Invariant Theoretical Interpretation of Interaction

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Some systems of fields have been considered which are invariant under a certain group of transformations depending on n parameters. A general rule is obtained for introducing a new field in a definite way with a definite type of interaction with the original fields by postulating the invariance of these systems under a wider group derived by replacing the parameters of the original group with a set of arbitrary functions. The transformation character of this new field under the wider group is determined from the invariance postulate. The possible types of the equations of the new fields can be also derived, giving rise to a certain conservation law owing to the invariance. As examples, the electromagnetic, the gravitational and the Yang-Mills fields are reconsidered following this line of approach.

INTRODUCTION

THE form of the interactions between some well known fields can be determined by postulating invariance under a certain group of transformations. For example, let us consider the electromagnetic interaction of a charged field $Q(x)$, $Q^*(x)$. The electromagnetic interaction appears in the Lagrangian through the expressions

$$\frac{\partial Q}{\partial x^\mu}-ieA_\mu Q \quad \text{or} \quad \frac{\partial Q^*}{\partial x^\mu}+ieA_\mu Q^*. \tag{1}$$

The gauge invariance of this system is easily verified in virtue of the combinations of Q, Q^*, and A_μ in (1), if this system is invariant under the phase transformation

$$Q\to e^{i\alpha}Q, \quad Q^*\to Q^*e^{-i\alpha}, \quad \alpha=\text{const.} \tag{2}$$

Reversing the argument, the combination (1) can be uniquely introduced by the following line of reasoning. In the first place, let us suppose that the Lagrangian $L(Q,Q_{,\mu})$ is invariant under the constant phase transformation (2). Let us replace this phase transformation with the wider one (gauge transformation) having the phase factor $\alpha(x)$ instead of the constant α. In order to make the Lagrangian still invariant under this wider transformation it is necessary to introduce the electromagnetic field through the combination (1). This combination and the transformation character of A_μ under the gauge transformation can be uniquely determined from the gauge invariance postulate of the Lagrangian $L(Q,Q_{,\mu},A_\mu)$.

This approach was taken by Yang and Mills[1] to introduce their new field $\mathbf{B}_\mu$ which interacts with fields having nonvanishing isotopic spins. The gravitational interaction also can be introduced in this fashion.

It may be worthwhile to investigate this approach for a more general case, for if there is a system of fields $Q^A(x)$ which is invariant under some transformation group depending on parameters ϵ_1, ϵ_2, $\cdots\epsilon_n$, then according to the aforementioned viewpoint we may have the possibility of introducing a new field, say $A(x)$, in a definite way. In addition, the transformation character of this new field and the interaction form with the Q's can be determined uniquely.

Let us tentatively call a family of the interactions derived in this way "the interactions of the first class," while other types of interactions are denoted as "the interactions of the second class." The electromagnetic, gravitational and $\mathbf{B}_\mu$-field interactions belong to the first class and the meson-nucleon interaction to the second class, at least at the present stage.

The main purpose of the present paper is to investigate the following problem. Let us consider a system of fields $Q^A(x)$, which is invariant under some transformation group G depending on parameters ϵ_1, ϵ_2, $\cdots\epsilon_n$. Suppose that the aforementioned parameter-group G is replaced by a wider group G', derived by replacing the parameters ϵ's by a set of arbitrary functions $\epsilon(x)$'s, and that the system considered is invariant under this wider group G'. Then, can we answer the following questions by using only the postulate of invariance stated above? (1) What kind of field, $A(x)$, is introduced on account of the invariance? (2) How is this new field A transformed under G'? (3) What form does the interaction between the field A and the original field Q take? (4) How can we determine the new Lagrangian $L'(Q,A)$ from the original one $L(Q)$? (5) What type of field equations for A are allowable?

The solution of these problems will be stated in Sec. 1. In Secs. 2, 3, and 4 the well-known examples of the interactions of the first class will be reconsidered following the line of reasoning of Sec. 1. We shall find an analogy between the transformation characters of the electromagnetic field A_μ, the Yang-Mills field $\mathbf{B}_\mu$, and Christoffel's affinity $\Gamma_{\mu\nu}{}^\lambda$ in the theory of the general relativity. Furthermore we shall understand the reason why in the Yang-Mills field strength the quadratic term, $\mathbf{B}_\mu\times\mathbf{B}_\nu$, appears which is quite similar to that occurring in the Riemann-Christoffel tensor $R^\lambda{}_{\mu\nu\rho}$, namely, to the term $\Gamma\Gamma-\Gamma\Gamma$ in R.

In the usual textbooks of general relativity the covariant derivative of any tensor is introduced by

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[1] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

using the concept of parallel displacement. On the other hand, we shall see in Sec. 4 that the covariant derivative of any tensor or spinor can be derived from the postulate of invariance under the "generalized Lorentz transformations" derived by replacing the six parameters of the usual Lorentz group with a set of six arbitrary functions of x. In deriving such covariant derivatives it is unnecessary to use explicitly the notion of parallel displacement.

Now the above stated classification of the interactions has only a tentative meaning. Some of the interactions of the second class might be translated to the first class if we could find a transformation group by means of which we can derive that interaction following the general scheme in Sec. 1. For example, if the interaction between mesons and nucleons could be reinterpreted in a fashion analogous to those of the first class, then one might presumably be able to get a wider viewpoint for interpreting the interactions between the new unstable particles and the nucleons.

1. GENERAL THEORY

Let us consider a set of fields $Q^A(x)$, $(A=1, 2, \cdots N)$, with the Lagrangian density

$$L(Q^A, Q^A{}_{,\mu}), \quad Q^A{}_{,\mu}=\partial Q^A/\partial x^\mu.$$

Now let us postulate that the action integral referred to some arbitrary four-dimensional domain Ω,

$$I=\int_\Omega L d^4x,$$

is invariant under the following infinitesimal transformation:

$$\begin{aligned} &Q^A \rightarrow Q^A+\delta Q^A,\\ &\delta Q^A=T_{(a),}{}^A{}_B\,\epsilon^a\,Q_B,\\ &\quad \epsilon^a=\text{infinitesimal parameter}\ (a=1, 2, \cdots n),\\ &T_{(a),}{}^A{}_B=\text{constant coefficient}. \end{aligned} \tag{1.1}$$

In addition, the transformation (1.1) is assumed to be a Lie group G depending on the n parameters ϵ^a.

Thus there must be a set of constants $f_b{}^a{}_c$ called the "structure constants," which are defined by

$$\begin{aligned}[][T_{(a)}, T_{(b)}]^A{}_B &= T_{(a),}{}^A{}_C\,T_{(b)}{}^C{}_B - T_{(b),}{}^A{}_C\,T_{(a),}{}^C{}_B\\ &= f_a{}^c{}_b\,T_{(c),}{}^A{}_B. \end{aligned} \tag{1.2}$$

These constants, $f_b{}^a{}_c$, have the following important properties:

$$\begin{aligned} f_a{}^m{}_b\,f_m{}^l{}_c+f_b{}^m{}_c\,f_m{}^l{}_a+f_c{}^m{}_a\,f_m{}^l{}_b&=0,\\ f_a{}^c{}_b&=-f_b{}^c{}_a. \end{aligned} \tag{1.3}$$

The relations (1.3) can be easily obtained from Jacobi's identity and the definition (1.2).

Now from the invariant character of I under the transformation (1.1) and from the fact that this invariance is always preserved for an arbitrary domain Ω, we have the invariance of the Lagrangian density itself. Namely we have

$$\delta L \equiv \frac{\partial L}{\partial Q^A}\delta Q^A+\frac{\partial L}{\partial Q^A{}_{,\mu}}\delta Q^A{}_{,\mu}\equiv 0. \tag{1.4}$$

The symbol $\equiv$ means that δL must vanish at any world point and further that this relation does not depend on the behavior of Q^A and $Q^A{}_{,\mu}$. Substituting (1.1) into (1.4) we get

$$\frac{\partial L}{\partial Q^A}T_{(a),}{}^A{}_B\,Q^B+\frac{\partial L}{\partial Q^A{}_{,\mu}}T_{(a),}{}^A{}_B\,Q^B{}_{,\mu}\equiv 0, \quad (a=1, 2, \cdots n) \tag{1.5}$$

since the ϵ's are independent of each other. These n identities are the necessary and sufficient conditions for the invariance of I under G.

If we take into account the field equation for Q^A, we obtain from (1.5) the following n conservation laws:

$$\partial J^\mu{}_a/\partial x^\mu=0, \quad J^\mu{}_a=\frac{\partial L}{\partial Q^A{}_{,\mu}}T_{(a),}{}^A{}_B Q^B. \tag{1.6}$$

This is so because (1.5) can be rewritten as follows:

$$\left\{\frac{\partial L}{\partial Q^A}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial Q^A{}_{,\mu}}\right)\right\}\delta Q^A+\frac{\partial}{\partial x^\mu}\left[\frac{\partial L}{\partial Q^A{}_{,\mu}}\delta Q^A\right]\equiv 0.$$

The first term, $\{\quad\}\delta Q^A$, vanishes on account of the field equation.

