

# Unification of Gravity and Standard Model: Weyl-Dirac-Born-Infeld action

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## Abstract

We construct a unified (quantum) description by the gauge principle of gravity and Standard Model (SM), that extends the Dirac-Born-Infeld action to the SM and Weyl geometry in  $d$  dimensions, hereafter called Weyl-Dirac-Born-Infeld action (WDBI). This action is a general gauge theory of SM and Weyl group (of dilatations and Poincaré symmetry), in the Weyl gauge covariant (metric) formulation of Weyl geometry. The theory is SM and Weyl gauge invariant in arbitrary  $d$  dimensions, hence there is no Weyl anomaly. The action has the unique elegant feature, not present in other gauge theories or even in string theory, that it is mathematically well defined in  $d$  dimensions, with no need to introduce a UV regulator (scale or field). The WDBI action actually *predicts* that gravity, through (Weyl covariant) space-time curvature, acts as UV regulator of both SM and gravity in  $d = 4$ . A series expansion of the WDBI action (in dimensionless couplings) recovers in the leading order the Weyl gauge invariant action of SM and Weyl (gauge theory of) quadratic gravity. The SM and Einstein-Hilbert gravity are recovered in the Stueckelberg broken phase of Weyl gauge symmetry, which also restores Riemannian geometry below Planck scale. Sub-leading orders are suppressed by powers of (dimensionless) gravitational coupling ( $\xi$ ) of Weyl quadratic gravity.

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# 1 Introduction

In this work we search for a unified (quantum) description, by the gauge principle [1], of Standard Model (SM) and gravity. On the SM side this principle was extremely successful. Since gravity “is” geometry, applying this principle to gravity means to consider a gauged space-time symmetry; this dictates the underlying geometry and gravity action as a gauge theory action. Then what space-time symmetry to consider beyond Poincaré symmetry?

Again, the SM points us in the right direction: SM with a vanishing Higgs mass *parameter* is scale invariant [2], which is a hint that this symmetry may be more fundamental<sup>1</sup>, so we could actually gauge it. This means gauging the Weyl group of dilatations and Poincaré symmetry [3–6]. Actually, there is not much else one can do: this is the only *true* gauge theory of a space-time symmetry beyond Poincaré [5] i.e. with dynamical/physical gauge boson<sup>2</sup>. In the absence of matter, the gauge theory of the Weyl group “is” Weyl geometry (WG) [12–14] which has this gauge symmetry by construction: WG is defined by classes of equivalence of the metric and Weyl gauge field ( $\omega_\mu$ ) of dilatations, related by Weyl gauge transformations. The associated gravity action is then constructed as a (vector-tensor) gauge theory of Weyl group [13], see [5, 6] for an update.

No prior knowledge of Weyl geometry is needed here. Given its gauged dilatation invariance beyond Poincaré, Weyl geometry can be regarded as Riemannian geometry “covariantised” with respect to gauged dilatation symmetry (also known as Weyl gauge symmetry) [11, 15]. More exactly, there exists a Weyl gauge covariant formulation of Weyl geometry, which is the only physical formulation, and is automatically *metric*,  $\hat{\nabla}_\mu g_{\alpha\beta} = 0$ , (but non-affine) [5, 15, 16]<sup>3</sup>, something overlooked for a century of Weyl geometry despite Dirac’s suggestion [19]. The associated gauge theory is quadratic in curvatures, known as Weyl gauge theory of quadratic gravity (“Weyl quadratic gravity”), see reviews in [11] and [20].

This action is spontaneously broken à la Stueckelberg [21], in which the Weyl gauge field of dilatations  $\omega_\mu$  acquires mass proportional to Planck mass  $M_p$  after eating the would-be-Goldstone  $\phi$  (or “dilaton” ghost) propagated by the higher derivative  $\hat{R}^2$  term in the action, and then decouples [22, 23]. As a result, Weyl geometry (connection) becomes Riemannian (Levi-Civita) and at large distances (scales below  $M_p$ ) one recovers the Einstein-Hilbert action [22] and  $\Lambda > 0$ . Thus, the phase transition where Weyl gauge symmetry is broken is interpreted as a change of the underlying geometry. No moduli fields are added ad-hoc for this breaking. All scales have geometric origin [18, 23] (in the vev of field  $\phi$  from geometric  $\hat{R}^2$  term), and since this mode was eaten by  $\omega_\mu$ , its vev does not need to be stabilised.

With these encouraging results, one can also add matter and consider the SM in Weyl geometry (SMW) [23]. This gives an interesting Weyl gauge invariant theory that describes gravity and SM, and respects current constraints, with Starobinsky-like inflation [24, 25], good fits of galaxy rotation curves [26] and black-hole solutions [27]. However, this theory

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<sup>1</sup>Also at high energies or in the early universe, states are effectively massless, endorsing this idea.

<sup>2</sup>One can also gauge the full conformal group (Weyl plus special conformal symmetry) [7] to obtain conformal gravity [8, 9]. But this is not a true gauge theory since its action cannot have dynamical (physical) gauge bosons of Weyl dilatations and of special conformal symmetry [7], thus this theory does not have the spectrum of a gauge theory. It is for this reason its action is actually a *particular limit* of the action of the gauge theory of (smaller) Weyl group discussed here, when the Weyl gauge boson is “pure gauge” [6, 10, 11].

<sup>3</sup>The norm of a vector is invariant under Weyl-gauge-covariant parallel transport [5, 17], [18] (Appendix B)

(SMW) does not seem most general or fully satisfactory since it is ultimately “gluing” together in a sum the actions of SM and of Weyl geometry i.e. Weyl quadratic gravity. It would be good to *derive* this action from a more fundamental one.

The goal of this paper is to find a more general, unified gauge theory action implementing the Weyl gauge symmetry, beyond the SMW scenario. The theory should make no distinction between SM and Weyl geometry field operators (curvatures, etc), in which these fields *and* their derivatives must transform *covariantly* with respect to both SM and Weyl gauge symmetries (much like we treat SM fields with respect to  $SU(3)\times SU(2)\times U(1)$ ). External and internal symmetries must be treated on equal footing in building the action.

Such unified gauge theory can be realized by a version of Dirac-Born-Infeld action [28–30] due to *both* SM and Weyl geometry, called here Weyl-Dirac-Born-Infeld (WDBI). The WDBI action is a  $d$ -dimensional space-time integral of  $\sqrt{\det A_{\mu\nu}}$  where  $A_{\mu\nu}$  is a linear combination of operators of mass dimension 2, that are SM and Weyl gauge invariant; these operators are products of fields of SM and of Weyl geometry (curvatures) and their covariant derivatives. This action gives a unified framework of internal (SM) and external (Weyl) gauge symmetries, with manifest covariance/invariance with respect to both symmetries. Obviously, this action is more general than a sum of Weyl invariant SM and Weyl quadratic actions. It also shows some (vague) similarities to a D-brane effective action.

This WDBI action is a truly special gauge theory: by construction, it is automatically SM and Weyl gauge invariant in *arbitrary*  $d$  dimensions - a feature due to Weyl geometry; the WDBI action is mathematically well-defined and does not require introducing an ultraviolet (UV) regulator (be it a field or scale<sup>4</sup>), and all couplings in the action remain dimensionless. The WDBI action actually *predicts* that the Weyl-covariant (!) space-time curvature  $\hat{R}^\epsilon$  i.e. gravity itself acts as UV regulator in  $d = 4 - 2\epsilon$  dimensions<sup>5</sup>, for SM and gravitational interactions. In other words, for the  $d = 4$  action the regularisation of the action is trivial, simply replace  $d = 4 \rightarrow d = 4 - 2\epsilon$ ; no DR scale (or field) is added! This is the only gauge theory with this property, showing the importance of our WDBI action.

This mechanism does not work in Riemannian geometry where Weyl gauge covariance does not exist. Comparatively, in ordinary gauge theories a UV regulator scale (DR scale  $\mu$ , etc) is added by hand; in conformal gravity action a dilaton is added as regulator to maintain its symmetry [9]. Even in string theory, a DR scale is introduced, and local Weyl invariance (on the Riemannian worldsheet, not in physical space-time as here) is broken by regularisation; it is restored by the condition of vanishing Ricci tensor (in target space) [31].

The Weyl gauge invariance in  $d$  dimensions of the WDBI action of SM and Weyl geometry is important, since it implies that this action is automatically Weyl anomaly-free. Hence, the WDBI action is a consistent (quantum) gauge theory of gravity and SM. This opens a new perspective on physics Beyond SM, based on the old successful gauge principle, that goes beyond the usual quadratic action of gauge theories.

The plan of the paper is this: Section 2 reviews the formalism of WG as a gauge theory. Section 3 constructs the WDBI action and shows how SM and Weyl quadratic gravity are obtained in a leading order expansion. Einstein-Hilbert action is recovered in the broken phase, with subleading order corrections suppressed by  $M_p$ . Conclusions are in Section 4.

<sup>4</sup>The introduction of such DR subtraction scale  $\mu$  would actually break Weyl gauge symmetry.

<sup>5</sup>This supports, a-posteriori, the regularisation used in SMW [23] showing it is Weyl-anomaly free [15].

