Spinors and gravity without Lorentz indices

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Abstract

Coupling spinor fields to the gravitational field, in the setting of general relativity, is standardly done via the introduction of a vierbein field and the (associated minimal) spin connection field. This makes three types of indices feature in the formalism: world/coordinate indices, Lorentz vector indices, and Lorentz spinor indices, respectively. This article will show, though, that it is possible to dispense altogther with the Lorentz indices, both tensorial ones and spinorial ones, obtaining a formalism featuring only world indices. This will be possible by having both the 'Dirac operator' and the generators of 'Lorentz' transformations become spacetime-dependent, although covariantly constant. The formalism is developed in the setting of complexified quaternions.

1 Introduction

According to standard wisdom, see for instance [1, Sec. 31.A] or [2, Sec. 12.1], spinor fields can be coupled to the gravitational field, in the setting of general relativity, only via the introduction of fields carrying Lorentz vector indices (in excess of world indices), more specifically, the vierbein field ${e^\mu}_a$, and the spin connection field ${\omega_\mu}^{ab}$. In this way, the resulting formalism ends up featuring, somewhat unsatisfactorily, three different types of indices: world indices, Lorentz vector indices, and Lorentz spinor indices, respectively, the latter of course being carried by the spinor field itself.

This article will present a formalism, though, using world indices only, i.e., a formalism in which no Lorentz indices feature, neither tensorial ones nor spinorial ones. Perhaps surprisingly, it will prove possible to have the spinor field carry a world index (transforming as such under coordinate transformations), rather than a Lorentz spinor index, while as a field still transforming in the standard spinor representation of the Lorentz group in any local Lorentz frame. This will be achieved by having both the 'Dirac operator' and the generators of 'Lorentz' transformations become spacetime-dependent, although covariantly constant. By carrying a world index, the spinor field may then readily be coupled to the gravitational field via the connection field ${\Gamma^\rho}_{\mu\nu}$ by

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which tensorial fields are coupled, thus implementing, it would seem, the equivalence principle in a more coherent way than in the standard vierbein formalism. The structure needed to set up the formalism will be constructed in terms of quantities valued in the complexified quaternions.

2 Preliminaries

The set of complexified quaternions is denoted $\mathbb{C}\otimes\mathbb{H}$, equal to $\mathbb{H}\otimes\mathbb{C}$ as any two elements from $\mathbb{C}$ and $\mathbb{H}$, respectively, are assumed to (multiplicatively) commute: $ch = hc$, for all $(c,h) \in \mathbb{C}\times\mathbb{H}$. Usual complex conjugation, $x \to x^*$, is assumed to act only on $\mathbb{C}$, and usual quaternionic conjugation, $x \to \overline{x}$, is assumed to act only on $\mathbb{H}$. More specifically,

$$\begin{aligned}(\mathbb{C}\otimes\mathbb{H})^* &= \{c^* h \,|c \in \mathbb{C}, h \in \mathbb{H}\}, \\ \overline{(\mathbb{C}\otimes\mathbb{H})} &= \left\{c\overline{h} \,|c \in \mathbb{C}, h \in \mathbb{H}\right\}.\end{aligned}$$

In conjunction, these two conjugations can be used to split $\mathbb{C}\otimes\mathbb{H}$ as $\mathbb{C}\otimes\mathbb{H} = (\mathbb{C}\otimes\mathbb{H})^+ \cup (\mathbb{C}\otimes\mathbb{H})^-$, where

$$\begin{aligned}(\mathbb{C}\otimes\mathbb{H})^+ &\equiv \{x \in \mathbb{C}\otimes\mathbb{H}\,|\overline{x}^* = +x\}, \\ (\mathbb{C}\otimes\mathbb{H})^- &\equiv \{x \in \mathbb{C}\otimes\mathbb{H}\,|\overline{x}^* = -x\},\end{aligned}$$

this being an almost disjoint union, zero being the only common element of $(\mathbb{C}\otimes\mathbb{H})^+$ and $(\mathbb{C}\otimes\mathbb{H})^-$. The scalar- and vector parts of $\mathbb{C}\otimes\mathbb{H}$, respectively, are denoted $\mathrm{Scal}\,(\mathbb{C}\otimes\mathbb{H}) \cong \mathbb{C}$ and $\mathrm{Vec}\,(\mathbb{C}\otimes\mathbb{H}) = (\mathbb{C}\otimes\mathbb{H})\backslash\mathrm{Scal}\,(\mathbb{C}\otimes\mathbb{H})$. A bilinear inner product $\langle\cdot,\cdot\rangle : (\mathbb{C}\otimes\mathbb{H})^2 \to \mathbb{C}$ is given by

$$\begin{aligned}2\,\langle x,y\rangle &\equiv x\overline{y} + y\overline{x} \\ &\equiv \overline{x}y + \overline{y}x.\end{aligned} \tag{1}$$

The literature disagrees on the presence or not of the factor of 2, this however being inconsequential as long as the same factor is consistently used throughout; for instance, if the factor figures in Eq. (1) above, then it will have to figure as well in Eq. (6) below. As $\mathbb{C}\otimes\mathbb{H}$ is a socalled composition algebra [3, 4], in fact a somewhat dull one as it is associative, this inner product satisfies the following relations, among other ones:

$$\langle x,y\rangle = \langle y,x\rangle\,, \tag{2}$$

$$\langle x,y\rangle = \langle \overline{x},\overline{y}\rangle\,, \tag{3}$$

$$\langle xy,z\rangle = \langle y,\overline{x}z\rangle = \langle x,z\overline{y}\rangle\,, \tag{4}$$

$$\langle x,yz\rangle = \langle \overline{y}x,z\rangle = \langle x\overline{z},y\rangle\,, \tag{5}$$

not all independent but listed nonetheless for completeness, and

$$\langle xu,yv\rangle + \langle xv,yu\rangle = 2\,\langle x,y\rangle\,\langle u,v\rangle\,, \tag{6}$$

for any $x, y, z, u, v \in \mathbb{C} \otimes \mathbb{H}$. As a basis (over $\mathbb{C}$) for $\mathbb{C} \otimes \mathbb{H}$, any four elements $q_\mu \in \mathbb{C} \otimes \mathbb{H}$ for which $\det\left(\langle q_\mu, q_\nu \rangle\right) \neq 0$, will suffice, as then $\mathbb{C} \otimes \mathbb{H} = \mathrm{Span}_{\mathbb{C}}\left(q_\mu\right)$. But a more specific choice of basis will be made: Let $s_\mu \in (\mathbb{C} \otimes \mathbb{H})^+$ for which $\det\left(\langle s_\mu, s_\nu \rangle\right) \neq 0$. Then $\langle s_\mu, s_\nu \rangle$ as a matrix will be symmetric, due to Eq. (2); real-valued, due to

$$\begin{aligned} \langle s_\mu, s_\nu \rangle^* &= \left\langle s_\mu^*, s_\nu^* \right\rangle \\ &= \left\langle \overline{s}_\mu, \overline{s}_\nu \right\rangle \\ &= \langle s_\mu, s_\nu \rangle , \end{aligned}$$

using $s_\mu^* = \overline{s}_\mu$ and Eq. (3); and non-singular, due to the determinantal condition. Furthermore, it has signature $(1,3)$, i.e., if it is diagonalized, then one diagonal element will be positive, and three diagonal elements will be negative; this is readily seen from the specific example $s_\mu = (1, \mathrm{i}e_i)$, where $e_i \in \mathrm{Vec}\,(\mathbb{H})$ are the standard quaternionic units obeying $e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k$. Having these properties, it is natural to identify this quantity with the metric of a signature $(1,3)$ Riemann-Cartan spacetime:

$$g_{\mu\nu} \equiv \langle s_\mu, s_\nu \rangle , \tag{7}$$

thus, at the same time, elevating s_μ to a type $(0,1)$ tensor field. The corresponding type $(1,0)$ tensor field s^μ is then, of course, given by $s^\mu = g^{\mu\nu} s_\nu$. With these two types of fields at hand, the following completeness relation may be shown to hold:

$$\langle x, y \rangle = \langle x, s_\mu \rangle \langle s^\mu, y \rangle \equiv g^{\mu\nu} \langle x, s_\mu \rangle \langle s_\nu, y \rangle , \tag{8}$$

for any $x, y \in \mathbb{C} \otimes \mathbb{H}$.