Now let us consider the following transformation:

$$\begin{aligned} &\delta Q^A(x)=T_{(a),}{}^A{}_B\,\epsilon^a(x)\,Q^B,\\ &T_{(a),}{}^A{}_B=\text{constant},\\ &\quad \epsilon^a(x)=\text{infinitesimal arbitrary function}, \end{aligned} \tag{1.1$'$}$$

instead of (1.1). In this case δL does not vanish but becomes

$$\delta L\equiv\{(1.5)\}_a\,\epsilon^a(x)+\frac{\partial L}{\partial Q^A{}_{,\mu}}T_{(a),}{}^A{}_B\,Q^B\frac{\partial \epsilon^a}{\partial x^\mu},$$

or

$$\delta L\equiv\frac{\partial L}{\partial Q^A{}_{,\mu}}T_{(a),}{}^A{}_B\,Q^B\frac{\partial \epsilon^a}{\partial x^\mu}, \tag{1.5$'$}$$

by virtue of the identity (1.5).

In order to preserve the invariance of the Lagrangian under (1.1)$'$, it is necessary to introduce a new field

$$A'^J(x), \quad J=1, 2, \cdots M,$$

in such a way that the right-hand side of (1.5)$'$ can be cancelled with the contribution from this new field A'^J.

Now let us denote the new Lagrangian by

$$L'(Q^A, Q^A{}_{,\mu}, A'^J),$$

and consider the following transformation:

$$\delta Q^A = T_{(a),}{}^A{}_B\, Q^B\, \epsilon^a(x),$$
$$\delta A'^J = U_{(a)}{}^J{}_K\, A'^K\, \epsilon^a(x) + C^J{}_{,}{}^\mu{}_a \frac{\partial \epsilon^a}{\partial x^\mu}, \tag{1.7}$$

where the coefficients U and C are unknown constants which will be determined later. In addition, let us propose that the new action integral I' is invariant under the transformation (1.7).

Our problem is to answer the five questions listed in the Introduction.

From the invariance postulate we get the following identity:

$$\delta L'(x) \equiv \frac{\partial L'}{\partial Q^A}\delta Q^A + \frac{\partial L'}{\partial Q^A{}_{,\mu}}\delta Q^A{}_{,\mu} + \frac{\partial L'}{\partial A'^J}\delta A'^J \equiv 0.$$

Inserting (1.7) into the above and taking account of the arbitrariness of choosing ϵ^a and $\partial\epsilon^a/\partial x^\mu$, we see that each coefficient of ϵ and $\partial\epsilon/\partial x$ must vanish independently. Namely, we have the identities

$$\frac{\partial L'}{\partial Q^A} T_{(a),}{}^A{}_B\, Q^B + \frac{\partial L'}{\partial Q^A{}_{,\mu}} T_{(a),}{}^A{}_B\, Q^B{}_{,\mu} + \frac{\partial L'}{\partial A'^J} U_{(a)}{}^J{}_{K,}\, A'^K \equiv 0, \tag{1.8}$$

$$\frac{\partial L'}{\partial Q^A{}_{,\mu}} T_{(a),}{}^A{}_B\, Q^B + \frac{\partial L'}{\partial A'^J} C^J{}_{,}{}^\mu{}_a \equiv 0. \tag{1.9}$$

Now, in order to be able to determine uniquely the A'^J-dependence of L', the number of the components of A'^J should be equal to the number of Eqs. (1.9), namely,

$$M = 4n$$

should hold. In addition, the matrix $C^J{}_{,}{}^\mu{}_a$ must be nonsingular. Thus there is the inverse of C defined by

$$C^J{}_{,}{}^\mu{}_a\, C^{-1a}{}_{\mu,K} = \delta^J{}_K, \quad C^{-1a}{}_{\mu,J}\, C^J{}_{,}{}^\nu{}_b = \delta^a{}_b\, \delta^\nu{}_\mu.$$

Then (1.9) can be rewritten as

$$\frac{\partial L'}{\partial A^a{}_\mu} + \frac{\partial L'}{\partial Q^A{}_{,\mu}} T_{(a),}{}^A{}_B\, Q^B \equiv 0,$$

where we have put

$$A^a{}_\mu = C^{-1a}{}_{\mu,J}\, A'^J.$$

Thus A'^J should be contained in L' only through the combination

$$\nabla_\mu Q^A \equiv \frac{\partial Q^A}{\partial x^\mu} - T_{(a),}{}^A{}_B\, Q^B\, C^{-1a}{}_{\mu,J}\, A'^J,$$

or

$$\nabla_\mu Q^A \equiv \frac{\partial Q^A}{\partial x^\mu} - T_{(a),}{}^A{}_B\, Q^B\, A^a{}_\mu. \tag{1.10}$$

By using $A^a{}_\mu$ in place of A'^J, the transformation character of A turns into

$$\delta A^a{}_\mu = S_{(c)}{}^{a,}{}_{\mu,}{}^\nu{}_b\, A^b{}_\nu\, \epsilon^c(x) + \partial\epsilon^a/\partial x^\mu, \tag{1.11}$$

where

$$S_{(c)}{}^{a,}{}_{\mu,}{}^\nu{}_b = C^{-1a}{}_{\mu,J}\, U_{(c)}{}^J{}_K\, C^K{}_{,}{}^\nu{}_b.$$

Now the new Lagrangian must have the form

$$L'(Q^A, Q^A{}_{,\mu}, A^a{}_\mu) = L''(Q^A, \nabla_\mu Q^A).$$

Therefore we have the relations

$$\frac{\partial L'}{\partial Q^A} = \left.\frac{\partial L''}{\partial Q^A}\right|_{\nabla Q=\text{const}} - \left.\frac{\partial L''}{\partial \nabla_\mu Q^B}\right|_{Q=\text{const}} T_{(a),}{}^B{}_A\, A^a{}_\mu,$$

$$\frac{\partial L'}{\partial Q^A{}_{,\mu}} = \left.\frac{\partial L''}{\partial \nabla_\mu Q^A}\right|_{Q=\text{const}},$$

$$\frac{\partial L'}{\partial A'^J} = -\left.\frac{\partial L''}{\partial \nabla_\nu Q^A}\right|_{Q=\text{const}} T_{(b),}{}^A{}_B\, Q^B\, C^{-1b}{}_{\nu,J}.$$

By using these relations, (1.8) becomes

$$\left.\frac{\partial L''}{\partial Q^A}\right|_{\nabla Q=\text{const}} T_{(a),}{}^A{}_B\, Q^B + \left.\frac{\partial L''}{\partial \nabla_\mu Q^A}\right|_{Q=\text{const}} T_{(a),}{}^A{}_B\, \nabla_\mu Q^B + \left.\frac{\partial L''}{\partial \nabla_\mu Q^A}\right|_{Q=\text{const}} Q^B A^b{}_\nu \{[T_{(a)}\, T_{(b)}]^A{}_B\, \delta^\nu{}_\mu - S_{(a)}{}^{d,}{}_{\mu,}{}^\nu{}_b\, T_{(d),}{}^A{}_B\} \equiv 0. \tag{1.12}$$

If we put[2]

$$L''(Q^A, \nabla_\mu Q^A) = L(Q^A, \nabla_\mu Q^A),$$

namely, put L'' to be what is obtained by replacing $\partial Q^A/\partial x^\mu$ in the original Lagrangian L with the "covariant derivative," $\nabla_\mu Q^A$, then on account of the identity (1.5), the first and the second terms in (1.12) cancel each other. The remaining terms of (1.12) can be rewritten in the following way owing to the group character (1.2):

$$\left.\frac{\partial L}{\partial \nabla_\mu Q^A}\right|_{Q=\text{const}} Q^B A^b{}_\nu \{f_a{}^d{}_b\, \delta^\nu{}_\mu - S_{(a)}{}^{d,}{}_{\mu,}{}^\nu{}_b\} T_{(d),}{}^A{}_B \equiv 0.$$

Therefore we can determine the unknown coefficient S as follows:

$$S_{(a)}{}^{c,}{}_{\mu,}{}^\nu{}_b = \delta^\nu{}_\mu\, f_a{}^c{}_b. \tag{1.13}$$

Using this expression for S, we can easily show the covariant character of the derivative $\nabla_\mu Q^A$, i.e.,

$$\delta\nabla_\mu Q^A = T_{(a),}{}^A{}_B\, \epsilon^a(x)\, \nabla_\mu Q^B. \tag{1.14}$$

[2] This particular choice of L'' is due to the requirement that when the field A is assumed to vanish, we must have the original Lagrangian L.