## 2 Weyl geometry as a gauge theory of Weyl group

Let us first review briefly Weyl geometry as a gauge theory of the Weyl group, in the Weyl gauge covariant (metric, non-affine) formulation [15], also [5, 6, 16] for more details. For generality we work in  $d$  dimensions. Weyl geometry is defined by classes of equivalence of the metric ( $g_{\mu\nu}$ ) and gauge field of dilatations ( $\omega_\mu$ ), related by a Weyl gauge transformation, shown below (in the absence of matter)

$$g'_{\mu\nu} = \Sigma^2 g_{\mu\nu}, \quad \omega'_\mu = \omega_\mu - \partial_\mu \ln \Sigma, \quad \sqrt{g'} = \Sigma^d \sqrt{g}, \quad g'^{\mu\nu} = \Sigma^{-2} g^{\mu\nu} \quad (1)$$

where  $\Sigma = \Sigma(x) > 0$ . The Weyl charge  $q$  of  $g_{\mu\nu}$  was set to  $q = 2$  - such normalization for an Abelian symmetry is a choice. To work with an arbitrary charge for  $g_{\mu\nu}$ , and also restore the Weyl gauge coupling  $\alpha$ , replace  $\Sigma^2 \rightarrow \Sigma^q$  and  $\omega_\mu \rightarrow (\alpha q/2) \omega_\mu$  in our results.

Transformation (1) defines Weyl gauge symmetry. The definition of the geometry is completed by the so-called “non-metricity” condition:

$$\tilde{\nabla}_\mu g_{\alpha\beta} + 2\omega_\mu g_{\alpha\beta} = 0, \quad \text{where} \quad \tilde{\nabla}_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}_{\lambda\mu}^\rho g_{\rho\nu} - \tilde{\Gamma}_{\lambda\nu}^\rho g_{\rho\mu}. \quad (2)$$

Assuming a symmetric connection  $\tilde{\Gamma}_{\mu\nu}^\rho = \tilde{\Gamma}_{\nu\mu}^\rho$ , from (2) one finds  $\tilde{\Gamma}$ , invariant under (1):

$$\tilde{\Gamma}_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho |_{\partial_\mu \rightarrow \partial_\mu + 2\omega_\mu} = \Gamma_{\mu\nu}^\rho + (\delta_\mu^\rho \omega_\nu + \delta_\nu^\rho \omega_\mu - g_{\mu\nu} \omega^\rho), \quad (3)$$

with  $\Gamma$  the familiar Levi-Civita (LC) connection  $\Gamma_{\mu\nu}^\rho = (1/2) g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$ .

Further, one associates a Riemann tensor of Weyl geometry  $\tilde{R}_{\mu\nu\sigma}^\rho$  to  $\tilde{\Gamma}$ , via the commutator of two  $\tilde{\nabla}_\mu(\tilde{\Gamma})$  acting on a vector  $v^\rho$ :  $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]v^\rho = \tilde{R}_{\sigma\mu\nu}^\rho v^\sigma$ . This gives a Riemann tensor of Weyl geometry, defined by  $\tilde{\Gamma}$ , by a formula similar to that in Riemannian geometry, but with  $\Gamma$  replaced by  $\tilde{\Gamma}$ <sup>6</sup>. One then computes the Ricci tensor of Weyl geometry  $\tilde{R}_{\mu\nu} = \tilde{R}_{\sigma\mu\nu}^\sigma$ , etc. This gives the well-known affine, non-metric ( $\tilde{\nabla}_\mu g_{\alpha\beta} \neq 0$ ) formulation of Weyl geometry. This formulation, used for a century, is not physical since it is *not* Weyl gauge covariant. Indeed, with  $\tilde{\Gamma}$  invariant under (1), one shows that  $\tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu}$  transforms like  $g^{\mu\nu}$  i.e.  $\tilde{R}' = \Sigma^{-2} \tilde{R}$ , but  $\tilde{\nabla}_\mu \tilde{R}$  is not Weyl covariant:  $\tilde{\nabla}'_\mu \tilde{R}' \neq \Sigma^{-2} \tilde{\nabla}_\mu \tilde{R}$ .

However, there does exist a Weyl gauge covariant formulation [15, 19] of this geometry, as required for a gauge theory. One defines a gauge covariant derivative  $\hat{\nabla}_\mu$  of a tensor field  $T_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_r}$  of space-time charge  $\tilde{q}_T$  with  $T' = \Sigma^{\tilde{q}_T} T$ , then [15]<sup>7</sup>

$$\hat{\nabla}_\mu T = [\tilde{\nabla}_\mu(\tilde{\Gamma}) + \tilde{q}_T \omega_\mu] T \quad \Rightarrow \quad \hat{\nabla}'_\mu T' = \Sigma^{\tilde{q}_T} \hat{\nabla}_\mu T. \quad (4)$$

where we did not display the indices of the tensor  $T_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_r}$ .

Since  $\hat{\nabla}_\mu$  depends on the charge  $\tilde{q}_T$  of field  $T$ , no connection  $\hat{\Gamma}$  can be associated to  $\hat{\nabla}$  for all fields on which it acts; hence, this Weyl covariant formulation is *non-affine*, but it is *metric* since we now have  $\hat{\nabla}_\mu g_{\alpha\beta} = 0$ . Thus one can do all calculations directly in

<sup>6</sup>One has  $\tilde{R}_{\sigma\mu\nu}^\rho = \partial_\mu \tilde{\Gamma}_{\nu\sigma}^\rho - \partial_\nu \tilde{\Gamma}_{\mu\sigma}^\rho + \tilde{\Gamma}_{\mu\lambda}^\rho \tilde{\Gamma}_{\nu\sigma}^\lambda - \tilde{\Gamma}_{\nu\lambda}^\rho \tilde{\Gamma}_{\mu\sigma}^\lambda$ .

<sup>7</sup>In general  $\tilde{q}_T = p - r + q_T$  where  $q_T$  is the tangent space charge, see e.g. the review in Section 2 of [16].

this geometry, without going to a (metric) Riemannian geometry as done in the past (for a modern, rigorous interpretation of Weyl geometry as a gauge theory see [5, 6, 16]).

One then defines the Riemann tensor of Weyl geometry  $\hat{R}^\lambda_{\mu\nu\sigma}$ , using  $\hat{\nabla}_\mu$  (instead of  $\tilde{\nabla}_\mu$ ) in the standard definition of this tensor:  $[\hat{\nabla}_\mu, \hat{\nabla}_\nu]v^\lambda = \hat{R}^\lambda_{\mu\nu\sigma}v^\mu$ , where  $v^\mu = e^\mu_a v^a$  is a vector with vanishing Weyl charge on the tangent space,  $q_{v^a} = 0$ <sup>8</sup>. With this, one can compute the Riemann tensor  $\hat{R}^\mu_{\nu\rho\sigma}$ , Ricci tensor  $\hat{R}_{\mu\sigma} = \hat{R}^\lambda_{\mu\lambda\sigma}$  and Ricci scalar  $\hat{R} = \hat{R}_{\mu\sigma}g^{\mu\sigma}$  of Weyl geometry, in terms of their Riemannian counterparts. These relations are presented in the Appendix, eqs.(A-1), and will be used later on<sup>9</sup>. One also shows [15] (eq.A-25) that in this Weyl gauge covariant formulation, the Weyl tensor  $\hat{C}^\mu_{\nu\rho\sigma}$  associated to  $\hat{R}_{\mu\nu\rho\sigma}$  is equal to its Riemannian geometry version ( $C^\mu_{\nu\rho\sigma}$ ), so  $\hat{C}^\mu_{\nu\rho\sigma} = C^\mu_{\nu\rho\sigma}$ .

In the Weyl gauge covariant (metric) formulation of Weyl geometry, under transformation (1) we have [15]

$$\begin{aligned} \hat{R}' &= \Sigma^{-2}\hat{R}, & \hat{R}'_{\mu\nu} &= \hat{R}_{\mu\nu}, & \hat{R}'^\sigma_{\mu\nu\rho} &= \hat{R}^\sigma_{\mu\nu\rho}, \\ \hat{\nabla}'_\mu \hat{R}' &= \Sigma^{-2}\hat{\nabla}_\mu \hat{R}, & \hat{\nabla}'_\alpha \hat{R}'_{\mu\nu} &= \hat{\nabla}_\alpha \hat{R}_{\mu\nu}, & \hat{\nabla}'_\alpha \hat{R}'^\sigma_{\mu\nu\rho} &= \hat{\nabla}_\alpha \hat{R}^\sigma_{\mu\nu\rho}, \\ X' &= \Sigma^{-4}X, & X &= \hat{R}^2_{\mu\nu\rho\sigma}, \hat{R}^2_{\mu\nu}, \hat{C}^2_{\mu\nu\rho\sigma}, \hat{G}, \hat{F}^2_{\mu\nu}, \end{aligned} \quad (5)$$

Here the square of a tensor denotes contraction by the metric of indices in the same position. The field strength of  $\omega_\mu$  is  $\hat{F}_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$ , also invariant under (1).  $\hat{G}$  is the Chern-Euler-Gauss-Bonnet term of Weyl geometry (hereafter Euler term), see Appendix, eq.(A-3).