3 'Modified Clifford algebra'

Consider the following complex-valued type $(2,1)$ tensor field:

$$M^{\mu\rho}{}_\sigma \equiv \langle s^\mu, s^\rho s_\sigma \rangle , \tag{9}$$

not to be confused with any connection field (which is not even a tensor field, of course). Under complex conjugation, it behaves as follows:

$$\begin{aligned} \left(M^{\mu\rho}{}_\sigma\right)^* &= \left\langle \overline{s}^\mu, \overline{s}^\rho \overline{s}_\sigma \right\rangle \\ &= \left\langle s^\mu, s_\sigma s^\rho \right\rangle \\ &= M^\mu{}_\sigma{}^\rho, \end{aligned} \tag{10}$$

using $s_\mu^* = \overline{s}_\mu$ and Eq. (3).

Proposition 1 *$M^{\mu\rho}{}_\sigma$ satisfies the following algebra:*

$$\begin{aligned} 2g^{\mu\nu}\delta^\rho_\sigma &= M^{\mu\rho}{}_\tau \left(M^{\nu\tau}{}_\sigma\right)^* + M^{\nu\rho}{}_\tau \left(M^{\mu\tau}{}_\sigma\right)^* &\quad (11) \\ &= \left(M^{\mu\rho}{}_\tau\right)^* M^{\nu\tau}{}_\sigma + \left(M^{\nu\rho}{}_\tau\right)^* M^{\mu\tau}{}_\sigma . &\quad (12) \end{aligned}$$

Proof. As the metric is real-valued, the assertion implied by the second equality follows immediately from the assertion of the first line. It is thus sufficient to prove the latter, say. By direct calculation:

$$\begin{aligned}
M^{\mu\rho}{}_{\tau}\left(M^{\nu\tau}{}_{\sigma}\right)^{*}+M^{\nu\rho}{}_{\tau}\left(M^{\mu\tau}{}_{\sigma}\right)^{*} &= \left\langle s^{\mu},s^{\rho}s_{\tau}\right\rangle\left\langle\overline{s}^{\nu},\overline{s}^{\tau}\overline{s}_{\sigma}\right\rangle+\left\langle s^{\nu},s^{\rho}s_{\tau}\right\rangle\left\langle\overline{s}^{\mu},\overline{s}^{\tau}\overline{s}_{\sigma}\right\rangle \\
&= \left\langle\overline{s}^{\rho}s^{\mu},s_{\tau}\right\rangle\left\langle\overline{s}^{\nu}s_{\sigma},\overline{s}^{\tau}\right\rangle+\left\langle\overline{s}^{\rho}s^{\nu},s_{\tau}\right\rangle\left\langle\overline{s}^{\mu}s_{\sigma},\overline{s}^{\tau}\right\rangle \\
&= \left\langle\overline{s}^{\rho}s^{\mu},s_{\tau}\right\rangle\left\langle s^{\tau},\overline{s}_{\sigma}s^{\nu}\right\rangle+\left\langle\overline{s}^{\rho}s^{\nu},s_{\tau}\right\rangle\left\langle s^{\tau},\overline{s}_{\sigma}s^{\mu}\right\rangle \\
&= \left\langle\overline{s}^{\rho}s^{\mu},\overline{s}_{\sigma}s^{\nu}\right\rangle+\left\langle\overline{s}^{\rho}s^{\nu},\overline{s}_{\sigma}s^{\mu}\right\rangle \\
&= 2\left\langle\overline{s}^{\rho},\overline{s}_{\sigma}\right\rangle\left\langle s^{\nu},s^{\mu}\right\rangle \\
&= 2\left\langle s^{\mu},s^{\nu}\right\rangle\left\langle s^{\rho},s_{\sigma}\right\rangle \\
&= 2g^{\mu\nu}\delta^{\rho}_{\sigma},
\end{aligned}$$

using several of the properties of the inner product listed in Sec. 2. ∎

In terms of 4×4 matrices $\mathbf{M}^{\mu}$ with components $\left(\mathbf{M}^{\mu}\right)^{\rho}{}_{\sigma}\equiv M^{\mu\rho}{}_{\sigma}$, this algebra may also be written concisely in matrix notation as

$$\begin{aligned}
2g^{\mu\nu}\mathbf{1} &= \mathbf{M}^{\mu}\mathbf{M}^{\nu*}+\mathbf{M}^{\nu}\mathbf{M}^{\mu*} &(13)\\
&= \mathbf{M}^{\mu*}\mathbf{M}^{\nu}+\mathbf{M}^{\nu*}\mathbf{M}^{\mu}, &(14)
\end{aligned}$$

where $\mathbf{1}$ is the identity matrix. Apart from the complex conjugations, this algebra is the $Cl\,(1,3)$ Clifford algebra, and it may thus perhaps be called a 'modified Clifford algebra' (the algebra may certainly have been studied somewhere in the literature, and thus have a specific name, but the author is not aware of any such). The relevance of this algebra will become clear shortly.

4 'Modified Lorentz algebra'

Consider the following complex-valued type $(3,1)$ tensor field:

$$\begin{aligned}
4S^{\mu\nu\rho}{}_{\sigma} &\equiv \left\langle\overline{s}^{\mu}s^{\rho},\overline{s}^{\nu}s_{\sigma}\right\rangle-\left\langle\overline{s}^{\nu}s^{\rho},\overline{s}^{\mu}s_{\sigma}\right\rangle &(15)\\
&\equiv \left\langle s^{\rho},\left(s^{\mu}\overline{s}^{\nu}-s^{\nu}\overline{s}^{\mu}\right)s_{\sigma}\right\rangle,
\end{aligned}$$

Proposition 2 *$S^{\mu\nu\rho}{}_{\sigma}$ may be written in terms of $M^{\mu}{}_{\rho\sigma}$ as follows:*

$$4S^{\mu\nu\rho}{}_{\sigma}=M^{\mu\rho}{}_{\tau}\left(M^{\nu\tau}{}_{\sigma}\right)^{*}-M^{\nu\rho}{}_{\tau}\left(M^{\mu\tau}{}_{\sigma}\right)^{*}. \quad (16)$$