Now let us investigate the possible type of the Lagrangian for the free A-field. Let it be denoted by

$$L_0(A^a{}_\mu, A^a{}_{\mu,\nu}), \quad A^a{}_{\mu,\nu}=\partial A^a{}_\mu/\partial x^\nu.$$

The invariance postulate for L_0 under the transformation (1.11) leads to

$$\frac{\partial L_0}{\partial A^a{}_\mu} f_b{}^a{}_c A^c{}_\mu + \frac{\partial L_0}{\partial A^a{}_{\mu,\nu}} f_b{}^a{}_c A^c{}_{\mu,\nu} \equiv 0, \tag{1.15}$$

$$\frac{\partial L_0}{\partial A^a{}_\mu} + \frac{\partial L_0}{\partial A^b{}_{\nu,\mu}} f_a{}^c{}_c A^c{}_\nu \equiv 0, \tag{1.16}$$

$$\frac{\partial L_0}{\partial A^a{}_{\mu,\nu}} + \frac{\partial L_0}{\partial A^a{}_{\nu,\mu}} \equiv 0. \tag{1.17}$$

From (1.17) we see that the derivative of A should be contained in L_0 through the combination

$$A^a{}_{[\mu,\nu]} \equiv \frac{\partial}{\partial x^\mu} A^a{}_\nu - \frac{\partial}{\partial x^\nu} A^a{}_\mu.$$

Thus (1.16) can be written as

$$\frac{\partial L_0}{\partial A^a{}_\mu} \equiv \frac{\partial L_0}{\partial A^c{}_{[\nu,\mu]}} f_a{}^c{}_b A^b{}_\nu. \tag{1.16$'$}$$

(1.16)$'$ means that the derivative of A appears in L_0, only through the particular combination

$$F^a{}_{\mu\nu} = \frac{\partial A^a{}_\nu}{\partial x^\mu} - \frac{\partial A^a{}_\mu}{\partial x^\nu} - \tfrac{1}{2} f_b{}^a{}_c (A^b{}_\mu A^c{}_\nu - A^b{}_\nu A^c{}_\mu). \tag{1.18}$$

Finally, substituting (1.16) into the first term of (1.15), we get

$$\frac{1}{2}\frac{\partial L_0}{\partial F^a{}_{\nu\mu}} \{ f_c{}^a{}_b A^b{}_{[\nu,\mu]} + \tfrac{1}{2}(f_d{}^a{}_b f_c{}^d{}_e - f_d{}^a{}_e f_c{}^d{}_b)(A^b{}_\nu A^e{}_\mu - A^b{}_\mu A^e{}_\nu)\} \equiv 0,$$

or by virtue of (1.3) we have

$$\frac{1}{2}\frac{\partial L_0}{\partial F^a{}_{\mu\nu}} f_c{}^a{}_b F^b{}_{\mu\nu} \equiv 0. \tag{1.19}$$

(See Appendix I.) Since L_0 must have the form

$$L_0(A, \partial A/\partial x) \equiv L_0'(A^a{}_\mu, F^a{}_{\mu\nu}),$$

we have the relations

$$\left.\frac{\partial L_0}{\partial A^a{}_{\nu,\mu}}\right|_{A=\mathrm{const}} = \left.\frac{\partial L_0'}{\partial F^a{}_{\mu\nu}}\right|_{A=\mathrm{const}},$$

$$\left.\frac{\partial L_0}{\partial A^a{}_\mu}\right|_{\partial A/\partial x=\mathrm{const}} = \left.\frac{\partial L_0'}{\partial A^a{}_\mu}\right|_{F=\mathrm{const}} - \left.\frac{\partial L_0'}{\partial F^b{}_{\mu\nu}}\right|_{A=\mathrm{const}} f_a{}^b{}_c A^c{}_\nu.$$

From these relations and (1.16), we have

$$\left.\frac{\partial L_0'}{\partial A^a{}_\mu}\right|_{F=\mathrm{const}} \equiv 0.$$

Namely L_0 must be a function of F alone and must satisfy the identity (1.19).

As may easily be seen, the transformation character of F is given by

$$\delta F^a{}_{\mu\nu} = \epsilon^b(x)\, f_b{}^a{}_c F^c{}_{\mu\nu}. \tag{1.20}$$

Equation (1.20) can be verified by using the relation (1.3).

Now let us define a set of matrices, $M_{(1)}$, $M_{(2)}$, $\cdots M_{(n)}$, in the following way:

(a,b)-element of $M_{(c)} \equiv M_{(c)}{}^a{}_b = f_c{}^a{}_b$,

$$(a,\, b,\, c=1,\, 2,\, \cdots n).$$

Then these matrices are a representation of degree n for the generators of the Lie group G, since the relation (1.3) can be written as

$$[M_{(a)}, M_{(b)}]^l{}_c = f_a{}^m{}_b M_{(m),}{}^l{}_c.$$

Therefore (1.20) shows that n quantities, $F^1{}_{\mu\nu}$, $F^2{}_{\mu\nu}$, $\cdots F^n{}_{\mu\nu}$, are transformed cogradiently to the transformation of Q.

So far we have not used the field equations of A and Q.

The variation of the total Lagrangian density

$$L_T = L_0(F) + L(Q, \nabla Q)$$

can be rewritten as

$$\frac{\delta L_T}{\delta Q^A}\delta Q^A + \frac{\delta L_T}{\delta A^a{}_\mu} f_b{}^a{}_c \epsilon^b A^c{}_\mu - \frac{\partial}{\partial x^\mu}\left(\frac{\delta L_T}{\delta A^a{}_\mu}\right)\epsilon^a$$
$$+\frac{\partial}{\partial x^\mu}\left\{\frac{\partial L}{\partial \nabla_\mu Q^A}\delta Q^A + \frac{\partial L_0}{\partial F^a{}_{\mu\nu}}\delta A^a{}_\nu + \frac{\delta L_T}{\delta A^a{}_\mu}\epsilon^a\right\} \equiv 0, \tag{1.21}$$

where the following abbreviations have been used:

$$\frac{\delta L_T}{\delta Q^A} = \frac{\partial L_T}{\partial Q^A} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L_T}{\partial Q^A{}_{,\mu}}\right), \quad \frac{\delta L_T}{\delta A^a{}_\mu} = \frac{\partial L_T}{\partial A^a{}_\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial L_T}{\partial A^a{}_{\mu,\nu}}\right).$$

Now let us choose the arbitrary function $\epsilon^a(x)$ in such a way that the values of all the ϵ's and $\partial\epsilon/\partial x$'s vanish on the boundary surface of the integration domain Ω. Then the integration of (1.21) over the domain Ω becomes

$$\int_\Omega K d^4x \equiv 0, \tag{1.22}$$

with the abbreviation

$$K = \frac{\delta L_T}{\delta Q^A}\delta Q^A + \frac{\delta L_T}{\delta A^a{}_\mu} f_b{}^a{}_c \epsilon^b A^c{}_\mu - \frac{\partial}{\partial x^\mu}\left(\frac{\delta L_T}{\delta A^a{}_\mu}\right)\epsilon^a,$$

because the integration of the divergence term in (1.21) vanishes on account of our special choice of the ϵ's. Since the ϵ's can be chosen arbitrarily within Ω, K must vanish at every point in Ω, as is easily seen from (1.22).

Consequently the identity (1.21) are separated into the following two relations:

$$K\equiv 0, \tag{1.23}$$

and

$$\frac{\partial}{\partial x^\mu}\left\{\frac{\partial L}{\partial\nabla_\mu Q^A}\delta Q^A+\frac{\partial L_0}{\partial F^a{}_{\mu\nu}}\delta A^a{}_\nu+\frac{\delta L_T}{\delta A^a{}_\mu}\epsilon^a\right\}\equiv 0. \tag{1.24}$$

From (1.24), we have

$$\frac{\partial}{\partial x^\mu}\left\{\frac{\partial L}{\partial\nabla_\mu Q^A}T_{(a),}{}^A{}_B Q^B+\frac{\partial L_0}{\partial F^b{}_{\mu\nu}}f_a{}^b{}_c A^c{}_\nu+\frac{\delta L_T}{\delta A^a{}_\mu}\right\}\equiv 0, \tag{1.25}$$

$$\frac{\partial L}{\partial\nabla_\mu Q^A}T_{(a),}{}^A{}_B Q^B+\frac{\partial L_0}{\partial F^b{}_{\mu\nu}}f_a{}^b{}_c A^c{}_\nu+\frac{\partial L_T}{\partial A^a{}_\mu}\equiv 0, \tag{1.26}$$

and

$$\frac{\partial L_0}{\partial F^a{}_{\mu\nu}}+\frac{\partial L_0}{\partial F^a{}_{\nu\mu}}\equiv 0.$$

Put

$$J^\mu{}_a=\partial L_T/\partial A^a{}_\mu. \tag{1.27}$$

Then (1.26) leads to

$$J^\mu{}_a\equiv-\left(\frac{\partial L}{\partial\nabla_\mu Q^A}T_{(a),}{}^A{}_B Q^B+\frac{\partial L_0}{\partial F^b{}_{\mu\nu}}f_a{}^b{}_c A^c{}_\nu\right), \tag{1.28}$$

and (1.25) becomes

$$\frac{\partial J^\mu{}_a}{\partial x^\mu}\equiv\frac{\partial}{\partial x^\mu}\left\{\frac{\delta L_T}{\delta A^a{}_\mu}\right\}. \tag{1.29}$$

If we use the field equation

$$\delta L_T/\delta A^a{}_\mu=0,$$

then we have the conservation of the "current," i.e.,

$$\partial J^\mu{}_a/\partial x^\mu=0 \quad (a=1,2,\cdots n). \tag{1.30}$$

Thus we have obtained a general rule for introducing a new field A in a definite way when there exists some conservation law such as (1.6) or there is a Lie group depending upon some parameters under which the system is invariant.

In the following sections we shall consider the following groups as examples of the original Lie group: (1) the phase transformation of a charged field, (2) the rotation group in the isotopic spin space, and (3) the Lorentz group.