We see that the curvature tensors/scalar *and*  $\hat{\nabla}_\mu$  acting on them transform covariantly under (1), with the same Weyl charge as the operator itself. This property of Weyl geometry operators is similar to the implementation of (internal) gauge symmetries of the SM with respect to which fields and their derivatives transform covariantly. If one is not familiar with Weyl geometry, one may simply regard eqs.(A-1) as re-definitions of Riemann and Ricci tensors and scalar of Riemannian geometry, such that these re-defined expressions and their derivative  $\hat{\nabla}_\mu$  transform covariantly, as in eqs.(5).

This formulation of Weyl geometry may be seen as a covariantised version of Riemannian geometry with respect to the gauged dilatation symmetry [11, 15]; since the formulation is metric, one can do calculations [16], compute quantum corrections [15], etc.

The action of Weyl gauge theory of quadratic gravity, associated to Weyl geometry and invariant under (1), is [13]<sup>10</sup>

$$S_w = \int d^4x \sqrt{g} \left\{ \frac{1}{4! \xi^2} \hat{R}^2 - \frac{1}{\eta^2} \hat{C}^2_{\mu\nu\rho\sigma} - \frac{1}{4\alpha^2} \hat{F}^2_{\mu\nu} + \hat{G} \right\} \quad (6)$$

with perturbative couplings  $\xi, \alpha, \eta < 1$ ; for more on topological terms like  $\hat{G}$  see [6, 15].

Action (6) undergoes a Stueckelberg breaking of Weyl gauge symmetry, in which  $\omega_\mu$  becomes massive and decouples. One is left at low scales with Riemannian geometry and

<sup>8</sup>This definition can be extended if the tangent-space charge of this vector is non-zero [16] (section 2).

<sup>9</sup>The relation of Weyl covariant (metric) formulation to the non-metric one is:  $\hat{R}^\rho_{\sigma\mu\nu} = \tilde{R}^\rho_{\sigma\mu\nu} - \delta^\rho_\sigma \hat{F}_{\mu\nu}$ .

<sup>10</sup>see also more recent developments in [5, 6, 11, 15, 16, 22, 23].

Einstein-Hilbert action and a positive cosmological constant [22,23] (we return to this action later in the text). Correspondingly, there is a conserved Weyl gauge current,  $j_\mu \propto \hat{\nabla}_\mu \hat{R}$  with  $\hat{\nabla}^\mu j_\mu = 0$  [16,22], which generalises a similar current in global scale invariant theories [32–36].

At a geometric level, one can actually define a more general Weyl gauge invariant action than (6), by a version of Dirac-Born-Infeld action [29,30] associated to Weyl geometry itself, in  $d$  dimensions. This action is [28]

$$S'_w = \int d^d x \left\{ -\det [a_0 \hat{R} g_{\mu\nu} + a_1 \hat{R}_{\mu\nu} + a_2 \hat{F}_{\mu\nu}] \right\}^{\frac{1}{2}}, \quad (7)$$

$S'_w$  is Weyl gauge invariant in arbitrary  $d$  dimensions (each term under det is invariant, see (5)), with  $a_{0,1,2}$  some dimensionless coefficients. Note that no UV regulator scale/field is needed here to make this action well-defined in  $d$  dimensions.

A particular expansion of  $S'_w$  in ratios of (dimensionless) couplings  $a_j/a_0$ ,  $j = 1, 2$ , recovers in the leading order the Weyl gauge theory of quadratic gravity, eq.(6), while sub-leading orders might account for some quantum corrections to (6) [28]. The immediate natural question is whether one can extend  $S'_w$  to include matter (Standard Model)?

### 3 WDBI action: unification of Gravity and SM

In this section we construct the Weyl-Dirac-Born-Infeld (WDBI) action of SM and Weyl geometry and study its properties, inspired by action (7). The goal is to write an action that includes SM interactions alongside the gravitational interaction, on equal footing, while respecting both SM and Weyl gauge symmetries in arbitrary  $d$  dimensions.

To achieve this goal, we must identify all operators constructed from SM and Weyl geometry (curvatures) fields, that have (in  $d$  dimensions) mass dimension 2, and are both SM and Weyl gauge invariant (Weyl charge  $q = 0$ ). Why operators of mass dimension 2? Using these operators, the square root of the  $d$ -dimensional determinant of their linear combination (denoted  $A_{\mu\nu}$ ) has mass dimension  $d$ . Therefore, the associated WDBI action is automatically dimensionless and mathematically well-defined in *arbitrary*  $d$  dimensions, with dimensionless couplings, without an additional regulator (like a DR scale  $\mu$ ) required in gauge theories in  $d$  dimensions. This aspect has important implications discussed later.

This approach sets on equal footing SM operators and Weyl geometry operators, internal and external gauge symmetries, and gives a unified description, by the gauge principle, of gravity and SM. This is a more general approach than considering the action of SM embedded in Weyl geometry [23].

First let us specify the transformation of SM scalars  $\phi$  and fermions  $\psi$  under (1)

$$\phi' = \Sigma^{q_\phi} \phi, \quad \psi' = \Sigma^{q_\psi} \psi, \quad q_\phi = -\frac{1}{2}(d-2), \quad q_\psi = -\frac{1}{2}(d-1), \quad \Sigma = \Sigma(x) \quad (8)$$

The Weyl charges of  $\phi$ ,  $\psi$  are found from their (invariant) kinetic terms in curved space-time in  $d$  dimensions (see e.g. the appendix in [23]). This is possible since SM with a vanishing Higgs mass *parameter* is scale invariant and gauging this scale symmetry (to obtain SM with

Weyl gauge symmetry) is then immediate [23]. If  $d = 4$  we have  $q_\phi = -1$  and  $q_\psi = -3/2$  i.e. Weyl charges coincide with their inverse mass dimension. The Weyl gauge covariant derivatives of  $\phi, \psi$  are covariantised versions of their Riemannian version with respect to the gauged dilatation symmetry and transform covariantly with same charge, as shown below.

### 3.1 Weyl invariant operators of mass dimension two

Let us write the operators defined by the fields of SM and Weyl conformal geometry, that in  $d$  dimensions have a mass dimension 2 and are both SM and Weyl gauge invariant (Weyl charge  $q = 0$ ). In doing so, we include operators suppressed by powers of the Weyl scalar curvature  $\hat{R}$ . Using (1), (5) and (8), the list of such operators includes:

- Weyl geometry operators

$$\hat{R} g_{\mu\nu}, \quad \hat{R}_{\mu\nu}, \quad \hat{F}_{\mu\nu}. \quad (9)$$

- SM gauge sector:

$$F_{\mu\nu}^{(1)}, \quad F_{\alpha\beta}^{(j)} F_{\rho\sigma}^{(j)} g^{\alpha\rho} g^{\beta\sigma} \hat{R}^{-1} g_{\mu\nu}. \quad (10)$$

Here  $F_{\mu\nu}^{(1)}$  is the field strength of SM hypercharge field  $B_\mu$ ,  $F_{\mu\nu}^{(1)} = \partial_\mu B_\nu - \partial_\nu B_\mu$ , and  $F_{\alpha\beta}^{(i)} F^{(i)\alpha\beta}$ ,  $i = 1, 2, 3$  are SM gauge kinetic terms for U(1), SU(2), SU(3), in this order.

- Higgs sector:

$$(\hat{\nabla}_\alpha H)(\hat{\nabla}^\alpha H)^\dagger \hat{R}^{1-d/2} g_{\mu\nu}, \quad H^\dagger H \hat{R}^{2-d/2} g_{\mu\nu}, \quad (H^\dagger H)^2 \hat{R}^{3-d} g_{\mu\nu}, \quad (11)$$

where

$$\hat{\nabla}_\alpha H = (D_\alpha + q_H \omega_\alpha) H, \quad q_H = -\frac{1}{2}(d-2). \quad (12)$$

Here  $D_\alpha H = (\partial_\alpha - i\mathcal{A}_\alpha)H$  is the SM covariant derivative of the Higgs doublet,  $\mathcal{A}_\alpha = (g/2)\vec{\sigma}\cdot\vec{\mathcal{A}}_\alpha + (g'/2)B_\alpha$ , with  $\vec{\mathcal{A}}_\alpha$  the SU(2) gauge boson,  $B_\alpha$  the U(1) of hypercharge, of gauge couplings  $g$  and  $g'$  respectively. One checks that  $\hat{\nabla}_\alpha H$  transforms covariantly under SM and Weyl gauge symmetry with the same charges as  $H$ .