Proof. By direct calculation:

$$\begin{aligned}
M^{\mu\rho}{}_{\tau}\left(M^{\nu\tau}{}_{\sigma}\right)^{*}-M^{\nu\rho}{}_{\tau}\left(M^{\mu\tau}{}_{\sigma}\right)^{*} &= \left\langle s^{\mu},s^{\rho}s_{\tau}\right\rangle\left\langle\overline{s}^{\nu},\overline{s}^{\tau}\overline{s}_{\sigma}\right\rangle-\left\langle s^{\nu},s^{\rho}s_{\tau}\right\rangle\left\langle\overline{s}^{\mu},\overline{s}^{\tau}\overline{s}_{\sigma}\right\rangle \\
&= \left\langle\overline{s}^{\rho}s^{\mu},s_{\tau}\right\rangle\left\langle\overline{s}^{\nu}s_{\sigma},\overline{s}^{\tau}\right\rangle-\left\langle\overline{s}^{\rho}s^{\nu},s_{\tau}\right\rangle\left\langle\overline{s}^{\mu}s_{\sigma},\overline{s}^{\tau}\right\rangle \\
&= \left\langle\overline{s}^{\rho}s^{\mu},s_{\tau}\right\rangle\left\langle s^{\tau},\overline{s}_{\sigma}s^{\nu}\right\rangle-\left\langle\overline{s}^{\rho}s^{\nu},s_{\tau}\right\rangle\left\langle s^{\tau},\overline{s}_{\sigma}s^{\mu}\right\rangle \\
&= \left\langle\overline{s}^{\rho}s^{\mu},\overline{s}_{\sigma}s^{\nu}\right\rangle-\left\langle\overline{s}^{\rho}s^{\nu},\overline{s}_{\sigma}s^{\mu}\right\rangle \\
&= \left\langle\overline{s}^{\mu}s^{\rho},\overline{s}^{\nu}s_{\sigma}\right\rangle-\left\langle\overline{s}^{\nu}s^{\rho},\overline{s}^{\mu}s_{\sigma}\right\rangle,
\end{aligned}$$

most of the steps being analogous to the ones taken in the proof of Proposition 1. ■

In terms of 4×4 matrices $\mathbf{S}^{\mu\nu}$ with components $(\mathbf{S}^{\mu\nu})^{\rho}{}_{\sigma} \equiv S^{\mu\nu\rho}{}_{\sigma}$, Eq. (16) may also be written concisely in matrix notation as

$$4\mathbf{S}^{\mu\nu} = \mathbf{M}^{\mu}\mathbf{M}^{\nu *} - \mathbf{M}^{\nu}\mathbf{M}^{\mu *}, \tag{17}$$

using as well the previously defined matrices $\mathbf{M}^{\mu}$.

Proposition 3 *In conjunction,* $\mathbf{S}^{\mu\nu}$ *and* $\mathbf{M}^{\rho}$ *satisfy the following identity:*

$$\mathbf{M}^{\rho}\mathbf{S}^{\mu\nu *} - \mathbf{S}^{\mu\nu}\mathbf{M}^{\rho} = g^{\rho\mu}\mathbf{M}^{\nu} - g^{\rho\nu}\mathbf{M}^{\mu}. \tag{18}$$

This is in the present formalism the analogue of the identity $[\gamma^{\rho}, S^{\mu\nu}] = (V^{\mu\nu})^{\rho}{}_{\sigma}\gamma^{\sigma}$ *from the standard Dirac formalism [5].*

Proof. By direct calculation:

$$\begin{aligned}
4\mathbf{S}^{\mu\nu}\mathbf{M}^{\rho} &\equiv (\mathbf{M}^{\mu}\mathbf{M}^{\nu *} - \mathbf{M}^{\nu}\mathbf{M}^{\mu *})\,\mathbf{M}^{\rho} \\
&= \mathbf{M}^{\mu}\left(-\mathbf{M}^{\rho *}\mathbf{M}^{\nu} + 2g^{\rho\nu}\mathbf{1}\right) - \mathbf{M}^{\nu}\left(-\mathbf{M}^{\rho *}\mathbf{M}^{\mu} + 2g^{\rho\mu}\mathbf{1}\right) \\
&= -\mathbf{M}^{\mu}\mathbf{M}^{\rho *}\mathbf{M}^{\nu} + \mathbf{M}^{\nu}\mathbf{M}^{\rho *}\mathbf{M}^{\mu} - 2g^{\rho\mu}\mathbf{M}^{\nu} + 2g^{\rho\nu}\mathbf{M}^{\mu} \\
&= -\left(-\mathbf{M}^{\rho}\mathbf{M}^{\mu *} + 2g^{\rho\mu}\mathbf{1}\right)\mathbf{M}^{\nu} + \left(-\mathbf{M}^{\rho}\mathbf{M}^{\nu *} + 2g^{\rho\nu}\mathbf{1}\right)\mathbf{M}^{\mu} - 2g^{\rho\mu}\mathbf{M}^{\nu} + 2g^{\rho\nu}\mathbf{M}^{\mu} \\
&= \mathbf{M}^{\rho}\left(\mathbf{M}^{\mu *}\mathbf{M}^{\nu} - \mathbf{M}^{\nu *}\mathbf{M}^{\mu}\right) - 4g^{\rho\mu}\mathbf{M}^{\nu} + 4g^{\rho\nu}\mathbf{M}^{\mu} \\
&\equiv 4\mathbf{M}^{\rho}\mathbf{S}^{\mu\nu *} - 4g^{\rho\mu}\mathbf{M}^{\nu} + 4g^{\rho\nu}\mathbf{M}^{\mu},
\end{aligned}$$

using Eqs. (13)-(14) and (17). ■

Proposition 4 *The matrices* $\mathbf{S}^{\mu\nu}$ *satisfy the following algebra:*

$$[\mathbf{S}^{\mu\nu}, \mathbf{S}^{\rho\sigma}] = -\left(g^{\mu\rho}\mathbf{S}^{\nu\sigma} - g^{\mu\sigma}\mathbf{S}^{\nu\rho} - g^{\nu\rho}\mathbf{S}^{\mu\sigma} + g^{\nu\sigma}\mathbf{S}^{\mu\rho}\right), \tag{19}$$

this being the Lorentz algebra with $\eta^{\mu\nu}$ *replaced by* $g^{\mu\nu}$ *(it may thus perhaps be called a 'modified Lorentz algebra').*

Proof. By direct calculation: From

$$\begin{aligned}
\mathbf{S}^{\mu\nu}\mathbf{M}^{\rho}\mathbf{M}^{\sigma *} &= \left[\mathbf{M}^{\rho}\mathbf{S}^{\mu\nu *} - (g^{\mu\rho}\mathbf{M}^{\nu} - g^{\nu\rho}\mathbf{M}^{\mu})\right]\mathbf{M}^{\sigma *} \\
&= \mathbf{M}^{\rho}\mathbf{S}^{\mu\nu *}\mathbf{M}^{\sigma *} - g^{\mu\rho}\mathbf{M}^{\nu}\mathbf{M}^{\sigma *} + g^{\nu\rho}\mathbf{M}^{\mu}\mathbf{M}^{\sigma *} \\
&= \mathbf{M}^{\rho}\left[\mathbf{M}^{\sigma *}\mathbf{S}^{\mu\nu} - (g^{\mu\sigma}\mathbf{M}^{\nu *} - g^{\nu\sigma}\mathbf{M}^{\mu *})\right] - g^{\mu\rho}\mathbf{M}^{\nu}\mathbf{M}^{\sigma *} + g^{\nu\rho}\mathbf{M}^{\mu}\mathbf{M}^{\sigma *} \\
&= \mathbf{M}^{\rho}\mathbf{M}^{\sigma *}\mathbf{S}^{\mu\nu} - g^{\mu\sigma}\mathbf{M}^{\rho}\mathbf{M}^{\nu *} + g^{\nu\sigma}\mathbf{M}^{\rho}\mathbf{M}^{\mu *} - g^{\mu\rho}\mathbf{M}^{\nu}\mathbf{M}^{\sigma *} + g^{\nu\rho}\mathbf{M}^{\mu}\mathbf{M}^{\sigma *},
\end{aligned}$$