2. PHASE TRANSFORMATION GROUP AND THE ELECTROMAGNETIC FIELD

Let us consider a charged field Q and Q^*. The Lagrangian of this system is assumed to be invariant under the phase transformation

$$\delta Q^A=i\alpha Q^A,\quad \delta Q^{A*}=-i\alpha Q^{A*},\quad \alpha=\text{a real constant}.$$

Since this one-parameter group is commutative, the structure constant, of course, vanishes. By replacing the constant α with a scalar function $\lambda(x)$, a vector field $A_\mu(x)$ is introduced. The transformation character of $A_\mu(x)$ is given by

$$\delta A_\mu=\partial\lambda/\partial x^\mu,$$

following the general formula (1.11). The new Lagrangian L' has the form

$$L'=L(Q,Q^*,\nabla_\mu Q,\nabla_\mu Q^*),$$

where $\nabla_\mu Q$ and $\nabla_\mu Q^*$ are given by

$$\nabla_\mu Q^A=\frac{\partial Q^A}{\partial x^\mu}-iA_\mu Q^A,\quad \nabla_\mu Q^{A*}=\frac{\partial Q^{A*}}{\partial x^\mu}+iA_\mu Q^{A*},$$

because in the present case

$$T^A{}_B=i\delta^A{}_B \quad\text{for}\quad Q^A,\quad T^A{}_B=-i\delta^A{}_B\quad\text{for}\quad Q^{A*}.$$

The Lagrangian L_0 for the free A_μ-field is

$$L_0=L_0(F_{\mu\nu}),$$

where

$$F_{\mu\nu}=\frac{\partial A_\nu}{\partial x^\mu}-\frac{\partial A_\mu}{\partial x^\nu}.$$

The current J^μ can be obtained from the two different expressions

$$J^\mu=\frac{\partial L_T}{\partial A_\mu}=-i\left(\frac{\partial L}{\partial\nabla_\mu Q^A}Q^A-\frac{\partial L}{\partial\nabla_\mu Q^{A*}}Q^{A*}\right).$$

3. ROTATION GROUP IN ISOTOPIC SPIN SPACE AND THE YANG-MILLS FIELD

As an example let us consider a system of proton and neutron fields;

$$\psi^\alpha=\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}\text{proton}\\ \text{neutron}\end{pmatrix}.$$

The Lagrangian in the charge-independent theory is invariant under the rotation in the three-dimensional isotopic spin space:

$$\delta\psi^\alpha=i\sum_{c=1}^{3}\epsilon^c\,\tau_{(c)}{}^\alpha{}_\beta\,\psi^\beta,\quad \delta\bar\psi_\alpha=-i\sum_{c=1}^{3}\epsilon^c\,\bar\psi_\beta\,\tau_{(c)}{}^\beta{}_\alpha, \tag{3.1}$$

where $\tau_{(1)}$, $\tau_{(2)}$, and $\tau_{(3)}$ are the usual isotopic spin matrices.

In this case the general notation T in Sec. 2 corresponds to τ as follows

$$T_{(c),}{}^{A}{}_{B}\rightarrow i\tau_{(c),}{}^{\alpha}{}_{\beta}, \quad \begin{pmatrix}\alpha,\beta=1,2\\ c=1,2,3\end{pmatrix}.$$

By replacing the parameters, ϵ^a, with a set of functions, $\epsilon^a(x)$, the Yang-Mills field

$$B^c{}_\mu(x) \quad (c=1,2,3)$$

is introduced, and this appears in the Lagrangian through the combination [see (1.10)]

$$\nabla_\mu\psi^\alpha=\partial\psi^\alpha/\partial x^\mu-i\tau_{(c),}{}^\alpha{}_\beta\,\psi^\beta\,B^c{}_\mu. \tag{3.2}$$

The variation of $B^c{}_\mu$ is given by [see (1.10) and (1.13)]

$$\delta B^c{}_\mu=f_a{}^c{}_b\,\epsilon^a(x)\,B^b{}_\mu+\frac{\partial\epsilon^c}{\partial x^\mu}, \tag{3.3}$$

where $f_a{}^c{}_b$ is defined by

$$[i\tau_{(a)},i\tau_{(b)}]=f_a{}^c{}_b\cdot i\tau_{(c)}. \tag{3.4}$$

The derivative of $B^a{}_\mu$ can appear only through the combination [see (1.18)]

$$F^a{}_{\mu\nu}=\frac{\partial B^a{}_\nu}{\partial x^\mu}-\frac{\partial B^a{}_\mu}{\partial x^\nu}-\frac{1}{2}f_b{}^a{}_c(B^b{}_\mu\,B^c{}_\nu-B^b{}_\nu\,B^c{}_\mu). \tag{3.5}$$

The variations of $\nabla_\mu\psi$ and $F^a{}_{\mu\nu}$ are as follows:

$$\delta\nabla_\mu\psi^\alpha=i\epsilon^c\,\tau_{(c),}{}^\alpha{}_\beta\,\nabla_\mu\psi^\beta$$

and

$$\delta F^a{}_{\mu\nu}=\epsilon^c\,f_c{}^a{}_b\,F^b{}_{\mu\nu}. \tag{3.6}$$

As was stated in Sec. 1, $F^a{}_{\mu\nu}$ is transformed under the rotation group as a vector, namely, the isotopic spin of this B-field is unity. The expression for the "current" has the form [see (1.25) and (1.24)]:

$$J^\mu{}_c=\frac{\partial L_T}{\partial B^c{}_\mu}=-i\frac{\partial L}{\partial\nabla_\mu\psi^\alpha}\tau_{(c),}{}^\alpha{}_\beta\,\psi^\beta-\frac{\partial L_0}{F^a{}_{\mu\nu}}f_c{}^a{}_b\,B^b{}_\nu.$$

4. LORENTZ GROUP AND THE GRAVITATIONAL FIELD

Let us consider a system of fields $Q^A(x)$ being defined with respect to some Lorentz frame. In addition, let us assume that the action integral

$$I=\int L(Q^A,Q^A{}_{,k})d^4x$$

is invariant under any Lorentz transformation.

Now besides the x-system, let us introduce an arbitrary system of the curvilinear coordinates u^μ ($\mu=1,2,3,4$). In what follows, the Latin and Greek indices represent quantities defined with respect to the x-system (or the local Lorentz frame) and to the u-system respectively.

The square of the invariant length of the infinitesimal line element is given by

$$ds^2=g^*{}_{ik}\,dx^i\,dx^k=g_{\mu\nu}\,du^\mu\,du^\nu,$$

where

$$g^*{}_{11}=g^*{}_{22}=g^*{}_{33}=-g^*{}_{44}=1, \quad g^*{}_{ik}=0 \quad \text{for} \quad i\neq k,$$

and

$$g_{\mu\nu}(u)=\frac{\partial x^i}{\partial u^\mu}\frac{\partial x^k}{\partial u^\nu}g^*{}_{ik}.$$

Let us introduce two sets of functions defined by

$$h^k{}_\mu(u)=\partial x^k/\partial u^\mu,$$

and

$$h_k{}^\mu(u)=\partial u^\mu/\partial x^k. \tag{4.1}$$

Then we have the following relations:

$$g^*{}_{kl}\,h^k{}_\mu\,h^l{}_\nu=g_{\mu\nu}(u), \quad g_{\mu\nu}\,h_k{}^\mu\,h_l{}^\nu=g^*{}_{kl}, \quad h_k{}^\mu\,h^l{}_\mu=\delta^l{}_k,$$

$$g^{kl*}\,h_k{}^\mu\,h_l{}^\nu=g^{\mu\nu}(u), \quad g^{\mu\nu}\,h^k{}_\mu\,h^l{}_\nu=g^{kl*}, \quad h_k{}^\mu\,h^k{}_\nu=\delta^\mu{}_\nu,$$

$$\det(g_{\mu\nu})=g=-h^2\equiv-[\det(h^k{}_\mu)]^2.$$

Raising or lowering of both kinds of suffices can be done by means of $g_{\mu\nu}$, $g^{\mu\nu}$ or $g^*{}_{kl}$ and g^{kl*}. The geometrical meaning of the sets of $h^k{}_\mu$ and $h_k{}^\mu$, is obvious. The introduction of the four-world vector[3] $h_1{}^\mu$, $h_2{}^\mu$, $h_3{}^\mu$, and $h_4{}^\mu$ assigns respectively a local Lorentz frame to every world point. Of course, the local frames at every world point are transformed in the same way under any Lorentz transformation, i.e.,

$$x^k\rightarrow x^k+\epsilon^k{}_l x^l,$$

$$h_k{}^\mu\rightarrow h_k{}^\mu+\delta h_k{}^\mu, \quad \epsilon^{kl}=-\epsilon^{lk}$$

$$\delta h_k{}^\mu=-\epsilon^l{}_k\,h_l{}^\mu.$$

On account of this geometrical meaning of the h's, we can transform the world tensor into the corresponding local tensor defined with respect to the local frame, or vice versa, using $h_k{}^\mu$ or $h^k{}_\mu$. For example,

$$Q^k(u)=h^k{}_\mu(u)\,Q^\mu(u), \quad Q^\mu(u)=h_k{}^\mu(u)\,Q^k(u),$$

where the abbreviation

$$Q^k(u)=Q^k\{x(u)\}$$

has been used.

