The spin effect on the circular motion of massive bodies and the violation of the Equivalence Principle

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Recent interest on studying possible violations of the Equivalence Principle has led to the development of space satellite missions testing it for bodies moving on circular orbits around Earth. This experiment establishes that the validity of the Equivalence Principle and gravitational acceleration is independent of the composition of bodies. However, the internal dynamics of the bodies (such as spin) has not yet been considered. In this work, it is shown that the circular orbit motion of test bodies do present a violation of the Equivalence Principle when spin effects are not negligible. An exact solution for the circular motion of spinning massive bodies is found showing that the violation manifests itself through different tangential velocities of the test bodies, depending on the orientation of its spin with respect to the total angular momentum of the satellite. Besides, the test bodies present no tangential acceleration, and the Eötvös ratio for tangential accelerations is not a useful parameter to determine the Equivalence Principle validity. We introduce a parameter to determine the difference of tangential velocities, estimating it for the circular motion of a satellite orbiting Earth. It is found that violation of the Equivalence Principle due to spin–gravity coupling may, in principle, be measured within the capabilities of current satellite missions.

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Introduction.- The Equivalence Principle (EP) is one of the cornerstones of General Relativity. Among all the different possible ways in which it has been stated, one of its simplest form (called its weak form) establishes that all bodies fall with the same acceleration in a given gravitational field [1], which is based on Galileo's results implying the equivalence between gravitational and inertial masses. Another more precise form to enunciate it is that all bodies moving under the influence of gravitational forces only follow geodesics [1]. Although the EP only applies in a region of spacetime small enough to neglect the inhomogeneities of gravitational fields [1], the complete framework of General Relativity is based on it. Therefore, it is clear that any violation of the EP can bring enormous consequences to our understanding of the Universe and to the future of research in physics.

In order to determine experimentally the validity of the EP, some experiments have been carried out recently in different settings [2–5]. They consider the trajectories of massive composed falling bodies, measuring their accelerations, and determining whether (or not) the Eötvös ratio parameter that characterizes the falling is non-zero. The Eötvös parameter Δ measures the relative difference between accelerations for falling test bodies, and according to the EP, it should vanish. For instance, in Ref. [2], ^{87}Rb atoms were studied in a vertical free–falling configuration, where the cluster spin was vertically aligned, pointing either up or down. This experiment determined that $\Delta \sim 10^{-7}$, establishing that the experimental results were not in agreement with any of the considered theoretical models for spin-curvature and spin-torsion couplings of Refs. [6–8]. In order to match the experimental conclusions of Ref. [2], a Lagrangian model for spin–gravity interaction was studied in Ref. [9]. It was shown that this Lagrangian theory [9–12] for spinning massive particles (tops) exactly predicts the results of ^{87}Rb atom experiment [2] for vertically free–fallling tops. Furthermore, in Ref. [2], a different and more concrete experimental setting was proposed for tops moving parabolically, where the EP is manifestly violated and where a measurment could be possibly performed. The main reason why the Lagrangian theory for tops coincides with the experimental results of [2] is because it predicts non–geodesics behavior of tops [10–12]. Spin introduces (tidal) forces that, in general, deviate any massive free–falling spinning body from a geodesic. However, in particular, it can rigourously shown that in vertical free–falling trajectory with aligned spins (to the trayectory), this force vanishes and the top does follow a geodesic [9]. This Lagrangian theory has been used to study tops in different contexts and gravitational fields [13–21], always finding new effects on the dynamics associated to the non–geodesics motion of tops due to its spin–gravity coupling.

This previous success in the agreement of the results which stem from Lagrangian theory for tops with experiment, leads to wonder what experimental settings can be appropriated to measure deviations of EP. It is the purpose of this work to study the possible modifications which could be implemented in the MICROSCOPE satellite (MS) mission [3] experiments, in order to test the

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validity of the Lagrangian theory for tops.