using Eq. (18) and its complex conjugate, follows

$$\begin{aligned}
4\mathbf{S}^{\mu\nu}\mathbf{S}^{\rho\sigma} &\equiv \mathbf{S}^{\mu\nu}\left(\mathbf{M}^{\rho}\mathbf{M}^{\sigma *} - \mathbf{M}^{\sigma}\mathbf{M}^{\rho *}\right) \\
&= \left(\mathbf{M}^{\rho}\mathbf{M}^{\sigma *}\mathbf{S}^{\mu\nu} - g^{\mu\sigma}\mathbf{M}^{\rho}\mathbf{M}^{\nu *} + g^{\nu\sigma}\mathbf{M}^{\rho}\mathbf{M}^{\mu *} - g^{\mu\rho}\mathbf{M}^{\nu}\mathbf{M}^{\sigma *} + g^{\nu\rho}\mathbf{M}^{\mu}\mathbf{M}^{\sigma *}\right) \\
&\quad - \left(\mathbf{M}^{\sigma}\mathbf{M}^{\rho *}\mathbf{S}^{\mu\nu} - g^{\mu\rho}\mathbf{M}^{\sigma}\mathbf{M}^{\nu *} + g^{\nu\rho}\mathbf{M}^{\sigma}\mathbf{M}^{\mu *} - g^{\mu\sigma}\mathbf{M}^{\nu}\mathbf{M}^{\rho *} + g^{\nu\sigma}\mathbf{M}^{\mu}\mathbf{M}^{\rho *}\right) \\
&= \left(\mathbf{M}^{\rho}\mathbf{M}^{\sigma *} - \mathbf{M}^{\sigma}\mathbf{M}^{\rho *}\right)\mathbf{S}^{\mu\nu} - g^{\mu\rho}\left(\mathbf{M}^{\nu}\mathbf{M}^{\sigma *} - \mathbf{M}^{\sigma}\mathbf{M}^{\nu *}\right) + g^{\mu\sigma}\left(\mathbf{M}^{\nu}\mathbf{M}^{\rho *} - \mathbf{M}^{\rho}\mathbf{M}^{\nu *}\right) \\
&\quad + g^{\nu\rho}\left(\mathbf{M}^{\mu}\mathbf{M}^{\sigma *} - \mathbf{M}^{\sigma}\mathbf{M}^{\mu *}\right) - g^{\nu\sigma}\left(\mathbf{M}^{\mu}\mathbf{M}^{\rho *} - \mathbf{M}^{\rho}\mathbf{M}^{\mu *}\right) \\
&= 4\mathbf{S}^{\rho\sigma}\mathbf{S}^{\mu\nu} - 4\left(g^{\mu\rho}\mathbf{S}^{\nu\sigma} - g^{\mu\sigma}\mathbf{S}^{\nu\rho} - g^{\nu\rho}\mathbf{S}^{\mu\sigma} + g^{\nu\sigma}\mathbf{S}^{\mu\rho}\right).
\end{aligned}$$

This proof is structurally quite analogous to the proof in the standard Dirac formalism that $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ satisfies the standard Lorentz algebra. ■

Consider as well the following type $(3,1)$ tensor field:

$$\begin{aligned} V^{\mu\nu\rho}{}_{\sigma} &\equiv \langle s^\mu, s^\rho \rangle \langle s^\nu, s_\sigma \rangle - \langle s^\nu, s^\rho \rangle \langle s^\mu, s_\sigma \rangle \\ &= g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma. \end{aligned} \tag{20}$$

The corresponding 4×4 matrices $\mathbf{V}^{\mu\nu}$ with components $(\mathbf{V}^{\mu\nu})^\rho{}_\sigma \equiv V^{\mu\nu\rho}{}_\sigma$ are readily shown to satisfy the very same algebra as do $\mathbf{S}^{\mu\nu}$, Eq. (19).

Note that in a local inertial frame in which $\langle s^\mu, s^\nu \rangle = \eta^{\mu\nu}$, the algebra Eq. (19), with spacetime-dependent structure constants, reduces to the standard Lorentz algebra, with spacetime-independent structure constants. So in a local inertial frame, where $g^{\mu\nu} = \eta^{\mu\nu}$, the quantities $\mathbf{S}^{\mu\nu}$ and $\mathbf{V}^{\mu\nu}$ are ordinary representations of the standard (unmodified) Lorentz algebra. More specifically, $\mathbf{S}^{\mu\nu}$ is a (spin $\frac{1}{2}$) spinor representation, and $\mathbf{V}^{\mu\nu}$ is a vector representation, these assertions being readily established by calculating their corresponding Casimir operators, $\frac{1}{2}\mathbf{S}^{ij}\mathbf{S}_{ij}$ and $\frac{1}{2}\mathbf{V}^{ij}\mathbf{V}_{ij}$, respectively, for the SO(3) subgroup of the Lorentz group. Note that in this article, a notation without explicit i's in the definition of the generators and, correspondingly, in the Lie algebra is used.

Proposition 5 *$(\mathbf{S}^{\mu\nu})^\rho{}_\sigma$ and $(\mathbf{V}^{\mu\nu})^\rho{}_\sigma$ satisfy the following identities:*

$$g_{\sigma\alpha}(\mathbf{S}^{\mu\nu})^\alpha{}_\beta g^{\beta\rho} = -(\mathbf{S}^{\mu\nu})^\rho{}_\sigma, \tag{21}$$

$$g_{\sigma\alpha}(\mathbf{V}^{\mu\nu})^\alpha{}_\beta g^{\beta\rho} = -(\mathbf{V}^{\mu\nu})^\rho{}_\sigma. \tag{22}$$

Proof. As Eq. (22) is just the generalization to curvilinear coordinates of the well-known identity from special relativity responsible for the invariance of the line element, only the proof of Eq. (21) will be given. By direct calculation:

$$\begin{aligned} 4g_{\sigma\alpha}(\mathbf{S}^{\mu\nu})^\alpha{}_\beta g^{\beta\rho} &= g_{\sigma\alpha}\langle s^\alpha, (s^\mu\overline{s}^\nu - s^\nu\overline{s}^\mu)\, s_\beta\rangle g^{\beta\rho} \\ &= \langle s_\sigma, (s^\mu\overline{s}^\nu - s^\nu\overline{s}^\mu)\, s^\rho\rangle \\ &= -\langle (s^\mu\overline{s}^\nu - s^\nu\overline{s}^\mu)\, s_\sigma, s^\rho\rangle \\ &= -\langle s^\rho, (s^\mu\overline{s}^\nu - s^\nu\overline{s}^\mu)\, s_\sigma\rangle \\ &= -(\mathbf{S}^{\mu\nu})^\rho{}_\sigma, \end{aligned}$$

using Eqs. (2) and (5). ■

Proposition 6 *The following identity holds:*

$$\begin{aligned} 2(\mathbf{S}^{\mu\nu})^{\rho\sigma} &= g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma} \pm \mathrm{i}\varepsilon^{\mu\nu\rho\sigma} \\ &\equiv (\mathbf{V}^{\mu\nu})^{\rho\sigma} \pm \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}, \end{aligned} \tag{23}$$

for either the plus or the minus sign. Here, $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor of type $(4,0)$ [6, p. 202].