In this way we can rewrite the action integral as follows:

$$I=\int \mathfrak{L}(Q^A(u),Q^A{}_{,\mu}(u),h^k{}_\mu(u))d^4u,$$

where $\mathfrak{L}$ is defined by

$$\mathfrak{L}=L(Q^A(u),h_k{}^\mu(u)\,Q^A{}_{,\mu}(u))\,h, \tag{4.2}$$

and $Q^A{}_{,\mu}$ stands for

$$\partial Q^A(u)/\partial u^\mu.$$

[3] The world vector means a vector which is defined with respect to the u-system.

The reason for the fact that the Q^A in (4.2) is not transformed into the corresponding world quantity is that if Q^A is a spinor this rewriting is not possible, because the spinor can be well defined only with respect to a Lorentz frame.

Now I is invariant under the following two kinds of transformations[4,5]:

(1) The Lorentz transformation

$$\begin{aligned} \delta h^k{}_\mu &= \epsilon^k{}_l\, h^l{}_\mu, \\ \delta Q^A &= \tfrac{1}{2} T_{(kl),}{}^A{}_B\, Q^B\, \epsilon^{kl}, \\ u^\mu &= \text{unchanged}, \end{aligned} \tag{4.3}$$

where $T_{(kl),}{}^A{}_B$ is the (A,B)-element of the $N\times N$ matrix $T_{(kl)}$ which is the representation of the generator of the Lorentz group. The matrix $T_{(kl)}$ satisfies the relation

$$[T_{(kl)},T_{(m,n)}]=\tfrac{1}{2} f_{kl,}{}^{ab}{}_{mn}\, T_{(ab)}, \quad T_{(kl)}=-T_{(lk)}.$$

(2) The general point transformation

$$\begin{aligned} &u^\mu \rightarrow u^\mu+\lambda^\mu(u)=u'^\mu, \\ &\lambda^\mu(u)=\text{an arbitrary function of } u, \\ &\delta h^k{}_\mu = -\frac{\partial \lambda^\nu}{\partial u^\mu} h^k{}_\nu, \\ &\delta Q^A(u)\equiv Q^{A\prime}(u')-Q^A(u)=0, \\ &\delta Q^A{}_{,\mu} = -\frac{\partial \lambda^\nu}{\partial u^\mu} Q^A{}_{,\nu}. \end{aligned} \tag{4.4}$$

Now our Lagrangian (4.2) has the suitable form for the application of the general method stated in I, if the given functions, $h^k{}_\mu$, are regarded as a set of field quantities satisfying the condition:

$$\partial h^k{}_\mu/\partial u^\nu = \partial h^k{}_\nu/\partial u^\mu, \tag{4.5}$$

and having the transformation character (4.3) under the Lorentz group. Though we will omit the condition (4.5), the invariance of I under the transformations (4.3) and (4.4) still holds. The only role of (4.5) is to guarantee the possibility of finding the simplest and most convenient system of coordinates $(x^1, \cdots, x^4)$. In fact if we replace the parameters, ϵ^{ik}, with a set of arbitrary functions, $\epsilon^{ik}(u)$, after the Lorentz transformation depending on such $\epsilon(u)$'s, the relation (4.5) is destroyed.

The condition (4.5) is inconsistent with the application of the general procedure of Sec. 1 to the present problem. Accordingly we shall consider the h's as a set of *16 independent given functions.*

Now following the prescription of Sec. 1, let us consider the "generalized Lorentz transformation" depending upon a set of arbitrary functions $\epsilon^{ik}(u)$ instead of the parameters ϵ^{ik}. Under this transformation, Q^A and $h^k{}_\mu$ are assumed to be transformed as

$$\begin{aligned} \delta Q^A &= \tfrac{1}{2}\epsilon^{kl}(u)\; T_{(kl),}{}^A{}_B\, Q^B, \\ \delta h^k{}_\mu &= \epsilon^k{}_l(u)\; h^l{}_\mu. \end{aligned} \tag{4.6}$$

Then, in order to retain the invariance of I under the transformation (4.6), it is necessary to introduce a new field

$$A^{kl}{}_\mu(u) = -A^{lk}{}_\mu(u),$$

which has the following transformation character according to (1.11):

$$\begin{aligned} \delta A^{kl}{}_\mu &= \tfrac{1}{4} f_{ab,}{}^{kl}{}_{hg}\, \epsilon^{ab}(u)\, A^{hg}{}_\mu + \frac{\partial \epsilon^{kl}}{\partial u^\mu} \\ &= \epsilon^k{}_m\, A^{ml}{}_\mu + \epsilon^l{}_m\, A^{km}{}_\mu + \frac{\partial \epsilon^{kl}}{\partial u^\mu}. \end{aligned} \tag{4.7}$$

Furthermore the new Lagrangian is given by

$$\mathfrak{L}(Q^A,\nabla_\mu Q^A,h^k{}_\mu) = hL\{Q^A,(h_k{}^\mu\, \nabla_\mu Q^A)\} \tag{4.8}$$

where[6]

$$\nabla_\mu Q^A = \frac{\partial Q^A}{\partial u^\mu} - \frac{1}{2} A^{kl}{}_\mu\, T_{(kl),}{}^A{}_B\, Q^B. \tag{4.9}$$

[see (1.10)].

The factors $\frac{1}{2}$ in (4.9) and $\frac{1}{4}$ in (4.7) are necessary because in summing up the terms in these expressions with respect to the dummy suffices the same contributions are counted twice or four times.

Because of the "general Lorentz transformation," under which each local frame at each world point is transformed differently, the relation (4.5) was abandoned. Since this relation is satisfied only when the basic world is flat, we are forced to take as our basic space-time some Riemannian space with the metric

$$g_{\mu\nu}(u) = h^k{}_\mu\, h_{k\nu},$$

and the affine connection

$$\Gamma_{\mu\nu}{}^\lambda = \tfrac{1}{2} g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\mu}}{\partial u^\nu} + \frac{\partial g_{\nu\sigma}}{\partial u^\mu} - \frac{\partial g_{\mu\nu}}{\partial u^\sigma}\right).$$

Accordingly we would expect that there exists some relationship between $A^{kl}{}_\mu$ and $h^k{}_\mu$.

In order to obtain this relationship let us consider, as an example, the local tensor

$$Q^{kl}(u)(=Q^A).$$

Then from (4.9) we have

$$\nabla_\mu Q^{kl} = \frac{\partial Q^{kl}}{\partial u^\mu} - A^{km}{}_\mu\, Q_m{}^l - A^{lm}{}_\mu\, Q^k{}_m.$$

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[4] R. Utiyama, Progr. Theoret. Phys. (Japan) 2, 38 (1947).
[5] L. Rosenfeld, Ann. Physik 5, 113 (1930).
[6] F. J. Belinfante, Physica 7, 305 (1940); K. Husimi, Proc. National Research Council of Japan 4, 81 (1943).

By using h, this can be rewritten as

$$\nabla_\mu Q^{k\nu}=\frac{\partial Q^{k\nu}}{\partial u^\mu}-A^{km}{}_\mu\, Q_m{}^\nu+\Gamma'_{\rho\mu}{}^\nu\, Q^{k\rho}, \qquad (4.10)$$

where $Q^{m\nu}$ and Γ' are defined by

$$Q^{k\nu}=h_m{}^\nu\, Q^{km},$$

and

$$\Gamma'_{\nu\mu}{}^\rho=h_l{}^\rho\frac{\partial h^l{}_\nu}{\partial u^\mu}-h_k{}^\rho\, h_{l\nu}\, A^{kl}{}_\mu. \qquad (4.11)$$

In general, the following relation is easily derived from (4.9):

$$\nabla_\mu Q^{kl\cdots,\rho\sigma\cdots}{}_{ab\cdots,\alpha\beta\cdots}=\frac{\partial}{\partial u^\mu}Q^{kl\cdots,\rho\sigma\cdots}{}_{ab\cdots,\alpha\beta\cdots}$$
$$-A^k{}_{i\mu}\, Q^{il\cdots,\rho\sigma\cdots}{}_{ab\cdots,\alpha\beta\cdots}-A^l{}_{i\mu}\, Q^{ki\cdots,\rho\sigma\cdots}{}_{ab\cdots,\alpha\beta\cdots}-\cdots$$
$$+A^i{}_{a\mu}\, Q^{kl\cdots,\rho\sigma\cdots}{}_{ib\cdots,\alpha\beta\cdots}+A^i{}_{b\mu}\, Q^{kl\cdots,\rho\sigma\cdots}{}_{ai\cdots,\alpha\beta\cdots}+\cdots$$
$$+\Gamma'_{\lambda\mu}{}^\rho\, Q^{kl\cdots,\lambda\sigma\cdots}{}_{ab\cdots,\alpha\beta\cdots}+\Gamma'_{\lambda\mu}{}^\sigma\, Q^{kl\cdots,\rho\lambda\cdots}{}_{ab\cdots,\alpha\beta\cdots}$$
$$+\cdots-\Gamma'_{\alpha\mu}{}^\lambda\, Q^{kl\cdots,\rho\sigma\cdots}{}_{ab\cdots,\lambda\beta\cdots}$$
$$-\Gamma'_{\beta\mu}{}^\lambda\, Q^{kl\cdots,\rho\sigma\cdots}{}_{ab\cdots,\alpha\lambda\cdots}-\cdots. \qquad (4.12)$$