The MS mission performed experiments with the aim of measuring the forces required to maintain two cylindrical test massive bodies in the same circular orbit around Earth. Masses made by equal and different compositions showed no difference on their trajectory behavior, finding an Eötvös ratio of the order $\Delta \sim 10^{-15}$. The MS mission was focused in determining if the atomic composition of massive bodies can produce any violation of EP, and their findings have a strong indication that it does not. However, according to the Lagrangian theory for tops, it is spin, and not body composition, which can induce such violation to EP. Thus, any experimental setting should also consider the angular momenta of test bodies in order to prove the validity of EP.

In the following letter, we will solve the equations of motion for tops in circular orbits, and show how the spin induces a breaking of the EP through the non–geodesic motion of the test bodies. We use those results, first to show how the Lagrangian theory agrees with the results of the MS mission when spin is neglected, and secondly, to suggest a simple modifications on future experiments, estimating that a violation of the EP can be measured within similar technological and statistical capabilities to those of the MS mission.

Theory for tops.- The Lagrangian model for spinning particles considers tops with mass m, spin J, energy E and total angular momentum j. The full theory is developed in Refs. [9–15], and we limit ourselves here to highlight its most relevant results. It is well-known that the velocity u^μ of a spinning particle is not parallel, in general, to the canonical momentum vector P^μ. The velocity vector may, under some circumstances, become spacelike [10–12]. However, the momentum vector remains always timelike and gives rise to the dynamical conservation law of mass $m^2 \equiv P^\mu P_\mu > 0$ [11, 15]. The spin of tops is defined through an antisymmetric tensor $S^{\mu\nu}$ (see below). The action $S = \int L\, d\lambda$ associated to the Lagrangian theory for tops is λ–reparametrization invariant, where the Lagrangian $L(a_1, a_2, a_3, a_4) = (a_1)^{1/2}\mathcal{L}\left(a_2/a_1, a_3/(a_1)^2, a_4/(a_1)^2\right)$ is an arbitrary function of four invariants a_1, a_2, a_3, a_4, and $\mathcal{L}$ is an arbitrary function of $a_1 \equiv u^\mu u_\mu$, $a_2 \equiv \sigma^{\mu\nu}\sigma_{\mu\nu} = -\mathrm{tr}(\sigma^2)$, $a_3 \equiv u_\alpha \sigma^{\alpha\beta}\sigma_{\beta\gamma}u^\gamma$, $a_4 \equiv \det(\sigma)$ [9, 11, 15], where u^μ and $\sigma^{\mu\nu}$ are the top's velocity and angular velocity respectively defined in terms of derivatives with respect to the arbitrary parameter λ (see Refs. [9–15]). The momentum vector P_μ and the antisymmetric spin tensor $S_{\mu\nu}$ are canonically conjugated to the position and orientation of the top, $P_\mu \equiv \partial L/\partial u^\mu$ and $S_{\mu\nu} \equiv \partial L/\partial \sigma^{\mu\nu} = -S_{\nu\mu}$. Explicit examples of such Lagrangians can be found in Refs. [10, 15]. Without this Lagrangian formulation the canonical momentum and the spin tensor cannot be appropriately defined. In this way, it is found that the dynamics of a top describes a non–geodesic behavior, seen through the momentum equation [11, 14, 15]

$$\frac{DP^\mu}{D\lambda} \equiv \dot{P}^\mu + \Gamma^\mu_{\alpha\beta}P^\alpha u^\beta = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}u^\nu S^{\alpha\beta}\,, \quad (1)$$

and the equation for the spin tensor

$$\frac{DS^{\mu\nu}}{D\lambda} \equiv \dot{S}^{\mu\nu} + \Gamma^\mu_{\alpha\beta}S^{\alpha\nu}u^\beta + \Gamma^\nu_{\alpha\beta}S^{\mu\alpha}u^\beta = P^\mu u^\nu - u^\mu P^\nu\,, \quad (2)$$

