field or a gradient of a scalar field, to the Lorentzian spacetime metric which couples to matter [30–37]. In this approach, an extra field distinguishes the direction of time allowing a mapping from the postulated Riemannian to a pseudo-Riemannian metric. In our pregeometric theory however, the metric is not postulated but constructed from the Lorentz-covariant derivatives of the khronon field ϕ^I as a composite field. Perhaps closest to our approach is the Wick rotation in tangent space implemented upon the zeroth component of tetrad frame [20] thus mapping real Lorentz metrics to real Euclidean metrics; in this proposal the time coordinate ambiguity is translated into Lorentz frame ambiguity[3].

In what follows, we develop the theoretical structure and physical implications of the $Spin(4)$ gauge theory. After introducing the basic mathematical ingredients, particularly the chiral decomposition and the construction of the covariant field strengths, we deduce the action principle that governs the dynamics of the Cartan khronon and the gauge connection. This sets the stage for a systematic exploration of the theory's vacuum structure, cosmological solutions, and spherically symmetric configurations. To demonstrate the theory in a concrete and tractable setting, we focus in section III on the case right-handed gravity, which serves as a proving ground for key conceptual features, including the emergence of time and the recovery of standard gravitational dynamics. Sections IV and V are devoted to cosmology, where we analyse the general theory in full generality, constructing the perturbation theory systematically up to linear order. Section VI turns to the spherically symmetric case, laying the groundwork for further exploration of exact inhomogeneous configurations. Matter couplings and spinor structures are addressed in VII. For a full overview of the material, we refer the reader to the table of contents above. As the examples will show, the proposed framework enables a consistent real-valued description of spacetime, gravitational dynamics, and dark matter phenomena. We close in section VIII with an outlook on near-term observational handles that could falsify or corroborate the theory.

II. FORMULATION OF THE THEORY

In this section, we first introduce the relevant mathematical formalism that allows to split the $Spin(4)$ gauge connection into two $SU(2) \times SU(2)$ connections, based on the reducibility of the Euclidean Lorentz algebra $\mathfrak{so}(4) = \mathfrak{so}(3) + \mathfrak{so}(3)$. Then we consider the possible action principles that can be constructed from the khronon field ϕ^I using the gauge-covariant derivative.

A. The reducibility of the Lorentz group

A generic object X in the adjoint representation of the Euclidean Lorentz algebra $\mathfrak{so}(4)$ is specified by its components $X^{IJ} = -X^{JI}$, where the indices $I, J, \dots$ run from 1 to 4. The object can be split into its self-dual and anti-self-dual components by using the appropriate projectors. These projectors are

$$P_{\pm}^{IJ}{}_{KL} \equiv \frac{1}{2}\left(\delta_K^{[I}\delta_L^{J]} \pm \frac{1}{2}\epsilon^{IJ}{}_{KL}\right) \equiv \frac{1}{2}\left(1 \pm \star\right)^{IJ}{}_{KL}\,, \tag{1}$$

[2] More explicitly, the triad and its conjugate would transform as $E_i^a \to iE_i^a$, $K_a^i \to -iK_a^i$. This can be interpreted as flipping the sign of the 'space space' part of the metric in order to change signature.

[3] This reminds of the situation in the context of the conjugate problem, "the problem of energy" in general-relativistic physics: though it was realised a long time ago that in a (teleparallel) tetrad formulation one obtains coordinate-invariant expressions for energy-momenta, the issue of frame-noncovariance was resolved only recently, see [38, 39] for some discussions and many references.

and we denote ${}^{\pm}X^{IJ} \equiv P_{\pm}^{IJ}{}_{KL}X^{KL}$. It follows that $\star\,{}^{\pm}X^{IJ} = \pm\,{}^{\pm}X^{IJ}$, which is why they are called self-dual and anti-self-dual, respectively. One can easily check that the $P_{\pm}^{IJ}{}_{KL}$ indeed are projectors, i.e. they are complete, $X = {}^{+}X + {}^{-}X$, orthogonal, ${}^{+-}X = {}^{-+}X = 0$ and idempotent, ${}^{++}X = {}^{+}X$, ${}^{--}X = {}^{-}X$. In the definitions (1), we have referred to the two $SO(4)$-invariant structures, the Kronecker metric δ^{I}_{J} and the totally antisymmetric Levi-Civita symbol $\epsilon^{IJ}{}_{KL}$, for which our conventions are such that $\delta^{I}_{J} = \mathrm{diag}(1,1,1,1)^{I}{}_{J}$ and $\epsilon^{12}{}_{34} = 1$. From now on, throughout this article, we may nonchalantly lower and raise $SO(4)$ indices as this is always understood wrt ($\equiv$ with respect to) the Kronecker metric δ_{IJ}.

The generators o_{IJ} of the algebra

$$[o_{IJ}, o_{KL}] = 4\delta_{[L[I}o_{J]K]} = 2\left(\delta_{L[I}o_{J]K} - \delta_{K[I}o_{J]L}\right) = \delta_{LI}o_{JK} - \delta_{LJ}o_{IK} - \delta_{KI}o_{JL} + \delta_{KJ}o_{IL}\,, \tag{2}$$

could be represented with the antisymmetric matrices $(o_{IJ})^{K}{}_{L} = 2\delta^{K}_{[I}\delta_{J]L}$; or as well like $o_{IJ} = 2x_{[I}\partial_{J]}$; in VII A we will recall the appropriate spinor representation; and in the appendix A we consider a quaternionic matrix representation.

There's no canonical split into rotations and boosts, the *Cartan khronon* will be needed for this. At this point, we'll just choose to denote rotations around the 4th axis in a different way, and introduce the small Latin letters for the other three indices, so that $i, j, \ldots$ run from 1 to 3. Let us also introduce 3-dimensional totally antisymmetric Levi-Civita symbol ϵ_{ijk} for which our convention is $\epsilon_{123} = 1$, and allow ourselves to freely raise and lower the 3-dimensional indices with the 3-dimensional Kronecker metric δ_{ij}. Then we define

$$o^{i} \equiv \frac{1}{2}\epsilon^{4ijk}o_{jk} = -\frac{1}{2}\epsilon^{ijk}o_{jk}\,, \tag{3a}$$

$$b^{i} \equiv -o_{4}{}^{i}\,, \tag{3b}$$

so that we can rewrite the algebra (2) as

$$[o^{i}, o^{j}] = \epsilon^{ijk}o_{k}\,, \quad [o^{i}, b^{j}] = \epsilon^{ijk}b_{k}\,, \quad [b^{i}, b^{j}] = \epsilon^{ijk}o_{k}\,. \tag{4}$$

Then as usual we define the linear combinations of the generators,

$$l^{i} \equiv \frac{1}{2}\left(o^{i} + b^{i}\right)\,, \tag{5a}$$

$$r^{i} \equiv \frac{1}{2}\left(o^{i} - b^{i}\right)\,, \tag{5b}$$

so that we can yet rewrite (4) as

$$[l^{i}, l^{j}] = \epsilon^{ijk}l_{k}\,, \quad [r^{i}, r^{j}] = \epsilon^{ijk}r_{k}\,, \quad [l^{i}, r^{j}] = 0\,. \tag{6}$$

Now going back to (1), we can verify that

$${}^{+}o^{4i} = \frac{1}{2}\left(o^{4i} - \frac{1}{2}\epsilon^{i}{}_{jk}o^{jk}\right) = \frac{1}{2}\left(-b^{i} + o^{i}\right) = r^{i}\,, \tag{7a}$$

$${}^{-}o^{4i} = \frac{1}{2}\left(o^{4i} + \frac{1}{2}\epsilon^{i}{}_{jk}o^{jk}\right) = \frac{1}{2}\left(-b^{i} - o^{i}\right) = -l^{i}\,. \tag{7b}$$

Since parity reverses boosts (electric/polar/odd) but not rotations (magnetic/axial/even), it is clear that the parity operator

$$P_{\leftrightarrow}^{IJ}{}_{KL} = \mathrm{diag}(-1,-1,-1,1)^{I}_{K}\mathrm{diag}(-1,-1,-1,1)^{J}_{L}\,, \tag{8}$$

flips $l^{i} \leftrightarrow r^{i}$. It then seems that $P_{\leftrightarrow}P_{\pm} = -P_{\mp}$, but it does hold that $P_{\leftrightarrow}\star P_{\pm} = \star P_{\mp}$. Note that there inevitably exist parity-duality-odd objects X and parity-duality-even objects $\star X$. Which is which may depend on specific conventions, but either choice can be implemented consistently.

To be explicit, we can decompose any element $X \in \mathfrak{so}(4)$ as

$$\begin{aligned} X &= \frac{1}{2}X^{IJ}o_{IJ} = \frac{1}{2}\left({}^{+}X^{IJ}\,{}^{+}o_{IJ} + {}^{-}X^{IJ}\,{}^{-}o_{IJ}\right) = {}^{+}X^{4i}\,{}^{+}o_{4i} + \frac{1}{2}\,{}^{+}X^{ij}\,{}^{+}o_{ij} + {}^{-}X^{4i}\,{}^{-}o_{4i} + \frac{1}{2}\,{}^{-}X^{ij}\,{}^{-}o_{ij} \\ &= \left({}^{+}X^{4k} - \frac{1}{2}\,{}^{+}X^{ij}\epsilon^{k}{}_{ij}\right){}^{+}o_{4k} + \left({}^{-}X^{4k} + \frac{1}{2}\,{}^{-}X^{ij}\epsilon^{k}{}_{ij}\right){}^{-}o_{4k} \equiv {}^{+}X^{i}r_{i} + {}^{-}X^{i}l_{i}\,, \end{aligned} \tag{9a}$$

where

$$ {}^{\pm}X^{i} = \pm 2\,{}^{\pm}X^{4i} = -\epsilon^{i}{}_{jk}\,{}^{\pm}X^{jk} = \pm X^{4i} - \frac{1}{2}\epsilon^{i}{}_{jk}X^{jk}\,. \tag{9b} $$

These are our conventions for representing the X equivalently in the adjoint representation of $\mathfrak{so}(3)+\mathfrak{so}(3)$ (which in this case is equivalent to the fundamental representation). We may refer to this representation as the left/right basis.

Had we chosen instead the $SO_{\mathbb{C}}(1,3)$ structure group, the differences would begin from (1), which would be modified like $\star \to -i\star$, the imaginary unit i being required for consistency of the projections in the Lorentzian regime. In this article we stick to the all-plusses signature and to a strictly real formulation; for a pedagogical review of formulations with different signatures, see [40].

B. The gauge field and field strength

The gauge field, also known as the connection, plays a main role in gauge theory. The connection is a 1-form, and we will denote as $\boldsymbol{A}$. For clarity, we use bold symbols for n-forms when $n>0$. Valued in the adjoint representation of the $\mathfrak{so}(4)$ algebra, $\boldsymbol{A} = \frac{1}{2}\boldsymbol{A}^{IJ}o_{IJ}$, and it is conventional to refer to $\boldsymbol{A}$ as the spin connection (though conventionally, it takes values in the $\mathfrak{so}(1,3)$ algebra). According to the development in subsection II A above, the components of the spin connection in the left/right basis are related to its components in the anti/self-dual basis as

$$ {}^{\pm}\boldsymbol{A}^{1} = \pm\boldsymbol{A}^{41} - \boldsymbol{A}^{23} = \pm 2\,{}^{\pm}\boldsymbol{A}^{41} = \mp 2\,{}^{\pm}\boldsymbol{A}^{23}\,, \tag{10a} $$
$$ {}^{\pm}\boldsymbol{A}^{2} = \pm\boldsymbol{A}^{42} + \boldsymbol{A}^{13} = \pm 2\,{}^{\pm}\boldsymbol{A}^{42} = \pm 2\,{}^{\pm}\boldsymbol{A}^{13}\,, \tag{10b} $$
$$ {}^{\pm}\boldsymbol{A}^{3} = \pm\boldsymbol{A}^{43} - \boldsymbol{A}^{12} = \pm 2\,{}^{\pm}\boldsymbol{A}^{43} = \mp 2\,{}^{\pm}\boldsymbol{A}^{12}\,. \tag{10c} $$

The connection defines the gauge-covariant derivative $\mathbf{D} = \mathbf{d} + \boldsymbol{A}$, wherein the $\mathbf{d}$ is the usual exterior derivative. For example, consider the $SO(4)$-covariant derivative of a particular component of an anti/self-dual scalar (meaning a 0-form) element ${}^{\pm}X^{IJ}$,

$$ \begin{aligned} 2\mathbf{D}\,{}^{\pm}X^{41} &= \mathbf{D}\left(\pm X^{41} - X^{23}\right) = \pm\left(\mathbf{d}\boldsymbol{X}^{41} + \boldsymbol{A}^{4}{}_{2}X^{21} + \boldsymbol{A}^{4}{}_{3}X^{31} + \boldsymbol{A}^{1}{}_{2}X^{42} + \boldsymbol{A}^{1}{}_{3}X^{43}\right) \\ &- \left(\mathbf{d}X^{23} + \boldsymbol{A}^{2}{}_{1}X^{13} + \boldsymbol{A}^{2}{}_{4}X^{43} + \boldsymbol{A}^{3}{}_{1}X^{21} + \boldsymbol{A}^{3}{}_{4}X^{24}\right) \\ &= \mathbf{d}\left(\pm X^{41} - X^{23}\right) + {}^{\pm}\boldsymbol{A}^{2}\left(X^{21} \pm X^{43}\right) + {}^{\pm}\boldsymbol{A}^{3}\left(X^{31} \pm X^{24}\right) \\ &= \mathbf{d}\,{}^{\pm}X^{1} + {}^{\pm}\boldsymbol{A}^{2}\,{}^{\pm}X^{3} - {}^{\pm}\boldsymbol{A}^{3}\,{}^{\pm}X^{2}\,. \end{aligned} \tag{11} $$

This demonstrates that for any $\boldsymbol{X}^{IJ}$ it holds that

$$ \pm 2\mathbf{D}\,{}^{\pm}\boldsymbol{X}^{4i} = {}^{\pm}\mathbf{D}\,{}^{\pm}\boldsymbol{X}^{i} = \mathbf{d}\,{}^{\pm}\boldsymbol{X}^{i} + \epsilon^{i}{}_{jk}\,{}^{\pm}\boldsymbol{A}^{j}\wedge{}^{\pm}\boldsymbol{X}^{k}\,. \tag{12} $$

We don't need different notations for the covariant derivative acting on different objects: the connection coefficients are determined by the covariance group of the respective object. So, for example

$$ \mathbf{D}\,{}^{\pm}\boldsymbol{X}^{i} = \mathbf{d}\,{}^{\pm}\boldsymbol{X}^{i} + \epsilon^{i}{}_{jk}\,{}^{\pm}\boldsymbol{A}^{j}\wedge{}^{\pm}\boldsymbol{X}^{k}\,, \tag{13a} $$
$$ \mathbf{D}\boldsymbol{X}^{I} = \mathbf{d}\boldsymbol{X}^{I} + \boldsymbol{A}^{I}{}_{J}\wedge\boldsymbol{X}^{J}\,, \tag{13b} $$

and this generalises to arbitrary tensors with mixed indices as well. With the connections, one has to recall that they're not covariant, and therefore the covariant derivatives of their components do not make sense, but the components of their covariant derivatives are well-defined. The latter $\boldsymbol{F}\equiv\mathbf{D}\boldsymbol{A}$ are called the gauge field strengths, or, in more geometrical language, the curvatures. The components of the curvatures are given as

$$ {}^{\pm}\boldsymbol{F}^{i} \equiv (\mathbf{D}\,{}^{\pm}\boldsymbol{A})^{i} = \mathbf{d}\,{}^{\pm}\boldsymbol{A}^{i} + \frac{1}{2}\epsilon^{i}{}_{jk}\,{}^{\pm}\boldsymbol{A}^{j}\wedge{}^{\pm}\boldsymbol{A}^{k} = \pm 2\,{}^{\pm}\boldsymbol{F}^{4i}\,, \tag{14a} $$

where, as usual[4]

$$ \boldsymbol{F}^{IJ} \equiv (\mathbf{D}\boldsymbol{A})^{IJ} \equiv \mathbf{d}\boldsymbol{A}^{IJ} + \boldsymbol{A}^{I}{}_{K}\wedge\boldsymbol{A}^{KJ}\,, \tag{14b} $$

and the covariant derivative of any curvature is zero, i.e. $\mathbf{D}\boldsymbol{F}=0$. This is due to the Jacobi identity satisfied by Lie algebras, and its consequences in geometry are known as Bianchi identities.

[4] The $\equiv$ symbol is appropriate between the 2nd and the 3rd expressions from a geometrical perspective, so we have included it for clarity, though from an algebraic perspective wherein multiplication is understood as commutation, i.e. $\mathbf{D}\boldsymbol{A} = [\mathbf{d},\boldsymbol{A}] + [\boldsymbol{A},\boldsymbol{A}]$, the equality of the 2nd = 3rd expressions follows from (2).

C. The action principle

As mentioned in the introduction, the fundamental field of the theory can be regarded as the (Euclidean) spinor ψ. However, for the purposes of the current article we formulate the theory directly in terms of $\phi^I = \bar{\psi}\gamma^I\phi$ (the spinor notations are clarified in section VII), so that the *Cartan khronon* field ϕ^I is a dimensionless scalar in the fundamental representation of the Euclidean Lorentz torsor. By torsor, we simply mean the group without an identity element, and the physical relevance of this distinction is that the field ϕ^I lives in a space without an origin. Since only differences in ϕ^I are physical, the action should not change under constant shifts of the field. In particular, we will interpret the khronon as a clock field which measures the lapse of time in terms of relative durations between events rather than giving an absolute reading associated with a given event. Thus, we require invariance of the action principle under global translations, defined as $\phi^I \rightarrow \xi^I$, wherein $\mathbf{D}\xi^I = \mathbf{d}\xi^I + \boldsymbol{A}^I{}_J\xi^J = 0$.

This restriction taken into account, the building blocks for the action principle are the 1-form $\mathbf{D}\phi^I$, and the two independent 2-forms ${}^{\pm}\boldsymbol{F}^{IJ}$. Of course, the action principle should be $SO(4)$-invariant as well, and in 4 spacetime dimensions, the action should be an integral over a 4-form. In the end, we are then allowed only a 5-parameter action. This action turns out to contain only even powers of the 1-form $\mathbf{D}\phi^I$, and therefore has no danger of breaking potential reflection symmetries in the matter sector, as it realises the Z_2 symmetry[5] under $\phi^I \rightarrow -\phi^I$.

Let us consider this in detail. Firstly, without the khronon field we may only write down the two independent 4-form invariants,

$$\begin{aligned} I_{(0)} &= g_G \int \epsilon_{IJKL}\boldsymbol{F}^{IJ}\wedge\boldsymbol{F}^{KL} + g_P\int \boldsymbol{F}^{IJ}\wedge\boldsymbol{F}_{IJ} \\ &= (g_P + 2g_G)\int {}^{+}\boldsymbol{F}^{IJ}\wedge{}^{+}\boldsymbol{F}_{IJ} + (g_P - 2g_G)\int {}^{-}\boldsymbol{F}^{IJ}\wedge{}^{-}\boldsymbol{F}_{IJ} \\ &= \oint (g_G\epsilon_{IJKL} + g_P\eta_{IK}\eta_{JL})\left(\boldsymbol{A}^{IJ}\wedge\boldsymbol{F}^{KL} - \frac{1}{3}\boldsymbol{A}^{IJ}\wedge\boldsymbol{A}^{K}{}_{M}\wedge\boldsymbol{A}^{ML}\right) . \end{aligned} \tag{15a}$$

In the second line we have expressed the action in terms of the anti/self-dual curvatures, and in the third line shown explicitly how the action reduces to a boundary term. Thus, the 4-form integrand in $I_{(0)}$ is a total derivative of a 3-form. The g_G-term is known as the Gauss-Bonnet term and the g_P-term is known as the Pontryagin term. Note that we cannot write these terms in the left/right basis, considering 4-forms like $\sim \int {}^{\pm}\boldsymbol{F}^i \wedge {}^{\pm}\boldsymbol{F}_i$, because no canonical split is available.

We saw that only topological field theories may be constructed without the building block $\mathbf{D}\phi^I$. With that at hand, we can build the 4-form integrand,

$$I_{(4)} = \frac{\lambda}{4!}\int \epsilon_{IJKL}\mathbf{D}\phi^I\wedge\mathbf{D}\phi^J\wedge\mathbf{D}\phi^K\wedge\mathbf{D}\phi^L\,. \tag{15b}$$

This will turn out to correspond to a cosmological constant term. To have dynamics for the spin connection, we need to consider yet the two quadratic terms contained in

$$I_{(2)} = \frac{1}{2}\int \epsilon_{IJKL}\mathbf{D}\phi^I\wedge\mathbf{D}\phi^J\left(g_+\,{}^{+}\boldsymbol{F}^{KL} + g_-\,{}^{-}\boldsymbol{F}^{KL}\right) = \int \mathbf{D}\phi^I\wedge\mathbf{D}\phi^J\wedge\left(g_+\,{}^{+}\boldsymbol{F}_{IJ} - g_-\,{}^{-}\boldsymbol{F}_{IJ}\right) . \tag{15c}$$

This parameterisation, introducing the two coupling constants $g_{\pm}$ for the anti/self-dual field strengths ${}^{\pm}\boldsymbol{F}^{IJ}$, respectively, was introduced in [41]. It is easy to see that (15) completely exhausts all possible actions which are compatible with the local $SO(4)$ symmetry and the global $T(4)$ invariance.

If we did not insist on the latter, the action densities in (15) could be multiplied by arbitrary polynomials of the scalar singlet $\phi^I\phi_I$. We also note that if the global shift invariance was allowed to be respected only up to a boundary term, we could also consider the actions

$$\tilde{I}_{(2)} = -\int \phi^I\phi^J\boldsymbol{F}^K{}_J\wedge\left(g_+\,{}^{+}\boldsymbol{F}_{IK} - g_-\,{}^{-}\boldsymbol{F}_{IK}\right) \overset{\mathrm{b}}{=} I_{(2)}\,, \tag{16a}$$

and

$$\tilde{I}_{(4)} = -\frac{\lambda}{8}\int \epsilon_{IJKL}\phi^I\phi^M\boldsymbol{R}^J{}_M\wedge\mathbf{D}\phi^K\wedge\mathbf{D}\phi^L \overset{\mathrm{b}}{=} I_{(4)}\,, \tag{16b}$$

[5] The clock field in the models of Mukohyama *et al* also implement both a shift symmetry and a reflection symmetry [33, 37].

where $\stackrel{\rm b}{=}$ means equivalence up to a boundary term. If we broke the shift invariance with functions of $\phi^I\phi_I$, this equivalence would also be broken, and such generalisations of (16) would thus introduce yet new classes of extended theories. However, in this article, we insist on the exact shift invariance. Note that this invariance may not be thought of as a symmetry. It is a transformation that leaves the action invariant (in the cases (16) only up to boundaries) even when the equations of motion don't apply, but the constraint $\mathbf{D}\xi^I = 0$ means that the parameters ξ^I may ultimately depend non-locally on the connection field $\boldsymbol{A}$, and this can manifest in the equations of motion not being invariant under this transformation [42]. Further, as constant ξ^I even exists only in some special cases.

To recapitulate, the most general action for the $Spin(4)$ gauge theory is dictated by the symmetry principle and includes the five dimensionless parameters g_G, g_P, $g_\pm$ and λ. The action is a functional of the two fields, $\boldsymbol{A}$ and ϕ. Adding the possible source terms for these fields, $\boldsymbol{t}_I$ and $\boldsymbol{O}_{IJ} = -\boldsymbol{O}_{JI}$ respectively, the action is

$$I = I_{(0)} + I_{(2)} + I_{(4)} - \int \mathbf{D}\phi^I \wedge \boldsymbol{t}_I - \int \boldsymbol{A}^{IJ} \wedge \boldsymbol{O}_{IJ}\,, \tag{17}$$

where the $I_{(0)}$, $I_{(2)}$ and $I_{(4)}$ are as in (15). The dynamics of the theory only depend on the coupling constants $g_\pm$ and λ.