It was this identity that originally inspired the investigations resulting in the findings reported in this article. A sign change of the Levi-Civita tensor term corresponds to the (parity) transformation $s_\mu \to \overline{s}_\mu = s_\mu^*$, which does not change the metric and thus neither $(\mathbf{V}^{\mu\nu})^{\rho\sigma}$ nor $\varepsilon^{\mu\nu\rho\sigma}$ (the latter dedending only on the determinant of the metric), but changes $(\mathbf{S}^{\mu\nu})^{\rho\sigma}$ into its complex conjugate. The choice of 'handedness' of s_μ is freely disposable, but once taken, it must of course be consistently adhered to throughout.

Before giving the proof of Eq. (23), first a little preparation, introducing some auxiliary machinery: Consider the map $X : (\mathbb{C} \otimes \mathbb{H})^3 \to \mathbb{C} \otimes \mathbb{H}$ given by

$$3!X(x, y, z) \equiv (x\overline{y} - y\overline{x})\, z + (y\overline{z} - z\overline{y})\, x + (z\overline{x} - x\overline{z})\, y, \tag{24}$$

which by construction is completely antisymmetric in its three arguments. An aside: It satisfies the following orthogonality property and (generalized) Pythagorean property, respectively:

$$\begin{aligned} 0 &= \langle X(x_1, x_2, x_3), x_i \rangle, \\ \det(\langle x_i, x_j \rangle) &= \langle X(x_1, x_2, x_3), X(x_1, x_2, x_3) \rangle, \end{aligned}$$

for any $x_i \in \mathbb{C} \otimes \mathbb{H}$, where $i = 1, 2, 3$, of course. It corresponds to the complexification of a triple cross product on $\mathbb{R}^4$, taking three vectors and producing a single vector. Such cross products with three factors, possessing these two properties, are possible only in $\mathbb{R}^4$ and $\mathbb{R}^8$ [7, Sec. 7.5], and by complexification in $\mathbb{C}^4$ and $\mathbb{C}^8$, the underlying reason being the existence of the division algebras $\mathbb{H}$ and $\mathbb{O}$.

The expanded right-hand side of Eq. (24) consists of six terms, of course. By using $2\langle x, y \rangle \equiv x\overline{y} + y\overline{x} \equiv \overline{x}y + \overline{y}x$, etc., repeatedly to rearrange all terms as the first term (plus some extra) yields the following equivalent expression:

$$X(x, y, z) = x\overline{y}z - \langle y, z \rangle\, x + \langle z, x \rangle\, y - \langle x, y \rangle\, z. \tag{25}$$

From it follows that (note interchange of z and u in the second relation as compared to the first)

$$\begin{aligned} \langle X(x, y, z), u \rangle &= \langle x\overline{y}z, u \rangle - \langle y, z \rangle \langle x, u \rangle + \langle z, x \rangle \langle y, u \rangle - \langle x, y \rangle \langle z, u \rangle, \\ \langle X(x, y, u), z \rangle &= \langle x\overline{y}u, z \rangle - \langle y, u \rangle \langle x, z \rangle + \langle u, x \rangle \langle y, z \rangle - \langle x, y \rangle \langle u, z \rangle, \end{aligned}$$

which added yields

$$\begin{aligned} \langle X(x, y, z), u \rangle + \langle X(x, y, u), z \rangle &= \langle x\overline{y}z, u \rangle + \langle x\overline{y}u, z \rangle - 2\langle x, y \rangle \langle z, u \rangle \\ &= \langle x\overline{y}, u\overline{z} + z\overline{u} \rangle - 2\langle x, y \rangle \langle z, u \rangle \\ &= 2\langle x\overline{y}, 1 \rangle \langle z, u \rangle - 2\langle x, y \rangle \langle z, u \rangle \\ &= 2\langle x, y \rangle \langle z, u \rangle - 2\langle x, y \rangle \langle z, u \rangle \\ &\equiv 0, \end{aligned}$$

using Eqs. (1) and (4), showing that $\langle X(x, y, z), u \rangle$ is antisymmetric in its last two arguments, z and u. As it is as well completely antisymmetric in its first three arguments, due to the complete antisymmetry of $X(x, y, z)$, as previously noted, it is in fact completely antisymmetric in all its arguments. Now to the proof of Eq. (23):

Proof. From Eqs. (25) follows, using Eq. (7),

$$\begin{aligned} X\left(s^{\mu},s^{\nu},s^{\rho}\right) &= s^{\mu}\overline{s}^{\nu}s^{\rho}-g^{\nu\rho}s^{\mu}+g^{\rho\mu}s^{\nu}-g^{\mu\nu}s^{\rho}, \\ X\left(s^{\nu},s^{\mu},s^{\rho}\right) &= s^{\nu}\overline{s}^{\mu}s^{\rho}-g^{\mu\rho}s^{\nu}+g^{\rho\nu}s^{\mu}-g^{\nu\mu}s^{\rho}, \end{aligned}$$

which subtracted, using the antisymmetry of X, yields

$$2X\left(s^{\mu},s^{\nu},s^{\rho}\right)=\left(s^{\mu}\overline{s}^{\nu}-s^{\nu}\overline{s}^{\mu}\right)s^{\rho}+2g^{\rho\mu}s^{\nu}-2g^{\rho\nu}s^{\mu},$$

from which it follows that

$$\begin{aligned} 2\left\langle X\left(s^{\mu},s^{\nu},s^{\rho}\right),s^{\sigma}\right\rangle &= \left\langle\left(s^{\mu}\overline{s}^{\nu}-s^{\nu}\overline{s}^{\mu}\right)s^{\rho},s^{\sigma}\right\rangle+2g^{\rho\mu}\left\langle s^{\nu},s^{\sigma}\right\rangle-2g^{\rho\nu}\left\langle s^{\mu},s^{\sigma}\right\rangle \\ &= \left\langle\left(s^{\mu}\overline{s}^{\nu}-s^{\nu}\overline{s}^{\mu}\right)s^{\rho},s^{\sigma}\right\rangle+2\left(g^{\rho\mu}g^{\nu\sigma}-g^{\rho\nu}g^{\mu\sigma}\right) \\ &= -\left\langle s^{\rho},\left(s^{\mu}\overline{s}^{\nu}-s^{\nu}\overline{s}^{\mu}\right)s^{\sigma}\right\rangle+2\left(g^{\rho\mu}g^{\nu\sigma}-g^{\rho\nu}g^{\mu\sigma}\right) \\ &= -4\left(\mathbf{S}^{\mu\nu}\right)^{\rho\sigma}+2\left(\mathbf{V}^{\mu\nu}\right)^{\rho\sigma}, \end{aligned}$$

using Eqs. (4); or, equivalently:

$$2\left(\mathbf{S}^{\mu\nu}\right)^{\rho\sigma}=\left(\mathbf{V}^{\mu\nu}\right)^{\rho\sigma}-\left\langle X\left(s^{\mu},s^{\nu},s^{\rho}\right),s^{\sigma}\right\rangle.$$

Now, due to the complete antisymmetry of $\left\langle X\left(x,y,z\right),u\right\rangle$ in all its arguments, as previously established, the second addend on the right-hand side is completely antisymmetric in $\mu\nu\rho\sigma$. As it is by construction also a type $(4,0)$ tensor, it must be proportional to $\varepsilon^{\mu\nu\rho\sigma}$. By plugging in some simple choices of s^{μ}, it is readily established that $\left\langle X\left(s^{\mu},s^{\nu},s^{\rho}\right),s^{\sigma}\right\rangle=\pm\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}$, the signs being mutually exclusive, of course. From this, the proof then follows. ■

It should be noted that the condition $\left\langle X\left(s^{\mu},s^{\nu},s^{\rho}\right),s^{\sigma}\right\rangle=\pm\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}$, for some specific choice of sign, is completely independent of the relation $g_{\mu\nu}=\left\langle s_{\mu},s_{\nu}\right\rangle$, Eq. (7), posing on s^{μ} only a *discrete* condition of 'handedness'.