- SM fermionic sector (sum over SM fermions understood):

$$(i\bar{\psi}\gamma^a e_a^\alpha \hat{\nabla}_\alpha \psi + \text{h.c.}) \hat{R}^{1-d/2} g_{\mu\nu}, \quad (13)$$

with

$$\hat{\nabla}_\alpha \psi = \left[ D_\alpha + q_\psi \omega_\alpha + \frac{1}{2} \tilde{s}_\alpha^{ab} \sigma_{ab} \right] \psi, \quad q_\psi = -\frac{1}{2}(d-1). \quad (14)$$

$D_\alpha$  is the SM covariant derivative of fermions,  $\sigma_{ab} = (1/4)[\gamma_a, \gamma_b]$ , ( $a, b$  are tangent space indices), and  $\tilde{s}_\mu^{ab}$  is the spin connection in Weyl geometry; this has an expression given by the covariantised version (with respect to gauged dilatations) of the Riemannian spin connection  $s_\alpha^{ab} = -e^{\lambda b} (\partial_\alpha e_\lambda^a - \Gamma_{\alpha\lambda}^\nu e_\nu^a)$  see e.g. [5, 23]:

$$\tilde{s}_\alpha^{ab} = s_\alpha^{ab} \Big|_{\partial_\alpha e_\nu^a \rightarrow [\partial_\alpha + \omega_\alpha] e_\nu^a} = s_\alpha^{ab} + (e_\alpha^a e^{\nu b} - e_\alpha^b e^{\nu a}) \omega_\nu, \quad (15)$$

where we used that  $e_\nu^a$  has Weyl charge  $q = 1$  (half of that of  $g_{\mu\nu}$ ). Note  $\tilde{s}_\alpha^{ab}$  is invariant under (1), therefore  $\hat{\nabla}_\alpha \psi$  transforms covariantly under (1) with the same Weyl charge as  $\psi$ . Further, one notices  $\gamma^\alpha \tilde{s}_\alpha^{ab} \sigma_{ab} = \gamma^\alpha s_\alpha^{ab} \sigma_{ab} + (d-1)\gamma^\alpha \omega_\alpha$  in  $d$  dimensions; therefore, in (14) the dependence on  $\omega_\alpha$  of the spin connection is cancelled by that from  $q_\psi \omega_\alpha$ ; then  $\gamma^\alpha \hat{\nabla}_\alpha \psi = \gamma^\alpha \nabla_\alpha \psi$  and then the expression in (13) invariant under (1) becomes

$$(i \bar{\psi} \gamma^a e_a^\alpha \nabla_\alpha \psi + \text{h.c.}) \hat{R}^{1-d/2} g_{\mu\nu}, \quad (16)$$

with Riemannian operator  $\nabla_\alpha = D_\alpha + (1/2)s_\alpha^{ab} \sigma_{ab}$ . So even though they are charged under (1), in  $d$  dimensions fermions do not couple to  $\omega_\alpha$  at tree-level, extending a similar result for  $d = 4$  [23, 37].

- Yukawa sector:

$$[(\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R) + \text{h.c.}] \hat{R}^{2-3d/4} g_{\mu\nu}. \quad (17)$$

with  $\tilde{H} = i\sigma_2 H^\dagger$  and  $Y, Y'$  Yukawa matrices. It is easily checked that the sum of the Weyl charges of the fields present is zero and this operator has mass dimension 2, for any  $d$ .

- Gauge kinetic mixing term ( $\omega_\mu$  - hypercharge):

$$\hat{F}_{\alpha\beta} F^{(1)\alpha\beta} \hat{R}^{-1} g_{\mu\nu}. \quad (18)$$

This is invariant under SM group; it is also Weyl gauge invariant, with mass dimension 2.

- Gauge kinetic term of  $\omega_\mu$

$$\hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} \hat{R}^{-1} g_{\mu\nu}, \quad (19)$$

which is also Weyl gauge invariant, with mass dimension 2.

Additional operators of Weyl charge  $q = 0$  and mass dimension two are possible and will be discussed later. Note also that we considered operators suppressed at most by one power of  $\hat{R}$  for  $d = 4$ ; higher suppression powers can be considered, but they will not introduce new terms in the leading order action (section 3.6).

### 3.2 WDBI action in $d$ dimensions

Using operators (9) to (19), we write a linear combination ( $A_{\mu\nu}$ ) of these and integrate  $\sqrt{\det A_{\mu\nu}}$  in  $d$  dimensions. This gives a version of Dirac-Born-Infeld action of both SM and Weyl geometry, which we call Weyl-Dirac-Born-Infeld action (WDBI). The action is then:

$$S_{\mathbf{d}} = \int d^d x \left[ -\det A_{\mu\nu} \right]^{\frac{1}{2}}, \quad (20)$$

$$\begin{aligned} A_{\mu\nu} = & a_0 \hat{R} g_{\mu\nu} + a_1 \hat{R}_{\mu\nu} + a_2 \hat{F}_{\mu\nu} + a_3 F_{\mu\nu}^{(1)} + a_4^{(i)} F_{\alpha\beta}^{(i)} F^{(i)\alpha\beta} g_{\mu\nu} \hat{R}^{-1} \\ & + a_5 |\hat{\nabla}_\alpha H|^2 \hat{R}^{1-d/2} g_{\mu\nu} + a_6 |H|^2 \hat{R}^{2-d/2} g_{\mu\nu} + a_7 |H|^4 \hat{R}^{3-d} g_{\mu\nu} \\ & + a_8 (i \bar{\psi} \gamma^a e_a^\alpha \hat{\nabla}_\alpha \psi + \text{h.c.}) \hat{R}^{1-d/2} g_{\mu\nu} \\ & + a_9 (\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R + \text{h.c.}) \hat{R}^{2-3d/4} g_{\mu\nu}, \\ & + a_{10} \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} \hat{R}^{-1} g_{\mu\nu} + a_{11} \hat{F}_{\alpha\beta} F^{(1)\alpha\beta} \hat{R}^{-1} g_{\mu\nu}, \end{aligned} \quad (21)$$

Action  $S_{\mathbf{d}}$  has both SM and Weyl gauge invariances in arbitrary  $d$  dimensions, with dimensionless coefficients  $a_0, \dots, a_{11}$ . Note that no UV regulator, DR subtraction scale  $\mu$  or field, etc, is present in this action (a scale, if present, would actually break Weyl symmetry). We return to this issue shortly. Next, define

$$X^\lambda{}_\nu = \frac{g^{\lambda\rho}}{a_0 \hat{R}} A_{\rho\nu} - \delta^\lambda{}_\nu, \quad (22)$$

and expand  $S_{\mathbf{d}}$  <sup>11</sup>

$$S_{\mathbf{d}} = \int d^d x \sqrt{g} (a_0 |\hat{R}|)^{d/2} \left\{ 1 + \frac{1}{2} \text{tr} X + \frac{1}{4} \left( \frac{1}{2} (\text{tr} X)^2 - \text{tr} X^2 \right) + \mathcal{O} \left[ \left( \frac{a_j}{a_0} \right)^3 \right] \right\}, \quad (23)$$

with  $g = -\det g_{\mu\nu}$ .  $X^\lambda{}_\nu$  depends on ratios of coefficients,  $a_j/a_0$  ( $j=1, \dots, 11$ ) assumed to be small  $|a_j/a_0| \ll 1$ , as required for phenomenological reasons, that we verify later<sup>12</sup>. We find

$$\begin{aligned} S_{\mathbf{d}} = & \int d^d x \sqrt{g} \left\{ \hat{R}^{d/2-2} \left[ c_0 \hat{R}^2 + c_1 (\hat{C}_{\mu\nu\rho\sigma}^2 - \hat{G}) + c_2 \hat{F}_{\mu\nu}^2 + c_3 \hat{F}^{\mu\nu} F_{\mu\nu}^{(1)} + c_4^{(j)} F_{\mu\nu}^{(j)} F^{(j)\mu\nu} \right] \right. \\ & + c_5 |\hat{\nabla}_\mu H|^2 + c_6 |H|^2 \hat{R} + c_7 |H|^4 \hat{R}^{2-d/2} + c_8 \left( \frac{i}{2} \bar{\psi}_L \gamma^a e_a^\alpha \nabla_\alpha \psi_R + \text{h.c.} \right) \\ & \left. + c_9 (\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R + \text{h.c.}) \hat{R}^{1-d/4} + \mathcal{O} \left( \frac{1}{\hat{R}^3} \right) \right\} + a_0^{d/2} \mathcal{O} \left( \frac{a_i}{a_0} \right)^3. \end{aligned} \quad (24)$$

The dimensionless coefficients  $c_j$ ,  $j = 1, \dots, 9$  are functions of  $a_k$  ( $k = 1, \dots, 11$ ), found in Appendix, eqs.(A-6) to (A-12) and show how terms in action (20) contribute to (24). The terms of coefficients  $a_{10}$  and  $a_{11}$  are redundant in the leading order, since they do not bring

<sup>11</sup>We use  $[\det(1+X)]^{1/2} = 1 + \frac{1}{2} \text{tr} X + \frac{1}{4} \left[ \frac{1}{2} (\text{tr} X)^2 - \text{tr} X^2 \right] + \left[ \frac{1}{48} (\text{tr} X)^3 - \frac{1}{8} \text{tr} X \text{tr} X^2 + \frac{1}{6} \text{tr} X^3 \right] + \mathcal{O}(X^4)$

<sup>12</sup>In (23) and below, to simplify notation we wrote  $\mathcal{O}[(a_j/a_0)^3]$  but we actually mean  $\mathcal{O}(a_j a_k a_m / a_0^3)$ .

new operators in the action. Similarly for the term of coefficient  $a_7$ , but its presence ensures coefficient  $c_7$  (of  $|H|^4 \hat{R}^{2-d/2}$ ) is independent of  $c_6$  (of  $|H|^2 \hat{R}$ ), for phenomenological reasons.