III. RIGHT-HANDED GRAVITY

In this section, we illustrate the workings of the pre-geometric theory in terms of a simplified model. We set $g_G = g_P = g_- = \lambda = 0$ and let $g_+ = 1$, and also ignore the possible material spin currents described by $\boldsymbol{O}_{IJ}$. Thus, we study the theory

$$I_+ = \int \mathbf{D}\phi^I \wedge \mathbf{D}\phi^J \wedge {}^+\boldsymbol{F}_{IJ} - \int \mathbf{D}\phi^I \wedge \boldsymbol{t}_I\,. \tag{18}$$

We will first investigate the most basic examples of solutions, the vacuum solutions reducing to the Minkowski and the case of flat FLRW (Friedmann-Lemaître-Robertson-Walker) cosmology. These cases have been solved in the previous literature on Lorentz gauge theory, but the important subtlety we introduce in this section is their derivation via Wick rotation from the Euclidean space. We then proceed to study more involved spacetimes and present the explicit Kerr solution in the Lorentz gauge theory.

A. Spacetime structure

The EoM (equations of motion) derived by varying I_+ wrt the two fields, the khronon and the connection, are respectively

$$\mathbf{D}\left(2\,{}^+\boldsymbol{F}_{IJ} \wedge \mathbf{D}\phi^J - \boldsymbol{t}_I\right) = 0\,, \tag{19a}$$

$$\mathbf{D}\,{}^+\left(\mathbf{D}\phi^I \wedge \mathbf{D}\phi^J\right) = 2\phi^{[I}\,{}^+\boldsymbol{F}^{J]} - \phi^{[I}\boldsymbol{t}^{J]}\,. \tag{19b}$$

The khronon field equation (19a) states that a certain 3-form is a (covariantly) closed form. We call this 3-form $\boldsymbol{M}_I$ and use it also in the connection EoM. Thus, we can rewrite the EoM (19) as

$$2\,{}^+\boldsymbol{F}_{IJ} \wedge \mathbf{D}\phi^J = \boldsymbol{t}_I + \boldsymbol{M}_I\,, \tag{20a}$$

$$\mathbf{D}\,{}^+\left(\mathbf{D}\phi^I \wedge \mathbf{D}\phi^J\right) = \phi^{[I}\boldsymbol{M}^{J]}\,, \tag{20b}$$

$$\mathbf{D}\boldsymbol{M}_I = 0\,. \tag{20c}$$

Until here, the equations look identical regardless of whether the gauge group $SO_{\mathbb{C}}(1,3)$ or $SO(4)$ is considered. When breaking the symmetry to $SO(3)$ by imposing that $\phi^I = \phi\delta^I_0$, some signs become different, and most importantly, the imaginary unit which is necessary in the $SO_{\mathbb{C}}(1,3)$ formulation, is now absent.

Using the formalism introduced in II A, we can express the system (20) in the symmetry-broken phase in the form

$${}^+\boldsymbol{F}_i \wedge \mathbf{D}\phi^i = \boldsymbol{t}_4 + \boldsymbol{M}_4\,, \tag{21a}$$

$${}^+\boldsymbol{F}_i \wedge \mathbf{d}\phi - \frac{1}{2}\epsilon_{ijk}\,{}^+\boldsymbol{F}^j \wedge \mathbf{D}\phi^k = -\boldsymbol{t}_i - \boldsymbol{M}_i\,, \tag{21b}$$

$$\left({}^+\boldsymbol{F}_i - {}^-\boldsymbol{F}_i\right) \wedge \mathbf{d}\phi + \epsilon_{ijk}\left({}^+\boldsymbol{F}^j - {}^-\boldsymbol{F}^j\right) \wedge \mathbf{D}\phi^k = 2\boldsymbol{M}_i = 0\,, \tag{21c}$$

$$\mathbf{D}\boldsymbol{M}_I = 0\,. \tag{21d}$$

The two equations in (21b,21c) correspond to the $(4i)$ and the (ij) components of the equation (20b), and their combination sets $\boldsymbol{M}^i = 0$ to zero. Thus, the 3-form $\boldsymbol{M}^I$ is always aligned with ϕ^I in the right-handed gravity model wherein $g_- = 0$. We return to this below.

1. The Bartels frame

At this point, it is convenient to take a further step toward the conventional spacetime structure. We have a distinguished direction for time measured by ϕ, the appearance of the three 1-forms $\mathbf{D}\phi^i$ suggests their identification with the Bartels frame[6], a spatial triad of frames that may span a 4-dimensional spatial metric $\sim \mathbf{D}\phi^i \otimes \mathbf{D}\phi_i$ on the 3-dimensional hypersurfaces orthogonal to the flow of time. The conventional (co)frame fields[7], however, are dimensionless, whereas $\mathbf{D}\phi^i = -\boldsymbol{A}^{4i}\phi$ has the dimension of energy (= the dimension of one per length or one per time unit). Let us simply introduce an arbitrary mass unit, let it be given by $\kappa^{-1/2}$, in terms of which to define the dimensionless Bartels frame. In terms of this mass unit, we can locally identify the khronon field with a coordinate τ in the conventional units of one per energy. Thus,

$$\mathbf{d}\phi \equiv \kappa^{-\frac{1}{2}}\mathbf{d}\tau\,, \tag{22a}$$

$$\mathbf{e}^i \equiv \sqrt{\kappa}\mathbf{D}\phi^i = -\frac{\tau}{2}\left({}^+\boldsymbol{A}^i - {}^-\boldsymbol{A}^i\right)\,. \tag{22b}$$

It follows that the torsion of the Bartels frame is a gauge-invariant measure of the chiral asymmetry,

$$\boldsymbol{T}^i \equiv \mathbf{D}\mathbf{e}^i = -\frac{\tau}{2}\left({}^+\boldsymbol{F}^i - {}^-\boldsymbol{F}^i\right)\,. \tag{23}$$

In the manifestly $SO(4)$-covariant form, the above definitions read

$$\mathbf{e}^I \equiv \sqrt{\kappa}\mathbf{D}\phi^I\,, \tag{24a}$$

$$\boldsymbol{T}^I \equiv \mathbf{D}\mathbf{e}^I = \sqrt{\kappa}\boldsymbol{F}^I{}_J\phi^J\,. \tag{24b}$$

The torsion is always orthogonal to the khronon, since $\phi_I\boldsymbol{T}^I = \sqrt{\kappa}\phi^I\phi^J\boldsymbol{F}_{IJ} = 0$. Note that these identities are kinematical and independent of the dynamics governed by the action chosen for the theory.

2. The Minkowski solution

Now we may return to the dynamics of the minimal right-handed gravity model. Though $\boldsymbol{M}_i = 0$, the conservation equations $\mathbf{D}\boldsymbol{M}^i = \boldsymbol{A}^{i4}\wedge\boldsymbol{M}_4 = -\kappa^{-1/2}\mathbf{e}^i\wedge\boldsymbol{M}_4 = 0$ nevertheless yield nontrivial constraints, and in particular restrict the possibly nonzero component $\boldsymbol{M}^4$ to be given by a single scalar function $\hat{\rho}$ such that (see section 3.1 of Ref.[46] for a detailed derivation)

$$\boldsymbol{M}_4 = -\sqrt{\kappa}\hat{\rho}\star\mathbf{d}\tau\,, \quad \text{where} \quad \star\mathbf{d}\tau = -\frac{1}{6}\epsilon_{ijk}\mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k\,. \tag{25}$$

The formula for $\star\mathbf{d}\tau$ follows directly from the identifications above when $\star$ is understood as the $SO(4)$ Hodge operator that was already used in (1). Now we may put the system of equations (21) in the form

$${}^+\boldsymbol{F}_i\wedge\mathbf{e}^i = \sqrt{\kappa}\boldsymbol{t}_4 - \kappa\hat{\rho}\star\mathbf{d}\tau\,, \tag{26a}$$

$${}^+\boldsymbol{F}_i\wedge\mathbf{d}\tau - \epsilon_{ijk}{}^+\boldsymbol{F}^j\wedge\mathbf{e}^k = -\sqrt{\kappa}\boldsymbol{t}_i\,, \tag{26b}$$

$$\boldsymbol{T}_i\wedge\mathbf{d}\tau + \epsilon_{ijk}\boldsymbol{T}^j\wedge\mathbf{e}^k = 0\,, \tag{26c}$$

$$\mathbf{d}\left(\hat{\rho}\star\mathbf{d}\tau\right) = 0\,. \tag{26d}$$

The conservation equation (26d) shows that $\hat{\rho}$ behaves like the energy density of an ideal dust. The connection EoM (26c) sets the torsion of the right-handed connection to vanish, and then (26a,26b) are expected to reduce to the effective Einstein equations in the presence of an ideal dust source.

---

[6] Martin Bartels (1769-1836) can be regarded as a key figure behind the discoveries of non-Euclidean geometries [43]. He also developed a method of moving frames and, for example, derived the Frenet-Serret formulas [44]. The $\mathbf{e}^i$ that we call the Bartels frame can be obtained as a spatial pullback of the 4-dimensional $\mathbf{e}^I$ which is generally known as the Cartan('s moving) frame, after Elié Cartan (1869-1951) who greatly developed further the existing geometrical methods [45].

[7] On terminology: in this article we call the 1-forms $\mathbf{e}^I$ frame fields. Sometimes tetrads, which would be the inverse vectors of those 1-forms are called the frame and the quartet $\mathbf{e}^I$ is then called the coframe or the cotetrad.

To begin with, it is pertinent to check how we recover the Minkowski spacetime. Consider the antiself-dual vacuum solution for the connection,

$$ {}^{+}\boldsymbol{A}^i = 0 \quad \Rightarrow \quad {}^{+}\boldsymbol{F}^i = 0\,, \tag{27a} $$

$$ {}^{-}\boldsymbol{A}^i = \frac{2}{\tau}\mathbf{d}x^i \quad \Rightarrow \quad {}^{-}\boldsymbol{F}^i = \frac{2}{\tau^2}\left(-\mathbf{d}\tau\wedge\mathbf{d}x^i + \epsilon^i{}_{jk}\mathbf{d}x^j\wedge\mathbf{d}x^k\right)\,. \tag{27b} $$

Due to (27a), the two first equations (26a,26b) are trivially satisfied. The torsion is now $\boldsymbol{T}^i = \frac{1}{2}\,{}^{-}\boldsymbol{F}^i\tau$, which satisfies (26c), and the Bartels frame $\mathbf{e}^i = \mathbf{d}x^i$ describes the Euclidean space. To make contact with the standard pseudo-Riemannian formalism in terms of a metric tensor, we have to reconsider $\mathbf{e}^4$. The strange part now is that we do not identify $\mathbf{e}^4$ with the differential of a time coordinate like $\mathbf{d}t$, but implement the Wick rotation $\mathbf{e}^4 = \mathbf{d}\tau \to i\mathbf{d}t$. Then we can identify the invariant $\mathbf{e}^I\otimes\mathbf{e}_I$ with the usual Minkowski metric. The imaginary unit only enters into the operational relation between a dynamical field and the time-like component of an observer's frame. All fields and all coordinates are real both before and after the Wick rotation[8]. In practical terms, this is equivalent to setting $\mathbf{d}t \equiv \mathbf{e}^0$, where $\mathbf{e}^0$ is a frame component in a tangent space with the Minkowski metric η_{AB}. Constructing the robust differential-geometric formalism for such a mapping between different frame bundles could be interesting. However, towards the aim of this article, we rather focus on demonstrating the practical workings of the theory with more nontrivial physical examples.

3. *Contrasting with SO(1,3)*

In the $SO_{\mathbb{C}}(1,3)$ gauge theory, the above Minkowski solution is not unique. One can also device the simple solution $(\phi^1,\phi^2,\phi^3,\phi^4) = \kappa^{-\frac{1}{2}}(x,y,z,\tau)$, identifying the khronon components directly with the spacetime coordinates [13]. This solution implies the vanishing of the connection coefficients $\boldsymbol{A}^{IJ} = 0$ and thus has also vanishing curvature and torsion; therefore it cannot be gauge-equivalent to the above solution. The simple solution exists in the SO(4) formulation, but it does not represent a spacetime, meaning that it cannot be Wick-rotated into a Minkowski space since the solution is not compatible with the first step of our recipe: it is not realised physically. Problems with the solution in the Lorentz-signature version of the theory, with possible implications to an alternative formulation of a Lorentz gauge theory [47–49], were discussed in Ref.[50].

Another, related point of comparison also arises from the investigations of Ref.[50] (see section V there), where it was discovered that the $SO_{\mathbb{C}}(1,3)$ theory admits an additional third dynamical phase with no counterpart in the underlying Euclidean formulation. Although mathematically consistent, this phase led to unphysical consequences, most notably the appearance of caustic singularities even in static, spherically symmetric configurations. Such behavior, reminiscent of pathologies in "mimetic matter" models [51], resulted in e.g. black hole solutions that failed to be asymptotically flat, indicating a breakdown of physical viability. These anomalies illustrate the dangers of enlarging the phase space by complexification without a firm foundational principle.

In contrast to the $SO_{\mathbb{C}}(1,3)$ formulation, the present Euclidean-based theory offers a more controlled and geometrically grounded framework, one that appears free of such spurious phases and singularities as those in the above two examples. This makes it a particularly reliable platform for the path integral approach, where only physically admissible configurations are summed over, and the structure of the theory naturally enforces consistency at both the classical and quantum levels.

B. Expanding spacetime

The case of the Minkowski solution is rather special, since therein the Bartels frame reduces to the constant delta function. It is crucial to check whether the prescription works in the general case with a dynamic Bartels frame. For this purpose, we will consider the cosmological Ansatz,

$$ \begin{aligned} \boldsymbol{A}^{4i} &= A(\tau)\mathbf{d}x^i \\ \boldsymbol{A}^{ij} &= B(\tau)\epsilon^{ij}{}_k\mathbf{d}x^k \end{aligned} \quad \Rightarrow \quad \begin{aligned} {}^{+}\boldsymbol{A}^i &= (A-B)\,\mathbf{d}x^i \\ {}^{-}\boldsymbol{A}^i &= -(A+B)\,\mathbf{d}x^i \end{aligned} \tag{28a} $$

[8] Though, from the pseudo-Riemannian viewpoint one could in hindsight argue that the connection field had complex components. However, since we first solve for the connection and Wick-rotate the thus obtained second order field equations, the imaginary unit does not enter into the equations at any stage. (A possible interpretation is that, prior to the collapse of the wave function, the connection need not satisfy on-shell conditions. We leave this line of thought aside in the present work.)

This Ansatz is homogeneous and isotropic, thus allowing only the two functions of τ (or equivalently, functions of ϕ). For simplicity, we have chosen the form corresponding to flat FLRW cosmology. We will denote derivatives wrt to τ with a dot. The curvature of the connection (28) is

$$^+\boldsymbol{F}^i = \left(\dot{A}-\dot{B}\right)\mathbf{d}\tau\wedge\mathbf{d}x^i + \frac{1}{2}\left(A-B\right)^2\epsilon^i{}_{jk}\mathbf{d}x^j\wedge\mathbf{d}x^k\,, \tag{29a}$$

$$^-\boldsymbol{F}^i = -\left(\dot{A}+\dot{B}\right)\mathbf{d}\tau\wedge\mathbf{d}x^i + \frac{1}{2}\left(A+B\right)^2\epsilon^i{}_{jk}\mathbf{d}x^j\wedge\mathbf{d}x^k\,, \tag{29b}$$

which, in the symmetry-broken phase $\phi^I = \delta^I_4\kappa^{-1/2}\tau$, corresponds to the torsion

$$\boldsymbol{T}^i = -\dot{A}\tau\mathbf{d}\tau\wedge\mathbf{d}x^i + AB\tau\epsilon^i{}_{jk}\mathbf{d}x^j\wedge\mathbf{d}x^k\,. \tag{30}$$

Plugging into (26), these three 3-form equations plus one 4-form equation become, respectively[9],

$$3A\tau\left(A-B\right)^2\star\mathbf{d}x^4 = \sqrt{\kappa}\boldsymbol{t}^4 + \kappa\hat{\rho}(A\tau)^3\star\mathbf{d}x^4\,, \tag{31a}$$

$$\left[\left(\dot{A}-\dot{B}\right)A\tau + \left(A-B\right)^2\right]\star\mathbf{d}x^i = -\sqrt{\kappa}\boldsymbol{t}^i\,, \tag{31b}$$

$$A\tau\left(\dot{A}\tau+B\right)\star\mathbf{d}x^i = 0\,, \tag{31c}$$

$$\left[\dot{\hat{\rho}}A\tau + 3\hat{\rho}\left(\dot{A}\tau+A\right)\right]\star 1 = 0\,. \tag{31d}$$

From (31c) we obtain $B=-\dot{A}\tau$. We can identify the scale factor as $a\equiv -A\tau$, since the cosmological Bartels frame is then $\mathbf{e}^i = a\mathbf{d}x^i$. Now, taking the standard form of the source terms $\boldsymbol{t}^4 = \sqrt{\kappa}\rho a^3\star\mathbf{d}x^4$ and $\boldsymbol{t}^i = -\sqrt{\kappa}pa^2\dot{\tau}\star\mathbf{d}x^i$, the two equations (31a,31b) become

$$3\left(\dot{a}/a\right)^2 = \kappa\left(\rho+\hat{\rho}\right)\,,\quad 2\ddot{a}/a + \left(\dot{a}/a\right)^2 = -\kappa p\,. \tag{32}$$

Fixing the scale $1/\kappa = -m_P^2$ for the standard gravitational coupling, the Wick-rotation $\tau\rightarrow it$ yields us the standard two Friedmann equations,

$$3H^2 = m_P^{-2}\left(\rho+\hat{\rho}\right)\,,\quad\text{and}\quad 2\partial_t H + 3H^2 = -m_P^{-2}p\,,\quad\text{where}\quad H\equiv\partial_t\log a\,, \tag{33}$$

and the final equation (31d) gives the expected conservation law for ideal dust in FLRW background, $\partial_t\hat{\rho}+3H\hat{\rho}=0$.

This worked only too well. Still, like it was in the Minkowski example, it remains unclear why the Wick rotation is needed. Up to interpretations of the variables, the equations (32) in terms of the Euclidean time τ are exactly the usual FLRW equations for an expanding universe, allowing the same physical interpretation as usual. So, effectively we have a "geometrodynamical" set-up with a stack of Bartels frame configurations $\mathbf{e}^i$ ordered according to an external parameter, and everything looks the same whether the parameter τ or t is considered.

The necessity of the Wick rotation in more generic spacetimes becomes apparent when considering perturbations around the FLRW background. We will demonstrate this with tensor fluctuations. The generic spin-2 fluctuation of the connection can be parameterised with two symmetric, transverse-traceless 3×3 tensors: an even-parity h_{ij} and an independent, odd-parity $\tilde{h}_{ij}$, which appear in the perturbed connection components as

$$\boldsymbol{A}^{4i} = A\left(\delta^i_j + h^i{}_j\right)\mathbf{d}x^j\,, \tag{34a}$$

$$\boldsymbol{A}^{ij} = B\left(\epsilon^{ij}{}_k + \epsilon^{ijl}\tilde{h}_{lk}\right)\mathbf{d}x^k\,. \tag{34b}$$

Plugging into the connection EoM (26c) and linearising, we obtain the three 3-form equations

$$\left[2\left(\tau\dot{A}+B\right)\delta^i_j - \tau A\dot{h}_{ij} + B\left(h^i{}_j - \tilde{h}^i{}_j\right) + \epsilon_{(i}{}^{kl}h_{j)k,l}\right]\star\mathbf{e}^i = 0\,, \tag{35}$$

which, plugging in the background solution, yields the solution for the odd tensor perturbation in terms of the even tensor perturbation,

$$\tilde{h}_{ij} = h_{ij} + \frac{1}{B}\left(\epsilon_{(i}{}^{kl}h_{j)k,l} + a\dot{h}_{ij}\right)\,. \tag{36}$$

---

[9] With slight abuse of notation, we have here exploited the $SO(4)$ Hodge symbol for the dual of the spacetime coordinate differentials. Covariant forms would be the $SO(4)$ duals of $SO(4)$-valued objects, like $\star\mathbf{e}^i$, or the spacetime duals of coordinate-indexed objects, like $*\mathbf{d}x^i$, where $\star$ is wrt the Kronecker metric and $*$ would be wrt the spacetime metric.

Tensor perturbations decouple from the energy equation (26a), but plugging (36) into the momentum equations gives (26b), in the absence of tensor perturbations in the matter source $\boldsymbol{t}_i$,

$$\left[\ddot{h}_{ij} + 3(\dot{a}/a)\dot{h}_{ij} + a^{-2}\nabla^2 h_{ij}\right] \star \mathbf{e}^j = 0 \,. \tag{37a}$$

The 3-gradient is defined as $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j$. In terms of the Euclidean time τ, this does not describe waves but rather an exponential instability. Thus it is clear that we should perform the Wick rotation to the standard Lorentzian time t to arrive at the standard equation describing the propagation of gravitational waves in FLRW background,

$$\partial_t^2 h_{ij} + 3H\partial_t h_{ij} - a^{-2}\nabla^2 h_{ij} = 0 \,. \tag{37b}$$

We will generalise these computations in section IV to scalar and vector perturbations in the most general 5-parameter theory (17). Here it might be instructive to check more explicitly how an exponential growth in the Euclidean picture is indeed seen as a wave motion at the Lorentzian stage. For simplicity, neglecting the expansion and setting $a = 1$, the canonically normalised solution to (37) for a Fourier mode with wave number k,

$$h_{ij}(t) = a_{ij}(k)\cos kt = a_{ij}(k)\cosh(k\tau) = h_{ij}(\tau)\,, \tag{38a}$$

is a real perturbation in both descriptions when restricting to even wave modes. However, in the sector of odd wave mode solutions,

$$m_P h_{ij}(t) = m_P b_{ij}(k)\sin(kt) \quad \Leftrightarrow \quad \kappa^{-1/2}h_{ij}(\tau) = \kappa^{-1/2}b_{ij}(k)\sinh(k\tau)\,, \tag{38b}$$

it is rather the canonically normalised perturbations which are real in both pictures of the same solution. This reflects two well-known facts pertaining to Wick rotated descriptions of physics. Firstly, periodic evolution in one may correspond to hyperbolic evolution in the other. Secondly, the correspondence is curious and far from being yet thoroughly understood.