The overdot represents the derivative with respect to an arbitrary parameter (λ), in such a way that velocity $u^\mu = \dot{x}^\mu$ is the derivative of coordinates. In addition, $\Gamma^\nu{}_{\rho\tau}$ are the Christoffel symbols for the metric field $g_{\mu\nu}$ (the speed of light is set equal to 1). The six independent components of the antisymmetric spin tensor generate Lorentz transformations, and in order to restrict them to generate three dimensional rotations we impose the Tulczyjew constraint $S^{\mu\nu}P_\nu = 0$ [10, 11, 22, 23]. This constraint has been shown to be important in the consistency of a theory for spinning massive particles [23], as it can be deduced as a constraint which emerges from the Lagrangian of the theory, and not an external imposition on the top dynamics [10]. Lastly, in this theory, the (square) top spin $J^2 \equiv \frac{1}{2}S^{\mu\nu}S_{\mu\nu}$ can be shown to be a conserved quantity [11–15].

The non–geodesic behavior of a top moving on a background gravitational field is determined by Eqs. (1) and (2), plus the constraint. As a result, the top can be interpreted as an extended object that feels tidal forces due to gravity. Spin gives internal structure to massive particles, and they cannot be longer described as pointlike objects. Due to the fact that any extended object is crossed by infinitely many geodesics (only a pointlike object is crossed by just one geodesic) the averaged motion does not align with any of the constituent geodesics, and the motion is, in general, non–geodesic. Similar effects have been studied for fields (which are naturally extended objects)[24] and electromagnetic waves [25, 26]. Thus, one should expect that the inclusion of spin in the dynamics of massive particles should lead to weak EP violation.

Circular motion solution.- Several different general and exact solutions of the Lagrangian theory for tops, and the above equations, have been found in Refs. [9, 11–19]. Here we present only the key steps to obtain the solution for a circular motion of the top with spin perpendicular to the plane of motion. We refer to readers to those references for a full and detailed procedure to get the solutions for the equations of motion derived from the Lagrangian theory.

Assuming a Schwarzschild field background (describing approximatelly the Earth gravitational field), the equatorial motion of a top can be solved exactly, as any equatorial plane can be defined for circular motion to take place. We write the metric in spherical coordinates $g_{tt} = 1 - 2r_0/r$, $g_{rr} = -\left(1 - 2r_0/r\right)^{-1}$, $g_{\theta\theta} = -r^2$, $g_{\phi\phi} = -r^2\sin^2\theta$, where $r_0 = GM$ with the gravitational constant G and the Earth mass M. The circular motion is defined as such by $\dot{r} = 0$. Besides, without any loss of generality, we can study the the motion in the plane defined by $\theta = \pi/2$. If the top is initially in that plane and $\dot{\theta} = 0$, then it remains in that equatorial plane, where $P^\theta = 0$ [9, 11, 15]. In this solution, spin can be chosen to be orthogonal to the equatorial plane $S^{r\theta} = S^{\theta\phi} = S^{0\theta} = 0$ [9, 11, 15], being parallel or antiparallel to the angular momentum of the top along the whole trajectory. In Refs. [9, 11, 15] is shown that the general solutions for the momenta equations (1) are $P_\phi = (-j \pm EJ/m)/(1-\eta)$, and $P_t = [E \mp jJr_0/(mr^3)]/(1-\eta)$, with the dimensionless parameter $\eta = J^2 r_0/(m^2 r^3)$. Here, the $\pm$ stands for two trajectories that depend on the spin orientation, parallel or antiparallel to the total angular momentum of the top, both of them remaining perpendicular to the plane of motion. These two momenta are conserved ($\dot{P}_t = 0$ and $\dot{P}_\phi = 0$) for circular motion [9, 15], and thus, the Eötvös ratio is meaningless for this particular orbit. As we will see, the EP violation appears in changes of the velocity. From the constant of motion $P_\mu P^\mu = m^2$, we get that $P^r = 0 = \left[P_t^2 - \left(P_\phi^2/r^2 + m^2\right)(1 - 2r_0/r)\right]^{1/2}$, in consistency with the circular motion solution, and the relation between the radial momentum and the radial velocity $\dot{r} = (1 - 2r_0/r)\,(P^r/P_t) = 0$, given by solutions of the Lagrangian theory for tops [9, 11, 15]. This constraint determines the energy of each trajectory