This relation is nothing but the usual covariant derivative with the exception that for the Latin indices the $A^{kl}{}_{,\mu}$ must be inserted in place of the usual affinity Γ, and for the Greek indices our Γ' must be used instead of the Γ. Therefore for the world tensor $Q^{\mu\nu\cdots}$ our "covariant derivative" agrees with the usual covariant derivative with the affinity Γ'. Namely, if we use the symbol δ_μ for the usual covariant derivative, we get

$$\nabla_\mu Q^{\rho\nu}=\frac{\partial Q^{\rho\nu}}{\partial u^\mu}+\Gamma'_{\sigma\mu}{}^\rho\, Q^{\sigma\nu}+\Gamma'_{\sigma\mu}{}^\nu\, Q^{\rho\sigma}\equiv\delta_\mu\, Q^{\rho\nu}.$$

The relationship between A and h can be derived from the following consideration:

$$\nabla_\mu g^{kl*}=-A^{kl}{}_\mu-A^{lk}{}_\mu=0$$
$$=h^k{}_\rho\, h^l{}_\nu\, \nabla_\mu g^{\rho\nu}=h^k{}_\rho\, h^l{}_\nu\, \delta_\mu g^{\rho\nu}.$$

From this expression we have

$$\delta_\mu g^{\rho\nu}=\frac{\partial g^{\rho\nu}}{\partial u^\mu}+\Gamma'_{\sigma\mu}{}^\rho\, g^{\sigma\nu}+\Gamma'_{\sigma\mu}{}^\nu\, g^{\rho\sigma}=0. \qquad (4.13)$$

If we assume that[7]

$$\Gamma'_{\mu\nu}{}^\rho=\Gamma'_{\nu\mu}{}^\rho,$$

then we can solve (4.13) for Γ'. The solution is

$$\Gamma'_{\mu\nu}{}^\rho=\tfrac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\sigma\mu}}{\partial u^\nu}+\frac{\partial g_{\nu\sigma}}{\partial u^\mu}-\frac{\partial g_{\mu\nu}}{\partial u^\sigma}\right)\equiv\Gamma_{\mu\nu}{}^\rho,$$

or

$$h_l{}^\rho\frac{\partial h^l{}_\nu}{\partial u^\mu}-A^\rho{}_{\nu,\mu}=\Gamma_{\nu\mu}{}^\rho, \qquad (4.14)$$

where

$$A^\rho{}_{\nu,\mu}=h_k{}^\rho\, h_{l\nu}\, A^{kl}{}_{,\mu}.$$

(4.14) is just the relation desired.

Now from (4.14) we see that $A^\rho{}_{\nu,\mu}$ is a world tensor of the third rank under the general point transformation (4.4) because the inhomogeneous term

$$-\frac{\partial^2\lambda^\rho}{\partial u^\nu\partial u^\mu}$$

arising from $\delta\Gamma$ is cancelled out in virtue of the term $\delta(h\partial h/\partial u)$. Consequently the $A^{kl}{}_\mu$ is a covariant world vector under the u-transformation (4.4). Then it is easily seen that our "covariant derivative" ∇_μ is in fact covariant under both kinds of transformations, namely, under any "general Lorentz transformation" and any u-transformation (4.4).

Thus we have obtained the general expression for the covariant derivative without using the concept of parallel displacement. For example, if the Q^A were the spinor field ψ, we have

$$\nabla_\mu\psi=\frac{\partial\psi}{\partial x^\mu}-\frac{i}{4}A^{kl}{}_\mu\,[\gamma_k,\gamma_l]\psi,$$

where the γ_k's are the usual Dirac γ matrices.

Now let us consider the Lagragian $\mathfrak{L}_0$ of the free A-field:

$$\mathfrak{L}_0(h^k{}_\mu,A^{kl}{}_\mu,\partial A^{kl}{}_\mu/\partial u^\nu),$$

where the $h^k{}_\mu$ is necessary to raise or lower both kinds of tensor suffices.

From the invariance postulate for $\mathfrak{L}_0$ under the "general Lorentz transformation," we see that $\mathfrak{L}_0$ must have the form

$$\mathfrak{L}_0(h^k{}_\mu,F^{kl}{}_{\mu\nu}),$$

[7] In general (4.13) gives the following expression for Γ', if the symmetry $\Gamma'_{\mu\nu}{}^\lambda=\Gamma'_{\nu\mu}{}^\lambda$ is not assumed:

$$\Gamma'_{\mu\nu}{}^\rho=\Gamma_{\mu\nu}{}^\rho-(B^\rho{}_{\mu,\nu}+B^\rho{}_{\nu,\mu})+B_{\mu\nu,}{}^\rho,$$

where Γ is the Christoffel affinity, while B is an arbitrary tensor with the symmetry character $B_{\mu\nu,\rho}=-B_{\nu\mu,\rho}$. [See E. Schrödinger, *Space-time Structure* (Cambridge University Press, Cambridge, 1950), p. 66.] Therefore our new field $A^{kl}{}_\mu$ (or $A_{\rho\nu,\mu}=h_{k\rho}\, h_{l\nu}\, A^{kl}{}_\mu$) can be represented in the following way in terms of h, $\partial h/\partial u$ and B:

$$A_{\rho\nu,\mu}=h_{l\rho}\frac{\partial h^l{}_\nu}{\partial u^\mu}-\Gamma_{\nu\mu,\rho}+B_{\rho\nu,\mu}+B_{\rho\mu,\nu}-B_{\nu\mu,\rho}.$$

The right-hand side of the above expression is antisymmetric in ρ and ν because of the fact that

$$\left(h_{l\rho}\frac{\partial h^l{}_\nu}{\partial u^\mu}-\Gamma_{\nu\mu,\rho}\right)+(\nu\text{ and }\rho\text{ interchanged})=\frac{\partial g_{\rho\nu}}{\partial u^\mu}-\Gamma_{\nu\mu,\rho}-\Gamma_{\rho\mu,\nu}=0.$$

Therefore the antisymmetry of $A_{\rho\nu,\mu}$ in ρ and ν does not give any restriction to the symmetry character of B. Now if B is assumed to vanish, we obtain the relation (4.14). On the other hand, if the basic space-time is flat, then A takes the form $A_{\rho\nu,\mu}=B_{\rho\nu,\mu}+B_{\rho\mu,\nu}-B_{\nu\mu,\rho}$, on account of the relation $\partial h^l{}_\nu/\partial u^\mu=\partial h^l{}_\mu/\partial u^\nu$, or, equivalently, $h_{l\rho}\,\partial h^l{}_\nu/\partial u^\mu=\Gamma_{\nu\mu,\rho}$.

where F is defined by

$$F^{kl}{}_{\mu\nu}=\frac{\partial A^{kl}{}_{\nu}}{\partial u^{\mu}}-\frac{\partial A^{kl}{}_{\mu}}{\partial u^{\nu}}-\frac{1}{4}f_{ab,}{}^{kl}{}_{mn}(A^{ab}{}_{\mu}A^{mn}{}_{\nu}-A^{ab}{}_{\nu}A^{mn}{}_{\mu})$$

$$=\frac{\partial A^{kl}{}_{\nu}}{\partial u^{\mu}}-\frac{\partial A^{kl}{}_{\mu}}{\partial u^{\nu}}+A^{kb}{}_{\mu}A^{l}{}_{b\nu}-A^{kb}{}_{\nu}A^{l}{}_{b\mu}. \quad (4.15)$$

(4.15) can be rewritten formally as follows:

$$F^{kl}{}_{\mu\nu}=\nabla_{\mu}A^{kl}{}_{\nu}-\nabla_{\nu}A^{kl}{}_{\mu}-A^{kb}{}_{\mu}A^{l}{}_{b\nu}+A^{kb}{}_{\nu}A^{l}{}_{b\mu}, \quad (4.15)'$$

where $\nabla_{\mu}A^{kl}{}_{\nu}$ does not behave like a local tensor of the second rank, but is a covariant world tensor as the suffices μ and ν show. Using the expression (4.15)′ we can prove the following relation (see Appendix II)

$$F^{kl}{}_{\mu\nu}=h^{l\lambda}h^{k}{}_{\alpha}R^{\alpha}{}_{\lambda\mu\nu}, \quad (4.16)$$

where R is the Riemann-Christoffel curvature tensor:

$$R^{\alpha}{}_{\lambda\mu\nu}=\frac{\partial\Gamma_{\lambda\mu}{}^{\alpha}}{\partial u^{\nu}}-\frac{\partial\Gamma_{\lambda\nu}{}^{\alpha}}{\partial u^{\mu}}+\Gamma_{\lambda\mu}{}^{\beta}\Gamma_{\beta\nu}{}^{\alpha}-\Gamma_{\lambda\nu}{}^{\beta}\Gamma_{\beta\mu}{}^{\alpha}. \quad (4.17)$$

Though $h^{k}{}_{\mu}$ is contained in our Lagrangian as well as A and $\partial A/\partial u$, we can still prove that $A^{kl}{}_{\mu}$ appears in $\mathfrak{L}_0$ only through the combination F.