Action (24) is brought to canonical form in Weyl geometry, shown below, with dimensionless physical perturbative couplings of gravity  $\xi, \eta, \alpha < 1$ , SM couplings  $\alpha_j < 1$ , ( $j = 1, 2, 3$ ), non-minimal coupling  $\xi_H < 1$ , correct signs of kinetic terms and no gauge kinetic mixing  $\omega_\mu$ -hypercharge (investigated elsewhere [23]); we assume below  $\xi \ll \eta \sim \alpha < 1$  for physical reasons detailed later. Then

$$\begin{aligned}
S_{\mathbf{d}} = & \int d^d x \sqrt{g} \left\{ \hat{R}^{d/2-2} \left[ \frac{1}{4! \xi^2} \hat{R}^2 - \frac{1}{\eta^2} (\hat{C}_{\mu\nu\rho\sigma}^2 - \hat{G}) - \frac{1}{4\alpha^2} \hat{F}_{\mu\nu}^2 - \frac{1}{4\alpha_j^2} F_{\mu\nu}^{(j)} F^{(j)\mu\nu} \right] \right. \\
& + |\hat{\nabla}_\mu H|^2 - \frac{\xi_H}{6} |H|^2 \hat{R} - \lambda |H|^4 \hat{R}^{2-d/2} + \left( \frac{i}{2} \bar{\psi}_L \gamma^a e_a^\alpha \nabla_\alpha \psi_R + \text{h.c.} \right) \\
& \left. + (\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R + \text{h.c.}) \hat{R}^{1-d/4} + \mathcal{O}\left(\frac{1}{\hat{R}^3}\right) \right\} + a_0^{d/2} \mathcal{O}\left(\frac{a_i}{a_0}\right)^3. \quad (25)
\end{aligned}$$

Demanding that coefficients  $c_j$  have the values shown in (25), (A-13), we find a solution for coefficients  $a_j$  in action (20) that brings (24) to canonical form (25). First,  $a_0, a_1$  are fixed by the two equations below

$$a_0^{d/2} = \frac{1}{\eta^2} \frac{16(d-3)}{d-2} \left(\frac{a_0}{a_1}\right)^2, \quad (26)$$

$$\frac{a_0}{a_1} = \frac{-1}{4} (1 \pm \sqrt{1+16\kappa}) \approx \mp \sqrt{\kappa}, \quad \kappa \equiv \frac{(d-2)}{16(d-1)} \left[ \frac{\eta^2}{24\xi^2} \frac{d-1}{d-3} - 1 \right] \gg 1. \quad (27)$$

so  $a_0 \sim \xi^{-4/d}$ . Assuming for simplicity  $a_{10} = 0$  (this is easily relaxed), then we find  $a_2$

$$\frac{a_2}{a_1} = \frac{d-2}{2} (-1 \pm \sqrt{1-z}) \sim \mathcal{O}(1), \quad z \equiv \frac{\eta^2}{4\alpha^2} \frac{1}{(d-2)(d-3)}. \quad (28)$$

$z < 1$  for  $\eta^2 < 4\alpha^2(d-2)(d-3)$ . The physical couplings  $\xi, \eta, \alpha$  in (25) are then fixed by  $a_{0,1,2}$  above. The rest of physical couplings are obtained for the following  $a_j, j = 4, \dots, 11$ :

$$\begin{aligned}
a_4^{(j)} = & \frac{1}{4f} \left[ -\frac{a_0^{2-d/2}}{\alpha_j^2} - a_3^2 \delta_{j1} \right], \quad (j = 1, 2, 3); \quad a_5 = a_8 = a_9 = \frac{a_0^{2-d/2}}{f}, \quad a_6 = -\frac{\xi_H a_0^{2-d/2}}{6f} \\
a_7 = & \frac{-1}{f} \left[ \lambda a_0^{2-d/2} + a_6^2 \frac{d(d-2)}{8} \right], \quad a_{11} = \frac{-1}{f} (2a_2 + a_1(d-2)), \quad f \equiv \frac{a_0 d}{2} + a_1 \frac{d-2}{4}. \quad (29)
\end{aligned}$$

We see that  $a_{1,2} \sim a_0 \xi \sim \xi^{1-4/d}$  and  $a_j \sim a_0^{1-d/2} \sim \xi^{2-4/d}, j = 4, \dots, 11$ ; next, we also impose this last relation to  $a_3$ , which is possible since the above  $a_{11}$  enforces  $c_3 = 0$ , leaving  $a_3$  arbitrary. To conclude,  $|a_{1,2}/a_0| \sim \xi \ll 1$ ,  $|a_j/a_0| \sim a_0^{-d/2} \sim \xi^2 \ll 1, j = 3, \dots, 11$ , and the convergence of expansion (23) is then assured for our solution for  $a_k$ , giving action (25).

### 3.3 Properties of WDBI action

Action (25) is still in the Weyl geometry formulation. To obtain this action in a Riemannian formulation, one simply replaces  $\hat{R}$  of Weyl geometry by its Riemannian expression shown in eq.(A-1). All other terms, except  $\hat{G}$ , are unchanged: indeed,  $\hat{F}_{\mu\nu}$  has the same expression in Riemannian and in flat case, and in the Weyl covariant formulation used here the term  $\hat{C}_{\mu\nu\rho\sigma}^2$  is equal to its Riemannian version, so  $\hat{C}_{\mu\nu\rho\sigma}^2 = C_{\mu\nu\rho\sigma}^2$ , eq.(A-2).

Regarding  $\hat{G}$  (Euler term), it is a topological term (total derivative) if  $d = 4$  (hence it does not affect the equations of motion), but this changes in  $d$  dimensions (for a discussion see [15]); its expression in Riemannian notation is found in (A-3) with  $\hat{R}_{\mu\nu\rho\sigma}$ ,  $\hat{R}_{\mu\nu}$  and  $\hat{R}$  replaced by their Riemannian counterparts, eqs.(A-1).

Action (25) has interesting properties:

(a) In the leading order of  $S_{\mathbf{d}}$  we obtained Weyl gauge invariant actions of the SM and Weyl quadratic gravity of eq.(6), analytically continued to  $d$  dimensions; there are also non-minimal couplings of SM to gravity ( $\xi_H$  and those induced by  $\hat{R}$  which contains  $\omega_\mu$ ). For  $d = 4$ , the geometric part of this action (first three terms in (25)) recovers Einstein-Hilbert gravity after a Stueckelberg mechanism [22, 23], as reviewed in the next section.

(b) In  $d = 4 - 2\epsilon$ , we see that the exact WDBI action *predicts* that the scalar curvature  $\hat{R}^{-\epsilon}$  i.e. geometry/gravity acts as the UV regulator<sup>13</sup> for the leading order action of expanded  $S_{\mathbf{d}}$ . This is possible due to the Weyl gauge covariance of  $\hat{R}$ . The leading order action is thus mathematically well-defined and needs no UV regulator (field or scale); the regularisation is “built-in” exact  $S_{\mathbf{d}}$ . Being Weyl gauge invariant in  $d$  dimensions, the leading order action is Weyl anomaly-free<sup>14</sup>, as discussed in [15] with the regularisation derived here. Actually, at each order in the expansion  $S_{\mathbf{d}}$  is Weyl gauge invariant and Weyl-anomaly free.

(c) The exact WDBI action, being Weyl gauge invariant in  $d$  dimensions, is also Weyl anomaly-free. Therefore the WDBI action is a consistent (quantum) gauge theory.

If one starts with the WDBI action in  $d = 4$  dimensions, its analytical continuation to  $d = 4 - 2\epsilon$  is trivial, simply replace  $d = 4 \rightarrow d = 4 - 2\epsilon$ ; the action is then mathematically well defined and Weyl gauge invariant, with no added UV regulator scale. Quantum calculations can be performed in this Weyl gauge invariant phase, respecting all the symmetries of the theory. This shows the power of Weyl geometry as a gauge theory.

This elegant behaviour is unique, not seen in other theories in Riemannian geometry, where a UV regulator (scale or field) is necessarily added “by hand”, to ensure the theory is mathematically well-defined in  $d$  dimensions. Also, in conformal gravity a dilaton field is added ad-hoc as regulator to maintain its symmetry [9] in  $d$  dimensions. Finally, even in string theory, local Weyl invariance (on the Riemannian worldsheet, not in physical space-time as here) is actually broken by the DR regularisation in  $d = 2 + \epsilon$ . It is restored by the condition of vanishing Ricci tensor in target space, e.g. [31]. (As a side-remark, this condition may not be necessary if worldsheet geometry is that of Weyl geometry where this symmetry is natural in  $d$  dimensions, see Appendix).

<sup>13</sup>This requires  $\hat{R}$  be non-zero, see later.

<sup>14</sup>For a discussion on Weyl anomaly in Riemannian case see [9, 38–42] and [15] for Weyl geometry.