$$\frac{\left(E_\pm \mp \frac{jJr_0}{mr^3}\right)^2}{(1-\eta)^2} = \left(1 - \frac{2r_0}{r}\right)\left[m^2 + \frac{\left(-j \pm \frac{E_\pm J}{m}\right)^2}{r^2(1-\eta)^2}\right], \tag{3}$$

of a top moving on a circular orbit of radius r. Furthermore, the non–trivial spin evolution equations (2) relevant to the circular motion in the plane $\theta = \pi/2$, reduce to $DS^{tr}/D\lambda = 0$ and $DS^{t\phi}/D\lambda = P^t\dot{\phi} - P^\phi$ [9, 11, 15]. These equations, together with the relations $S^{tr} = -S^{\phi r}P_\phi/P_t$ and $(S^{\phi r})^2 = J^2\,(P_t)^2/(m^2 r^2)$ that can derived from the two constraints and the Tulczyjew condition [9, 11, 15], allow us to get the angular velocity $\dot{\phi}_\pm$ for the two possible trajectories of this motion [9, 11, 15]

$$\dot{\phi}_\pm = \frac{1}{r^2}\left(1 - \frac{2r_0}{r}\right)\left(\frac{2\eta+1}{\eta-1}\right)\left(\frac{-j \pm E_\pm J/m}{E_\pm \mp jJr_0/(mr^3)}\right), \tag{4}$$

where the energy E is given by solutions of (3). Tops can have two tangential velocities $r\dot{\phi}_\pm$, according to Eq. (4). The interplay of its spin with gravity, introduces corrections in this tangential velocity, which depends on the spin orientation.

For an experiment such as the MS mission, a possible measurement of a maximal violation of the EP can be achieved if two test bodies (with equal composition) are set to rotate in order to have opposite (internal) angular momenta directions, parallel and antiparallel to the total angular momentum of the circular motion of the satellite. In such cases, any deviation of the EP must be reflected in different measurements of the angular velocities of the test bodies. Thus, the dimensionless ratio

$$\begin{aligned}\delta &= \frac{\dot{\phi}_- - \dot{\phi}_+}{\dot{\phi}_- + \dot{\phi}_+} \\ &= \frac{\left[-j + \frac{E_+ J}{m}\right]\left[E_- + \frac{jJr_0}{mr^3}\right] + \left[j + \frac{E_- J}{m}\right]\left[E_+ - \frac{jJr_0}{mr^3}\right]}{\left[j - \frac{E_+ J}{m}\right]\left[E_- + \frac{jJr_0}{mr^3}\right] + \left[j + \frac{E_- J}{m}\right]\left[E_+ - \frac{jJr_0}{mr^3}\right]},\end{aligned} \tag{5}$$

is non–zero in case of an EP violation.

Notice that for these circular motions described by solution (4), tops do not present tangential accelerations $\ddot{\phi} = 0$, as the radius remains constant. Therefore, there is no relative acceleration between the two test bodies. Furthermore, when spin is neglected $J = 0$ ($\eta = 0$), a massive particle can only have a one angular velocity $\dot{\phi} = j/(r^2 E)$ and one energy, yielding the usual result $\delta = 0$ for geodesic motion in the Schwarzschild field [27]. The approximately vanishing Eötvös ratio and $\delta = 0$ are the results measured in the MS mission [3], which is in agreement with the Lagrangian theory for tops.