C. Rotating spacetime

We now turn to a particularly nontrivial example of an exact solution describing a rotating spacetime - specifically, the Kerr solution - as a test case for examining the universality of the proposed Euclidean formulation. Since the Kerr geometry is not static, extending it to a Euclidean regime is highly nontrivial and challenges the applicability or ate least the interpretation of conventional methods which involve complexified metrics [24, 52]. The presence of rotation introduces off-diagonal metric components and prevents the existence of a global, hypersurface-orthogonal timelike Killing vector field. Although a synchronous coordinate system should, in principle, be attainable[10], we choose to adopt a convenient ansatz informed by Doran's metric [56], which incorporates nontrivial lapse and shift functions. Thus, we will study a generic axially symmetric $Spin(4)$ connection given as follows:

$$\boldsymbol{A}^{41} = -\frac{1}{\tau}\left(\beta\mathbf{d}\tau + \frac{\sigma}{f}\mathbf{d}r - \alpha\sin^2\theta\mathbf{d}\varphi\right), \quad \boldsymbol{A}^{42} = -\frac{\sigma}{\tau}\mathbf{d}\theta\,, \quad \boldsymbol{A}^{43} = -\frac{f\sin\theta}{\tau}\mathbf{d}\varphi\,, \tag{39a}$$

$$\boldsymbol{A}^{12} = O\mathbf{d}\tau + P\mathbf{d}r + Q\mathbf{d}\theta + R\mathbf{d}\varphi\,, \quad \boldsymbol{A}^{13} = S\mathbf{d}\tau + T\mathbf{d}r + U\mathbf{d}\theta + V\mathbf{d}\varphi\,, \quad \boldsymbol{A}^{23} = W\mathbf{d}\tau + X\mathbf{d}r + Y\mathbf{d}\theta + Z\mathbf{d}\varphi\,. \tag{39b}$$

The 'electric' components of the connection involve the 4 functions $f(r)$, $\sigma(r,\theta)$, $\alpha(r,\theta)$, $\beta(r,\theta)$ which conveniently parameterise an axially symmetric metric geometry, and the completely generic 'magnetic' components of the connection then involve the 3×4=12 free functions $O(r,\theta),\ldots,Z(r,\theta)$. Then the anti/self-dual connection is

$${}^{\pm}\boldsymbol{A}^1 = -\left(\pm\frac{\beta}{\tau} + W\right)\mathbf{d}\tau - \left(\pm\frac{\sigma}{\tau f} + X\right)\mathbf{d}r - Y\mathbf{d}\theta - \left(Z \mp \frac{\alpha}{\tau}\sin^2\theta\right)\mathbf{d}\varphi\,, \tag{40a}$$

$${}^{\pm}\boldsymbol{A}^2 = S\mathbf{d}\tau + T\mathbf{d}r + U\mathbf{d}\theta + \left(V \mp \frac{\sigma}{\tau}\right)\mathbf{d}\varphi\,, \tag{40b}$$

$${}^{\pm}\boldsymbol{A}^3 = -O\mathbf{d}\tau - P\mathbf{d}r - Q\mathbf{d}\theta - \left(R \pm \frac{f}{\tau}\sin\theta\right)\mathbf{d}\varphi\,. \tag{40c}$$

[10] We note three apparently independent attempts (which do not seem to agree with each other) at deriving a synchronous coordinate system for the Kerr metric [53–55].

In the absence of matter source, the connection EoM (26c) reduce to

$$
\begin{aligned}
&-\left[\frac{\alpha P\sin\theta+\beta f'}{\sigma}+\frac{2}{\tau}-\frac{U}{\sigma}+\frac{\alpha}{f\sigma^2}\left(2\sigma\cos\theta-\sigma\text{'}\sin\theta\right)+\frac{\alpha\text{'}\sin\theta+\sigma R\csc\theta}{f\sigma}+\frac{f\beta\sigma'}{\sigma^2}\right]\star\mathbf{e}^1\\
&-\left[\frac{f}{\sigma}(\beta P+T)-\frac{\alpha'}{\sigma}\sin\theta-O\right]\star\mathbf{e}^2-\left[\frac{f}{\sigma}(P-\beta T)-\sigma\text{'}+S\right]\star\mathbf{e}^3\\
&-\left[\frac{\alpha T\sin\theta+Q+f'}{\sigma}+\frac{f\sigma'}{\sigma^2}+\frac{V}{f}\csc\theta\right]\mathbf{e}^4=0\,, \qquad (41a)\\
&\left[\frac{\alpha\sigma'}{\sigma^2}\sin\theta-\frac{f\beta}{\sigma}P+O-\frac{\beta\text{'}}{\sigma}+\frac{\beta\sigma\text{'}}{\sigma^2}-\frac{Y}{\sigma}\right]\star\mathbf{e}^1-\left[\frac{\alpha P\sin\theta+\beta f'}{\sigma}+\frac{2}{\tau}+\frac{f}{\sigma}\left(\beta'+X\right)+\frac{R}{f}\csc\theta\right]\star\mathbf{e}^2\\
&-\left[f\left(\frac{\sigma'}{\sigma}-\beta X\right)+Q+\sigma W\right]\star\mathbf{e}^3-\left[\frac{\alpha}{\sigma}X\sin\theta+\frac{1}{\sigma}\cot\theta+\frac{\sigma\text{'}}{\sigma^2}-\frac{f}{\sigma}P+\frac{Z}{f}\csc\theta\right]\star\mathbf{e}^4=0\,, \qquad (41b)\\
&-\left[\frac{\alpha X-\alpha'\beta+\alpha\beta'}{\sigma}\sin\theta-S+\frac{1}{\sigma}\cot\theta+\frac{f\beta}{\sigma}T+\frac{Z}{f}\csc\theta\right]\star\mathbf{e}^1+\left[\frac{\alpha}{\sigma}T\sin\theta+W+\frac{f'}{\sigma}-\frac{f\beta}{\sigma}X+\frac{V}{f}\csc\theta\right]\star\mathbf{e}^2\\
&+\left[\frac{f\beta\sigma'}{\sigma^2}+\frac{f\left(\beta'+X\right)-U}{\sigma}+\frac{2}{\tau}\right]\star\mathbf{e}^3+\left[\left(\frac{\alpha'}{\sigma}+\frac{\alpha\sigma'}{\sigma^2}\right)\sin\theta-\frac{fT+Y}{\sigma}\right]\star\mathbf{e}^4=0\,, \qquad (41c)
\end{aligned}
$$

where $'$ and ' respectively denote ∂_r and ∂_θ. These 3×4 equations yield us the solutions for the 3×4 functions in the 'magnetic' part of the connection in terms of the 4 functions that appear in the 'electric' part of the connection:

$$
\begin{aligned}
&O=\frac{\beta\text{'}\sigma+\beta\sigma\text{'}}{2\sigma^2}+\frac{\alpha'\beta^2-\alpha\beta\beta'}{2\sigma}\sin\theta\,,\quad P=\frac{\sigma\text{'}}{f\sigma}+\frac{\alpha'\beta-\alpha\beta'}{2f}\sin\theta\,,\\
&Q=-\frac{f\sigma'}{\sigma}\,,\quad R=-\frac{\alpha}{2\sigma}\left(\alpha'\beta-\alpha\beta'\right)\sin^3\theta-\frac{\alpha\text{'}\sigma+\alpha\sigma\text{'}}{2\sigma^2}\sin^2\theta-\frac{1}{\sigma}\left(\alpha\cos\theta+\beta ff'\right)\sin\theta-\frac{f}{\tau}\sin\theta\,. \qquad (42a)\\
&S=\frac{\alpha'\beta+\alpha\beta'}{2\sigma}\sin\theta+\frac{\beta\beta\text{'}\sigma-\beta^2\sigma\text{'}}{2\sigma^2}\,,\quad T=\frac{\alpha'}{f}\sin\theta+\frac{\beta\text{'}\sigma-\beta\sigma\text{'}}{2f\sigma}\,,\\
&U=\frac{\alpha\text{'}\sigma-\alpha\sigma\text{'}}{2f\sigma}\sin\theta+\frac{\alpha}{f}\cos\theta+\frac{f\beta\sigma'}{\sigma}+\frac{\sigma}{\tau}\,,\quad V=-\frac{\alpha\alpha'}{\sigma}\sin^3\theta-\frac{\alpha\beta\text{'}\sigma-\alpha\beta\sigma\text{'}}{2\sigma^2}\sin^2\theta-\frac{ff'}{\sigma^2}\sin\theta\,. \qquad (42b)\\
&W=\frac{\alpha\text{'}\sigma-\alpha\sigma\text{'}}{2f\sigma^2}\beta\sin\theta+\frac{\alpha}{f\sigma}\beta\cos\theta-\frac{f\beta\beta'}{\sigma}-\frac{\beta}{\tau}\,,\\
&X=\frac{\alpha\text{'}\sigma-\alpha\sigma\text{'}}{2f^2\sigma}\sin\theta+\frac{\alpha}{f^2}\cos\theta-\beta'-\frac{\sigma}{\tau f}\,,\quad Y=\frac{\alpha\sigma'}{\sigma}\sin\theta-\frac{\beta\text{'}\sigma-\beta\sigma\text{'}}{2\sigma}\,,\\
&Z=-\frac{\alpha\alpha\text{'}\sigma-\alpha^2\sigma\text{'}}{2f\sigma^2}\sin^3\theta-\frac{\alpha^2}{f\sigma}\sin^2\theta\cos\theta+\frac{\tau f\alpha'\beta-\tau f\alpha\beta'-2\alpha\sigma}{2\tau\sigma}\sin^2\theta-\frac{f}{\sigma}\cos\theta\,. \qquad (42c)
\end{aligned}
$$

Plugging this solution into energy-momentum equations (26a, 26b), we finally obtain the solutions for the remaining 4 functions:

$$
f=\pm\sqrt{r^2-a_E^2}\,,\quad \sigma=\pm\sqrt{r^2-a_E^2\cos^2\theta}\,,\quad \alpha=\pm a_E\beta\,,\quad \beta=\pm\sqrt{\frac{\kappa m_S r}{4\pi\sigma^2}}\,. \qquad (43)
$$

Two integration constants appear in the solution, a_E and m_S. When considering the emergent pseudo-Riemannian geometry, it is crucial to note that the parameter $a_E\equiv J_E/m_S$ will correspond to the angular momentum J per unit mass m_S in the rotating spacetime, characterising the intrinsic rotation or "twist" of the geometry, conventionally encoded in off-diagonal metric components that mix space and time directions. When performing a Wick rotation, $J_E\sim m_S r^2\dot\varphi\to -i m_S r^2\partial_t\varphi\sim -iJ$, the angular momentum parameter would formally become imaginary, $a_E\to -ia$. This change occurs because the rotation, fundamentally tied to temporal flow, does not straightforwardly translate into the Euclidean regime; the analytic continuation alters the nature of the time coordinate and thus affects quantities dependent on it. This situation is quite analogous to what we already encountered with gravitational waves in section III B above: in Lorentzian spacetime, they represent oscillatory, propagating disturbances, but in the Euclidean regime, their equations become elliptic, leading to exponentially growing or decaying solutions rather than oscillations. Our approach ensures that the metric remains real (or equivalently, the khronon field and the Bartels frame remain real) and well-defined in both Lorentzian and Euclidean descriptions, meaning that the angular momentum as well retains a consistent, physically meaningful interpretation across the analytic continuation without becoming complex. This reflects a significant advantage, as it allows for a unified treatment of rotation that avoids the complications of complexified parameters common in standard Euclideanisations. To be explicit, the solution (43) gives us the pseudo-Riemannian geometry described by the Cartan frame

$$\mathbf{e}^0 = \mathbf{d}t\,, \tag{44a}$$
$$\mathbf{e}^1 = \sqrt{\frac{r^2+a^2\cos^2\theta}{r^2+a^2}}\mathbf{d}r + \sqrt{\frac{m_S r}{4\pi m_P^2\left(r^2+a^2\cos^2\theta\right)}}\left(\mathbf{d}t - a\sin^2\theta\mathbf{d}\varphi\right)\,, \tag{44b}$$
$$\mathbf{e}^2 = \sqrt{r^2+a^2\cos^2\theta}\,\mathbf{d}\theta\,, \tag{44c}$$
$$\mathbf{e}^3 = \sqrt{r^2+a^2}\sin\theta\mathbf{d}\varphi\,, \tag{44d}$$

which is the same as the one reported in Ref.[56] (we have chosen the first plus signs and two last minus signs at (43) to match the conventions of Ref.[56]). Now the pseudo-Riemannian (Lorentzian) dynamics are understood as the manifestation of a deeper, fundamentally Euclidean structure. The familiar causal and dynamical features of spacetime arise as analytic continuations - appearances - of the underlying real Riemannian geometry,

$$\mathbf{e}^4 = \mathbf{d}\tau\,, \tag{45a}$$
$$\mathbf{e}^1 = \sqrt{\frac{r^2-a_E^2\cos^2\theta}{r^2-a_E^2}}\mathbf{d}r + \sqrt{\frac{\kappa r}{4\pi\left(r^2-a_E^2\cos^2\theta\right)}}\left(\mathbf{d}\tau + a_E\sin^2\theta\mathbf{d}\varphi\right)\,, \tag{45b}$$
$$\mathbf{e}^2 = \sqrt{r^2-a_E^2\cos^2\theta}\,\mathbf{d}\theta\,, \tag{45c}$$
$$\mathbf{e}^3 = \sqrt{r^2-a_E^2}\sin\theta\mathbf{d}\varphi\,. \tag{45d}$$

Moreover, this Riemannian geometry emerges from the configuration of the khronon field ϕ, whose magnitude, measured in conventionally assigned units of κ, sets the structure of time in ultimately arbitrary units of mass-like m_P.

IV. COSMOLOGY

We shall now investigate the generic 5-parameter theory and begin by focusing on its cosmological solutions. Since the part $I_{(0)}$ in (17) is a surface integral, for the purposes of classical dynamics the action we consider reduces to

$$I = \int \mathbf{D}\phi^I\wedge\mathbf{D}\phi^J\wedge\left(g_+\,{}^+\boldsymbol{F}_{IJ} - g_-\,{}^-\boldsymbol{F}_{IJ}\right) + \int\star\lambda - \int\left(\mathbf{D}\phi^I\wedge\boldsymbol{t}_I + \boldsymbol{A}^{IJ}\wedge\boldsymbol{O}_{IJ}\right)\,. \tag{46}$$

From this, we obtain the EoM

$$\mathbf{D}\left[2\left(g_+\,{}^+\boldsymbol{F}_{IJ} - g_-\,{}^-\boldsymbol{F}_{IJ}\right)\wedge\mathbf{D}\phi^J + \lambda\star\mathbf{D}\phi_I - \boldsymbol{t}_I\right] = 0\,, \tag{47a}$$
$$\begin{aligned}\mathbf{D}\left[g_+\,{}^+\left(\mathbf{D}\phi_I\wedge\mathbf{D}\phi_J\right) - g_-\,{}^-\left(\mathbf{D}\phi_I\wedge\mathbf{D}\phi_J\right)\right] &- 2\left(g_+\phi_{[I}\,{}^+\boldsymbol{F}_{J]K} - g_-\phi_{[I}\,{}^-\boldsymbol{F}_{J]K}\right)\wedge\mathbf{D}\phi^K\\ &= -\phi_{[I}\boldsymbol{t}_{J]} + \lambda\phi_{[I}\star\mathbf{D}\phi_{J]} + \boldsymbol{O}_{IJ}\,,\end{aligned} \tag{47b}$$

which we can rearrange in the more convenient form,

$$2\,{}^{g\pm}\boldsymbol{F}_{IJ}\wedge\mathbf{D}\phi^J = \boldsymbol{t}_I - \lambda\star\mathbf{D}\phi_I + \boldsymbol{M}_I\,, \tag{48a}$$
$$\mathbf{D}\,{}^{g\pm}\left(\mathbf{D}\phi_I\wedge\mathbf{D}\phi_J\right) = \phi_{[I}\boldsymbol{M}_{J]} + \boldsymbol{O}_{IJ}\,, \tag{48b}$$
$$\mathbf{D}\boldsymbol{M}_I = 0\,, \tag{48c}$$

by defining the projector

$${}^{g\pm}X_{IJ} \equiv g_+\,{}^+X_{IJ} - g_-\,{}^-X_{IJ}\,. \tag{49}$$

In the symmetry-broken phase, we obtain for the khronon EoM

$$\left(g_+\,{}^+\boldsymbol{F}_i + g_-\,{}^-\boldsymbol{F}_i\right)\wedge\mathbf{D}\phi^i = \boldsymbol{t}^4 - \lambda\star\mathbf{D}\phi + \boldsymbol{M}^4\,, \tag{50a}$$
$$\left(g_+\,{}^+\boldsymbol{F}^i + g_-\,{}^-\boldsymbol{F}^i\right)\wedge\mathbf{d}\phi - \epsilon^i{}_{jk}\left(g_+\,{}^+\boldsymbol{F}^j - g_-\,{}^-\boldsymbol{F}^j\right)\wedge\mathbf{D}\phi^k = -\boldsymbol{t}^i + \lambda\star\mathbf{D}\phi^i - \boldsymbol{M}^i\,, \tag{50b}$$

and for the connection EoM

$$g_+\kappa^{-1/2}\left(\boldsymbol{T}^i\wedge\mathbf{d}\phi + \epsilon^i{}_{jk}\boldsymbol{T}^j\wedge\mathbf{D}\phi^k\right) = -\frac{1}{2}\phi\boldsymbol{M}^i - {}^+\boldsymbol{O}^i\,, \tag{50c}$$
$$g_-\kappa^{-1/2}\left(\boldsymbol{T}^i\wedge\mathbf{d}\phi - \epsilon^i{}_{jk}\boldsymbol{T}^j\wedge\mathbf{D}\phi^k\right) = \frac{1}{2}\phi\boldsymbol{M}^i - {}^-\boldsymbol{O}^i\,. \tag{50d}$$

Here we have taken the anti/self-dual projections of (48b), but it is sometimes more convenient to instead consider the (4-i) and the (i-j) components of (48b),

$$(g_+ - g_-)\, \boldsymbol{T}^i \wedge \mathbf{e}^4 + (g_+ + g_-)\, \epsilon^i{}_{jk} \boldsymbol{T}^j \wedge \mathbf{e}^k \;=\; -\kappa\phi \boldsymbol{M}^i - 2\kappa \boldsymbol{O}^{4i}\,, \tag{51a}$$

$$(g_+ + g_-)\, \boldsymbol{T}^i \wedge \mathbf{e}^4 + (g_+ - g_-)\, \epsilon^i{}_{jk} \boldsymbol{T}^j \wedge \mathbf{e}^k \;=\; \epsilon^i{}_{jk}\kappa \boldsymbol{O}^{jk}\,, \tag{51b}$$

as the effective source term from the integration form appears only for the former components. The two khronon field equations (50a) and (50b) specialised to the homogeneous and isotropic background now give, respectively,

$$-\frac{3}{a^2}\left[(g_+ + g_-)\left(A^2 + B^2\right) - 2\,(g_+ - g_-)\, AB\right] \star \mathbf{d}\tau \;=\; \sqrt{\kappa}\left(\boldsymbol{t}^4 - \kappa^{-3/2}\lambda \star \mathbf{d}\tau + \boldsymbol{M}^4\right)\,, \tag{52a}$$

$$\frac{1}{a^2}\left[(g_+ + g_-)\left(A^2 + B^2 - 2a\dot{A}\right) - 2\,(g_+ - g_-)\left(AB - a\dot{B}\right)\right] \star \mathbf{e}^i \;=\; -\sqrt{\kappa}\left(\boldsymbol{t}^i - \kappa^{-3/2}\lambda \star \mathbf{e}^i + \boldsymbol{M}^i\right)\,, \tag{52b}$$

whereas the two connection field equations (50c,50d) can be lumped together as

$$g_\pm \left(\pm \dot{A}\tau + B\right) \star \mathbf{e}^i = \pm \frac{a\kappa}{4}\phi \boldsymbol{M}^i + \frac{a\kappa}{2}\, {}^\pm\boldsymbol{O}^i\,. \tag{53}$$

It can be first instructive to check the two cases $g_\mp = 0$ in the absence of spin currents, but taking into account the perfect fluid source,

$$\boldsymbol{t}^i \;=\; \sqrt{\kappa} p \star \mathbf{e}^i\,, \tag{54a}$$

$$\boldsymbol{t}^4 \;=\; -\sqrt{\kappa}\rho \star \mathbf{d}\tau\,. \tag{54b}$$

In these special limits, the two torsion equations $g_\pm\left(\dot{A}\tau \pm B\right) = 0$ imply that $\boldsymbol{M}^i = 0$ and that $B = \mp\dot{A}\tau$. Plugging this solution into (52) yields in both cases the Friedmann equations

$$3g_\pm \left(\frac{\dot{a}}{a}\right)^2 \;=\; \kappa\left(\rho + \hat{\rho}\right) - \Lambda \quad \Leftrightarrow \quad 3g_\pm H^2 = m_P^{-2}\left(\rho + \hat{\rho}\right) + \Lambda \tag{55a}$$

$$g_\pm \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \;=\; -\kappa p - \Lambda \quad \Leftrightarrow \quad g_\pm\left(2\partial_t H + 3H^2\right) = -m_P^{-2} p + \Lambda\,, \tag{55b}$$

where $\Lambda \equiv \kappa^{-1}\lambda = -m_P^2\lambda$, and the effective energy density $\hat{\rho}$ again obeys the ideal dust conservation equation, implying the conservation of the matter source with the equation of state p/ρ. As expected, both the cases ($g_\mp = 0$, $g_\pm = 1$) reproduce the system already found in III B. However, now that the parameter λ has been included, we note that the cosmological constant has the different sign in the Lorentzian regime. Let us remark in passing that this could potentially be the key to unlock a new physical relevance to the AdS/CFT correspondence.

For generic $g_\pm$, in the absence of the integration forms $\boldsymbol{M}^i = 0$ and spin currents $\boldsymbol{O}^{IJ} = 0$, the two torsion equations (53) together with (30) imply that $\dot{A} = 0$ and $B = 0$. Nontrivial cosmological solutions thus in general require nonvanishing $\boldsymbol{M}^i \neq 0$. We make an Ansatz for the dark matter form,

$$\boldsymbol{M}^i \;=\; \sqrt{\kappa}\hat{p} \star \mathbf{e}^i\,, \tag{56a}$$

$$\boldsymbol{M}^4 \;=\; -\sqrt{\kappa}\hat{\rho} \star \mathbf{d}\tau\,. \tag{56b}$$

In the absence of material spin currents, two torsion equations (53)

$$g_\pm\left(\dot{A}\tau \pm B\right) = \frac{a}{4}\tau\kappa\hat{p}\,, \tag{57}$$

are together solved by

$$B \;=\; -\frac{g_+ - g_-}{g_+ + g_-}\tau\dot{A}\,, \tag{58a}$$

$$\kappa\hat{p} \;=\; \frac{8g_+g_-}{g_+ + g_-}\frac{\dot{A}}{a}\,. \tag{58b}$$

This solution requires that $g_+ \neq -g_-$, and we will check the special case of equal $g = \pm g_\pm$ later. The Euler equation $\mathbf{D}\boldsymbol{M}^i = \mathbf{d}\boldsymbol{M}^i + \boldsymbol{A}^i{}_j \wedge \boldsymbol{M}^j - \boldsymbol{A}^4{}_i \wedge \boldsymbol{M}_4 = 0$ is trivially satisfied, since each of the three terms vanishes identically for the chosen cosmological Ansatz. However, the continuity equation, $\mathbf{D}\boldsymbol{M}_4 = \mathbf{d}\boldsymbol{M}_4 + \boldsymbol{A}_{4i} \wedge \boldsymbol{M}^i = 0$ gives now

$$\dot{\hat{\rho}} + 3\frac{\dot{a}}{a}\hat{\rho} = -3\frac{\hat{p}}{\tau} = \frac{24g_+g_-}{\kappa\left(g_+ + g_-\right)}\frac{A\dot{A}}{a^2} = \frac{24g_+g_-}{\kappa\left(g_+ + g_-\right)\tau^2}\left(\frac{\dot{a}}{a} - \frac{1}{\tau}\right)\,. \tag{59}$$

The effective pressure term can be seen as a source of the effective energy density. The homogeneous equation has a solution that contributes an ideal dust part into the energy density $\hat{\rho}_{\rm CDM} \in \hat{\rho}$, where $\hat{\rho}_{\rm CDM} \sim a^{-3}$, and in this sense a cold dark matter contribution still arises as an integration constant. However, the total effective fluid cannot be ideal dust and thus potential problems with caustics could be avoided. Moreover, dark matter can be now be associated with the spin current $\hat{\boldsymbol{O}^{IJ}} = \phi_{[I}\boldsymbol{M}_{J]}$, an exotic possibility not often explored in the literature [57–59].

It will be convenient to use the short-hand notations for the parameter combinations

$$\alpha \equiv \frac{(g_+ - g_-)^2}{g_+ + g_-}\,, \tag{60a}$$

$$\beta \equiv -\frac{4g_+g_-}{g_+ + g_-}\,, \tag{60b}$$

$$\gamma \equiv \frac{g_+ + g_-}{g_+ - g_-} = \frac{g_+ - g_-}{\alpha}\,. \tag{60c}$$

Note that we have defined the parameter combination γ such that it corresponds to the well-known Barbero-Immirzi parameter in the context of loop quantum gravity [60, 61]. We may rewrite the action (15c) as

$$I_{(2)} = \frac{1}{2}\left(g_+ + g_-\right)\int \mathbf{D}\phi^I \wedge \mathbf{D}\phi^J \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IK}\delta_{JL}\right) \wedge \boldsymbol{F}^{KL}\,, \tag{61}$$

which upon the substitution $\mathbf{D}\phi^I \to \kappa^{-\frac{1}{2}}\mathbf{e}^I$ would become (Euclideanised) action for loop quantum gravity, with the Barbero-Immirzi parameter γ controlling the relative coupling strengths of the first, "Palatini" term and the second, "Holst" term. Since now torsion plays a crucial role, both these terms are relevant to dynamics. After this remark, let us proceed with the cosmological equations.