### 3.4 WDBI action in d=4 and the broken phase

Let us consider now the limit of  $d = 4$  dimensions of action (20), (25). We have

$$S_4 = \int d^4x \sqrt{g} \left[ -\det A_{\mu\nu} \right]^{1/2} \quad (30)$$

$$\begin{aligned} \text{with } A_{\mu\nu} = & a_0 \hat{R} g_{\mu\nu} + a_1 \hat{R}_{\mu\nu} + a_2 \hat{F}_{\mu\nu} + a_3 F_{\mu\nu}^{(1)} + a_4^{(j)} F_{\alpha\beta}^{(j)} F^{(j)\alpha\beta} g_{\mu\nu} \hat{R}^{-1} \\ & + a_5 |\hat{\nabla}_\alpha H|^2 \hat{R}^{-1} g_{\mu\nu} + a_6 |H|^2 g_{\mu\nu} + a_7 |H|^4 \hat{R}^{-1} g_{\mu\nu} \\ & + a_8 (i \bar{\psi} \gamma^a e_a^\alpha \hat{\nabla}_\alpha \psi + \text{h.c.}) \hat{R}^{-1} g_{\mu\nu} \\ & + a_9 (\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R + \text{h.c.}) \hat{R}^{-1} g_{\mu\nu} \\ & + a_{10} \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} \hat{R}^{-1} g_{\mu\nu} + a_{11} \hat{F}_{\alpha\beta} F^{(1)\alpha\beta} \hat{R}^{-1} g_{\mu\nu}. \end{aligned} \quad (31)$$

Action (25) becomes

$$\begin{aligned} S_4 = & \int d^4x \sqrt{g} \left\{ \frac{1}{4! \xi^2} \hat{R}^2 - \frac{1}{\eta^2} \hat{C}_{\mu\nu\rho\sigma}^2 - \frac{1}{4\alpha^2} \hat{F}_{\mu\nu}^2 \right. \\ & - \frac{1}{4\alpha_j^2} F_{\mu\nu}^{(j)} F^{(j)\mu\nu} + |\hat{\nabla}_\mu H|^2 - \frac{\xi_H}{6} |H|^2 \hat{R} - \lambda |H|^4 + \left( \frac{i}{2} \bar{\psi}_L \gamma^a e_a^\alpha \nabla_\alpha \psi_R + \text{h.c.} \right) \\ & \left. + (\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R + \text{h.c.}) + \mathcal{O}\left(\frac{1}{\hat{R}^3}\right) \right\} + \mathcal{O}\left(\frac{a_i}{a_0}\right)^3 \end{aligned} \quad (32)$$

provided that

$$\begin{aligned} a_1 &= \pm \frac{2\sqrt{2}}{\eta}, \quad a_0 = \frac{-a_1}{4} (1 \pm \sqrt{1 + 16\kappa}), \quad \kappa = \frac{1}{24} \left[ \frac{\eta^2}{8\xi^2} - 1 \right] \gg 1, \\ a_2 &= (-1 \pm \sqrt{1-z}) a_1; \quad z = \frac{\eta^2}{8\alpha^2}, \quad a_4^{(j)} = \frac{-1}{4f} \left[ \frac{1}{\alpha_j^2} - a_3^2 \delta_{j1} \right]; \quad j = 1, 2, 3. \\ a_5 &= a_8 = a_9 = \frac{1}{f}, \quad a_6 = -\frac{\xi_H}{6f}, \quad a_7 = \left[ -\lambda - \frac{\xi_H^2}{36f^2} \right] \frac{1}{f}, \quad a_{11} = \frac{-2}{f} (a_2 + a_1). \end{aligned} \quad (33)$$

where  $f = 2a_0 + a_1/2$ . The Weyl gauge covariant derivatives of Higgs and fermions in (32) are immediate from their  $d$ -dimensional expressions, eqs.(12), (14), (16) evaluated for  $d = 4$ . The topological term  $\hat{G}$  was removed from  $S_4$ , being a total derivative.

As mentioned,  $S_4$  of (30) has an immediate analytical continuation (regularisation), by simply replacing  $d = 4 \rightarrow d = 4 - 2\epsilon$ , to obtain the exact WDBI action  $S_d$  which is Weyl gauge invariant in  $d$  dimensions and Weyl anomaly-free. No regulator scale is introduced.

The leading order of  $S_4$  contains the SM action with a mild change in the Higgs sector to make it Weyl gauge invariant, with non-minimal gravitational couplings, plus the Weyl quadratic gravity action, first line in (32). We thus recovered in this leading order the action of SM in Weyl geometry (SMW), studied in [23]. The exact WDBI action is however more general and has additional contributions: in its series expansion, these appear as sub-leading orders, which are higher dimensional (non-polynomial) operators (see section 3.5).

It is well-known that the leading order action shown in (32) has a Stueckelberg breaking of the Weyl gauge symmetry [22, 23]. Since this is relevant for the sub-leading orders of  $S_4$ , we briefly review this mechanism by considering only the geometric part of  $S_4$ , shown in (32)<sup>15</sup>, which is

$$S_{\mathbf{w}} = \int d^4x \sqrt{g} \left\{ \frac{1}{4! \xi^2} \hat{R}^2 - \frac{1}{\eta^2} \hat{C}_{\mu\nu\rho\sigma}^2 - \frac{1}{4\alpha^2} \hat{F}_{\mu\nu}^2 \right\}. \quad (34)$$

First, replace in this action  $\hat{R}^2 \rightarrow -2\phi^2 \hat{R} - \phi^2$ , to obtain a new action which gives an equation of motion for  $\phi$  of solution:  $\phi^2 = -\hat{R}$  ( $\hat{R} < 0$ )<sup>16</sup> which replaced back in the action recovers  $S_{\mathbf{w}}$ ; hence the two actions are equivalent. Next, one goes to the Riemannian picture, using (A-1) for  $d = 4$ , to write  $\hat{R}$  in terms of Riemannian scalar curvature  $R$ . After some arrangements the action in Riemannian geometry notation becomes [23]

$$S_{\mathbf{w}} = \int d^4x \sqrt{g} \left\{ \frac{-1}{2\xi^2} \left[ \frac{1}{6} \phi^2 R + (\partial_\mu \phi)^2 \right] - \frac{\phi^4}{4! \xi^2} + \frac{\alpha^2 q^2}{8 \xi^2} \phi^2 \left[ \omega_\mu - \partial_\mu \ln \phi \right]^2 - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right\} \quad (35)$$

where  $F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu = \hat{F}_{\mu\nu}$  and we used eq.(A-2). The action remains invariant under (1). By applying transformation (1) with  $\Sigma = \phi^2 / \langle \phi^2 \rangle$  one is fixing  $\phi$  to its vev, assumed to exist. Naively, one sets  $\phi \rightarrow \langle \phi \rangle$  in  $S_{\mathbf{w}}$ . In terms of transformed (“primed”) fields the above action gives in the broken phase

$$S_{\mathbf{w}} = \int d^4x \sqrt{g'} \left[ -\frac{1}{2} M_p^2 R' + \frac{1}{2} m_\omega^2 \omega'_\mu \omega'^\mu - \Lambda M_p^2 - \frac{1}{4} \hat{F}'_{\mu\nu}{}^2 - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right], \quad (36)$$

where we rescaled  $\omega_\mu \rightarrow \alpha \omega_\mu$  and introduced the cosmological constant, Planck scale and the mass of  $\omega_\mu$

$$\Lambda \equiv \frac{1}{4} \langle \phi \rangle^2, \quad M_p^2 \equiv \frac{\langle \phi^2 \rangle}{6 \xi^2}, \quad m_\omega^2 \equiv 6 \alpha^2 M_p^2. \quad (37)$$

All mass scales have geometric origin due to the field  $\phi$  (from the  $\hat{R}^2$  geometric term) that generates them [11, 18]. As seen from (35), the gauge field  $\omega_\mu$  becomes massive in a Stueckelberg mechanism, by eating the derivative of  $\ln \phi$  field which is the would-be-Goldstone of gauged dilatations<sup>17</sup>. This is the Weyl gauge symmetry breaking in the absence of matter. In the presence of the SM, the new would-be-Goldstone is a mixing (radial direction in the field space) of  $\phi$  and the (neutral) Higgs field ( $h$ ), since now both contribute to the Planck mass and  $m_\omega$  in (37); in this sense, in action (35) one replaces  $(1/\xi^2) \phi^2 \rightarrow (1/\xi^2) \phi^2 + \xi_H h^2$ . The real (neutral) Higgs field is then the angular direction in the field space of initial  $\phi$  and  $h$ . For details see [23] (section 2.5 and Appendix C). This ends our review of the of breaking of Weyl gauge symmetry.

<sup>15</sup>To see the breaking including the effects from SM action shown in (32), see section 2.5 in [23].

<sup>16</sup> $\hat{R} < 0$  is consistent with  $R = -12H_0^2$  ( $\Lambda = 3H_0^2$ ) obtained for a Friedmann-Robertson-Walker metric.

<sup>17</sup> $\ln \phi$  transforms with a shift under (1).

Since we have  $\xi^2 \sim \Lambda/M_p^2$ , this explains our initial assumption  $\xi \ll 1$ . One has  $m_\omega \sim M_p$  for  $\alpha$  not far below 1, so massive  $\omega_\mu$  decouples below  $M_p$  and Weyl connection (3) and geometry become Levi-Civita connection and Riemannian geometry, respectively [22, 23]<sup>18</sup>. Further, for  $\eta$  near 1, we also have that the spin-two state due to the  $C_{\mu\nu\rho\sigma}^2$  term in the presence of the Einstein term in (36), has a mass  $\eta M_p$  [43] and thus it also decouples not far below  $M_p$ . Thus, for  $\eta \sim \alpha < 1$  not too small, as we assumed, one is left below  $M_p$  with the Einstein-Hilbert action, with  $\Lambda > 0$  and SM action with a Higgs sector with a coupling to  $\omega_\mu$ . The phenomenology of this action was discussed in [23].

To conclude, the WDBI action in  $d = 4$ , which is Weyl anomaly free, recovers in the leading order a Weyl gauge invariant action of SM and Weyl quadratic gravity. This symmetry is broken in the leading order of the WDBI action, the massive Weyl gauge boson and spin-two state decouple near  $M_p$ , and one recovers Einstein-Hilbert gravity plus SM action.