Estimations of the trajectory deviations.- The inclusion of spin into the the test bodies is essential for experiments carried out to demonstrate the validity of the EP. Let us assume two tops in an experiment orbiting near the Earth surface in a circular motion. For the sake of simplicity, let us consider an orbit with $r \sim 7 \times 10^6$[m] $\gg r_0 \sim 4.4 \times 10^{-3}$[m], and the total angular momentum of the satellite has the form $j \sim mrv$, where v is the satellite's velocity much smaller than the speed of light. The satellite's velocity can written as $v = r\Omega$, where Ω is the angular frequency of the satellite's trajectory. This angular frequency is related to the trajectory radius by Kepler's law $\Omega^2 = GM/r^3$, where G is the universal gravitational constant and M is the Earth mass. Therefore, the total angular momentum can be written as $j \sim m\sqrt{GMr} = m\sqrt{r_0 r}$. Additionally, estimate the intrinsic spin of each top as $J \sim md^2\omega$, where d is a characteristic length of the experimental test top body and ω is its internal angular frequency of rotation. Let us assume that our setting allows to have an small intrinsic spin compared to the total angular momentum of the satellite, $j \gg J$. In this way, we can neglect $\eta \ll 1$, being able to find that $P_{\phi\pm} \approx -mr^2\Omega \pm J(1+v^2/2-r_0/r)$, and $P_{t\pm} \approx E_\pm \approx m\left(1 + v^2/2 - r_0/r\right) \mp \Omega J$. This allows us to estimate the tangential satellite's speed as

$$\dot{\phi}_\pm \approx \frac{v}{r}\left(1 - \frac{v^2}{2} - \frac{r_0}{r}\right) \mp \frac{J}{mr^2}\,. \tag{6}$$

The top's spin induce different tangential velocities, such that the top with antiparallel spin is faster than the one with the parallel spin to the total angular momentum, $\dot{\phi}_- > \dot{\phi}_+$. Thus, this correction to the circular trajectory of two massive test objects with different spin orientations can be estimated through the ratio (5) to be simply

$$\delta \approx \frac{J}{mvr} = \frac{d^2\omega}{r^2\Omega} = \frac{d^2\omega}{\sqrt{GMr}}\,. \tag{7}$$

The approximated δ–ratio (7) depends on the the satellite parameters Ω and r, and on the experimental test tops desing ω and d. Assuming that the test tops have dimensions of the order of $d \sim 10^{-2}$[m], then the ratio (7) gives the relation

$$\delta \approx 1.9 \times 10^{-15}\ \omega\,. \tag{8}$$

where the internal angular frequency of tops is measured in [Hz]. This implies that each test top can rotate, in opposite direction, as slow as $\omega \sim 10$[Hz] for an EP violation to be detected within the current MS mission experimental measurement capabilities [3]. For a higher intrinsic angular velocity, a larger violation of the EP can be achieved.

The above result shows that it is the spin (internal dynamics) and not the composition, size or shape of the test bodies, the key for detecting a possible EP violation. As the MS results indicate [3], different body composition does not represent a violation of the EP.

Conclusions.- The main purpose of this work is to bring attention to notion of EP, and the conditions required for its violation. As it was presented above, any natural object with physical properties that cannot be described using a pointlike formalism (which presents features such as spin) will not follow geodesics, and therefore the EP cannot be applied to them.

In the simplest case of a circular motion for a spinning massive particle, the complete dynamics can be solved exactly, and it describes a non-geodesics motion whose deviations depend on the magnitude of the particle's spin. Notice that the solution presented above has constant momenta, and thus, the EP cannot be detected by measuring the Eötvös ratio. The breaking of the EP is manifested only in the change of the velocities of test bodies. Of course, this is not a general rule, and more complicated motions in different spacetimes may indeed present non–vanishing Eötvös ratios [9, 15, 17].

Any experimental setup designed to measured the EP must be constructed in order to capture the spin-gravity coupling and its effects. In particular, for experiments around Earth (as the one of the MS mission) the δ–ratio has enough freedom to adjust the parameters of the orbit radius of the satellite and the characteristic length and inner angular velocity of test body, depending on the accuracy of the setup.

As the δ-ratio show in Eqs. (5) and (7), when spin is negligible, geodesic motion is expected. Therefore, it comes as no surprise if the EP is repeatedly confirmed for pointlike test bodies.

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