So far we have assumed that $h^{k}{}_{\mu}$ is a given function. The behavior of $h^{k}{}_{\mu}$ in general relativity is defined by the field equation derived by the variational principle.

The total Lagrangian density is now given by

$$\mathfrak{L}_T(Q^A,\nabla_\mu Q^A,h^k{}_\mu,\partial h^k{}_\mu/\partial u^\nu,\partial^2 h^k{}_\mu/\partial u^\nu\partial u^\lambda)$$
$$=\mathfrak{L}(Q^A,\nabla_\mu Q^A,h^k{}_\mu)+\mathfrak{L}_0(h^k{}_\mu,F^{kl}{}_{\mu\nu}).$$

The field equations for Q and h are[8]

$$\delta\mathfrak{L}/\delta Q^A=0$$

and

$$\frac{\delta\mathfrak{L}_T}{\delta h^i{}_\mu}=\frac{\partial\mathfrak{L}_T}{\partial h^i{}_\mu}-\frac{\partial}{\partial u^\nu}\left(\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\mu,\nu}}\right)+\frac{\partial^2}{\partial u^\nu\partial u^\lambda}\left(\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\mu,\nu\lambda}}\right)=0,$$

with the abbreviations

$$h^i{}_{\mu,\nu}=\partial h^i{}_\mu/\partial u^\nu,\quad h^i{}_{\mu,\nu\lambda}=\partial^2 h^i{}_\mu/\partial u^\nu\partial u^\lambda.$$

[8] If the $B_{\mu\nu,\rho}$ is taken into account, the Riemann-Christoffel tensor $R^{\rho}{}_{\lambda\mu\nu}$ is the same function as (4.17) but Γ' must be inserted in place of Γ in (4.17). Here Γ' is the affinity given in reference 7. In this case, in addition to the equations for the Q and h fields, we have the following equation:

$$\frac{\delta\mathfrak{L}_T}{\delta B_{\mu\nu,\lambda}}=\frac{1}{2}\frac{\partial\mathfrak{L}}{\partial A^{kl}{}_\rho}\frac{\partial A^{kl}{}_\rho}{\partial B_{\mu\nu,\lambda}}+\frac{1}{4}\frac{\partial\mathfrak{L}_0}{\partial F^{kl}{}_{\alpha\beta}}\frac{\partial F^{kl}{}_{\alpha\beta}}{\partial B_{\mu\nu,\lambda}}-\frac{1}{4}\frac{\partial}{\partial u^\rho}\left\{\frac{\partial\mathfrak{L}_0}{\partial F^{kl}{}_{\alpha\beta}}\frac{\partial F^{kl}{}_{\alpha\beta}}{\partial\left(\frac{\partial B_{\mu\nu,\lambda}}{\partial u^\rho}\right)}\right\}=0.$$

Of course, in all these equations for the Q, h, and B fields, the affinity Γ' must be used instead of Γ. (4.18) and (4.19) also hold in this case, since B is invariant under the "general Lorentz transformations." Consequently this B field is of no use in avoiding the trivial result: $\mathfrak{M}^{\mu}{}_{ik}\equiv 0$.

Now corresponding to (1.23) and (1.24) we have the following identities:

$$\frac{\delta\mathfrak{L}_T}{\delta Q^A}\delta Q^A+\frac{\delta\mathfrak{L}_T}{\delta h^i{}_\mu}\delta h^i{}_\mu\equiv 0, \quad (4.18)$$

and

$$(\partial/\partial u^\mu)\mathfrak{M}^\mu\equiv 0, \quad (4.19)$$

where $\mathfrak{M}^\mu$ is

$$\mathfrak{M}^\mu=\frac{\partial\mathfrak{L}}{\partial\nabla_\mu Q^A}\delta Q^A+\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\rho,\mu}}\delta h^i{}_\rho+\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\rho,\mu\nu}}\delta h^i{}_{\rho,\nu}-\frac{\partial}{\partial u^\nu}\left(\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\rho,\mu\nu}}\right)\cdot\delta h^i{}_\rho. \quad (4.20)$$

Accordingly, the coefficient of $\partial^2\epsilon/\partial u^2$ in (4.19) identically vanishes:

$$\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\rho,\mu\nu}}h_{k\rho}-\frac{\partial\mathfrak{L}_T}{\partial h^k{}_{\rho,\mu\nu}}h_{i\rho}\equiv 0.$$

Thus $\mathfrak{M}^\mu$ becomes

$$\mathfrak{M}^\mu=\tfrac{1}{2}\epsilon^{ik}\mathfrak{M}^\mu{}_{ik},$$

$$\mathfrak{M}^\mu{}_{ik}=\frac{\partial\mathfrak{L}}{\partial\nabla_\mu Q^A}T_{(ik),}{}^A{}_B\,Q^B+\left[\left\{\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\rho,\mu}}h_{k\rho}+\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\rho,\mu\nu}}h_{k\rho,\nu}-\frac{\partial}{\partial u^\nu}\left(\frac{\partial\mathfrak{L}_T}{\partial h^i{}_{\rho,\mu\nu}}\right)h_{k\rho}\right\}-\{k\text{ and }i\text{ interchanged}\}\right].$$

Inserting this expression into (4.19), we have the trivial result

$$\partial\mathfrak{M}^\mu{}_{ik}/\partial u^\mu\equiv 0, \quad (4.21)$$

as

$$\mathfrak{M}^\mu{}_{ik}\equiv 0.$$

Since the Eulerian derivative $\delta L/\delta A$ appeared in (1.24), the nonvanishing "current" could be derived in the general theory in Sec. 1. In the present case, however, we have no Eulerian derivatives in $\mathfrak{M}$. Thus the field equations do not play any role in deriving the "current."

Now the usual equation for the gravitational field is derived by taking a particular Lagrangian

$$\mathfrak{L}_0=hR,$$

where R is defined as follows:

$$R=g^{\mu\nu}R_{\mu\nu}=h_l{}^\mu h_k{}^\nu F^{kl}{}_{\mu\nu},\quad R_{\mu\nu}=R^\lambda{}_{\mu\nu\lambda}.$$

Taking the variation with regard to h, we get

$$\frac{\delta\mathfrak{L}_0}{\delta h^i{}_\mu}+\frac{\delta\mathfrak{L}}{\delta h^i{}_\mu}=0.$$

The following relation is now easily verified:

$$\frac{\delta\mathfrak{L}_0}{\delta h^i{}_\mu}\delta h^i{}_\mu=\frac{\delta\mathfrak{L}_0}{\delta g_{\rho\mu}}h_{i\rho}\,\delta h^i{}_\mu+\frac{\partial}{\partial w^\mu}\left\{\frac{\partial\mathfrak{L}_0}{\partial g_{\rho\sigma,\,\mu}}h_{i\rho}\,\delta h^i{}_\sigma\right\},$$

where[9]

$$\delta\mathfrak{L}_0/\delta g_{\rho\sigma}=-h(R^{\rho\sigma}-\tfrac{1}{2}g^{\rho\sigma}R)=-\mathfrak{G}^{\rho\sigma}.$$

Thus we have

$$h_{i\sigma}\,\mathfrak{G}^{\rho\sigma}=-\mathfrak{T}^\rho{}_i,$$

with

$$\mathfrak{T}^\rho{}_i=-\delta\mathfrak{L}/\delta h^i{}_\rho,$$

or we have

$$\mathfrak{G}^{\rho\sigma}=-\mathfrak{T}^{\rho\sigma}.$$

Here $\mathfrak{T}$ is the symmetric energy-momentum-tensor density of the original field Q. The symmetry character of $\mathfrak{T}$ can be proved in the following way.

Since the Lagrangian $\mathfrak{L}$ for the Q-field is also invariant under the "general Lorentz transformation," we have an identity similar to (4.18):

$$\frac{\delta\mathfrak{L}}{\delta Q^A}\delta Q^A+\frac{\delta\mathfrak{L}}{\delta h^i{}_\mu}\delta h^i{}_\mu\equiv 0.$$

Inserting the field equation for the Q-field into this identity, we get

$$\mathfrak{T}_{ik}=\frac{\delta\mathfrak{L}}{\delta h^i{}_\mu}h_{k\mu}=\frac{\delta\mathfrak{L}}{\delta h^k{}_\mu}h_{i\mu}=\mathfrak{T}_{ki}.$$

From this relation,

$$\mathfrak{T}^{\mu\rho}=h^{i\mu}\,h^{k\rho}\,\mathfrak{T}_{ik}=\mathfrak{T}^{\rho\mu}$$

can be easily derived.

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APPENDIX I. CONDITION (1.19)

Here we shall show how to construct an invariant in terms of $F^a{}_{\mu\nu}$.

Consider a quantity G_a, the transformation character of which is contragradient to that of $F^a{}_{\mu\nu}$ under the transformation (1.20).