### 3.5 Sub-leading orders

What about the sub-leading orders of the expanded WDBI action? These are Weyl gauge invariant operators  $\mathcal{O}(1/\hat{R}^3)$  (part of  $\mathcal{O}[(a_i/a_0)^2]$ ) and  $\mathcal{O}[(a_i/a_0)^3]$ , see (24), (25), (32).

Concerning  $\mathcal{O}(1/\hat{R}^3)$  terms, their origin is in  $\text{tr}X^2$  and  $(\text{tr}X)^2$ ; they arise from multiplying two SM-like operators of coefficients  $a_j \propto a_0^{1-d/2} = \xi^{2-4/d}$ , ( $j = 3, 4, \dots, 11$ ). They have extra suppression relative to other terms  $\mathcal{O}[(a_i/a_0)^2]$  due to mixed contributions SM-gravity, shown in (24), (25), (32). Examples of such operators are

$$\frac{a_4 a_6}{a_0^2} |H|^2 F_{\mu\nu}^{(i)2} \hat{R}^{-1-d/2}, \quad \frac{a_6 a_7}{a_0^2} |H|^6 \hat{R}^{3-3d/2}, \quad \frac{a_6 a_9}{a_0} |H|^2 \bar{\Psi}_L Y_\psi H \psi_R \hat{R}^{2-5d/4}, \quad (38)$$

The coefficients of these operators are of order  $\sim \xi^4$ . The first operator gives a term in  $S_{\mathbf{d}}$

$$S_{\mathbf{d}} \sim \xi^2 \int d^d x \sqrt{g} \frac{|H|^2 F_{\mu\nu}^{(i)2}}{\hat{R}} \rightarrow \frac{1}{M_p^2} \int d^d x \sqrt{g} |H|^2 F_{\mu\nu}^{(i)2}. \quad (39)$$

In the last step we used the broken phase in  $d = 4$  with  $M_p$  of (37). Relative to the rest of  $\mathcal{O}(a_j^2/a_0^2)$  operators that we kept in the leading order action, this contribution is strongly suppressed by  $\xi^2 \ll 1$ , (or by  $M_p^2$  in the broken phase). Similar for the other two operators above. In general,  $\mathcal{O}(1/\hat{R}^3)$  operators bring  $\mathcal{O}(\xi^2)$  corrections to the physical couplings of the terms shown in the leading order action (recall  $\xi^2 \sim \Lambda/M_p^2$ ).

Concerning  $\mathcal{O}[(a_i/a_0)^3]$  operators, they generate corrections such as  $\mathcal{O}(a_6^3/a_0^3)$  that contributes to the action a term like

$$S_{\mathbf{d}} \sim \xi^4 \int d^d x \sqrt{g} \frac{|H|^6}{\hat{R}^{d-3}} \rightarrow \frac{\xi^2}{M_p^2} \int d^d x \sqrt{g} |H|^6, \quad (40)$$

which is more suppressed than (39). Since such operators respect the gauge symmetry, they may be generated as quantum corrections, if one computed these starting from the leading order action as tree-level action. In other words, the WDBI action may include

<sup>18</sup>If one is tuning  $\alpha$  to ultra-weak values ( $\ll 1$ ),  $\omega_\mu$  can in principle be light (TeV scale or even lower) [23].

some quantum effects, at least on geometric side [28]. To conclude, the expansion of the WDBI action generates sub-leading orders which are higher dimensional operators strongly suppressed by powers of gravitational coupling,  $\xi^2 \ll 1$  (or by  $M_p^2$  in the broken phase).

### 3.6 Other corrections

The list of Weyl gauge invariant operators of mass dimension 2, used to build the WDBI action was minimal, sufficient to recover in the leading order a Weyl gauge invariant SM action and Weyl quadratic gravity action. Additional similar operators could be present in  $A_{\mu\nu}$ , with new dimensionless coefficients. For example another operator is

$$\hat{R}^{\alpha\beta} \hat{F}_{\alpha\beta} \hat{R}^{-1} g_{\mu\nu} \propto \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} \hat{R}^{-1} g_{\mu\nu} \quad (41)$$

since the antisymmetric part of  $\hat{R}_{\alpha\beta}$  is  $\hat{F}_{\alpha\beta}$ . This operator generates a gauge kinetic term for  $\omega_\mu$  in the leading order action, already present in our action; up to a redefinition of Weyl gauge coupling, this operator brings no additional physics. Similarly, the operator obtained from the lhs of (41) with  $\hat{F} \rightarrow F^{(1)}$ , generates a gauge kinetic mixing (hypercharge -  $\omega_\mu$ ), already discussed in the leading order and it can also be ignored.

A more general form of  $A_{\mu\nu}$  is

$$A'_{\mu\nu} = A_{\mu\nu} [a_k g_{\mu\nu} \rightarrow a_k (g_{\mu\nu} + z_k \kappa_{\mu\nu})] \quad (42)$$

where  $k = 4, 5, \dots, 11$ , and  $z_4, \dots, z_{11}$  are new dimensionless coefficients, with  $\kappa_{\mu\nu} \equiv \hat{R}_{\mu\nu} \hat{R}^{-1}$  which transforms under (1) just like the metric. With the new  $A'_{\mu\nu}$  one shows that the same action is found in the leading order, up to a redefinition of coefficients  $c_k$ , without generating new terms. One can also extend  $\kappa_{\mu\nu}$  to include corrections to it like  $(1/\hat{R}^2) \hat{R}_{\alpha\beta} \hat{R}^{\alpha\beta} g_{\mu\nu}$ , which has the same Weyl charge as the metric, and so on. Such corrections do not bring new terms in the leading order action discussed, but this may change in higher orders of the expanded action.

## 4 Conclusions

In this work we constructed a general gauge theory beyond SM and gravity, based on Weyl gauge group of dilatations and Poincaré symmetries. The natural framework for such gauge symmetry is Weyl geometry. We used the Weyl gauge covariant (metric!) formulation of this geometry, which we reviewed. The action we found is a version of the Dirac-Born-Infeld action for SM and Weyl geometry, which we called Weyl-Dirac-Born-Infeld (WDBI) action.

To find the action, one constructs a linear combination ( $A_{\mu\nu}$ ) of all operators of mass dimension two, that are products of SM and Weyl geometry operators, and have SM and Weyl gauge invariance in  $d$  dimensions. The space-time integral of  $\sqrt{\det A_{\mu\nu}}$  in  $d$  dimensions gives the WDBI action. This is a general gauge theory of SM and Weyl group, valid in  $d$  dimensions. This action is obviously more general than the Weyl gauge invariant action of SM and Weyl quadratic gravity, studied previously.

By construction, the WDBI action is mathematically well-defined in  $d$  dimensions, with SM and Weyl gauge invariance, so it needs no UV regulator (DR scale  $\mu$  or field, etc). This action actually *predicts* that in  $d = 4 - 2\epsilon$  the Weyl gauge covariant scalar curvature  $\hat{R}^\epsilon$  i.e. geometry/gravity acts as a UV regulator for the  $d = 4$  theory (as seen in particular in a leading order of its series expansion). In other words, the regularisation of the WDBI action in  $d = 4$  is trivial, just replace  $d = 4 \rightarrow d = 4 - 2\epsilon$ ; no need to introduce a UV regulator scale/field! This is a special feature of the WDBI action, not seen in other theories, showing that this action is more fundamental.

This special behaviour is not possible in Riemannian geometry where Weyl gauge covariance does not exist; in ordinary gauge theories a regulator (DR scale  $\mu$ , etc) is added by hand; also, in conformal gravity action a dilaton is added as regulator field, to preserve its symmetry in  $d = 4 - 2\epsilon$ . Even in string theory, local Weyl invariance (on Riemannian worldsheet, not in space-time as here) is actually broken by the added DR scale (the symmetry is restored by a condition of vanishing Ricci tensor; this may not be necessary if the worldsheet geometry is Weyl geometry).

Since the WDBI action has manifest Weyl gauge symmetry in  $d$  dimensions, there is no Weyl anomaly, so this action is a consistent (quantum) gauge theory. In the leading order of a series expansion (in  $\xi$ ) of the WDBI action, one recovers the Weyl gauge invariant action of SM and Weyl quadratic gravity, mentioned above; this theory undergoes a Stueckelberg mechanism in which the Weyl gauge boson  $\omega_\mu$  becomes massive and Weyl gauge symmetry is broken. After  $\omega_\mu$  decouples below Planck scale, Riemannian geometry is recovered in the broken phase, together with the Einstein-Hilbert gravity and SM action.

Regarding the sub-leading orders of the series expansion of WDBI action, these are operators suppressed by powers of dimensionless gravitational coupling ( $\xi$ ), with a structure that has some similarities to quantum corrections to the leading order action. In other words, the WDBI action may encode some quantum corrections. In the broken phase, these operators are higher dimensional operators suppressed by powers of  $M_p$ , familiar in the SM.

To conclude, the WDBI action is a general gauge theory of SM and gravity, mathematically well-defined and Weyl gauge invariant in  $d$  dimensions and thus Weyl anomaly-free. This is an interesting unified (quantum) description, by the gauge principle, of SM and gravity, that deserves further study.