A. The Friedmann equations

Plugging now the solution (58a) into the field equations (52) we obtain

$$3\alpha\left(\frac{\dot{a}}{a}\right)^2 - 3\frac{\beta}{\tau^2} = \kappa\left(\rho + \hat{\rho}\right) - \Lambda\,, \tag{62a}$$

$$2\alpha\frac{\ddot{a}}{a} + \alpha\left(\frac{\dot{a}}{a}\right)^2 - \frac{\beta}{\tau^2} = -\kappa p - \Lambda\,. \tag{62b}$$

These equations are consistent with

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0\,, \tag{63a}$$

$$\dot{\hat{\rho}} + 3\frac{\dot{a}}{a}\left(\hat{\rho} + \frac{2\beta}{\kappa\tau^2}\right) = \frac{6\beta}{\kappa\tau^3}\,. \tag{63b}$$

Now let $\kappa^{-\frac{1}{2}}\tau \to m_P t$. Then the above equations read

$$3\alpha H^2 - 3\beta t^{-2} = m_P^{-2}\left(\rho + \hat{\rho}\right) + \Lambda\,, \tag{64a}$$

$$\alpha\left(3H^2 + 2H'\right) - \beta t^{-2} = -m_P^{-2} p + \Lambda\,, \tag{64b}$$

and

$$\partial_t\rho + 3H\left(\rho + p\right) = 0\,, \tag{65a}$$

$$\partial_t\hat{\rho} + 3H\hat{\rho} = 6\beta m_P^2\left(t^{-3} - Ht^{-2}\right)\,. \tag{65b}$$

It is clear that the Euclidean solutions can be mapped to Lorentzian solutions and vice versa, taking into account the effective flip of the sign of Λ noticed earlier.

We will first have a look at the vacuum solutions. There are two sets of vacuum solutions when $\rho = p = \Lambda = 0$,

$$m_P^{-2}\hat{\rho}(t) = \frac{2}{3t^2}\left(\alpha - 3\beta \pm \sqrt{\alpha^2 + 3\alpha\beta}\right)\,, \quad a(t) \sim t^{\frac{1}{3}\left(1 \pm \sqrt{1 + 3\frac{\beta}{\alpha}}\right)}\,. \tag{66}$$

Only in the case $\beta = 0$ do we have either the Minkowski vacuum or the ideal dust solution. Assume $\alpha = 1$ and $\beta = -\epsilon$ a very small parameter. Then, in the almost-Minkowski-branch $M_P^{-2}\hat{\rho} \approx 3\epsilon t^{-2}$ and $a \sim t^{\frac{\epsilon}{2}}$, whereas in the almost-dust-branch $m_P^{-2}\hat{\rho} \approx (4/3 + \epsilon)t^{-2}$ and $a \sim t^{\frac{2}{3} - \frac{\epsilon}{2}}$. Depending on the sign of g_-, we may thus render the effective dark matter equation of state slightly negative or slightly positive.

B. The special case $g_+ = -g_-$

Let us then check the special case $g_+ = -g_- \equiv g$. For simplicity, let $\lambda = 0$, since the cosmological constant can always be absorbed in the source term and interpreted as a part of ρ. The connection equation (57) forces then to set $\dot{A} = 0$, and we denote the constant A as $A = -a_0$. Thus, the khronon is essentially the scale factor $\phi = a/a_0$. The other condition we obtain is

$$\kappa\hat{p} = \frac{4g}{a\tau}B = \frac{4g\sqrt{\kappa}}{a_0\phi^2}B\,. \tag{67}$$

This implies the strange Friedmann equations (52)

$$12gB \;=\; a_0\phi^2\kappa^{3/2}\left(\rho+\hat{\rho}\right)\,, \tag{68a}$$

$$4g\left(\tau\dot{B}+2B\right) \;=\; -a_0\phi^2\kappa^{3/2}p\,. \tag{68b}$$

In this case, it occurs that both $a\dot{\rho}+3\dot{a}(\rho+p) = 0$ and $a\dot{\hat{\rho}}+3\dot{a}(\hat{\rho}+\hat{p}) = 0$, because now $\dot{a}/a = 1/\tau$, which means $H = 1/t$ in the Lorentzian picture. In achiral gravity, the universe always expands like it would be curvature-dominated. The vacuum solution is given by $\hat{p} = \hat{\rho}/3$, whilst the equation of state would have the opposite sign in the standard theory in order to drive the same expansion. If there is matter with the equation of state $w = p/\rho$, implying that $\rho \sim \phi^{-3(1+w)}$, then the integration form has to be chosen such that

$$\hat{\rho} = -\frac{\rho}{1-3w} + \frac{m_0}{\phi^4}\,, \tag{69}$$

where the first piece compensates the effect of the matter source, and the integration constant m_0 determines the energy density of the "dark radiation" which drives the $\dot{a}/a \sim 1/\tau$ expansion.

C. The special case $g_+ = g_-$

Another special case is $g_+ = g_- \equiv g$, for which $\alpha = 0$, $\beta = -2g$. We note that the solution (66) breaks down in this limit. Therefore we go back to the Friedmann equations (52) which reduce to

$$2g\tau^{-2} = \frac{\kappa}{3}\left(\rho+\hat{\rho}\right) = -\kappa p\,. \tag{70}$$

We see that now $\hat{\rho} = -(1+3w)\rho$, and this solution only exists if it is sourced by matter with pressure[11]. Then the expansion rate is given by the matter equation of state as usual, $a \sim \phi^{\frac{2}{3(1+w)}}$. Yet, the system is very unusual. To check its consistency, we should verify the conservation equation for the dark matter. For the RHS of the equation (59) we get

$$\dot{\hat{\rho}} + 3\frac{\dot{a}}{a}\hat{\rho} = 3w\left(1+3w\right)\frac{\dot{a}}{a}\rho\,, \tag{71a}$$

and for the LHS we get

$$-3\frac{\hat{p}}{\tau} = 12g\frac{A\dot{A}}{\kappa a^2} = -\frac{4g(1+3w)}{(1+w)\kappa\tau^3} = 3w\left(1+3w\right)\frac{\dot{a}}{a}\rho\,. \tag{71b}$$

These match, confirming that the system, though weird, is consistent. The $g_+ = g_-$ theory was called "quasi-topological" [50] because, on the one hand, the action does not reduce to a boundary term but, on the other hand, there are no local degrees of freedom [41]. The latter property reflects the emergence of an extra symmetry as $g_- \to g_+$ [41] (see also appendix A of [50]). This would, in an extension of the present work that would allow varying coupling constants (- "the dymaxion" -) in terms of running parameters or dynamical scalars, suggest a scenario wherein the dynamical universe emerges from the "quasi-topological" phase of enhanced symmetry [50].

Action principles, which are essentially equivalent to our case $g_+ = g_-$, appear in the works of several different authors in the literature [62–67]; however, the physical contents of the theory have not been understood in these works, since the action principles have been assumed to be dynamically equivalent to general relativity.

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[11] This may change in the presence of spatial curvature, but in this article we restrict to the flat FLRW cosmology.

V. LARGE-SCALE STRUCTURE

We shall now proceed to investigate the cosmological physics of the $Spin(4)$ theory, focusing on the evolution of structure in the universe. Rather than restrict ourselves to a specific background or gauge choice from the outset, we will develop the formalism of cosmological perturbation theory in full generality (up to linear order), allowing a unified treatment of scalar, vector, and tensor modes on arbitrary (flat) cosmological spacetimes. This will provide the tools necessary for analysing a wide range of physical phenomena, from generation of primordial fluctuations and their imprint on the cosmic microwave background to the propagation of gravitational waves and the dynamics of structure formation at late times.

Linear perturbation theory offers a crucial and highly nontrivial testing ground for the consistency of the theory. Unlike the homogeneous and isotropic background, the perturbed universe generically admits no symmetries, exposing potential inconsistencies in the proposed underlying structure. Thus, already the successful reproduction of standard results in cosmology a stringent check on the theory. Beyond merely reproducing the standard results of cosmological perturbation theory in the presence of ideal dust in the case $\beta = 0$, the generalised $\beta \neq 0$ cases reveal qualitatively new features absent in any conventional setting. This enriches the theoretical landscape, but most importantly, it also offers new observational signatures.

We consider the generic fluctuations and provide a summary of the parameterisations below in Table I. There are nice recent papers on cosmological perturbation[12] formalism with general connections [68, 69]; those should be consistent with the formulation below.

Fields	scalars	pseudos.	vectors	pseudov.	tensors	pseudot.
Cartan khronon	φ					
total # d.o.f. 1	1×1	0	0	0	0	0
$Spin(4)$ connection	c, r, s, ψ	$\tilde{c}, \tilde{r}, \tilde{s}, \tilde{\psi}$	v_i, u_i, w_i	$\tilde{u}_i, \tilde{v}_i, \tilde{w}_i$	h_{ij}	$\tilde{h}_{ij}$
total # d.o.f. 24	4×1	4×1	3×2	3×2	1×2	1×2
Dark matter	$n, m, \chi, \delta\hat{\rho}, \delta\hat{p}$	$\tilde{m}$	m_i, n_i, y_i	$\tilde{m}_i$	m_{ij}	
total # d.o.f. 16	5×1	1×1	3×2	1×2	1×2	0
Bartels frame	c, r, Ψ	$\tilde{s}$	v_i, w_i	$\tilde{u}_i$	h_{ij}	
total # d.o.f. 12	3×1	1×1	2×2	1×2	1×2	0
Metric	$\partial\varphi/\partial\phi, \sqrt{\kappa}\varphi + a^2\dot{c}, r, \Psi$		v_i, w_i		h_{ij}	
total # d.o.f. 10	4×1	0	2×2	0	1×2	0

TABLE I: Classification of fields by their transformation properties under the little Lorentz group. The two top rows are the theory's fundamental fields; the dark matter field arises from integrating the equations of motion; the two bottom rows are composite fields.

A. Spin-2 perturbations

In the general theory, we should generalise the Ansatz (34) for tensor perturbations by taking into account transverse-traceless fluctuations also in the integration 3-form $\boldsymbol{M}_I$, besides those in the connection 1-form. The generic Ansatz can then be given as

$$\boldsymbol{A}^{4i} \;=\; A\left(\delta^i_j + h^i{}_j\right)\mathbf{d}x^j\,, \quad \boldsymbol{A}^{ij} = B\left(\epsilon^{ij}{}_k + \epsilon^{ijl}\tilde{h}_{lk}\right)\mathbf{d}x^k\,, \tag{72a}$$

$$\boldsymbol{M}^4 \;=\; -\sqrt{\kappa}\hat{\rho}\star\mathbf{e}^4\,, \quad \boldsymbol{M}^i = \left(\sqrt{\kappa}\hat{p}\delta^i_j + \kappa^{-3/2}m^i{}_j\right)\star\mathbf{e}^j\,, \tag{72b}$$

where the new perturbation field m_{ij} is dimensionless, symmetric, transverse and traceless. From the set of connection equations (51a), we can derive an expression for $\tilde{h}_{ij}$ in terms of the other perturbations,

$$\tilde{h}_{ij} = h_{ij} + \frac{1}{B}\left(\gamma a\dot{h}_{ij} + \epsilon_{(i}{}^{kl}h_{j)k,l} - a\alpha\gamma\kappa^{-1}\tau m_{ij}\right)\,. \tag{73}$$

[12] For an introduction to and a reference for cosmological perturbation theory we recommend Kurki-Suonio's lecture notes at https://www.mv.helsinki.fi/home/hkurkisu/.

We note that this consistently generalises (36). Plugging this into the other set of connection equations (51b), we obtain a solution for the perturbation m_{ij},

$$m_{ij} = \frac{\beta\gamma\kappa\dot{h}_{ij}}{\tau}\,. \tag{74}$$

As expected, this vanishes when $\beta = 0$ or $\gamma = 0$. Using these results in the khronon EoM (50), we arrive at the evolution equation for the gravitational waves,

$$\ddot{h}_{ij} + 3(\dot{a}/a)\dot{h}_{ij} + (\gamma/a)^2\nabla^2 h_{ij} = 0\,, \tag{75}$$

consistently with [41]. As in subsection III B, we obtain the standard wave equation in the Lorentzian stage, wherein now, however, the gradient term is modulated by the prefactor γ^2. Thus, the speed of propagation of gravitational waves is strictly equal to the speed of light only in the case of strictly right-handed (or strictly left-handed) gravity.

B. Spin-1 perturbations

In the irreducible decomposition of the connection, half of the degrees of freedom, i.e. 6×2 components, are encoded into spin-1 perturbations in terms of 6 transverse (pseudo)vectors. We parameterise those as:

$$\boldsymbol{A}^4{}_i = Av_i\mathbf{d}\tau + A\left(\delta_{ij} + 2w_{(i,j)} + \epsilon_{ijk}\tilde{u}^k\right)\mathbf{d}x^j\,, \tag{76a}$$

$$\boldsymbol{A}^{ij} = B\epsilon^{ijk}\tilde{v}_k\mathbf{d}\tau + B\left(\epsilon^{ij}{}_k + 2\delta^{[i}_k u^{j]} + \epsilon^{ijl}\tilde{w}_{(l,k)}\right)\mathbf{d}x^k\,. \tag{76b}$$

Here $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are the independent pseudovectors; u, v and w are the independent vectors. We have freedom to adjust the frame by a rotation given by a pseudovector $\tilde{r}^i$, $\delta_{\tilde{r}}\boldsymbol{A}^{ij} = \epsilon^{ij}{}_k\mathbf{D}\tilde{r}^k$, which allows to set $\tilde{v}^k = 0$. In addition, we have the freedom to perform arbitrary diffeomorphisms, and a convenient coordinate system could be obtained by choosing a gauge s.t. $v^i = 0$, because then we would obtain the metric in the synchronous gauge. For generality, we shall proceed without any gauge fixing of the coordinates nor of the $SO(3)$ frame. However, because the Lorentz frame is now fixed indeed up to $SO(3)$ rotations, $\delta\phi^i = 0$, there can now be no vector perturbations in the khronon field. In the generic case, the 3-form $\boldsymbol{M}^I$ features four independent vector modes, and can be parameterised as

$$\boldsymbol{M}^4 = -\sqrt{\kappa}\hat{\rho}\star\mathbf{e}^4 + q_i\star\mathbf{e}^i\,, \tag{77a}$$

$$\boldsymbol{M}^i = n^i\star\mathbf{e}^4 + \left(\sqrt{\kappa}\hat{p}\delta^i_j + m^i{}_{,j} + m_j{}^{,i} + \epsilon^i{}_{jk}\tilde{m}^k\right)\star\mathbf{e}^j\,. \tag{77b}$$

Using the sourced connection equations (51a), we obtain expressions for the (pseudo)vector perturbations contained in the 'magnetic' connection (76b) as

$$u^i = \tilde{u}^i - \frac{1}{2B}\left(\nabla^2 w^i + \epsilon^{ijk}\tilde{u}_{j,k}\right) + \frac{\sqrt{\kappa}\tau^2 A}{2\alpha\gamma^2 B}n^i\,, \tag{78a}$$

$$\tilde{v}^i = -v^i + \frac{1}{B}\left(\dot{\tilde{u}}^i - \frac{1}{2}\epsilon^{ijk}v_{j,k}\right) + \frac{\tau\sqrt{\kappa}}{\alpha\gamma^2 B}\left(\frac{1}{2\gamma}n^i - \tilde{m}^i\right)\,, \tag{78b}$$

$$\tilde{w}^i = w^i - \frac{1}{2B}\left[\tilde{u}^i - \epsilon^{ijk}w_{j,k} + \gamma a\left(2\dot{w}^i + v^i\right) - \frac{2\tau\sqrt{\kappa}}{\alpha\gamma}m^i\right]\,. \tag{78c}$$

Using then the unsourced connection equations (51b) we obtain the expressions for vector perturbations contained in the spatial components of the integration 3-form $\boldsymbol{M}^i$ (77b) as

$$n^i = 0\,, \tag{79a}$$

$$m^i = \frac{\beta}{\tau\sqrt{\kappa}}\left(\dot{w}^i + \frac{1}{2}v^i\right)\,, \tag{79b}$$

$$\tilde{m}^i = 0\,. \tag{79c}$$

While continuity equation is trivially satisfied, the Euler equations $\mathbf{D}\boldsymbol{M}^i = 0$ identify the perturbation in the temporal part of integration form $\boldsymbol{M}^4$.

$$q^i = -\frac{\beta}{\sqrt{\kappa}\tau A}\nabla^2\left(\dot{w}^i + \frac{1}{2}v^i\right) \tag{80}$$

Using these results in the field equations (50), they reduce to expressions in terms of the coordinate-invariant perturbation $V^i \equiv 2\dot{w}^i + v^i$. In the absence of vector matter sources we obtain

$$\alpha \nabla^2 V^i \;=\; 0\,, \tag{81a}$$
$$\alpha\left(\dot{V}^i + 3\frac{\dot{a}}{a}V^i\right) \;=\; 0\,. \tag{81b}$$

It is clear that vector perturbations do not propagate.

C. Spin-0 perturbations

For generality, we don't fix into the synchronous coordinate system, but let the khronon ϕ depend on all the coordinates. At the level of cosmological background, the khronon may only be some function of τ, and for simplicity we choose the linear function, but there may be some generic perturbation φ. So,

$$\phi = \kappa^{-1/2}\tau + \varphi\,. \tag{82}$$

We introduce 4 independent scalars c, r, s and ψ, and 4 pseudoscalars $\tilde{c}$, $\tilde{r}$, $\tilde{s}$ and $\tilde{\psi}$ to describe the generic scalar perturbation of the connection

$$\boldsymbol{A}^4{}_i \;=\; A\dot{c}_{,i}\mathbf{d}\tau + A\left(\left(1-\psi\right)\delta_{ij} + \Delta^i_j r + \epsilon_{ij}{}^k\tilde{s}_{,k}\right)\mathbf{d}x^j\,, \tag{83a}$$
$$\boldsymbol{A}^{ij} \;=\; B\epsilon^{ijk}\dot{\tilde{c}}_{,k}\mathbf{d}\tau + B\left(\epsilon^{ij}{}_k\left(1-\tilde{\psi}\right) + \epsilon^{ijl}\Delta_{lk}\tilde{r} + 2s_{,[i}\delta_{j]k}\right)\mathbf{d}x^k\,, \tag{83b}$$

where we introduced the traceless $\Delta^i_i = 0$ double-derivative operator $\Delta_{ij}f \equiv f_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2 f$. It follows that the frame is

$$\mathbf{e}^4 \;=\; \left(1+\sqrt{\kappa}\dot{\varphi}\right)\mathbf{d}\tau + \sqrt{\kappa}\varphi_{,i}\mathbf{d}x^i\,, \tag{84a}$$
$$\mathbf{e}^i \;=\; a\dot{c}^{,i}\mathbf{d}\tau + a\left[\left(1+\Psi\right)\delta^i_j + \Delta^i_j r + \epsilon^i{}_{jk}\tilde{s}^k\right]\mathbf{d}x^j\,, \tag{84b}$$

where we have introduced $\Psi \equiv \frac{\sqrt{\kappa}}{\tau}\varphi - \psi$. There are 4 additional scalar degrees of freedom in the integration form

$$\boldsymbol{M}^4 \;=\; -\sqrt{\kappa}\left(\hat{\rho}+\delta\hat{\rho}\right)\star\mathbf{e}^4 - \chi_{,i}\star\mathbf{e}^i\,, \tag{85a}$$
$$\boldsymbol{M}_i \;=\; n_{,i}\star\mathbf{e}^4 + \left(\sqrt{\kappa}\left(\hat{p}+\delta\hat{p}\right)\delta_{ij} + \Delta_{ij}m + \epsilon_{ij}{}^k\tilde{m}_{,k}\right)\star\mathbf{e}^j\,. \tag{85b}$$

As it is well-known, not all the scalar perturbations are physical, since some of them can be eliminated by a coordinate transformation. Under a diffeomorphism parameterised by $\boldsymbol{\xi} = \xi^4\mathbf{d}\tau + a^2\xi_{,i}\mathbf{d}x^i$, with the action $\delta_{\boldsymbol{\xi}} \equiv \{\mathbf{d}, \sharp\boldsymbol{\xi}\lrcorner\}$, the time component of $\boldsymbol{\xi}$ shifts the khronon perturbation,

$$\delta_{\boldsymbol{\xi}}\varphi = \kappa^{-\frac{1}{2}}\xi^4\,. \tag{86a}$$

The perturbed $Spin(4)$ connection is transformed as

$$\delta_{\boldsymbol{\xi}}c = \delta_{\boldsymbol{\xi}}\tilde{c} = \delta_{\boldsymbol{\xi}}r = \delta_{\boldsymbol{\xi}}\tilde{r} = \xi\,,\quad \delta_{\boldsymbol{\xi}}\psi = -\frac{\dot{A}}{A}\xi^4 - \frac{1}{3}\nabla^2\xi\,,\quad \delta_{\boldsymbol{\xi}}\tilde{\psi} = -\frac{\dot{B}}{B}\xi^4 - \frac{1}{3}\nabla^2\xi\,,\quad \delta_{\boldsymbol{\xi}}s = \delta_{\boldsymbol{\xi}}\tilde{s} = 0\,. \tag{86b}$$

The perturbations in the integration form $\boldsymbol{M}^4$ and $\boldsymbol{M}^i$ transform as, respectively,

$$\delta_{\boldsymbol{\xi}}\delta\hat{\rho} \;=\; \left(\dot{\hat{\rho}} + 3\frac{\dot{a}}{a}\hat{\rho}\right)\xi^4 + \hat{\rho}\nabla^2\xi\,,\quad \delta_{\boldsymbol{\xi}}\chi = -a\sqrt{\kappa}\hat{\rho}\dot{\xi}\,, \tag{86c}$$
$$\delta_{\boldsymbol{\xi}}n \;=\; -\frac{\sqrt{\kappa}\hat{p}}{a}\xi^4\,,\quad \delta_{\boldsymbol{\xi}}\hat{p} = \frac{\partial\left(a^2\hat{p}\xi\right)}{a^2\partial\tau} - \frac{4}{3}\hat{p}\nabla^2\xi\,,\quad \delta_{\boldsymbol{\xi}}m = -2\sqrt{\kappa}\hat{p}\xi\,,\quad \delta_{\boldsymbol{\xi}}\tilde{m} = 0\,. \tag{86d}$$

This completes the setting-up of the scalar degrees of freedom and their transformations.