5 The spinor Lagrangian

Before presenting the spinor Lagrangian, first in flat spacetime, and later in nonflat spacetime, some preliminaries: Consider the following complex-valued type $(1,1)$ tensor field:

$$N^{\rho}{}_{\sigma}\equiv\left\langle s^{\rho},s_{\sigma}\kappa\right\rangle, \tag{26}$$

where $\kappa\in\mathrm{Vec}\left(\mathrm{i}\mathbb{H}\right)$ with $\kappa^{2}=1$ is spacetime-independent. It is antisymmetric, $N_{\rho\sigma}=-N_{\sigma\rho}$, because

$$\begin{aligned} N^{\rho}{}_{\sigma} &= -\left\langle s^{\rho}\kappa,s_{\sigma}\right\rangle \\ &= -\left\langle s_{\sigma},s^{\rho}\kappa\right\rangle \\ &= -N_{\sigma}{}^{\rho}, \end{aligned} \tag{27}$$

using $\overline{\kappa}=-\kappa$, and Eqs. (2) and (5).

Proposition 7 *$N^\rho{}_\sigma$ satisfies the following algebra (note the single complex conjugation):*

$$\begin{aligned} 0 &= M^{\mu\rho}{}_\tau \left(N^\tau{}_\sigma\right)^* + N^\rho{}_\tau M^{\mu\tau}{}_\sigma, && (28)\\ \delta^\rho_\sigma &= N^\rho{}_\tau N^\tau{}_\sigma, && (29)\end{aligned}$$

with $M^{\mu\rho}{}_\sigma$ being, of course, the previously defined tensor field, Eq. (9).

Proof. By direct calculation:

$$\begin{aligned} M^{\mu\rho}{}_\tau \left(N^\tau{}_\sigma\right)^* + N^\rho{}_\tau M^{\mu\tau}{}_\sigma &= -\left\langle s^\mu, s^\rho s_\tau\right\rangle \left\langle \overline{s}^\tau, \overline{s}_\sigma \kappa\right\rangle + \left\langle s^\rho, s_\tau \kappa\right\rangle \left\langle s^\mu, s^\tau s_\sigma\right\rangle \\ &= \left\langle \overline{s}^\rho s^\mu, s_\tau\right\rangle \left\langle s^\tau, \kappa s_\sigma\right\rangle - \left\langle s^\rho \kappa, s_\tau\right\rangle \left\langle s^\tau, s^\mu \overline{s}_\sigma\right\rangle \\ &= \left\langle \overline{s}^\rho s^\mu, \kappa s_\sigma\right\rangle - \left\langle s^\rho \kappa, s^\mu \overline{s}_\sigma\right\rangle \\ &= \left\langle \overline{s}^\rho s^\mu, \kappa s_\sigma\right\rangle + \left\langle \overline{s}^\mu s^\rho, \overline{s}_\sigma \kappa\right\rangle \\ &= \left\langle \overline{s}^\rho s^\mu, \kappa s_\sigma\right\rangle - \left\langle \overline{s}^\rho s^\mu, \kappa s_\sigma\right\rangle \\ &\equiv 0, \end{aligned}$$

and

$$\begin{aligned} N^\rho{}_\tau N^\tau{}_\sigma &= \left\langle s^\rho, s_\tau \kappa\right\rangle \left\langle s^\tau, s_\sigma \kappa\right\rangle \\ &= -\left\langle s^\rho \kappa, s_\tau\right\rangle \left\langle s^\tau, s_\sigma \kappa\right\rangle \\ &= -\left\langle s^\rho \kappa, s_\sigma \kappa\right\rangle \\ &= \left\langle s^\rho, s_\sigma \kappa^2\right\rangle \\ &= \left\langle s^\rho, s_\sigma\right\rangle \\ &= \delta^\rho_\sigma; \end{aligned}$$

using $\overline{\kappa} = -\kappa$ and $\kappa^* = -\kappa$, and several of the properties of the inner product listed in Sec. 2. ■

In terms of the 4×4 matrix $\mathbf{N}$ with components $\mathbf{N}^\rho{}_\sigma \equiv N^\rho{}_\sigma$, and the previously defined matrices $\mathbf{M}^\mu$, these relations may also be written concisely in matrix notation as

$$\begin{aligned} \mathbf{0} &= \mathbf{M}^\mu \mathbf{N}^* + \mathbf{N}\mathbf{M}^\mu, && (30)\\ \mathbf{1} &= \mathbf{N}^2, && (31)\end{aligned}$$

where $\mathbf{0}$ is the zero matrix.

Proposition 8 *$\mathbf{S}^{\mu\nu}$ and $\mathbf{N}$ commute:*

$$\mathbf{0} = \left[\mathbf{S}^{\mu\nu}, \mathbf{N}\right]. \qquad (32)$$

Proof. The assertion follows immediately from the following two expressions:

$$\begin{aligned} 4\left(\mathbf{S}^{\mu\nu}\right)^\rho{}_\tau \mathbf{N}^\tau{}_\sigma &\equiv \left\langle s^\rho, \left(s^\mu \overline{s}^\nu - s^\nu \overline{s}^\mu\right) s_\tau\right\rangle \left\langle s^\tau, s_\sigma \kappa\right\rangle \\ &= -\left\langle \left(s^\mu \overline{s}^\nu - s^\nu \overline{s}^\mu\right) s^\rho, s_\tau\right\rangle \left\langle s^\tau, s_\sigma \kappa\right\rangle \\ &= -\left\langle \left(s^\mu \overline{s}^\nu - s^\nu \overline{s}^\mu\right) s^\rho, s_\sigma \kappa\right\rangle \\ &= -\left\langle \overline{s}_\sigma \left(s^\mu \overline{s}^\nu - s^\nu \overline{s}^\mu\right) s^\rho, \kappa\right\rangle, \end{aligned}$$

and

$$\begin{aligned}
4\mathbf{N}^{\rho}{}_{\tau}\left(\mathbf{S}^{\mu\nu}\right)^{\tau}{}_{\sigma} &\equiv \left\langle s^{\rho}, s_{\tau}\kappa\right\rangle \left\langle s^{\tau}, \left(s^{\mu}\overline{s}^{\nu} - s^{\nu}\overline{s}^{\mu}\right) s_{\sigma}\right\rangle \\
&= -\left\langle s^{\rho}\kappa, s_{\tau}\right\rangle \left\langle s^{\tau}, \left(s^{\mu}\overline{s}^{\nu} - s^{\nu}\overline{s}^{\mu}\right) s_{\sigma}\right\rangle \\
&= -\left\langle s^{\rho}\kappa, \left(s^{\mu}\overline{s}^{\nu} - s^{\nu}\overline{s}^{\mu}\right) s_{\sigma}\right\rangle \\
&= -\left\langle \overline{s}^{\rho}\left(s^{\mu}\overline{s}^{\nu} - s^{\nu}\overline{s}^{\mu}\right) s_{\sigma}, \kappa\right\rangle \\
&= -\left\langle \overline{s}_{\sigma}\left(s^{\mu}\overline{s}^{\nu} - s^{\nu}\overline{s}^{\mu}\right) s^{\rho}, \kappa\right\rangle ,
\end{aligned}$$

using several of the properties of the inner product listed in Sec. 2, the last equality, in particular, following from Eq. (3) and $\overline{\kappa} = -\kappa$. ■