Since

$$G_aF^a{}_{\mu\nu}$$

is invariant by definition, δG_a is given by

$$\delta G_a=-\epsilon^c\,f_c{}^b{}_a\,G_b.$$

For example,

$$K_{a,\,\mu\nu}=f_a{}^l{}_m\,f_l{}^m{}_b\,F^b{}_{\mu\nu}$$

is transformed contragradiently to $F^a{}_{\mu\nu}$, for $\delta K_{a,\,\mu\nu}$ is

$$\delta K_{a,\,\mu\nu}=f_a{}^l{}_m\,f_l{}^m{}_b\,\delta F^b{}_{\mu\nu}$$
$$=f_a{}^l{}_m\,f_l{}^m{}_b\,f_j{}^b{}_k\,F^k{}_{\mu\nu}\,\epsilon^j.$$

By taking into account the relation (1.3), the above expression becomes

$$\delta K_{a,\,\mu\nu}=f_a{}^l{}_m(f_k{}^b{}_l\,f_b{}^m{}_j+f_l{}^b{}_j\,f_b{}^m{}_k)\,F^k{}_{\mu\nu}\,\epsilon^j.$$

Using (1.3) again, we can rewrite the first term of this expression as follows:

$$\delta K_{a,\,\mu\nu}=f_k{}^b{}_l(f_j{}^m{}_a\,f_m{}^l{}_b+f_a{}^m{}_b\,f_m{}^l{}_j)\,F^k{}_{\mu\nu}\epsilon^j-f_k{}^m{}_b\,f_a{}^l{}_m\,f_l{}^b{}_j\,F^k{}_{\mu\nu}\,\epsilon^j.$$

Since the second term is cancelled with the last term, we have

$$\delta K_{a,\,\mu\nu}=-\epsilon^j\,f_j{}^m{}_a\,f_m{}^l{}_b\,f_l{}^b{}_k\,F^k{}_{\mu\nu}=-\epsilon^j\,f_j{}^m{}_a\,K_{m,\,\mu\nu}.$$

Now let us call $F^a{}_{\mu\nu}$ a contravariant vector, and $K_{a,\,\mu\nu}$ a covariant vector with regard to the transformation (1.20). In addition, let us propose that $f_b{}^a{}_c$ is contravariant with respect to the suffix a and covariant with respect to b and c. Then we see that $f_b{}^a{}_c$ is a constant and invariant tensor owing to the following fact that

$$\delta f_b{}^a{}_c=\epsilon^j(f_j{}^a{}_k\,f_b{}^k{}_c-f_j{}^k{}_b\,f_k{}^a{}_c-f_j{}^k{}_c\,f_b{}^a{}_k),$$

which vanishes in virtue of the relation (1.3). Hence this proposal concerning the transformation character of $f_b{}^a{}_c$ is compatible with the covariant character of $K_{a,\,\mu\nu}$.

Using the quantity

$$g_{ab}=f_a{}^l{}_m\,f_l{}^m{}_b=g_{ba}$$

and its inverse g^{ab}, we can easily construct a tensor algebra similar to that used in the theory of relativity. For example, we have invariants

$$H_{\mu\nu,\,\rho\sigma}=g_{ab}\,F^a{}_{\mu\nu}\,F^b{}_{\rho\sigma}=H_{\rho\sigma,\,\mu\nu}.$$

In the case of the rotation group in three-dimensional isotopic spin space (see Sec. 3), $f_b{}^a{}_c$ has the following values:

$$\begin{cases}f_1{}^3{}_2=f_3{}^2{}_1=f_2{}^1{}_3=-1,\\ f_i{}^l{}_k=-f_k{}^l{}_i,\\ \text{otherwise } f=0.\end{cases}$$

Therefore, we have

$$g_{ab}=2\delta_{ab},$$

and

$$H_{\mu\nu,\,\rho\sigma}=2\delta_{ab}\,F^a{}_{\mu\nu}\,F^b{}_{\rho\sigma}.$$

Another familiar example is the case of the Lorentz group. Here we have

$$\tfrac{1}{4}f_{jk,}{}^{ab}{}_{cd}\,f_{ab,}{}^{cd}{}_{lm}=g^*{}_{jl}\,g^*{}_{km}-g^*{}_{jm}\,g^*{}_{kl},$$

[9] W. Pauli, Encyklopaedie der Mathematischen Wissenschaften (B. G. Teubner, Leipzig, 1904–1922), Vol. 5, Chap. 19, p. 621.

and

$$H_{\mu\nu,\rho\sigma}=\frac{1}{2^4}F^{ik}{}_{\mu\nu}\, f_{jk,}{}^{ab}{}_{cd}\, f_{ab,}{}^{cd}{}_{lm}\, F^{lm}{}_{\rho\sigma}$$

$$=\tfrac{1}{2}F^{jk}{}_{\mu\nu}\, F^{lm}{}_{\rho\sigma}\, g^*{}_{jl}\, g^*{}_{km}=\tfrac{1}{2}F^{jk}{}_{\mu\nu}\, F_{jk,\rho\sigma}.$$

If L_0 is a function of the invariant $H_{\mu\nu,\beta\sigma}$ alone, we can easily prove the identity

$$\frac{\partial L_0}{\partial F^a{}_{\mu\nu}} f_c{}^a{}_b\, F^b{}_{\mu\nu}\equiv 0. \tag{1.19}$$

Namely, the left-hand side can be written as

$$\frac{1}{4}\frac{\partial L_0}{\partial H_{\rho\sigma,\alpha\beta}}\frac{\partial H_{\rho\sigma,\alpha\beta}}{\partial F^a{}_{\mu\nu}} f_c{}^a{}_b\, F^b{}_{\mu\nu}$$

$$=\frac{1}{4}\frac{\partial L_0}{\partial H_{\rho\sigma,\mu\nu}} F^d{}_{\rho\sigma}\, F^b{}_{\mu\nu}(g_{ad}\, f_c{}^a{}_b+g_{ab}\, f_c{}^a{}_d).$$

The factor in the bracket vanishes on account of the relation (1.3). Consequently there exists, in fact, a family of invariant Lagrangians, L_0, which are functions of $F^a{}_{\mu\nu}$ alone and satisfy the condition (1.19).

APPENDIX II. PROOF OF THE RELATION $F^{kl}{}_{\mu\nu}=h^{l\lambda}h^k{}_\alpha R^\alpha{}_{\lambda\mu\nu}$

$F^{kl}{}_{\mu\nu}$ is given by

$$F^{kl}{}_{\mu\nu}=\nabla_\mu A^{kl}{}_\nu-\nabla_\nu A^{kl}{}_\mu-A^{kb}{}_\mu\, A^l{}_{b\nu}+A^{kb}{}_\nu\, A^l{}_{b\mu}. \tag{A.1}$$

Now according to the general rule (4.12)

$$\nabla_\mu A^{kl}{}_\nu=h^k{}_\rho\, h^l{}_\sigma\, \delta_\mu A^{\rho\sigma}{}_\nu, \tag{A.2}$$

where δ_μ is the usual covariant derivative with the Christoffel affinity.

$A^{\rho\sigma}{}_{,\nu}$ is by virtue of (4.14) rewritten as

$$A^{\rho\sigma}{}_{,\nu}=g^{\sigma\lambda}\, A^\rho{}_{\lambda,\nu}$$

$$=g^{\sigma\lambda}\, h_l{}^\rho\left(\frac{\partial h^l{}_\lambda}{\partial u^\nu}-\Gamma_{\nu\lambda}{}^\alpha\, h^l{}_\alpha\right).$$

If we suppose $h^l{}_\lambda$ as a simple covariant world vector and ignore the local suffix l, then the factor in the parenthesis in the above expression is just the usual covariant derivative of $h^l{}_\lambda$. Therefore we get

$$A^{\rho\sigma}{}_{,\nu}=g^{\sigma\lambda}\, h_l{}^\rho\, \delta_\nu h^l{}_\lambda.$$

On the other hand from (4.14) we have the relation

$$\delta_\nu h^l{}_\lambda=A^l{}_{m,\nu}\, h^m{}_\lambda. \tag{A.3}$$

Thus we have

$$\delta_\mu A^{\rho\sigma}{}_{,\nu}=g^{\sigma\lambda}\, \delta_\mu(h_l{}^\rho\, \delta_\nu h^l{}_\lambda).$$

By using (A.3) this becomes

$$\delta_\mu A^{\rho\sigma}{}_{,\nu}=g^{\sigma\lambda}\, h_l{}^\rho\, \delta_\mu\delta_\nu h^l{}_\lambda-A^m{}_{l,\mu}\, A^l{}_{k,\nu}\, h_m{}^\rho\, h^{k\sigma}.$$

Inserting this expression into (A.1) with (A.2), we have

$$F^{kl}{}_{\mu\nu}=h^{l\lambda}(\delta_\mu\delta_\nu-\delta_\nu\delta_\mu)h^k{}_\lambda.$$

As is well known, the Riemann-Christoffel tensor is defined by

$$(\delta_\mu\delta_\nu-\delta_\nu\delta_\mu)V^\lambda=R^\rho{}_{\lambda\mu\nu}\, V_\rho$$

for an arbitrary covariant vector V_ρ.

Thus we get

$$F^{kl}{}_{\mu\nu}=h^{l\lambda}\, h^k{}_\alpha\, R^\alpha{}_{\lambda\mu\nu}.$$