1. Derivation of the linear EoM

From the sourced connection EoM's (51a), we obtain the (pseudo)scalars in the 'magnetic' part of the connection,

$$\begin{aligned}
\dot{\tilde{c}} &= \dot{c} - \frac{\tilde{s}}{B} - \frac{\sqrt{\kappa}\tau}{2\alpha\gamma^3 B}\left(n + 2\gamma\tilde{m}\right) , && \text{(87a)}\\
\tilde{r} &= r - \frac{\tilde{s}}{B} + \gamma\frac{\tau A}{B}\left(\dot{c} - \dot{r} + \frac{\sqrt{\kappa}\tau}{\alpha\gamma^2}m\right) , && \text{(87b)}\\
s &= -\tilde{s} + \frac{1}{B}\left(\frac{\sqrt{\kappa}\dot{A}}{A}\phi + \psi + \frac{1}{3}\nabla^2 r - \frac{\sqrt{\kappa}\tau^2 A}{2\alpha\gamma^2}n\right) , && \text{(87c)}\\
\tilde{\psi} &= \frac{\sqrt{\kappa}\tau^2 A}{4\alpha\gamma B}\left(\sqrt{\kappa}\delta\hat{p} + \frac{2}{3}\nabla^2 m + 2\beta\frac{\dot{A}}{\tau^2 A}\varphi\right) - \frac{1}{2}\left(\Psi - \frac{1}{3}\nabla^2 r\right) - \frac{\gamma\tau A}{2B}\left(\sqrt{\kappa}\frac{\dot{A}}{A}\dot{\varphi} + \dot{\psi} + \frac{1}{3}\nabla^2\dot{r}\right) . && \text{(87d)}
\end{aligned}$$

From the unsourced connection equations (51b), we obtain 3×4 independent equations. Combined with (87), these identify the scalar perturbation of the integration 3-form (85).

$$\begin{aligned}
n &= 0 , && \text{(88a)}\\
m &= \frac{\beta}{\sqrt{\kappa}\tau}\left(\dot{c} + \dot{r}\right) , && \text{(88b)}\\
\tilde{m} &= 0 , && \text{(88c)}\\
\delta\hat{p} &= -2\frac{\beta}{\kappa\tau}\left(\frac{\sqrt{\kappa}\dot{A}}{\tau A}\varphi + \sqrt{\kappa}\frac{\dot{A}}{A}\dot{\varphi} + \dot{\psi} + \frac{1}{3}\nabla^2\dot{c}\right) . && \text{(88d)}
\end{aligned}$$

The Euler equation $\mathbf{D}\boldsymbol{M}_i = 0$ further identify

$$\chi = \frac{2\beta}{\sqrt{\kappa}\tau A}\left(\sqrt{\kappa}\left(\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2}\right)\varphi + \sqrt{\kappa}\frac{\dot{A}}{A}\dot{\varphi} + \dot{\psi} + \frac{1}{3}\nabla^2\dot{r}\right) , \tag{89}$$

and the continuity equation $\mathbf{D}\boldsymbol{M}_4 = 0$ yields the relation between the perturbation in density $\delta\hat{\rho}$ and in pressure $\delta\hat{p}$. Applying the background continuity equation (59) and the previous solutions for perturbations (87,88,89), we obtain

$$\begin{aligned}
\delta\dot{\hat{\rho}} + 3\left(\frac{1}{\tau} + \frac{\dot{A}}{A}\right)\delta\hat{\rho} &= \sqrt{\kappa}\hat{\rho}'\dot{\varphi} + 3\left(\frac{\sqrt{\kappa}}{\tau^2}\varphi + \sqrt{\kappa}\frac{\dot{A}}{A}\dot{\varphi} + \dot{\psi} + \frac{1}{3}\nabla^2\dot{c}\right) + \frac{6\beta}{\kappa\tau^2}\left(2\sqrt{\kappa}\frac{\dot{A}}{\tau A}\varphi + \frac{\dot{A}}{A}\dot{\varphi} + \dot{\psi} + \frac{1}{3}\nabla^2\dot{c}\right) && \text{(90)}\\
&\quad - \frac{2\beta}{\kappa\tau^2 A^2}\nabla^2\left[\sqrt{\kappa}\left(\frac{\dot{A}^2}{A^2} - \frac{\ddot{A}}{A}\right)\varphi - \sqrt{\kappa}\frac{\dot{A}}{A}\dot{\varphi} - \dot{\psi} - \frac{1}{3}\nabla^2\dot{r}\right] .
\end{aligned}$$

We include the matter source

$$\begin{aligned}
\boldsymbol{t}^4 &= -\sqrt{\kappa}\left(\rho + \delta\rho\right)\star\mathbf{e}^4 + \left(\rho + p\right)\left(\sqrt{\kappa}u_{,i} + a^{-1}\kappa\varphi\right)_{,i}\star\mathbf{e}^i , && \text{(91a)}\\
\boldsymbol{t}^i &= \left(\rho + p\right)\left(\sqrt{\kappa}u + a^{-1}\kappa\varphi\right)^{,i}\star\mathbf{e}^4 + \sqrt{\kappa}\left[\left(p + \delta p\right)\delta^i_j + \Delta^i{}_j\Pi\right]\star\mathbf{e}^j . && \text{(91b)}
\end{aligned}$$

This form of the source term corresponds to the standard parameterisation of a fluid stress energy, as shown in the appendix B. With these ingredients, the perturbation equations derived from the khronon EoM's are as below.

$$\delta\rho = -\delta\hat{\rho} - \frac{2\alpha\gamma^2}{\kappa\tau^2 A^2}\nabla^2\left(\sqrt{\kappa}\frac{\dot{A}}{A}\varphi + \psi + \frac{1}{3}\nabla^2 r\right) - \frac{6\alpha}{\kappa}\left(\frac{1}{\tau} + \frac{\dot{A}}{A}\right)\left(\frac{\sqrt{\kappa}}{\tau^2}\varphi + \sqrt{\kappa}\frac{\dot{A}}{A}\dot{\varphi} + \frac{\sqrt{\kappa}}{\tau^2}\varphi + \dot{\psi} + \frac{1}{3}\nabla^2\dot{c}\right), \quad (92a)$$

$$(\rho + p)\, u_{,i} = -\frac{\sqrt{\kappa}}{\tau A}(\rho + p)\,\varphi_{,i} - \frac{2\alpha}{\kappa\tau A}\left[\sqrt{\kappa}\left(\frac{\dot{A}^2}{A^2} - \frac{\ddot{A}}{A}\right)\varphi_{,i} - \sqrt{\kappa}\frac{\dot{A}}{A}\dot{\varphi}_{,i} - \dot{\psi}_{,i} - \frac{1}{3}\nabla^2\dot{r}_{,i}\right], \quad (92b)$$

$$\delta p = \frac{2\alpha}{\kappa}\left[3\sqrt{\kappa}\frac{\dot{A}}{\tau^2 A}\varphi + \sqrt{\kappa}\left(3\frac{\dot{A}}{\tau A} + 2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2}\right)\dot{\varphi} + 3\left(\frac{1}{\tau} + \frac{\dot{A}}{A}\right)\left(\dot{\psi} + \frac{1}{3}\nabla^2\dot{c}\right) + \sqrt{\kappa}\frac{\dot{A}}{A}\ddot{\varphi} + \ddot{\psi} + \frac{1}{3}\nabla^2\ddot{c}\right] \quad (92c)$$

$$+\frac{2(\alpha - \beta)}{3\kappa}\left[\frac{3\sqrt{\kappa}}{\tau^3}\varphi + 3\sqrt{\kappa}\frac{\dot{A}^2}{A^2}\dot{\varphi} + \frac{1}{\tau^2 A^2}\nabla^2\left(\sqrt{\kappa}\frac{\dot{A}}{A}\varphi + \psi + \frac{1}{3}\nabla^2\dot{r}\right)\right],$$

$$\Pi_{,i,j} = -\frac{\alpha - \beta}{\kappa\tau^2 A^2}\left(\sqrt{\kappa}\frac{\dot{A}}{A}\varphi_{,i,j} + \psi_{,i,j} + \frac{1}{3}\nabla^2 r_{,i,j}\right) - \frac{3\alpha}{\kappa}\left(\frac{1}{\tau} + \frac{\dot{A}}{A}\right)\left(\dot{c}_{,i,j} - \dot{r}_{,i,j}\right) - \frac{3\alpha}{\kappa}\left(\ddot{c}_{,i,j} - \ddot{r}_{,i,j}\right). \quad (92d)$$

There are only four independent (sets of) equations in the end, because the spatial components of the energy equation and time components of the momentum equations yield the identical velocity perturbation constraint (92b). Since u is potential for the velocity of the fluid i.e. its gradient is the fluid's motion, u is regarded as (proportional to) a time derivative. This should be taken into account when translating the physics into the Lorentzian "time frame", as we recall from the discussion of angular velocity in section III C. Thus, our Wick rotation implies $u \to -iv$, where v is the usual spacetime velocity perturbation potential. Denoting derivatives wrt t by a prime, recalling that $\Psi = \frac{\varphi}{m_P t} - \psi$ and defining $\Phi \equiv \frac{\varphi}{m_p}$, the full set of field equations (92) are translated into the Lorentzian framework as

$$\delta\rho + \delta\hat{\rho} = -6\alpha m_P^2 H\left(H\Phi' - \Psi' + \frac{1}{3}\nabla^2 c'\right) + 2(\alpha - \beta)\frac{m_P^2}{a^2}\nabla^2\left(H\Phi - \Psi + \frac{1}{3}\nabla^2 r\right) + 6\beta\frac{m_P^2}{t^3}\Phi, \quad (93a)$$

$$(\rho + p)\, v_{,i} = -\frac{\Phi_{,i}}{a}\hat{\rho} + 2\alpha\frac{m_P^2}{a}\left(H\Phi'_{,i} - \Psi'_{,i} + \frac{1}{3}\nabla^2 r'_{,i}\right) - 2\beta\frac{m_P^2}{at^2}\Phi_{,i}, \quad (93b)$$

$$\delta p = 2\alpha m_P^2\left[\left(3H^2 + 2H'\right)\Phi' + H\left(\Phi'' - 3\Psi' + \nabla^2 c'\right) - \left(\Psi'' - \frac{1}{3}\nabla^2 c''\right)\right] \quad (93c)$$

$$-\frac{2}{3}(\alpha - \beta)\frac{m_P^2}{a^2}\nabla^2\left(H\Phi - \Psi + \frac{1}{3}\nabla^2 r\right) - 2\beta\frac{m_P^2}{t^3}\Phi,$$

$$\Pi_{,i,j} = (\alpha - \beta)\frac{m_P^2}{a^2}\left(H\Phi_{,i,j} - \Psi_{,i,j} + \frac{1}{3}\nabla^2 r_{,i,j}\right) - 3\alpha m_P^2 H\left(c'_{,i,j} - r'_{,i,j}\right) - \alpha m_P^2\left(c''_{,i,j} - r''_{,i,j}\right). \quad (93d)$$

Some consistency checks are pertinent at this point, and we have verified that these equations imply the two matter EoM (B7) given in Appendix B. First, we solve $\delta\rho$ and $\delta\rho'$ from (93a) and its first time derivative, use (93b) as an expression for v and use (93c) as an expression for δp, and when we plug these into the continuity equation for matter (B7a), it becomes

$$\delta\hat{\rho}' + 3H\delta\hat{\rho} \;=\; \Phi'\hat{\rho}' + 3\left(H\Phi' - \Psi' + \frac{1}{3}\nabla^2 c'\right)\hat{\rho} + 6\beta\frac{m_P^2}{t^2}\left[\left(\frac{2H}{t} - \frac{3}{t^2}\right)\Phi + H\Phi' - \Psi' + \frac{1}{3}\nabla^2 c'\right] \quad (94)$$

$$-2\beta\frac{m_P^2}{a^2}\nabla^2\left[H'\Phi + H\Phi' - \Psi' + \frac{1}{3}\nabla^2 r'\right].$$

As it should, this is precisely the continuity equation for dark matter (94) translated into the Lorentzian "time frame". Another check is given by the Euler equation for matter (B7b), wherein we now plug v and v' solved from (93b) and its first time derivative, again use (93c) as an expression for δp and now also (93d) as an expression for Π. We find that (B7b) is now identically satisfied, as it should since we have already solved the Euler equations for dark matter.

2. *Newtonian gauge description*

Up to now, we have considered the generic perturbations without any gauge fixing. It can be useful to specialise into the conformal Newtonian gauge which is probably the most conventional gauge in cosmological perturbation theory.

However, the novel pregeometric approach suggests just a tiny bit unconventional notation for the metric perturbations in the Newtonian gauge. This gauge is reached by implementing the diffeomorphism $\delta_{\boldsymbol{\xi}}$ with $\xi^4 = -a(\dot{c} - \dot{r}) + \sqrt{\kappa}\varphi$ and $\xi = -a^{-2}r$. According to (86), the perturbations of the khronon and the connection then become

$$
\begin{aligned}
\Phi^N &= -a\left(c' - r'\right) , && (95a)\\
\Psi^N &= \Psi - H\left(\Phi - ac' + ar'\right) - \frac{1}{3}\nabla^2 r\,, && (95b)\\
c^N &= c - r\,, && (95c)\\
r^N &= 0\,, && (95d)
\end{aligned}
$$

where the superscript indicates that the variable is evaluated in the Newtonian gauge. Similarly, the dark matter and matter perturbations are transformed into $\delta\hat{\rho}^N$, $\delta\rho^N$, and so on. We omit the superscripts for convenience, and state the field equations (93) in the Newtonian gauge:

$$
\begin{aligned}
m_P^{-2}\left(\delta\rho + \delta\hat{\rho}\right) &= 6\alpha H\left(\Psi' - H\Phi'\right) - \frac{2}{a^2}\left(\alpha - \beta\right)\nabla^2\Psi - \frac{2\beta}{a^2}\nabla^2\Phi + \frac{6\beta}{t^3}\Phi\,, && (96a)\\
m_P^{-2}a\left(\rho + p\right)v &= -m_P^{-2}\hat{\rho}\Phi + 2\alpha\left(H\Phi' - \Psi'\right) - 2\beta\frac{\Phi}{t^2}\,, && (96b)\\
m_P^{-2}\delta p &= 2\alpha\left[\left(3H^2 + 2H'\right)\Phi' + H\left(\Phi'' - \Psi'' - 3\Psi'\right) + \frac{1}{3a^2}\nabla^2\left(\Psi + \Phi'\right)\right] + \frac{2\beta}{3a^2}\nabla^2\left(H\Phi - \Psi\right) - \frac{2\beta}{t^3}\Phi\,, && (96c)\\
m_P^{-2}a^2\Pi &= -\alpha\left(\Psi + \Phi'\right) + \beta\left(\Psi - H\Phi\right) . && (96d)
\end{aligned}
$$

It is interesting to note that the standard dimensionless gravitational perturbations that appear in the longitudinalised metric

$$
\mathbf{d}s^2 = -\left(1 + 2\Phi'\right)\mathbf{d}t \otimes \mathbf{d}t + a^2\left(1 + 2\Psi\right)\delta_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j\,, \tag{97}
$$

are now interpreted in solely in terms of the khronon field plus the diagonal perturbation of the spatial connection, like

$$
\Phi' = \frac{\partial\varphi}{\partial\phi}\,, \quad \Psi = \frac{\varphi}{\phi} - \psi\,. \tag{98}
$$

For this reason we call the dimensionless lapse perturbation Φ' (instead of Φ as usual). Now also Φ appears in theory, and it plays the role of the dark matter velocity perturbation potential, "$\Phi \sim \hat{v}$". In this way, no new fields enter into the system besides the dark matter density perturbation, consistently with the knowledge that the dynamical phase of the theory introduces only one scalar degree of freedom beyond general relativity. It is well known to those familiar with cosmological perturbation theory that the synchronous coordinate system leaves a remnant gauge freedom which is conventionally fixed by setting the cold dark matter velocity perturbation to zero. In our pregeometric theory, there is a fundamental relation between the temporal component of metric geometry and the dark matter velocity field; namely, the former is the derivative of the latter; fluctuations in time lapse are a manifestation of acceleration in dark matter.

3. *Clustering of dark matter*

Adopting this interpretation, let us consider an analogue of the usual comoving density perturbation $\delta\rho^C$, which we would now obtain as

$$
\delta\rho^C = \delta\rho + \delta\hat{\rho} + 3aH\left(\rho + p\right)v + 3\left(H\hat{\rho} + \frac{\hat{p}}{t}\right)\Phi\,. \tag{99}
$$

Using equations (96a) and (96b), we get

$$
-\nabla^2\left[\left(\alpha - \beta\right)\Psi + \beta H\Phi\right] = \frac{a^2\delta\rho^C}{2m_P^2}\,. \tag{100a}
$$

We see from (96d) that there will also generically occur effective anisotropic stress proportional to β/α i.e. the two gravitational potentials (98) will not be equal to minus each other. Using (96d), we now obtain the Poisson equation in its standard form (the effective Newton's constant being determined directly by the parameter α)

$$
\nabla^2\Phi' = \frac{a^2\delta\rho^C}{2\alpha m_P^2}\,. \tag{100b}
$$

The dark matter clustering is governed by (94), which is rewritten in the Newtonian gauge as

$$\delta\hat{\rho}' + 3H\delta\hat{\rho} - \rho'\Phi' + \rho\left(3\Psi' - 3H\Phi' - \frac{\nabla^2}{a^2}\Phi\right) = \frac{2\beta m_P^2}{a^2}\nabla^2\left(\Psi' - H\Phi' - H'\Phi\right) + \frac{6\beta m_P^2}{t^2}\left[-\Psi' + H\Phi' + \frac{1}{3}\left(\frac{\nabla^2}{a^2} + \frac{6Ht-9}{t^2}\right)\Phi\right]. \tag{101}$$

At this point, we shall make some approximations to facilitate the analysis.

It is helpful to focus on scales inside the horizon, such that $\nabla^2 f \gg H^2 f$. This is often called the 'quasi-static' limit, because it allows to neglect time derivatives, as they reflect the cosmological evolution, $f' \sim Hf$, $f'' \sim H^2 f$. Also, as we see from (100), the gravitational potentials are negligible compared to the density perturbations, $\delta\rho, \delta\hat{\rho} \gg \rho\Phi', \rho\Psi$. In the quasi-static limit (101) reduces to

$$\delta\hat{\rho}' + 3H\delta\hat{\rho} = \frac{\nabla^2}{a^2}\left[2\beta m_P^2\left(\Psi' - H\Phi'\right) + \left(\hat{\rho} - 2\beta m_P^2 H' + \frac{2\beta m_P^2}{t^2}\right)\Phi\right]. \tag{102}$$

The time derivative of this equation yields a second order equation for the dark matter density perturbation, sourced by Φ, Ψ and their derivatives. If we neglect the standard matter terms, which is justified in a vacuum and in regions dominated by dark matter, we can then use the field equations (96) to obtain expressions for the Φ, Ψ and their derivatives (Φ' being already obtained at (100b), since at small scales $\delta\hat{\rho}^C \approx \delta\hat{\rho}$ in the Newtonian gauge), in order to set the second order equation into an autonomous form. We report only the final result, in terms of the fractional energy density perturbation $\hat{\delta} \equiv \delta\hat{\rho}/\hat{\rho}$,

$$\hat{\delta}'' + \left[2H - \frac{2\beta m_P^2}{t^3}\left(\frac{6-6Ht}{\hat{\rho}} + \frac{t^2}{\hat{\rho}t^2 + 2\beta m_P^2}\right)\right]\hat{\delta}' = \left\{\frac{\hat{\rho}}{2\alpha m_P^2} + 2\beta\frac{m_P^2\hat{\rho}t^2\left[3\alpha Ht(5Ht-7) + 9\alpha - 5\beta\right] + 6\beta m_P^4\left[\alpha Ht(5Ht-8) + 4\alpha - \beta\right] - \hat{\rho}^2 t^4}{\alpha\hat{\rho}t^4\left(2\beta m_P^2 + \hat{\rho}t^2\right)}\right\}\hat{\delta}. \tag{103}$$

When $\beta = 0$, we recover the standard evolution equation for density perturbations in cold dark matter. In a generic theory, both the friction term and the effective Newton's constant (as read from the curly bracket) are modulated by time-dependent background terms. The result verifies the existence of a single scalar (extra) degree of freedom without any instabilities or other pathologies. It is remarkable that in the $\beta \neq 0$ case the effective fluid has pressure $\hat{p} \neq 0$, but nevertheless there doesn't appear any gradient term in (103). In other words, the generalised dark matter is still characterised by vanishing sound speed[13]. This is only possible because the effective fluid is associated a with spin current.

In addition to the modified growth rate of clustering, one of the various avenues for testing the theory is through the presence of an effective anisotropic stress [71–73], which manifests as a difference between the two scalar gravitational potentials (98). This difference is captured by the "gravitational slip" parameter $\eta \equiv -\Psi/\Phi'$ [74–76]. Expanding to first order in β in the present quasi-static approximation,

$$\eta = 1 + \left[\frac{2m_P^2 H}{\hat{\rho}}(\log\hat{\delta})' + \frac{1}{\alpha}\right]\beta + \mathcal{O}(\beta^2), \tag{104}$$

confirming that this effect is absent in the right-handed limit. The gravitational slip can be observationally constrained by combining galaxy clustering (which probes the Newtonian potential Φ') with weak lensing (which probes the lensing potential $\Phi' - \Psi$).

VI. SPHERICAL SYMMETRY

We now turn to the study of spherically symmetric solutions, which provide a crucial testing ground for the nonperturbative structure of the theory. Unlike the cosmological setting, where deviations from homogeneity are treated perturbatively, spherical symmetry allows for fully nonlinear configurations while still permitting significant

[13] This property of Lorentz gauge theory was anticipated in a completely different approach of phenomenological fluid Lagrangians [70]. A nonzero sound speed does not imply a physical pathology (though an imaginary sound speed implies an instability); however, it compromises the phenomenological viability of a wide range of alternative dark sectors from Cardassian and Chaplygin models to interacting quintessence scalars, 3-forms and vectors to Palatini-f(R) and various other modified gravities, see [70] for references.

analytic control. This sector is particularly well-suited for probing the theory's strong-field regime, examining the nature of horizons, and assessing the viability of black hole solutions. It also offers a sharp diagnostic for potential pathologies, such as singularities, asymptotic non-flatness, or unphysical branches that may arise in generalised gravity theories. In the present framework, grounded in a Euclidean $Spin(4)$ gauge structure, we will see that spherically symmetric solutions behave in a controlled and physically reasonable manner, further reinforcing the consistency of the theory beyond linear perturbations. While a full classification of solutions—especially beyond the restricted "one-handed gravity" sector—remains an open problem, this initial exploration reveals no inconsistencies and shows encouraging agreement with the cosmological sector, suggesting a coherent and unified underlying structure.

After first deriving the generic field equations in the generic spherically symmetric case, and further analysing them in the absence of material spin currents, we both reproduce the known static solutions when $\beta = 0$ and show their absence when $\beta \neq 0$ in VI B, and then briefly check the simplest, homogeneous solutions when $\beta \neq 0$ in VI C.

A. The general field equations

Without loss of generality, a spherically symmetric configuration can be parameterised as[14]:

$$\boldsymbol{A}^{41} = -\frac{F}{\tau}\mathbf{d}\rho\,, \quad \boldsymbol{A}^{42} = -\frac{r}{\tau}\mathbf{d}\theta\,, \quad \boldsymbol{A}^{43} = -\frac{r\sin\theta}{\tau}\mathbf{d}\varphi\,, \tag{105a}$$

$$\boldsymbol{A}^{12} = A\mathbf{d}\theta + B\sin\theta\mathbf{d}\varphi\,, \quad \boldsymbol{A}^{13} = -B\mathbf{d}\theta + A\sin\theta\mathbf{d}\varphi\,, \quad \boldsymbol{A}^{23} = C\mathbf{d}\tau + D\mathbf{d}\rho - \cos\theta\mathbf{d}\varphi\,. \tag{105b}$$

There are now 6 functions F, r, A, B, C and D, of 2 variables, τ and ρ. Note that because of the angular parameterisation, the connection functions A and B are now dimensionless, and also F is a dimensionless function. Due to spherical symmetry, all the 6 functions depend on the 2 variables via $r(\tau,\rho)$, and if we impose staticity, then all the functions may depend only on r. In the following, a dot denotes derivative wrt τ as previously, and a prime denotes derivative wrt ρ.