5.1 Flat spacetime

Consider globally flat spacetime $\langle s_\mu, s_\nu\rangle = \eta_{\mu\nu}$ in Cartesian coordinates, assuming s_μ to be spacetime-independent. Consider at the classical level the following spinor Lagrangian:

$$\mathcal{L} = \frac{\mathrm{i}}{2}\psi^*_\rho M^{\mu\rho}{}_{\sigma}\partial_\mu\psi^\sigma - \frac{\mathrm{i}}{2}\left(\partial_\mu\psi_\rho\right)^* M^{\mu\rho}{}_{\sigma}\psi^\sigma + \frac{m}{2}\left[\psi^*_\rho N^{\rho}{}_{\sigma}\psi^{\sigma *} - \psi_\rho\left(N^{\rho}{}_{\sigma}\right)^*\psi^\sigma\right], \tag{33}$$

where, as advertised in the Introduction, the complex Grassmann-valued spinor field ψ carries a world index, rather than a standard spinor index. Beware not to confuse ψ^ρ, the components of that field, with a Rarita-Schwinger/gravitino field [1, Sec. 31.3]. The Majorana-like mass term is properly nontrivial due to Eq. (27), and it is complex self-conjugate (hermitian), by construction. The kinetic term is complex self-conjugate (hermitian) due to Eq. (10). A Lorentz invariant Lagrangian with a Dirac-like mass term does not seem possible. The corresponding Euler-Lagrange equations of motion are given by

$$0 = \left(\mathrm{i}M^{\mu\rho}{}_{\sigma}\partial_\mu + mN^{\rho}{}_{\sigma}K\right)\psi^\sigma, \tag{34}$$

where K is the operator of complex conjugation; or, equivalently, in matrix/vector notation:

$$\begin{aligned}
0 &= \left(\mathrm{i}\mathbf{M}^\mu\partial_\mu + m\mathbf{N}K\right)\boldsymbol{\psi} \\
&\equiv \left(\mathrm{i}\mathbf{M}^\mu K\partial_\mu + m\mathbf{N}\right)K\boldsymbol{\psi},
\end{aligned} \tag{35}$$

where $\boldsymbol{\psi}$ is a four-column (vector) with components $\boldsymbol{\psi}^\mu = \psi^\mu$.

A word of warning: Beware not to mistake ψ^*_μ (as in the above Lagrangian, for instance) for being the components of the four-row (vector) $\boldsymbol{\psi}^\dagger$, for they are not due to $\left(\boldsymbol{\psi}^\dagger\right)_\mu \equiv (\boldsymbol{\psi})^{\mu *} = \psi^{\mu *} \neq \psi^*_\mu$, the inequality being the result of the metric being indefinite; this can be contrasted with the standard Dirac formalism where the analogous relation would read $\left(\boldsymbol{\psi}^\dagger\right)_a \equiv (\boldsymbol{\psi})^{a *} = \psi^{a *} = \psi^*_a$ (now equality), because there is no difference between having upper or lower *spinor* indices. Therefore, $\psi^*_\rho M^{\mu\rho}{}_{\sigma}\partial_\mu\psi^\sigma \neq \boldsymbol{\psi}^\dagger\mathbf{M}^\mu\partial_\mu\boldsymbol{\psi}$ (inequality), for instance, which is the reason why the above Lagrangian is given in tensor notation rather than in matrix/vector notation. Of course, some operator, ‡ say, could be defined so that $\psi_\rho^* M^{\mu\rho}{}_\sigma \partial_\mu \psi^\sigma = \boldsymbol{\psi}^\ddagger \mathbf{M}^\mu \partial_\mu \boldsymbol{\psi}$ (equality), etc., but it does not seem to be a very fruitful strategy.

Now, Eq. (35) is readily seen to be 'Klein-Gordon compatible' in the sense that a plane wave solution $\boldsymbol{\psi} = \boldsymbol{\psi}_0 \exp(-\mathrm{i}p \cdot x)$ can be (but need not be) a solution to it only if it is on mass shell, $p^2 = m^2$:

$$\begin{aligned} 0 &= (\mathrm{i}\mathbf{M}^\mu K \partial_\mu + m\mathbf{N})(\mathrm{i}\mathbf{M}^\nu K \partial_\nu + m\mathbf{N}) \\ &= \mathbf{M}^\mu \mathbf{M}^{\nu *} \partial_\mu \partial_\nu + \mathrm{i}m(\mathbf{M}^\mu \mathbf{N}^* + \mathbf{N}\mathbf{M}^\mu) K \partial_\mu + m^2 \mathbf{N}^2 \\ &= g^{\mu\nu} \partial_\mu \partial_\nu + m^2 \mathbf{1}, \end{aligned}$$

using Eq. (13) and Eqs. (30)-(31). Note that the assumed spacetime-independency of s_μ is used to freely move derivatives through $\mathbf{M}^\mu$ and $\mathbf{N}$.

But Klein-Gordon compatibility of the equations of motion is of course not near sufficient to have a sensible theory. The Lagrangian must also at least be globally Lorentz invariant, a subject to which is now turned: Assume that under a global infinitesimal 'Lorentz' transformation, ψ^ρ and s^ρ transform as

$$\delta\psi^\rho = -\frac{1}{2}(d\theta_{\alpha\beta})\left(\mathbf{S}^{\alpha\beta}\right)^\rho{}_\sigma \psi^\sigma, \tag{36}$$

$$\delta s^\rho = -\frac{1}{2}(d\theta_{\alpha\beta})\left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\sigma s^\sigma \equiv -(d\theta^\rho{}_\sigma) s^\sigma, \tag{37}$$

with $\left(\mathbf{S}^{\alpha\beta}\right)^\rho{}_\sigma$ and $\left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\sigma$ as previously defined. They are equivalent to

$$\delta\psi_\rho = \frac{1}{2}(d\theta_{\alpha\beta})\left(\mathbf{S}^{\alpha\beta}\right)^\sigma{}_\rho \psi_\sigma, \tag{38}$$

$$\delta s_\rho = \frac{1}{2}(d\theta_{\alpha\beta})\left(\mathbf{V}^{\alpha\beta}\right)^\sigma{}_\rho s_\sigma \equiv (d\theta^\sigma{}_\rho) s_\sigma, \tag{39}$$

due to Eqs. (21)-(22). The infinitesimal parameters $d\theta_{\alpha\beta} = -d\theta_{\beta\alpha} \in \mathbb{R}$ are assumed to be spacetime-independent, of course, as befits a global transformation (in flat spacetime in Cartesian coordinates). The overall sign of $d\theta_{\alpha\beta}$ has been chosen with foresight to have the standard vierbein to be introduced as an auxiliary/calculational device in Sec. 6 transform standardly under Lorentz transformations, compare Eq. (53). Due to the defining Eqs. (9) and (26), the variations δs^ρ and δs_ρ induce the following variations:

$$\delta M^{\mu\rho}{}_\sigma = -\frac{1}{2}(d\theta_{\alpha\beta})\left[\left(\mathbf{V}^{\alpha\beta}\right)^\mu{}_\tau M^{\tau\rho}{}_\sigma + \left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\tau M^{\mu\tau}{}_\sigma - \left(\mathbf{V}^{\alpha\beta}\right)^\tau{}_\sigma M^{\mu\rho}{}_\tau\right], \tag{40}$$

$$\delta N^\rho{}_\sigma = -\frac{1}{2}(d\theta_{\alpha\beta})\left[\left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\tau N^\tau{}_\sigma - \left(\mathbf{V}^{\alpha\beta}\right)^\tau{}_\sigma N^\rho{}_\tau\right]. \tag{41}$$