## Appendix

### • Weyl geometry formulae in $d$ dimensions

The relations of Weyl geometry curvature tensors/scalar (with a hat) in the Weyl gauge covariant formulation, to their Riemannian geometry counterparts, are found by using their definitions in the text, see [15] (Appendix) and [6, 16]:

$$\begin{aligned}
\hat{R}_{\alpha\mu\nu\sigma} &= R_{\alpha\mu\nu\sigma} + \left\{ g_{\alpha\sigma} \nabla_\nu \omega_\mu - g_{\alpha\nu} \nabla_\sigma \omega_\mu - g_{\mu\sigma} \nabla_\nu \omega_\alpha + g_{\mu\nu} \nabla_\sigma \omega_\alpha \right\} \\
&\quad + \left\{ \omega^2 (g_{\alpha\sigma} g_{\mu\nu} - g_{\alpha\nu} g_{\mu\sigma}) + \omega_\alpha (\omega_\nu g_{\sigma\mu} - \omega_\sigma g_{\mu\nu}) + \omega_\mu (\omega_\sigma g_{\alpha\nu} - \omega_\nu g_{\alpha\sigma}) \right\} \\
\hat{R}_{\mu\sigma} &= R_{\mu\sigma} + \left[ \frac{1}{2} (d-2) F_{\mu\sigma} - (d-2) \nabla_{(\mu} \omega_{\sigma)} - g_{\mu\sigma} \nabla_\lambda \omega^\lambda \right] + (d-2) (\omega_\mu \omega_\sigma - g_{\mu\sigma} \omega_\lambda \omega^\lambda) \\
\hat{R} &= g^{\mu\sigma} \hat{R}_{\mu\sigma} = R - 2(d-1) \nabla_\mu \omega^\mu - (d-1)(d-2) \omega_\mu \omega^\mu.
\end{aligned} \tag{A-1}$$

with  $\hat{R}_{\alpha\mu\nu\sigma} = g_{\alpha\lambda} \hat{R}^\lambda_{\mu\nu\sigma}$ . Here  $R_{\alpha\mu\nu\sigma} = g_{\alpha\lambda} R^\lambda_{\mu\nu\sigma}$ ,  $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ ,  $R = g^{\mu\nu} R_{\mu\nu}$  are the Riemann and Ricci tensor and scalar of Riemannian geometry, respectively, in  $d$  dimensions. The rhs of these equations is in Riemannian notation, with  $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho$ , and  $\Gamma$  the Levi-Civita connection. Note  $\hat{R}_{\mu\nu} - \hat{R}_{\nu\mu} = (d-2) \hat{F}_{\mu\nu}$ , so  $\hat{R}_{\mu\nu}$  is not symmetric if  $d \neq 2$ . The field strength  $F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu = \hat{F}_{\mu\nu}$  has the same expression as in Weyl geometry.

One shows that in the Weyl gauge covariant formulation used in this work, the Weyl tensor  $\hat{C}^\mu_{\nu\rho\sigma}$  associated to the Riemann tensor of Weyl geometry ( $\hat{R}^\mu_{\nu\rho\sigma}$ ) is actually equal to its Riemannian counterpart ( $C^\mu_{\nu\rho\sigma}$ ) [15] (eq.A-25)

$$\hat{C}^\mu_{\nu\rho\sigma} = C^\mu_{\nu\rho\sigma}. \tag{A-2}$$

In the text we used the following identities of Weyl conformal geometry (in the "hat" notation) that are similar to those of Riemannian geometry, but in a Weyl gauge covariant form [15], [16]

$$\hat{G} = \hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\rho\sigma\mu\nu} - 4 \hat{R}_{\mu\nu} \hat{R}^{\nu\mu} + \hat{R}^2 \tag{A-3}$$

and

$$\hat{C}^2_{\mu\nu\rho\sigma} = \hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\rho\sigma\mu\nu} - \frac{4}{d-2} \hat{R}_{\mu\nu} \hat{R}^{\nu\mu} + \frac{2}{(d-1)(d-2)} \hat{R}^2, \tag{A-4}$$

giving

$$\hat{R}_{\mu\nu} \hat{R}^{\nu\mu} = \frac{d-2}{4(d-3)} (\hat{C}^2_{\mu\nu\rho\sigma} - \hat{G}) + \frac{d}{4(d-1)} \hat{R}^2. \tag{A-5}$$

The last equation is used in eq.(23) to replace the dependence on the Ricci tensor ( $\hat{R}_{\mu\nu}$ ) of Weyl geometry by that on the Weyl tensor of Weyl geometry in the covariant formulation ( $\hat{C}_{\mu\nu\rho\sigma}$ ), since this is identical to the Weyl tensor of Riemannian geometry,  $C_{\mu\nu\rho\sigma}$ .

• **Coefficients  $c_j$**

The coefficients  $c_j$  in action (24) have the following expressions in terms of  $a_j$ :

$$c_0 = \left[ a_0^2 + \frac{1}{2} a_1 a_0 + a_1^2 \frac{d-2}{16(d-1)} \right] a_0^{d/2-2} \quad (\text{A-6})$$

$$c_1 = -\frac{a_1^2 (d-2)}{16(d-3)} a_0^{d/2-2} \quad (\text{A-7})$$

$$c_2 = \frac{a_2}{4} \left[ a_2 + a_1 (d-2) + a_{10} f \right] a_0^{d/2-2} \quad (\text{A-8})$$

$$c_3 = \frac{a_3}{4} \left[ 2a_2 + a_1 (d-2) + a_{11} f \right] a_0^{d/2-2}, \quad (\text{A-9})$$

$$c_4^{(j)} = \left[ a_4^{(j)} f + \delta_{1j} \frac{a_3^2}{4} \right] a_0^{d/2-2}, \quad j = 1, 2, 3, \quad (\text{A-10})$$

$$c_k = \left[ a_k f + \frac{1}{8} d (d-2) \delta_{k7} a_6^2 \right] a_0^{d/2-2}, \quad k = 5, 6, \dots, 9. \quad (\text{A-11})$$

$$\text{with the notation:} \quad f = a_0 \frac{d}{2} + a_1 \frac{(d-2)}{4}. \quad (\text{A-12})$$

The physical couplings in (25) are related to  $c_j$  as seen by comparing actions (24) and (25)

$$\begin{aligned} c_0 &= \frac{1}{4! \xi^2}, & c_1 &= \frac{-1}{\eta^2}, & c_2 &= \frac{-1}{4\alpha^2}, & c_3 &= 0, & c_4^{(i)} &= \frac{-1}{4\alpha_i^2}, \quad (i = 1, 2, 3) \\ c_5 &= c_8 = c_9 = 1, & c_6 &= \frac{-\xi_H}{6}, & c_7 &= -\lambda, \end{aligned} \quad (\text{A-13})$$

with  $\alpha_i$  ( $i = 1, 2, 3$ ) the couplings of the SM and  $\alpha$  the Weyl gauge coupling of dilatations.

From (A-13) with (A-6) to (A-12) one finds the values of initial  $a_k$  that lead to physical couplings shown in action (25); these values are presented in eqs.(26) to (29).

• **Weyl invariance in strings**

While this is not important for our study, let us develop a remark at the end of section 3.3. Consider the string action below, with  $\sigma^\alpha$ ,  $g_{\alpha\beta}$  ( $\alpha, \beta = 1, 2$ ) as worldsheet coordinates and metric. The action has local Weyl invariance (no gauge field  $\omega_\mu$ ). In a standard notation

$$S_s = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X). \quad (\text{A-14})$$

At one-loop, in a DR scheme in  $d = 2 + \epsilon$ , a regularised  $S_s$  is found by replacing  $d^2\sigma \rightarrow d^{2+\epsilon}\sigma \mu^\epsilon$  in (A-14). The DR scale  $\mu$  ensures  $S_s$  is dimensionless, but the symmetry of  $S_s$  is broken, since  $\sqrt{g} g^{\alpha\beta}$  has now a non-zero Weyl charge  $d - 2 = \epsilon$ , see (1). Then the counterterms will also break it, via renormalized  $G_{\mu\nu}(X)$  which receives a correction  $\alpha' \mathcal{R}_{\mu\nu}(X) \ln \mu$ , with  $\mathcal{R}_{\mu\nu}(X)$  the Ricci tensor in target space. The symmetry is restored by a condition of vanishing beta function of  $G_{\mu\nu}(X)$ ,  $\alpha' \mathcal{R}_{\mu\nu}(X) = 0$  [31].

However, if the worldsheet geometry is Weyl geometry, a Weyl invariant regularised  $S_s$  exists, found by replacing in (A-14):  $d^2\sigma \rightarrow d^{2+\epsilon}\sigma \hat{R}^{\epsilon/2}$ . With  $\hat{R}$  as the worldsheet scalar curvature of Weyl charge  $-2$ , (eq.(5)), this regularised action (with no regulator scale added) is Weyl invariant in  $d = 2 + \epsilon$ . Thus there is no need to demand  $\alpha' \mathcal{R}_{\mu\nu}(X) = 0$  to maintain local Weyl symmetry in  $d$  dimensions. Note from (A-1) that for  $d = 2$ :  $\hat{R}_{\alpha\beta} - \frac{1}{2} \hat{R} g_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0$ , as in Riemannian case.

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