The connection (105) can equivalently be presented as

$${}^{\pm}\boldsymbol{A}^{1} = -C\mathbf{d}\tau - \left(\pm\frac{F}{\tau} + D\right)\mathbf{d}\rho + \cot\theta\mathbf{d}\tilde{\varphi}\,, \tag{106a}$$

$${}^{\pm}\boldsymbol{A}^{2} = -\left(\pm\frac{r}{\tau} + B\right)\mathbf{d}\theta + A\mathbf{d}\tilde{\varphi}\,, \tag{106b}$$

$${}^{\pm}\boldsymbol{A}^{3} = -A\mathbf{d}\theta - \left(\pm\frac{r}{\tau} + B\right)\mathbf{d}\tilde{\varphi}\,, \tag{106c}$$

where we introduced the short-hand notation $\mathbf{d}\tilde{\varphi} \equiv \sin\theta\mathbf{d}\varphi$. We first compute the torsion,

$$\boldsymbol{T}^{1} = \left(\dot{F} - \frac{F}{\tau}\right)\mathbf{d}\tau\wedge\mathbf{d}\rho - 2Br\mathbf{d}\theta\wedge\mathbf{d}\tilde{\varphi}\,, \tag{107a}$$

$$\boldsymbol{T}^{2} = \left(\dot{r} - \frac{r}{\tau}\right)\mathbf{d}\tau\wedge\mathbf{d}\theta + Cr\mathbf{d}\tau\wedge\mathbf{d}\tilde{\varphi} + (r' + FA)\,\mathbf{d}\rho\wedge\mathbf{d}\theta + (FB + Dr)\,\mathbf{d}\rho\wedge\mathbf{d}\tilde{\varphi}\,, \tag{107b}$$

$$\boldsymbol{T}^{3} = -Cr\mathbf{d}\tau\wedge\mathbf{d}\theta + \left(\dot{r} - \frac{r}{\tau}\right)\mathbf{d}\tau\wedge\mathbf{d}\tilde{\varphi} - (FB + Dr)\,\mathbf{d}\rho\wedge\mathbf{d}\theta + (r' + FA)\,\mathbf{d}\rho\wedge\mathbf{d}\tilde{\varphi}\,, \tag{107c}$$

and then the anti/self-dual field strengths,

$${}^{\pm}\boldsymbol{F}^{1} = \left[\mp\frac{\partial}{\partial\tau}\left(\frac{F}{\tau}\right) + C' - \dot{D}\right]\mathbf{d}\tau\wedge\mathbf{d}\rho + \left[A^2 - 1 + \left(\frac{r}{\tau} \pm B\right)^2\right]\mathbf{d}\theta\wedge\mathbf{d}\tilde{\varphi}\,, \tag{108a}$$

$$\begin{aligned}{}^{\pm}\boldsymbol{F}^{2} = & -\left[\pm\frac{\partial}{\partial\tau}\left(\frac{r}{\tau}\right) + AC + \dot{B}\right]\mathbf{d}\tau\wedge\mathbf{d}\theta + \left[\dot{A} - \left(\pm\frac{r}{\tau} + B\right)C\right]\mathbf{d}\tau\wedge\mathbf{d}\tilde{\varphi} \\ & - \left[\pm\frac{r'}{\tau} + A\left(\pm\frac{F}{\tau} + D\right) + B'\right]\mathbf{d}\rho\wedge\mathbf{d}\theta + \left[A' - \left(\frac{r}{\tau} \pm B\right)\left(\frac{F}{\tau} \pm D\right)\right]\mathbf{d}\rho\wedge\mathbf{d}\tilde{\varphi}\,,\end{aligned} \tag{108b}$$

$$\begin{aligned}{}^{\pm}\boldsymbol{F}^{3} = & -\left[\dot{A} - \left(\pm\frac{r}{\tau} + B\right)C\right]\mathbf{d}\tau\wedge\mathbf{d}\theta - \left[\pm\frac{\partial}{\partial\tau}\left(\frac{r}{\tau}\right) + AC + \dot{B}\right]\mathbf{d}\tau\wedge\mathbf{d}\tilde{\varphi} \\ & - \left[A' - \left(\frac{r}{\tau} \pm B\right)\left(\frac{F}{\tau} \pm D\right)\right]\mathbf{d}\rho\wedge\mathbf{d}\theta - \left[\pm\frac{r'}{\tau} + A\left(\pm\frac{F}{\tau} + D\right) + B'\right]\mathbf{d}\rho\wedge\mathbf{d}\tilde{\varphi}\,.\end{aligned} \tag{108c}$$

[14] The coefficients $\boldsymbol{A}^{4i}$ have been put into a convenient form by exhausting the freedom to choose coordinates, whilst the coefficients $\boldsymbol{A}^{ij}$ are completely generic.

The third set of field equation (50c,50d) for the spherically symmetric torsion (107) reduces to

$$\frac{g_\pm}{\sqrt{\kappa} r}\left[\left(\dot{r}-\frac{r}{\tau}\mp B\right)\star\mathbf{e}^1-\left(\frac{r'}{F}+A\right)\star\mathbf{e}^4\right] = \frac{1}{2}\tau\boldsymbol{M}^1\mp{}^\pm\boldsymbol{O}^1\,, \tag{109a}$$

$$\frac{g_\pm}{\sqrt{\kappa}Fr}\left[\left(r\dot{F}+\dot{r}F-2F\frac{r}{\tau}\mp FB\mp Dr\right)\star\mathbf{e}^2+(\pm r'\pm FA+FCr)\star\mathbf{e}^3\right] = \tau\boldsymbol{M}^2\mp 2\,{}^\pm\boldsymbol{O}^2\,, \tag{109b}$$

$$\frac{g_\pm}{\sqrt{\kappa}Fr}\left[(\pm r'\pm FA+FCr)\star\mathbf{e}^2-\left(r\dot{F}+\dot{r}F-2F\frac{r}{\tau}\mp FB\mp Dr\right)\star\mathbf{e}^3\right] = \tau\boldsymbol{M}^3\pm 2\,{}^\pm\boldsymbol{O}^3\,, \tag{109c}$$

The spherically symmetric Ansatz suggested for the integration form $\boldsymbol{M}^I$ by (109) is of the form

$$\boldsymbol{M}^1 = -\frac{\sqrt{\kappa}\hat{p}_r}{2}\star\mathbf{e}^1-\frac{\hat{v}}{2\kappa^{\frac{3}{2}}}\star\mathbf{e}^4\,, \tag{110a}$$

$$\boldsymbol{M}^2 = -\frac{\sqrt{\kappa}\hat{p}_\theta}{2}\star\mathbf{e}^2+\hat{\pi}\star\mathbf{e}^3\,, \tag{110b}$$

$$\boldsymbol{M}^3 = \frac{\hat{\pi}}{\kappa^{\frac{3}{2}}}\star\mathbf{e}^2+\frac{\sqrt{\kappa}\hat{p}_\theta}{2}\star\mathbf{e}^3\,, \tag{110c}$$

$$\boldsymbol{M}^4 = \frac{\hat{w}}{2\kappa^{\frac{3}{2}}}\star\mathbf{e}^1+\frac{\sqrt{\kappa}\hat{\rho}}{2}\star\mathbf{e}^4\,. \tag{110d}$$

Thus, yet 6 new functions of 2 variables appear in the system. To make progress from here, we need to make some assumptions about the matter sources.

We will first restrict to the case of spinless matter. In the absence of spin currents, ${}^\pm\boldsymbol{O}^i=0$, we obtain the set of 2×4 equations

$$4g_\pm\left(\dot{r}-\frac{r}{\tau}\mp B\right) = -\tau r\kappa\hat{p}_r\,, \tag{111a}$$

$$2g_\pm\left(\frac{r'}{F}+A\right) = \tau\kappa^{-1/2}\hat{v}\,, \tag{111b}$$

$$2g_\pm\left(\frac{\dot{F}}{F}+\frac{\dot{r}}{r}-\frac{2}{\tau}\mp\frac{B}{r}\mp\frac{D}{F}\right) = -\tau\kappa\hat{p}_\theta\,, \tag{111c}$$

$$g_\pm\left(\pm\frac{r'}{F}\pm A+Cr\right) = r\tau\kappa^{-1/2}\hat{\pi}\,. \tag{111d}$$

Solving these equations, we find that two of the imperfect terms in $\boldsymbol{M}^i$ vanish, $\hat{v}=\hat{\pi}=0$, and the effective pressure is determined as

$$\kappa\hat{p}_r = \frac{2\beta}{\tau}\left(\frac{\dot{r}}{r}-\frac{1}{\tau}\right)\,, \tag{112a}$$

$$\kappa\hat{p}_\theta = \frac{\beta}{\tau}\left(\frac{\dot{F}}{F}+\frac{\dot{r}}{r}-\frac{2}{\tau}\right)\,. \tag{112b}$$

The equations (111) also give solutions to the 4 of the 6 functions parameterising the connection,

$$A=-\frac{r'}{F}\,,\quad B=\gamma^{-1}\left(\dot{r}-\frac{r}{\tau}\right)\,,\quad C=0\,,\quad D=\gamma^{-1}\left(\dot{F}-\frac{F}{\tau}\right)\,. \tag{113}$$

The field strengths (108) then become

$$\begin{aligned}
{}^{\pm}\boldsymbol{F}^1 &= -\left[\gamma^{-1}\ddot{F} - \left(\gamma^{-1}\mp 1\right)\left(\frac{\dot{F}}{\tau} - \frac{F}{\tau^2}\right)\right]\mathbf{d}\tau\wedge\mathbf{d}\rho + \left[\left(\frac{r'}{F}\right)^2 + \left(\left(\gamma^{-1}\mp 1\right)\frac{r}{\tau^2} - \gamma^{-1}\frac{\dot{r}}{\tau}\right)^2 - 1\right]\mathbf{d}\theta\wedge\mathbf{d}\tilde{\varphi}\,, \quad (114a)\\
{}^{\pm}\boldsymbol{F}^2 &= -\left[\gamma^{-1}\ddot{r} - \left(\gamma^{-1}\mp 1\right)\left(\frac{\dot{r}}{\tau} - \frac{r}{\tau^2}\right)\right]\mathbf{d}\tau\wedge\mathbf{d}\theta + \left(\frac{\dot{F}r'}{F^2} - \frac{\dot{r}'}{F}\right)\mathbf{d}\tau\wedge\mathbf{d}\tilde{\varphi}\\
&+ \gamma^{-1}\left(\frac{r'\dot{F}}{F} - \dot{r}'\right)\mathbf{d}\rho\wedge\mathbf{d}\theta + \left[\frac{F'r'}{F^2} - \frac{r''}{F} - \left(\left(\gamma^{-1}\mp 1\right)\frac{F}{\tau} - \gamma^{-1}\dot{F}\right)\left(\left(\gamma^{-1}\mp 1\right)\frac{r}{\tau} - \gamma^{-1}r\right)\right]\mathbf{d}\rho\wedge\mathbf{d}\tilde{\varphi}\,, \quad (114b)\\
{}^{\pm}\boldsymbol{F}^3 &= -\left(\frac{\dot{F}r'}{F^2} - \frac{\dot{r}'}{F}\right)\mathbf{d}\tau\wedge\mathbf{d}\theta - \left[\gamma^{-1}\ddot{r} - \left(\gamma^{-1}\mp 1\right)\left(\frac{\dot{r}}{\tau} - \frac{r}{\tau^2}\right)\right]\mathbf{d}\tau\wedge\mathbf{d}\tilde{\varphi}\\
&- \left[\frac{F'r'}{F^2} - \frac{r''}{F} - \left(\left(\gamma^{-1}\mp 1\right)\frac{F}{\tau} - \gamma^{-1}\dot{F}\right)\left(\left(\gamma^{-1}\mp 1\right)\frac{r}{\tau} - \gamma^{-1}r\right)\right]\mathbf{d}\rho\wedge\mathbf{d}\theta + \gamma^{-1}\left(\frac{r'\dot{F}}{F} - \dot{r}'\right)\mathbf{d}\rho\wedge\mathbf{d}\tilde{\varphi}\,. \quad (114c)
\end{aligned}$$

Thus, in the end we have arrived at the system where there are only four unknown functions, $F(\tau,\rho)$ which determines the metric of (eventually, pseudo)Riemannian geometry; $r(\tau,\rho)$ which plays the role of a radial coordinate as usual; $\hat{\rho}(\tau,\rho)$ which represents the weight of space that can be interpreted as the energy density of effective dark matter; and $\hat{w}(\tau,\rho)$ which describes a non-standard flow in the effective dark matter field (which cannot be mimicked by any standard fluid since it would require an asymmetric energy-momentum tensor). Again, to make progress from here we have to make further assumptions about the matter sources, in particular we have to specify the material energy current $\boldsymbol{t}^I$.

B. Static solutions require $\beta = 0$

We shall restrict to vacuum, $\boldsymbol{t}^I = 0$. Also, for simplicity we set $\lambda = 0$ so that there is no effective cosmological constant term. The energy constraint (50a) now gives

$$\begin{aligned}
\left(g_+ + g_-\right)\left[\left(1 - A^2\right)F + 2A'\right] - \alpha\left(F\dot{r}^2 + 2\dot{F}\dot{r}r\right) + 3\beta\frac{Fr^2}{\tau^2} &= -Fr^2\kappa\hat{\rho}\,, \quad (115a)\\
\left(g_+ + g_-\right)\frac{\partial}{\partial\tau}\left(\frac{r'}{F}\right) &= \frac{r}{2\kappa}\hat{w}\,. \quad (115b)
\end{aligned}$$

The radial momentum constraint from (50b) gives

$$\begin{aligned}
\left(g_+ + g_-\right)\left(A^2 - 1\right) + \alpha\left(\dot{r}^2 + 2r\ddot{r}\right) &= \beta\frac{r^2}{\tau^2}\,, \quad (115c)\\
\alpha\left(\dot{F}r' - F\dot{r}'\right) &= 0\,. \quad (115d)
\end{aligned}$$

To facilitate the analysis, we shall first look for static solutions. Then the geometrical functions should only depend of the radial function $r(\tau,\rho)$, i.e. we can consider that $F = F(r(\tau,\rho))$ and $\hat{\rho} = \hat{\rho}(r(\tau,\rho))$. Since we exclude the special case $g_+ = g_-$ from the present analysis, (115d) implies that A is a constant, $\dot{A} = A' = 0$ and we have that $r' = -AF$. Then we see from (115b) that the remaining imperfect term $\hat{w} = 0$ vanishes along with the others. The final field equations, the angular parts of momentum constraints from (50b) gives, after the simplifications allowed by the constancy of A, only one nontrivial equation,

$$\alpha\left(\ddot{F}r + \dot{F}\dot{r} + F\ddot{r}\right) = \beta\frac{Fr}{\tau^2}\,. \quad (115e)$$

We should then check the consistency of the field equations (115) by confirming their consistency with the conservation laws. The radial component of the equation $\mathbf{D}\boldsymbol{M}^i = 0$ reads

$$\hat{p}_r' = \frac{2}{r}\left(\hat{p}_\theta - \hat{p}_r\right) + \frac{F\hat{w}}{\kappa^{\frac{3}{2}}\tau^2}\,. \quad (116a)$$

With the imperfect term $\hat{w} = 0$ and the pressures given by (112), this equation is identically satisfied when A is a constant. Also, the radial components of the conservation equation $\mathbf{D}\boldsymbol{M}^2 = \mathbf{D}\boldsymbol{M}^3 = 0$ are identically satisfied. It remains to consider the fourth component, $\mathbf{D}\boldsymbol{M}^4 = 0$, which gives, for $\hat{w} = 0$,

$$\frac{\partial}{\partial \tau}\left(Fr^2\hat{\rho}\right) = -\frac{Fr^2}{\tau}\left(\hat{p}_r + 2\hat{p}_\theta\right) = \frac{2\beta r}{\kappa\tau^2}\left(\frac{3Fr}{\tau} - 2F\dot{r} - \dot{F}r\right), \tag{116b}$$

where we used the solution (112) in the second equality.

At this stage, we identify the τ with a time coordinate on a pseudo-Riemannian manifold by a Wick rotation. The transformation of the Euclidean line element into a pseudo-Riemannian line-element in the familiar radial Schwarzschild coordinates,

$$\mathbf{e}^I \otimes \mathbf{e}_I = \mathbf{d}\tau \otimes \mathbf{d}\tau + \mathbf{e}^i \otimes \mathbf{e}_i \to -f^2(r)\mathbf{d}t \otimes \mathbf{d}t + g^2(r)\mathbf{d}r \otimes \mathbf{d}r + r^2\mathbf{d}\theta \otimes \mathbf{d}\theta + r^2\mathbf{d}\tilde{\varphi} \otimes \mathbf{d}\tilde{\varphi} = \mathbf{e}^A \otimes \mathbf{e}_A\,, \tag{117}$$

is achieved by

$$-i\mathbf{d}\tau \;\to\; A\mathbf{d}t \pm \frac{Fg}{f}\mathbf{d}r\,, \tag{118a}$$

$$\mathbf{d}\rho \;=\; \mathbf{d}t \pm \frac{Ag}{Ff}\mathbf{d}r\,. \tag{118b}$$

The function f is given by $f^2 = A^2 - F^2$, and the function g can be chosen freely in general. However, our field equations have now constrained that $r' = -AF$, which implies that $g = \mp 1/f$ in (117,118). Thus, we already see that the Eddington parameter will retain its general-relativistic value regardless of the theory parameters $g_\pm$. Using now the coordinate transformation (118), we can re-express the derivatives wrt the khronon field τ as derivatives wrt to the radial coordinate r,

$$\dot{r} \;\to\; -iF\,, \quad \ddot{r} \to -F_{,r}F\,, \tag{119a}$$

$$\dot{F} \;\to\; -iF_{,r}F\,, \quad \ddot{F} \to -(F_{,r})^2F - F_{,rr}F^2\,. \tag{119b}$$

The nontrivial field equations (115a), (115e) and (116a) then become, respectively,

$$(g_+ + g_-)\left(1 - A^2\right) + \alpha\left(F + 2F_{,r}r\right)F + 3\beta\frac{r^2}{\tau^2} \;=\; -r^2\kappa\hat{\rho}\,, \tag{120a}$$

$$(g_+ + g_-)\left(1 - A^2\right) + \alpha\left(F^2 + 2F_{,r}Fr\right) \;=\; -\beta\frac{r^2}{\tau^2}\,, \tag{120b}$$

$$\alpha\left[F_{,rr}Fr + (F_{,r})^2r + 2F_{,r}F\right] \;=\; -\beta\frac{r}{\tau^2}\,. \tag{120c}$$

The difference of the two last equations yields a simple second-order differential equation with the solution

$$F^2 = \gamma^2\left(A^2 - 1\right) + \frac{r_S}{r} - \hat{\Lambda}r^2\,, \tag{121}$$

where r_S and $\hat{\lambda}$ are the integration constants. On the other hand, combining the two first equations gives $\hat{\rho} = -2\beta\tau^{-2}$. We promptly check that this effective energy density is compatible with the continuity equation (116b). However, we have only checked two linear combinations of the equations (119). To solve the full system of equations, we find that we must set $\hat{\Lambda} = \beta\tau^{-2}/3$. But this is not a constant unless $\beta = 0$: consistent static spherically symmetric solutions[15] exist only in the case $\beta = 0$, and then we recover the Schwarzschild solution, the special limit $a \to 0$ of the Kerr solution studied in section III C. It is not surprising that static spherically symmetric solutions are excluded by nonzero β, since in section IV we established that only the special case of maximal chiral asymmetry $g_\pm = 0$ allows the exact Minkowski solution. On the other hand, we already derived exact spherically symmetric (even homogeneous) but nonstatic vacuum solutions describing the expanding universe in the $\beta \neq 0$ class of theories, recall (66). We should at least recover these solutions also in the present set-up, then giving up the requirement of staticity.

[15] We note that observational implications of a black hole metric given by (121) have been considered recently [77, 78].

C. On solutions in the generic theory

So, let us take some steps back and return to the spherically symmetric Ansatz[16] (115), keeping the connection coefficients as, instead of functions of only $r(\tau,\rho)$, generic functions of ρ and τ. For example, the cosmological solutions obtained at (66) should correspond to

$$r(\tau,\rho)=\rho F(\tau,\rho)\,,\quad F(\tau,\rho)=\left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\left(1\pm\sqrt{1+3\frac{\beta}{\alpha}}\right)}\,, \tag{122}$$

where τ_0 is some constant.

The radial momentum constraint component (115d) again sets $\dot{A}=0$, but now this does not imply that $A'=0$, so we should take $A=A(\rho)$. The energy constraint component (115b) still eliminates the imperfect term $\hat{w}$. The three nontrivial khronon EoM we are then left with are

$$\left(g_++g_-\right)\left[\left(1-A^2\right)F+2A'\right]-\alpha\left(F\dot{r}^2+2\dot{F}\dot{r}r\right)+3\beta\frac{Fr^2}{\tau^2} \;=\; -Fr^2\kappa\hat{\rho}\,, \tag{123a}$$

$$\left(g_++g_-\right)\left(A^2-1\right)+\alpha\left(\dot{r}^2+2r\ddot{r}\right) \;=\; \beta\frac{r^2}{\tau^2}\,, \tag{123b}$$

$$\left(g_++g_-\right)A'+\alpha\left(\ddot{F}r+\dot{F}\dot{r}+F\ddot{r}\right) \;=\; \beta\frac{Fr}{\tau^2}\,, \tag{123c}$$

and we should also take into account the constraint

$$r'=-AF\,. \tag{123d}$$

We see that the $r(\tau,\rho)$ given in (122) solves (123b) when setting $A=-1$; then the corresponding $F(\tau,\rho)$ given in (122) solves (123c); and when plugging (122) into (123a) we obtain expressions for $\kappa\hat{\rho}$ which are equivalent to (66). Thus, the homogeneous vacuum solutions are reproduced consistently for $A=-1$, which are probably the simplest nontrivial solutions for the vacuum equations (123). In fact, when $A=\mp1$, a generic two-parameter solution for the scale factor can be obtained as

$$r(\tau,\rho)=g(\rho)F(\tau,\rho)\,,\quad F(\tau,\rho)=\pm g'(\rho)\left(\frac{\tau}{\tau_0}\right)^{\frac{1}{3}\left(1-\sqrt{1+3\frac{\beta}{\alpha}}\right)}\left(\left(\frac{\tau}{\tau_0}\right)^{\sqrt{1+3\frac{\beta}{\alpha}}}+F_0\right)^{\frac{2}{3}}\,, \tag{124a}$$

where F_0 is a constant, and $g(\rho)$ is an arbitrary function. This corresponds to a more complicated than simple power-law expansion. Consequently, the expression for the effective dark matter density is also more complicated,

$$\kappa\hat{\rho}=\frac{2\alpha\left(1+\sqrt{\frac{3\beta}{\alpha}+1}\right)(\frac{\tau}{\tau_0})^{2\sqrt{1+\frac{3\beta}{\alpha}}}-6\beta\left[(\frac{\tau}{\tau_0})^{2\sqrt{1+\frac{3\beta}{\alpha}}}+4F_0(\frac{\tau}{\tau_0})^{\sqrt{1+\frac{3\beta}{\alpha}}}+F_0^2\right]+2\alpha F_0^2\left(1-\sqrt{1+\frac{3\beta}{\alpha}}\right)}{3\tau^2\left[(\frac{\tau}{\tau_0})^{\sqrt{1+\frac{3\beta}{\alpha}}}+F_0\right]^2}\,. \tag{124b}$$

We have checked that this solution satisfies the continuity equation (116b). We see that the function $g(\rho)$ does not enter into the expression for the dark matter energy density. In the cases $A=\pm1$, the function $g(\rho)$ in (124a) only reflects the gauge freedom in choosing a radial coordinate. For some function $A(\rho)\neq\pm1$ we would expect to find inhomogeneous geometries, and in particular, time-evolving black hole solutions. However, it is not easy to see the form of these hypothetical solutions from (123).

Black holes, the properties of their horizons and their potential singularities in the generic $\beta\neq0$ theory is one of the important topics left open for future studies.

VII. MATTER COUPLINGS

Thus far, our exploration has focused on the emergence of spacetime and gravitational dynamics from the gauge-theoretic structure based on *Spin*(4), with the Cartan khronon playing a central role in breaking symmetry and

[16] This remains the generic spherically symmetric geometry. The familiar form of the metric (117) with functions $f(r)\to f(r,t)$ and $g(r)\to g(r,t)$ can be transformed into the Lemaître form using (118) wherein we did not make the assumption of static geometry. The resulting metric is known as the Lemaître-Tolman-Bondi metric.

establishing temporality. The formalism has proven to consistently reproduce both static and cosmological solutions, as well as nontrivial configurations such as the Kerr geometry.

We now turn to an equally essential aspect of any realistic theory, its coupling to matter. The introduction of matter fields into the $Spin(4)$ gauge theory brings into focus some of the most delicate aspects of the framework - chiefly, how to formulate spinor dynamics in a setting where spacetime and even its signature are emergent rather than fundamental. While spinor fields are traditionally defined on Lorentzian spacetimes, here they must be incorporated within a manifestly real Euclidean formalism, with the Lorentzian physics understood to arise via a nontrivial analytic continuation.

In this section, we construct the spinor representation of the Euclidean Clifford algebra, defining the associated gamma matrices in a way compatible with the chiral structure of the theory. We then proceed to define spinor fields, their transformation properties under the $Spin(4)$ gauge symmetry, and the form of their covariant derivatives. Importantly, we show how spinors source not only the composite frame field through energy-momentum, but also the gauge field through intrinsic spin currents. These spin currents may be both right- and left-handed, enriching the geometric content of the theory. The main result of this section is the derivation of these matter currents, which provides the crucial link between matter and geometry in the full theory.

A. Gamma matrices

The 2×2 Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{125a}$$

obey the algebra $\sigma^{[i}\sigma^{j]} = i\epsilon^{ij}{}_k\sigma^k$ and can be used to construct the 4×4 Dirac matrices,

$$\gamma^4 = \sigma^1 \otimes \sigma^0 , \quad \gamma^i = \sigma^2 \otimes \sigma^i , \quad \gamma^5 = \sigma^3 \otimes \sigma^0 = \gamma^1\gamma^2\gamma^3\gamma^4 . \tag{125b}$$

The Euclidean metrics are obtained as $\delta^{ij}\sigma^0 = \sigma^{(i}\sigma^{j)}$ and $\delta^{IJ}\mathbb{1} = \gamma^{(I}\gamma^{J)}$, where we denote the 4×4 unit matrix as $\mathbb{1} \equiv \sigma^0 \otimes \sigma^0$. The generators

$$o_{IJ} \equiv \frac{1}{2}\gamma_{[I}\gamma_{J]} = \frac{1}{4}[\gamma_I, \gamma_J] = \frac{1}{4}\left(\gamma_I\gamma_J - \gamma_J\gamma_I\right) , \tag{126}$$

then realise the $\mathfrak{so}(4)$ algebra (2). It is useful to note that

$$\gamma^I\gamma^J\gamma^K = 2\delta^{I[J}\gamma^{K]} + \delta^{JK}\gamma^I - \epsilon^{IJKL}\gamma_L\gamma^5 , \tag{127}$$

which can be checked e.g. by using (125b) and $\sigma^i\sigma^j\sigma^k = 2\delta^{k[j}\sigma^{i]} + \delta^{ij}\sigma^k + i\epsilon^{ijk}\sigma^0$.

B. Spinors

Let ψ be a 4-component column spinor. The conjugate is defined as $\bar{\psi} \equiv \psi^\dagger\gamma$, where γ is a Hermitian matrix $\gamma^\dagger = \gamma$ s.t. $(\gamma^I)^\dagger = \gamma\gamma^I\gamma$. In the Lorentzian case, we'd use the $\gamma^0 \equiv -i\gamma^4$, and the appropriate choice for the γ would be γ^0, but since working in the Euclidean signature it is possible to choose simply $\gamma = \mathbb{1}$. An $SO(4)$ transformation given by an orthonormal matrix with components $\Lambda^I{}_J$ is represented for spinors with the 4×4 matrix Λ such that

$$\psi \to \Lambda\psi , \quad \bar{\psi} \to \bar{\psi}\Lambda^{-1} , \quad \Lambda^{-1}\gamma^I\Lambda = \Lambda^I{}_J\gamma^J . \tag{128}$$

A consistency check is that $\gamma^{-1}\Lambda^\dagger\gamma = \Lambda^{-1}$, where $\Lambda = \exp\lambda^{IJ}o_{IJ}/2$ is the transformation parameterised by λ^{IJ}. For an infinitesimal transformation, $\Lambda \approx 1 + \lambda^{IJ}o_{IJ}/2$, one readily checks with a brief Clifford algebra that the matrix $\Lambda^I{}_J$ in (128) is given by $\Lambda^I{}_J \approx \delta^I{}_J + \lambda^I{}_J$. Thus, the transformation (128) requires[17] that $\Lambda^{-1} = \gamma\Lambda^\dagger\gamma$, which follows from that the generators of Λ satisfy $\gamma o^\dagger_{IJ}\gamma = -o_{IJ}$ as the direct consequence of $(\gamma^I)^\dagger = \gamma\gamma^I\gamma$. These properties hold in an arbitrary basis $\gamma^I \to U^\dagger\gamma^I U$, where U is a unitary matrix. Consider the two Weyl projections

$$\psi_\pm \equiv \frac{1}{2}\left(\mathbb{1} \mp \gamma\right)\psi \quad \Rightarrow \quad \psi = \psi_+ + \psi_- , \quad (\psi_\pm)_\pm = \psi_\pm , \quad (\psi_\pm)_\mp = 0 . \tag{129}$$

[17] A consistent choice would also seem to be $\gamma = -\gamma^5$, in which case $(\gamma^I)^\dagger = -\gamma\gamma^I\gamma$.

It is worth stressing that these projectors are nothing but (1) adapted for the spinor representation. This can be verified explicitly by noting that

$$\gamma o^{IJ} = -\frac{1}{2}\epsilon^{IJKL}o_{KL}\,, \tag{130}$$

which follows by multiplying (127) with γ_K and using that $\gamma_I\gamma^I = 4\mathbb{1}$ and that $\gamma_I\gamma^J\gamma^K\gamma^I = 4\delta^{JK}\mathbb{1}$. Thus, the projected generators can be expressed in two equivalent ways,

$$^{\pm}o^{IJ} \equiv P_{\pm}^{IJKL}o_{KL} = \frac{1}{2}o^{IJ} \pm \frac{1}{4}\epsilon^{IJKL}o_{KL} = \frac{1}{2}\left(\mathbb{1}\mp\gamma\right)o^{IJ} \equiv o^{IJ}_{\pm}\,. \tag{131}$$

We recall from (7) that $^{+}o^{4i} = r^i$ are the 3 right-handed and $^{-}o^{41} = -l^i$ are the 3 left-handed rotations. For the projections of the conjugates, the convention

$$\bar{\psi}_{\pm} \equiv \frac{1}{2}\bar{\psi}\left(\mathbb{1}\mp\gamma\right)\,, \tag{132}$$

is logical in the way that $\overline{\psi_{\pm}} = \bar{\psi}_{\mp}$. However, then the property $\psi_{\pm}\psi_{\pm} = 0$ familiar from the Lorentzian spinor geometry is lost. We readily check that now we have, instead,

$$\bar{\psi}\psi \;=\; \bar{\psi}_+\psi_+ + \bar{\psi}_-\psi_- \in \mathbb{R}\,, \quad \bar{\psi}_{\pm}\psi_{\mp} = 0\,, \tag{133a}$$
$$\bar{\psi}\gamma^I\psi \;=\; \bar{\psi}_+\gamma^I\psi_- + \bar{\psi}_-\gamma^I\psi_+ \in \mathbb{R}\,, \quad \bar{\psi}_{\pm}\gamma^I\psi_{\pm} = 0\,. \tag{133b}$$

The two null vectors $\bar{\psi}_+\gamma^I\psi_-$ and $\bar{\psi}_-\gamma^I\psi_+$ are the complex conjugates of each other, so only their sum is real. Thus, we can form real scalars and vectors from spinors as usual, but the flipping of the respective signs in (133) reflects the crucial difference to the Lorentzian case: now the conjugates of the right-handed spinors are still right-handed spinors. We remind that in the conventional $Spin(1,3)\times\mathbb{C} = SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ formulation there are two Lorentz groups, one of which acts on the right-handed and one which acts on the left-handed spinors, and the conjugates belong to the inequivalent representations. One identifies the physical Lorentz group by a holomorphic elimination of the spurious half of the complex transformations, i.e. via imposing that the action of the $SL(2,\mathbb{C})$ from the left is the Hermitian conjugate of the action of the $SL(2,\mathbb{C})$ from the right. In contrast, in present $Spin(4)$ formulation, no complexification is co required. The left- and the right-handed transformations are naturally identified with the irreducible subgroups of the one and the same real group, the conjugates of the spinors being accommodated into the conjugate representations of the respective subgroup.

Due to this crucial difference, the Euclidean Dirac action cannot be the straightforward translation of the well-known Lorentzian Dirac action. The kinetic term for a Dirac field in Minkowski background involves the Hermitian term $i\bar{\psi}\partial\!\!\!/\psi = i\bar{\psi}_+\partial\!\!\!/\psi_+ + i\bar{\psi}_-\partial\!\!\!/\psi_-$. The operator $\partial\!\!\!/$ is translated nicely into Euclidean regime, since the $\gamma^0\partial_t = \gamma^4\partial_\tau$. However, if we naively replace here the spinors we just constructed in the Euclidean geometry, we do not obtain quadratic kinetic terms for the two Weyl components, but according to (133), we get $\bar{\psi}\partial\!\!\!/\psi = \bar{\psi}_+\partial\!\!\!/\psi_- + \bar{\psi}_-\partial\!\!\!/\psi_+$. One way to deal with this is by "doubling" the degrees of freedom by taking $\bar{\psi}$ as another, independent field, not related to ψ by conjugation [1]. Then, however, the Hermitian property is lost.

Schwinger had written down a consistent Euclidean action for Dirac spinors already in 1959 [79]. It is explained in Ref.[80] that the action can be understood as a Wick rotation of the Lorentzian action which not only switches to imaginary time but also rotates the spinor indices. Geometrically, the rotation matrix $S = \exp\left(\gamma^4\gamma^5\theta/2\right)$ can be understood as a boost along a fifth dimension. It acts on spinors like $\psi \to S\psi$ and on conjugates like $\psi^\dagger \to \psi^\dagger S$. (This action is analogous to the conventional Wick rotation of spacetime vectors, whose contra- and covariant zeroth components are rotated identically.) Let us write the Dirac Lagrangian including a mass term:

$$i\psi^\dagger S\gamma^0\left(\partial\!\!\!/+m\right)S\psi = \psi^\dagger S\gamma^4\left(\partial\!\!\!/+m\right)S\psi = \psi^\dagger\gamma^4 S^{-1}\left(\partial\!\!\!/+m\right)S\psi = \psi^\dagger\gamma^4\left(\delta^\mu_I\gamma^I_E\partial_\mu + m\right)\psi\,. \tag{134}$$

The first form is manifestly the usual Dirac Lagrangian when $\theta = 0$ that is, $S = 1$. In the second form we just recalled that $\gamma^4 = i\gamma^0$. In the third form we used the property of the S-matrix that $S\gamma^4 = \gamma^4 S^{-1}$ [80]. In the final form we wrote the $\partial\!\!\!/_E$ explicitly in terms of the S-rotated gamma-matrices, $\gamma^I_E \equiv S\gamma^I S^{-1}$. For $\theta = \pi/2$, we obtain

$$\gamma^I_E = \gamma^I\,, \quad \gamma^4_E = \gamma^5\,, \quad \gamma^5_E = \gamma^1_E\gamma^2_E\gamma^3_E\gamma^4_E = -\gamma^4\,. \tag{135}$$

The kinetic term (134) suggests identifying the conjugate spinor as $\bar{\psi}_E = \psi^\dagger_E\gamma^5_E = -\psi^\dagger_E\gamma^4$, i.e. $\gamma_E = \gamma^5_E$. Note that the rotation of the spinor indices rotates also the Lorentz generators. It still holds that $\gamma_E o^{IJ}_E\gamma_E = -(o^{IJ}_E)^\dagger$, and we note that now

$$(\gamma^I_E)^\dagger = -\gamma_E\gamma^I_E\gamma_E\,, \quad (\gamma^5_E)^\dagger = \gamma_E\gamma^5_E\gamma_E\,. \tag{136}$$

The projected spinors in the rotated basis are

$$\psi_{E\pm} = \frac{1}{2}\left(1 \mp \gamma^5_E\right)\psi_E = \frac{1}{2}\left(1 \pm \gamma^4\right)\psi_E\,. \tag{137}$$

To recapitulate, the equivalent action to Schwinger's,

$$L_E = \bar\psi_E\left(\partial\!\!\!/_E + m\right)\psi_E = \bar\psi_{E+}\partial\!\!\!/_E\psi_{E-} + \bar\psi_{E-}\partial\!\!\!/_E\psi_{E+} + m\left(\bar\psi_{E+}\psi_{E+} + \bar\psi_{E-}\psi_{E-}\right)\,, \tag{138}$$

is manifestly Hermitian and admits the interpretation as a Wick rotation of the Dirac action taking into account the rotation of the spinor indices. The latter part of the transformation reconciles the apparent difference of the conjugacy property of Euclidean versus Lorentzian spinors $\overline{\psi_{L\pm}} = \bar\psi_{L\mp}$.

It is also possible to construct kinetic terms for Euclidean Majorana spinors without a "doubling" of the degrees of freedom. Instead of Hermitian conjugation, one should then turn to possible alternative complex structures, since reality in the Lorentzian regime may correspond to Osterwalder-Schrader positivity in the Euclidean regime [1]; see Wetterich [81] for a systematic discussion.

An interesting interpretation of Weyl spinors was proposed recently [82]. In fact, this interpretation is suggested by the matter coupling we had already adopted for the right-handed ($g_- = 0$) gravity model [83] along the lines of the matter coupling proposed in self-dual loop quantum gravity [84]. In the Lorentzian regime, one can consider the kinetic term

$$L_+ = \frac{i}{2}\left(\bar\psi\partial\!\!\!/\psi_+ - \bar\psi\overleftarrow{\partial\!\!\!/}\psi_-\right)\,, \tag{139}$$

wherein the actions of the derivative operator on the components ψ_- and $\bar\psi_-$ are projected out, and thus we will obtain only the couplings of the self-dual connection with matter. We perform the same spinorial Wick rotation to (139) as earlier in the case of (134),

$$L_+ = \frac{i}{2}\left(\psi^\dagger S\gamma^0\partial\!\!\!/S\psi_+ - \partial\!\!\!/(\psi^\dagger S\gamma^0)S\psi\right) = \frac{1}{2}\left(\psi^\dagger S\gamma^4\partial\!\!\!/S\psi_+ - (\psi^\dagger S\gamma^4)\overleftarrow{\partial\!\!\!/}S\psi\right) = \frac{1}{2}\left(\bar\psi_E\partial\!\!\!/_E\psi_{E+} - \bar\psi_E\overleftarrow{\partial\!\!\!/}_E\psi_{E-}\right)\,. \tag{140}$$

This coupling is Hermitian, $L_+^\dagger = L_+$, and allows the consistent minimal coupling to self-dual gravity the results of which were reported in Ref.[83]. An improvement due to the Euclidean formulation would seem to be that the self-dual coupling prescription within the context of the complexified Lorentz group [83, 84] is not manifestly Hermitean.

When restricting this prescription to pure Weyl spinors, say to the right-handed ones, an apparent problem would be that the component ψ_- appears in both pieces of the above action (140). A simple resolution is to consider it instead as transforming in the conjugate representation of ψ_+. Thus, the would-be left-handed component is rather interpreted as the conjugate of the right-handed component [82]. In this way, we obtain a consistent kinetic term involving solely the right-handed Weyl spinor. Thus, Weyl spinors as well are naturally incorporated into the (Euclidean) Lorentz gauge theory.

C. Energy and spin currents

Our aim is to construct the gravitational generalisation of the Schwinger Lagrangian for fermions (138). We drop the indices of the Euclidean basis in γ^I_E, ψ_E and write just γ^I, ψ, since the following derivations will be basis-independent.

The covariant derivative of a spinor and a conjugate spinor are

$$\mathbf{D}\psi = \mathbf{d}\psi + \frac{1}{2}\boldsymbol{A}^{IJ}o_{IJ}\psi\,, \tag{141a}$$

$$\mathbf{D}\bar\psi = \mathbf{d}\bar\psi - \frac{1}{2}\boldsymbol{A}^{IJ}\bar\psi o_{IJ}\,. \tag{141b}$$

Since the connection coefficients are real, $\mathbf{D}\bar\psi = \overline{\mathbf{D}\psi}$. Defining $\boldsymbol{A} \equiv {}^+\boldsymbol{A} + {}^-\boldsymbol{A}$ where, consistently with definitions in section II A,

$${}^+\boldsymbol{A} \equiv {}^+\boldsymbol{A}^i r_i\,, \quad \text{where} \quad r^i = -\frac{1}{8}\epsilon^{ijk}\gamma_j\gamma_k - \frac{1}{4}\gamma^{[4}\gamma^{i]}\,, \tag{142a}$$

$${}^-\boldsymbol{A} \equiv {}^-\boldsymbol{A}^i l_i\,, \quad \text{where} \quad l^i = -\frac{1}{8}\epsilon^{ijk}\gamma_j\gamma_k + \frac{1}{4}\gamma^{[4}\gamma^{i]}\,, \tag{142b}$$

we can expand (141) as

$$\mathbf{D}\psi = \mathbf{d}\psi + \boldsymbol{A}\psi = \mathbf{D}\psi_+ + \mathbf{D}\psi_- = \left(\mathbf{d} + {}^+\boldsymbol{A}\right)\psi_+ + \left(\mathbf{d} + {}^-\boldsymbol{A}\right)\psi_- \,, \tag{143a}$$

$$\mathbf{D}\bar{\psi} = \mathbf{d}\bar{\psi} - \bar{\psi}\boldsymbol{A} = \mathbf{D}\bar{\psi}_+ + \mathbf{D}\bar{\psi}_- = \mathbf{d}\bar{\psi}_+ - \bar{\psi}_+{}^+\boldsymbol{A} + \mathbf{d}\bar{\psi}_- - \bar{\psi}_-{}^-\boldsymbol{A} \,. \tag{143b}$$

In the construction of a kinetic Lagrangian, we consider the Hermitian 1-form,

$$\begin{aligned} \frac{1}{2}\left[\left(\bar{\psi}\gamma^I\mathbf{D}\psi\right) + \left(\bar{\psi}\gamma^I\mathbf{D}\psi\right)^\dagger\right] &= \frac{1}{2}\left(\bar{\psi}\gamma^I\mathbf{D}\psi - \mathbf{D}\bar{\psi}\gamma^I\psi\right) \\ &= \frac{1}{2}\left(\bar{\psi}\gamma^I\mathbf{d}\psi - \mathbf{d}\bar{\psi}\gamma^I\psi\right) + \frac{1}{4}\boldsymbol{A}_{JK}\left\{\gamma^I, o^{JK}\right\} \\ &= \bar{\psi}\gamma^I\mathbf{d}\psi - \frac{1}{2}\mathbf{d}\left(\bar{\psi}\gamma^I\psi\right) - \frac{1}{2}\star\boldsymbol{A}^{IJ}\bar{\psi}\gamma_J\gamma^5\psi \,. \end{aligned} \tag{144}$$

In the last step, we used partial integration and the formula (127). The Lagrangian 4-form density can then be formed as

$$\begin{aligned} \boldsymbol{L}_\psi = \frac{1}{2}\star\mathbf{e}_I\wedge\left(\bar{\psi}\gamma^I\mathbf{D}\psi - \mathbf{D}\bar{\psi}\gamma^I\psi\right) + \star\left(m\bar{\psi}\psi\right) &= \frac{1}{2}\star\mathbf{e}_I\wedge\left(\bar{\psi}\gamma^I\mathbf{d}\psi - \mathbf{d}\bar{\psi}\gamma^I\psi - \star\boldsymbol{A}^{IJ}\bar{\psi}\gamma_J\gamma^5\psi\right) + \star\left(m\bar{\psi}\psi\right) \\ \text{assuming non-degeneracy:} &= \frac{1}{2}\star\left[\left(\bar{\partial\!\!\!/}\bar{\psi}\right)\psi - \bar{\psi}\bar{\partial\!\!\!/}\psi - \frac{1}{2}\epsilon^{IJKL}A_{IJK}\bar{\psi}\gamma_L\gamma^5\psi + 2m\bar{\psi}\psi\right] . \end{aligned} \tag{145}$$

In the second line we have assumed the existence of inverse $\boldsymbol{\partial}_I$, s.t. we can write $\bar{\partial\!\!\!/} = \gamma^I\boldsymbol{\partial}_I \lrcorner \mathbf{d}$ and $A_{IJK} = \boldsymbol{\partial}_I \lrcorner \boldsymbol{A}_{JK}$. This density generates the source 3-forms

$$\boldsymbol{t}^I_\psi = -\frac{1}{2}\star\left(\mathbf{e}^I\wedge\mathbf{e}^J\right)\wedge\left(\bar{\psi}\gamma_J\mathbf{D}\psi - \mathbf{D}\bar{\psi}\gamma_J\psi\right) + \star\mathbf{e}^I m\bar{\psi}\psi \,, \tag{146a}$$

$$\boldsymbol{O}^{IJ}_\psi = -\frac{1}{4}\epsilon^{IJ}{}_{KL}\left(\star\mathbf{e}^K\right)\bar{\psi}\gamma^L\gamma^5\psi \,, \tag{146b}$$

which contribute, respectively, an energy current and a spin current. The latter decomposes into anti/self-dual pieces according to (1) as

$$^\pm\boldsymbol{O}^{IJ}_\psi = \mp\bar{\psi}\,^\pm\left(\star\mathbf{e}^{[I}\gamma^{J]}\right)\gamma^5\psi \,. \tag{147}$$

In the case of the right-handed model in section III, this is problematical: since the field equations enforce $^-\boldsymbol{O}^{IJ}_\psi = 0$, extra constraints are imposed on the fermion fields which obstruct the viable dynamics for these fields. In the generic models with $g_- \neq 0$, the antiself-dual spin current has a novel impact to the spacetime structure, whereas the self-dual spin current contributes to the gravitational dynamics in some analogy with the axial spin current familiar from the Einstein-Cartan model.

To close this final section, let us note that what was left beyond the scope of the current article is the incorporation of the rest of the gauge interactions. A first-order pregeometric theory for the Yang-Mills fields of the standard model, compatible with the Lorentz gauge theory of gravity, has been considered by Gallagher *et al* [83, 85, 86], and it might be useful to verify also the consistent embedding of this reformulation of the Yang-Mills theory into the new Euclidean framework. However, the *Spin*(4) gravity suggests a different, twistorial path towards unification as we will briefly point out in the outlook VIII A below. The Schwinger Lagrangian [79], following the formulation of Ref. [80], is underpinned by a rotation along a fifth dimension, indicating an extension of the geometric framework with extra imaginary dimension. In the hypothetical twistorial theory, this construction is completed by introducing a sixth gamma matrix, and the khronon is elevated from a 4-vector to a biquaternion - placing space and time on even more symmetric footing, and bringing forth, almost as a free byproduct, the structure of the electromagnetic, weak, and strong interactions[18], the distinction between these internal gauge interactions and the gravitational dynamics arising from the breaking of symmetry between the 3 spatial dimensions and the 3 extra dimensions which collapse into an external time.

[18] For related mathematical insight, see [87].

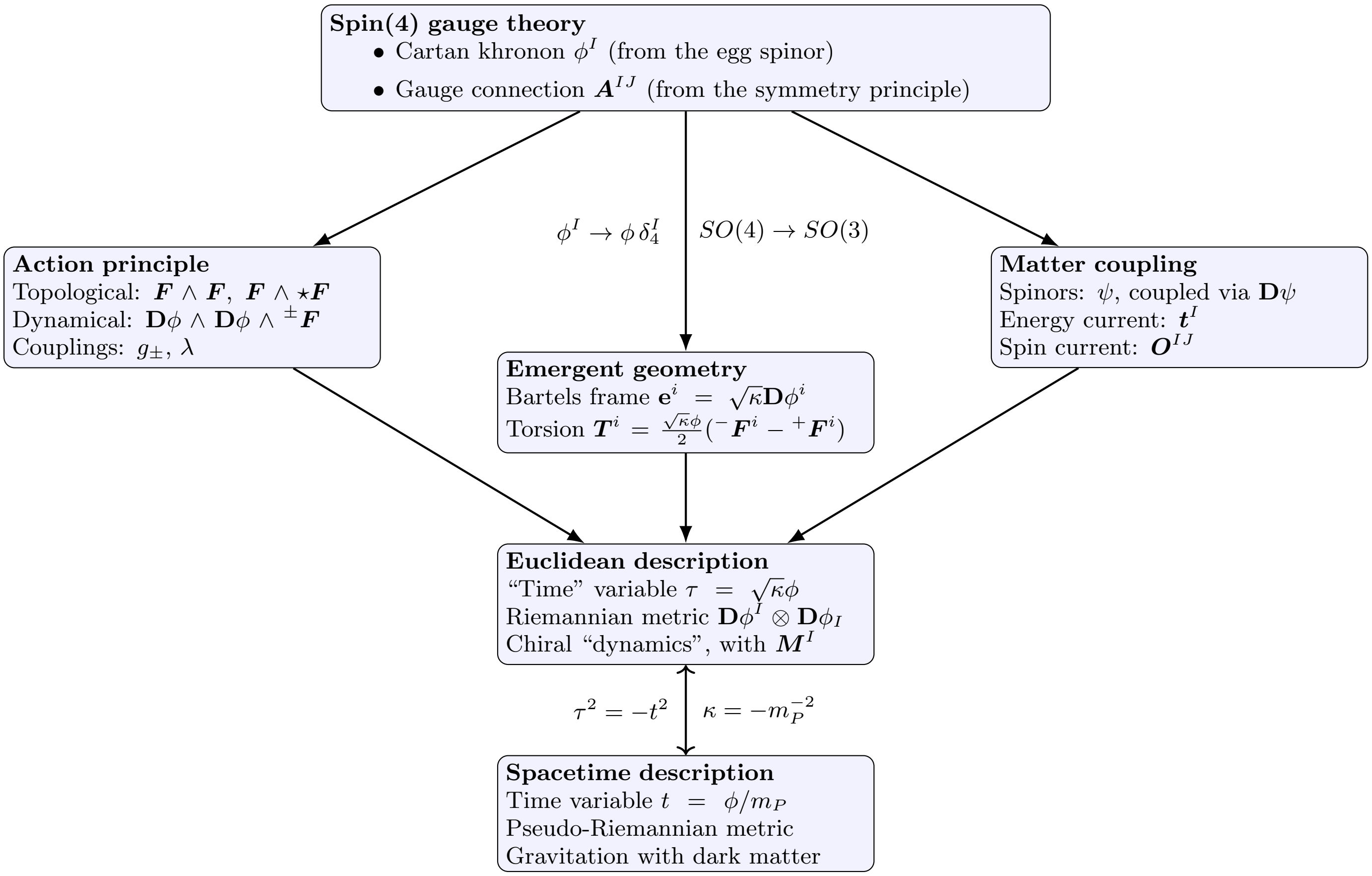

FIG. 1: Summary of the proceedings in this article.

VIII. CONCLUSIONS AND PERSPECTIVES

We proposed and developed a novel $Spin(4)$ gauge-theoretic framework in which the structure of spacetime, the arrow of time, gravitation, and even dark matter phenomena emerge from a fundamentally Euclidean and pregeometric origin. Central to this formulation is the Cartan khronon field, a real scalar in the fundamental representation, whose symmetry breaking dynamically selects a preferred temporal direction. This mechanism provides a technically minimal and conceptually transparent resolution to the longstanding "problem of time" in classical and quantum gravity.

By working entirely within a real-valued theory, we avoid the formal and interpretational ambiguities associated with complexification or ad hoc prescriptions. Instead, we have shown that the Wick rotation arises naturally and consistently as a reinterpretation of units - bridging the Euclidean and Lorentzian descriptions without modifying the underlying field content. The same field configurations admit dual interpretations, which become physically equivalent once the role of the khronon in setting time and energy scales is properly recognised.

The structure of the theory is heuristically illustrated in figure 1, highlighting the two important steps of our recipe, the symmetry breaking and the Wick rotation.

A. Overview and open questions

As emphasised in the introduction I, neither of these steps is new but rather ubiquitous in modern physics. What distinguishes our approach is the claim that the proposed implementation in $Spin(4)$ theory is unique, consistent and universal: if correct, we believe this could have far-reaching consequences. Our initial motivation arose from the path integral formulation of quantum field theory, where a compact and real-valued gauge structure offers a natural and potentially rigorous foundation. It is therefore an obvious and exciting next step to initiate a concrete study of the quantum theory and its Euclidean path integral.

We reviewed the formulation of the theory in section II in the language of differential forms. The structure, exploiting the reducibility of the Lorentz group, suggests a twistorial reformulation that would seem to naturally point to a unifying extension that emerges in a considerably more minimal and simpler - yet subtler - fashion than in the conventional $SO(N)$ or $Spin(N)$ (where $N > 4$) attempts at unified gauge theories [88]. As already hinted by the Wick rotation of the spinor basis, time - like space in the current formulation of the theory - would also be a purely imaginary quaternion. A possible physical interpretation of enlarging the Lorentz gauge group to de Sitter has already been established as incorporating scale invariance and introducing a second fundamental constant emerging from a symmetry breaking [89], while the conformal extension envisaged in [90, 91] would add a third constant of Nature, completing the triad of natural scales that govern the Universe.

Setting this speculation aside, the workings of the current pregeometric were successfully demonstrated in section III taking previous studies of exact solutions in Lorentz gauge theory substantially further. Future work should clarify how the Euclidean–Lorentzian duality reshuffles dynamical character - why, for instance, gravitational-wave amplitudes that are monotonic in one description emerge as oscillatory in the other, whereas a Euclidean cosmology with a steadily growing scale factor preserves its monotonicity across the dual frame. A promising clue lies in the 'electric-magnetic' split of geometry: the 'electric' sector (associated with tidal effects) and the 'magnetic' sector (linked to frame-dragging) may map differently under this duality, selectively converting growth into oscillation. Another intriguing aspect is the Euclidean Kerr geometry, which appears to break down for $r < a$; this invites further investigation of its topological and geometrical structure.

We also derived the Friedmann equations in the most general, 5-parameter theory (17). It will be interesting to study their implications to potentially both early and late cosmology, beyond the very simplest case of vacuum that was solved analytically in section IV A. At the level of background cosmology, it is very feasible to explore extensions of the minimal 5-parameter theory. For instance, one may allow a violation of the torsorial property of the khronon, or promote the $g_\pm$ from constants to dynamical parameters, implemented via scale-dependent running or by simply treating them as dynamical scalar fields. The new $Spin(4)$ framework might admit new modifications of gravity[19], some of which might be well-motivated or phenomenologically relevant. Both these qualities may likewise characterise the particularly minimal starting point for modifications offered by the ΛCDM theory of cosmology, in which unimodularisation eliminates the effect of λ and places both the cosmological constant and cold dark matter on a comparable footing [93].

However, already the theory at hand invites application in cosmology, as it inherently modifies the behaviour of gravitational and matter sectors in a way that impacts structure formation. The derivation of cosmological perturbation theory in section V is one of the main results of this article. The result (103) shows that our framework smoothly reproduces standard cold dark matter growth when $\beta = 0$, while for $\beta \neq 0$ the growth is subject to a modified friction and a time-dependent effective Newton's constant, yet retains a single, stable scalar degree of freedom with vanishing sound speed - made possible by the (effective) fluid's underlying spin currents. The next step is to confront these distinctive signatures with cosmological observations: forecasts for weak lensing, cosmic microwave background temperature and polarisation spectra, matter power spectra from large-scale structure surveys, and cross-correlations can reveal whether the model eases the current H_0 and σ_8 tensions in the ΛCDM model [76]. With the theoretical machinery now in place, a systematic data-driven campaign can decisively test the viability of this generalised dark matter sector.

An important question that was yet left open in the preliminary exploration of spherically symmetric exact solutions in section VI is the existence of black holes in the case of $\beta \neq 0$, which does not admit even an exact Minkowski solution as the pregeometric structure forces a dynamical behaviour to the emergent metric unless gravity is strictly one-handed. Of course, if the absence of solutions sufficiently resembling black holes could be proven, it would establish that $\beta = 0$. On the other hand, were such solutions found, they would necessarily be different from those in general relativity, and thus quite interesting from both observational and theoretical perspectives.

Finally, in section VII we reviewed the consistent coupling of spinor fields to the Euclidean theory, drawing on established constructions in the literature. While this ensures a geometrically coherent treatment of fermionic matter, the analysis also revealed new features: in models with $\beta \neq 0$, the antiself-dual spin current introduces novel effects on spacetime structure, while the self-dual component contributes to the gravitational dynamics in a way reminiscent of the axial spin current in Einstein–Cartan theory. With cosmological (and to some extent the more general spherically symmetric) backgrounds already under control, these provide timely and physically relevant arenas in which to explore the dynamical role of spin currents. Last but not least, features such as non-Minkowskian vacuum and novel matter couplings - even under the minimal coupling principle - open the possibility of testing the theory using the wealth of high-precision data constraining local (in this case effective) Lorentz violations from various laboratory experiments

[19] In view of modifying gravity, it may seem promising that all of the 5 terms in (17) are physically viable, in contrast to the conventional framework of metric-affine gravity which allows an infinite number of terms, yet the most often-studied quadratic bunch of terms containing (practically) only pathological modifications [92].

[94]. This aspect of the theory calls for a more detailed scrutiny than the preliminary, back-of-the-envelope calculation given in [13].

Independent of the new perspective on space and time that arises specifically from the Euclidean signature of the *Spin*(4) framework, our study offers an intriguing extension of the Lorentz gauge theory. In particular, we have shown that

gravitation is right-handed

iff dark matter is cold

iff the $\Lambda = 0$ vacuum is Minkowski

iff the speed of graviton is the speed of light.

Even if $\beta = 0$ in our Universe, these unexpected and tightly interwoven connections between its seemingly unrelated foundational aspects offer revelatory insights into its pregeometric structure.

B. A philosophical epilogue

> Space, the extensive medium of the material world, is clearly the seat of the group of coordinate transformations; but the group [of physical automorphisms] seems to have its origin in the ultimate elementary particles of matter. [95]

Although scholars have acknowledged the significance of the relationship between physical and mathematical automorphisms in Hermann Weyl's thought [96], its more precise and far-reaching implications may have remained insufficiently understood, particularly in the context of spacetime and gravitation theory. The mainstream approach of "gauging translations" in (metric-affine, Poincaré, teleparallel, etc) gravity theories[20] conflates auxiliary mathematical structure with physical substance, a move that betrays the physicist's lingering Newtonian presupposition that space, independently of any observer, furnishes an arena wherein phenomena take place [99].

Though Newton had published a proper theory of gravitation in the *Principia* (1687), the beginning of the modern science of space and time might be traced to Kant's considerations in the 18th century, on what we could nowadays call spontaneous breaking of chiral symmetry, from which he was eventually led to the view that Euclidean space and time are *a priori* forms of intuition [100]. In the 19th century, Riemann and Clifford developed a theory of gravitation and matter as manifestations of geometry and curvature, connecting physical phenomena to the structure and dynamics of space. Clifford's algebra already laid the mathematical groundwork for the unification of space and time - and, of the chiral aspects that have been elucidated in geometry (Atiyah) and gravity (Plebanski) more recently. The advent of quantum mechanics in the 20th century ultimately motivates an informational perspective, with space and time as fundamental categories of cognition agreeing with Kant's utterly compelling[21] conclusions. The basic units are bits, which because they are quantum-mechanical, are qubits. Moreover, because the world is only intelligible to us in space and in time, the atoms of information are *gauged qubits* - represented by spinors subject to Clifford's algebra.

In the *Spin*(4) theory, these atoms of information are counted as quaternions and thus in a perhaps less known matrix basis for spinors. The sense of causality requires an "imaginary" field that provides an elementary organising principle, time, born from the primordial *egg spinor* ψ. One can regard time as an illusion, but it is an illusion that definitely exists, or more to the point: nothing can exist without time. The Lorentz gauge theory gives a concrete form to an idea expressed in Weyl's characteristically unrivalled eloquence in the quotation above: external spacetime is but an intrinsic reflection of matter.

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[20] In 1974, C.-N. Yang introduced a Lorentz gauge theory of gravity, which he described as 'conceptually superior', likely due to its alignment with Hermann Weyl's foundational ideas [97]. The contrast between the mainstream approach and what may be termed the Weyl–Yang approach to gauge gravity is particularly apparent in Chapter 19 of [98].

[21] Though of course Gödel was allowed to suggest that the theory of gravity may help one to peek behind the veil of Maya.

PRG2608 "Space - Time - Matter".

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Appendix A: SU(2)×SU(2) formulation

A quaternion representation of the khronon is

$$\phi = i\begin{pmatrix} i\phi^4+\phi^3 & \phi^1-i\phi^2 \\ \phi^1+i\phi^2 & i\phi^4-\phi^3\end{pmatrix} = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^*\end{pmatrix}\,, \tag{A1}$$

so 4 real numbers are rewritten as the 2 complex numbers $z_1 = i\phi^4+\phi^3$ and $\phi^1+i\phi^2$ and they're packed into 1 quaternion. A property of a quaternions is that $\det\phi = \delta_{IJ}\phi^I\phi^J$, and that $\phi\phi^\dagger = \det\phi\mathbb{1}$. The action of $(G_-, G_+) \in SU(2)\times SU(2)$ on ϕ is

$$\phi \to G_-\phi G_+^\dagger\,. \tag{A2}$$

Let's see if this works, with the standard representation $l^i = r^i = i - \sigma^i/2$ for the su(2) algebras. If we perform an infinitesimal Lorentz transformation, $\phi^I \to \phi^I + g^I{}_J\phi^J$, the corresponding left-right transformations should be given by, according to (9b),

$$g^i_\pm \;=\; \pm g^{4i} - \frac{1}{2}\epsilon^i{}_{jk}g^{jk}\,. \tag{A3}$$

Plugging into (A2),

$$\phi \to \phi + \frac{i}{2}\left(-g^i_-\sigma_i\phi + g^i_+\phi\sigma_i\right) = \phi + i\begin{pmatrix} (ig^4{}_I+g^3{}_I)\phi^I & (g^1{}_I-ig^2{}_I)\phi^I \\ (g^1{}_I+ig^2{}_I)\phi^I & (ig^4{}_I-g^3{}_I)\phi^I\end{pmatrix}\,, \tag{A4}$$

verifies that our definitions are consistent. The transformation law (A2) suggests that the covariant derivative of the quaternion field is given as

$$\mathbf{D}\phi = \mathbf{d}\phi + {}^-\boldsymbol{A}\phi + \phi\,{}^+\boldsymbol{A}^\dagger = \mathbf{d}\phi + \frac{i}{2}\left({}^+\boldsymbol{A}_i\phi\sigma^i - {}^-\boldsymbol{A}_i\sigma^i\phi\right)\,, \tag{A5}$$

since the connections transform as

$${}^\pm\boldsymbol{A} \to G_\pm\mathbf{D}G_\pm^\dagger = -\left(\mathbf{d}G_\pm\right)G_\pm^\dagger + G_\pm\,{}^\pm\boldsymbol{A}G_\pm^\dagger\,. \tag{A6}$$

In the second equality we took into account that in a unitary matrix representation, $\mathbf{d}G^\dagger = -G^\dagger\left(\mathbf{d}G\right)G^\dagger$. Check:

$$\begin{aligned}\mathbf{D}\phi \;&\to\; \mathbf{d}\left(G_-\phi G_+^\dagger\right) + \left(G_-\mathbf{D}G_-^\dagger\right)\left(G_-\phi G_+^\dagger\right) + \left(G_-\phi G_+^\dagger\right)\left(G_+\mathbf{D}G_+^\dagger\right)^\dagger \\ &=\; \left(\mathbf{d}G_-\right)\phi G_+^\dagger + G_-\left(\mathbf{d}\phi\right)G_+^\dagger + G_-\phi\mathbf{d}G_+^\dagger - \left(\mathbf{d}G_-\right)\phi G_+^\dagger + G_-\,{}^-\boldsymbol{A}\phi G_+^\dagger + G_-\phi G_+^\dagger\left(-G^+\mathbf{d}G_+^\dagger G_+ + {}^+\boldsymbol{A}^\dagger G_+^\dagger\right) \\ &=\; G_-\left(\mathbf{D}\phi\right)G_+^\dagger\,.\end{aligned} \tag{A7}$$

When taking derivatives of odd forms, we have to notice that the right-handed connection is wedged from the RHS and thus with a minus sign,

$$\begin{aligned}\mathbf{D}(\mathbf{D}\phi) \;&=\; \mathbf{d}(\mathbf{D}\phi) + {}^-\boldsymbol{A}\wedge(\mathbf{D}\phi) - (\mathbf{D}\phi)\wedge{}^+\boldsymbol{A}^\dagger \\ &=\; {}^-\boldsymbol{F}\phi + \phi\,{}^+\boldsymbol{F}^\dagger\,, \quad \text{where} \quad {}^\pm\boldsymbol{F} = \mathbf{d}\,{}^\pm\boldsymbol{A} + {}^\pm\boldsymbol{A}\wedge{}^\pm\boldsymbol{A}\,.\end{aligned} \tag{A8a}$$

As a consistency check,

$$\begin{aligned}{}^\pm\boldsymbol{F} \;&=\; -\frac{i}{2}\mathbf{d}\,{}^\pm\boldsymbol{A}^i\sigma_i - \frac{1}{4}\,{}^\pm\boldsymbol{A}^i\wedge{}^\pm\boldsymbol{A}^j\sigma_i\sigma_j = -\frac{i}{2}\mathbf{d}\,{}^\pm\boldsymbol{A}^i\sigma_i - \frac{1}{8}\,{}^\pm\boldsymbol{A}^i\wedge{}^\pm\boldsymbol{A}^j[\sigma_i,\sigma_j] \\ &=\; -\frac{i}{2}\left({}^\pm\boldsymbol{A}^k + \frac{1}{2}\epsilon_{ij}{}^k\,{}^\pm\boldsymbol{A}^i\wedge{}^\pm\boldsymbol{A}^j\right)\sigma_k = -\frac{i}{2}\sigma_i\,{}^\pm\boldsymbol{F}^i\,,\end{aligned} \tag{A8b}$$

we confirm that the components match with (14a).

Appendix B: Matter current

We don't derive the cosmological source term from first principles, but use a phenomenological fluid parameterisation. The usual fluid energy-momentum (1,1)-tensor $\boldsymbol{T}$ can be given componentwise as

$$\begin{aligned}\boldsymbol{T} &= \left[-(\rho+\delta\rho)\,\partial_\tau + (\rho+p)\left(a^{-1}v + a^{-2}\sqrt{\kappa}\varphi + \dot{c}\right)^{,i}\partial_i\right]\otimes \mathbf{d}\tau \\ &+\; a\,(\rho+p)\,v_{,i}\partial_\tau\otimes\mathbf{d}x^i + \left[(p+\delta p)\,\delta^i_j + \Delta^i{}_j\Pi\right]\partial_i\otimes\mathbf{d}x^j\,, \end{aligned} \tag{B1}$$

where ρ and $\delta\rho$ are the energy density and its perturbation; p and δp are the pressure and its perturbation; v is the scalar velocity potential; and Π is the scalar anisotropic stress potential. The khronon perturbation φ and the connection perturbation $\dot{c}$ enter into the off-diagonal components by raising and lowering an index with the metric read from the line element

$$\mathbf{d}s^2 = (1+2\varphi')\,\mathbf{d}\tau\otimes\mathbf{d}\tau + \left(\sqrt{\kappa}\varphi + a^2\dot{c}\right)_{,i}\left(\mathbf{d}\tau\otimes\mathbf{d}x^i + \mathbf{d}x^i\otimes\mathbf{d}\tau\right) + a^2\left[(1+2\Psi)\,\delta_{ij} + 2\Delta_{ij}r\right]\mathbf{d}x^i\otimes\mathbf{d}x^j\,. \tag{B2}$$

Since $\boldsymbol{T}$ is a vector-valued 1-form, we can construct a 1-form $\boldsymbol{T}^I = \boldsymbol{T}\lrcorner\mathbf{e}^I$ valued in the fundamental representation of $SO(4)$ as

$$\boldsymbol{T}^4 = -(\rho+\delta\rho-\rho\varphi')\,\mathbf{d}\tau + \left[a\,(\rho+p)\,v + p\sqrt{\kappa}\varphi\right]_{,i}\mathbf{d}x^i\,, \tag{B3a}$$

$$\boldsymbol{T}^i = \left[(\rho+p)\left(v + a^{-1}\sqrt{\kappa}\varphi\right) + p\dot{c}\right]^{,i}\mathbf{d}\tau + a\left[(p+\delta p+\Psi)\,\delta^i_j + \delta^i_j\,(\Pi + r)\right]\mathbf{d}x^j\,. \tag{B3b}$$

Finally, we can form the 3-form current $\boldsymbol{t}^I = \sqrt{\kappa} * \boldsymbol{T}^I = \sqrt{\kappa}T^I{}_J \star \mathbf{e}^J$, wherein the scalar components $T^I{}_J = \mathbf{\partial}_J\lrcorner\boldsymbol{T}^I$ are obtained by using the frame field (i.e. the inverse of the Cartan frame),

$$\mathbf{\partial}_4 = (1-\varphi')\,\partial_\tau - \dot{c}^{,i}\partial_i\,, \tag{B4a}$$

$$\mathbf{\partial}_i = -a^{-1}\sqrt{\kappa}\varphi_{,i}\partial_\tau + a^{-1}\left[(1-\Psi)\,\delta^j_i - \Delta^j_i + \epsilon_i{}^{jk}\tilde{s}_{,k}\right]\partial_j\,. \tag{B4b}$$

The result

$$T^4{}_4 = -\rho-\delta\rho\,,\quad T^4{}_i = \delta_{ij}T^j{}_4 = (\rho+p)\left(v + a^{-1}\sqrt{\kappa}\varphi\right)_{,i}\,,\quad T^i{}_j = (p+\delta p)\,\delta^i_j + \Delta^i{}_j\Pi\,, \tag{B5}$$

agrees with (91). Note that there the $\star\mathbf{e}^I$ contain the linear perturbations; were replaced by only the background 3-forms $\star\mathbf{e}^I$, the extra perturbations terms appearing in $\boldsymbol{T}^I$ at (B3) had to be included in the formula.

The material energy current $\boldsymbol{t}^I$ satisfies conservation laws which are dictated by symmetry. In section 5.1. of Ref.[46] the Lorentz invariance and the coordinate invariance were shown to yield the two identities, respectively:

$$\mathbf{D}\boldsymbol{O}^{IJ} = \mathbf{D}\phi^{[I}\wedge\boldsymbol{t}^{J]}\,, \tag{B6a}$$

$$m_P\mathbf{D}\boldsymbol{t}_I = \mathbf{\partial}_I\lrcorner\boldsymbol{F}^{JK}\wedge\left(\boldsymbol{O}_{JK} - \phi_{[J}\boldsymbol{t}_{K]}\right)\,. \tag{B6b}$$

The first of these identities is now trivial, since we do not include spin currents $\boldsymbol{O}^{IJ} = 0$, and the energy-momentum tensor is by construction symmetric. The second identity yields important information, though it is redundant with the field equations. The background part yields (65a), which is extensively used in our cosmological investigations. The linear part gives two nontrivial equations for the matter source, called the continuity equation,

$$\delta\rho' + 3H\,(\delta\rho+\delta p) = (\rho+p)\left[\nabla^2\left(\frac{v}{a} + c' - \frac{\Phi}{a^2}\right) - 3\Psi'\right]\,, \tag{B7a}$$

and the Euler equation,

$$a\,(\rho+p)\,(v' + 4aHv) + (\rho+p)'\,v = \delta p - \frac{2}{3}\nabla^2\Pi\,. \tag{B7b}$$