Substituting Eqs. (36), (38) and (40)-(41) into $\delta\mathcal{L}$, it is readily seen that $\delta\mathcal{L} = 0$ if and only if the following conditions are satisfied:

$$\begin{aligned} 0 &= \left[\left(\mathbf{S}^{\alpha\beta}\right)^\rho{}_\tau - \left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\tau\right]^* N^\tau{}_\sigma - \left[\left(\mathbf{S}^{\alpha\beta}\right)^\tau{}_\sigma - \left(\mathbf{V}^{\alpha\beta}\right)^\tau{}_\sigma\right]^* N^\rho{}_\tau, \\ 0 &= \left[\left(\mathbf{S}^{\alpha\beta}\right)^\rho{}_\tau - \left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\tau\right]^* M^{\mu\tau}{}_\sigma - \left[\left(\mathbf{S}^{\alpha\beta}\right)^\tau{}_\sigma - \left(\mathbf{V}^{\alpha\beta}\right)^\tau{}_\sigma\right] M^{\mu\rho}{}_\tau - \left(\mathbf{V}^{\alpha\beta}\right)^\mu{}_\tau M^{\tau\rho}{}_\sigma; \end{aligned}$$

or, equivalently:

$$
\begin{aligned}
0 &= \left(\mathbf{S}^{\alpha\beta}\right)^{\rho}{}_{\tau} N^{\tau}{}_{\sigma} - \left(\mathbf{S}^{\alpha\beta}\right)^{\tau}{}_{\sigma} N^{\rho}{}_{\tau}, && (42)\\
0 &= \left(\mathbf{S}^{\alpha\beta}\right)^{\rho}{}_{\tau} M^{\mu\tau}{}_{\sigma} - \left[\left(\mathbf{S}^{\alpha\beta}\right)^{\tau}{}_{\sigma}\right]^{*} M^{\mu\rho}{}_{\tau} + \left(\mathbf{V}^{\alpha\beta}\right)^{\mu}{}_{\tau} M^{\tau\rho}{}_{\sigma}, && (43)
\end{aligned}
$$

using Eq. (23) and its complex conjugate, as well as the real-valuedness of $\mathbf{V}^{\alpha\beta}$; or, equivalently:

$$
\begin{aligned}
\mathbf{0} &= \mathbf{S}^{\alpha\beta}\mathbf{N} - \mathbf{N}\mathbf{S}^{\alpha\beta} \equiv \left[\mathbf{S}^{\alpha\beta}, \mathbf{N}\right], && (44)\\
\mathbf{0} &= \mathbf{S}^{\alpha\beta}\mathbf{M}^{\mu} - \mathbf{M}^{\mu}\mathbf{S}^{\alpha\beta *} + \left(\mathbf{V}^{\alpha\beta}\right)^{\mu}{}_{\tau}\mathbf{M}^{\tau}\\
&\equiv \mathbf{S}^{\alpha\beta}\mathbf{M}^{\mu} - \mathbf{M}^{\mu}\mathbf{S}^{\alpha\beta *} + \left(g^{\alpha\mu}\delta^{\beta}_{\tau} - g^{\beta\mu}\delta^{\alpha}_{\tau}\right)\mathbf{M}^{\tau}\\
&= \mathbf{S}^{\alpha\beta}\mathbf{M}^{\mu} - \mathbf{M}^{\mu}\mathbf{S}^{\alpha\beta *} + g^{\mu\alpha}\mathbf{M}^{\beta} - g^{\mu\beta}\mathbf{M}^{\alpha}. && (45)
\end{aligned}
$$

These conditions are indeed satisfied due to Eqs. (18) and (32), thus proving that $\mathcal{L}$ is globally 'Lorentz' invariant (the use of quotation marks here and previously is as a reminder that the generators obey the 'modified Lorentz algebra', Eq. (19)).

5.2 Nonflat spacetime

In going to nonflat spacetime (or employing curvilinear coordinates in flat spacetime, for that matter), the previously given Lagrangian, Eq. (33), must be generalized to

$$
\mathcal{L} = \frac{\mathrm{i}}{2}\psi^{*}_{\rho} M^{\mu\rho}{}_{\sigma}\nabla_{\mu}\psi^{\sigma} - \frac{\mathrm{i}}{2}\left(\nabla_{\mu}\psi_{\rho}\right)^{*} M^{\mu\rho}{}_{\sigma}\psi^{\sigma} + \frac{m}{2}\left[\psi^{*}_{\rho} N^{\rho}{}_{\sigma}\psi^{\sigma *} - \psi_{\rho}\left(N^{\rho}{}_{\sigma}\right)^{*}\psi^{\sigma}\right]. \quad (46)
$$

In order to retain the previously derived 'Klein-Gordon compatibility' of the equations of motion (for the case of flat spacetime in Cartesian coordinates), it is necessary that $\nabla_{\nu} M^{\mu\rho}{}_{\sigma} = 0$ and $\nabla_{\nu} N^{\rho}{}_{\sigma} = 0$, i.e., that $M^{\mu\rho}{}_{\sigma}$ and $N^{\rho}{}_{\sigma}$ are covariantly constant. In view of Eqs. (9) and (26), this will certainly be the case if s_{μ} itself is identically covariantly constant:

$$
0 \equiv \nabla_{\nu} s_{\mu} \equiv \partial_{\nu} s_{\mu} - \Gamma^{\rho}{}_{\mu\nu} s_{\rho}, \quad (47)
$$

a condition that is uniquely satisfied by

$$
\Gamma^{\rho}{}_{\mu\nu} = \left\langle s^{\rho}, \partial_{\nu} s_{\mu}\right\rangle, \quad (48)
$$

the uniqueness being due to s_{μ} being a basis of $\mathbb{C}\otimes\mathbb{H}$, i.e., $\mathbb{C}\otimes\mathbb{H} = \mathrm{Span}_{\mathbb{C}}\left(s_{\mu}\right)$, as noted in Sec. 2. These connection coefficients are real-valued (as they should be in order to be sensible in the realm of a Riemann-Cartan spacetime):

$$
\begin{aligned}
\left(\Gamma^{\rho}{}_{\mu\nu}\right)^{*} &= \left\langle \overline{s}^{\rho}, \partial_{\nu}\overline{s}_{\mu}\right\rangle\\
&= \left\langle s^{\rho}, \partial_{\nu} s_{\mu}\right\rangle\\
&= \Gamma^{\rho}{}_{\mu\nu},
\end{aligned}
$$

using $s_\mu^* = \overline{s}_\mu$ and Eq. (3). The covariant derivative of the components of the spinor field are then obviously taken to be

$$\nabla_\nu \psi^\rho \equiv \partial_\nu \psi^\rho + \Gamma^\rho{}_{\mu\nu}\psi^\mu, \tag{49}$$

as alluded to in the Introduction. Note that $\nabla_\nu s_\mu \equiv 0$ does not only imply covariant constancy of $M^{\mu\rho}{}_\sigma$ and $N^\rho{}_\sigma$, but in view of Eqs. (15) and (20) also, quite satisfactorily, covariant constancy of $S^{\alpha\beta\rho}{}_\sigma$ and $V^{\alpha\beta\rho}{}_\sigma$ (and thus as well of the corresponding 'Lorentz' generators):

$$0 \equiv \nabla_\nu S^{\alpha\beta\rho}{}_\sigma, \tag{50}$$

$$0 \equiv \nabla_\nu V^{\alpha\beta\rho}{}_\sigma, \tag{51}$$

Concerning global 'Lorentz' invariance: Assume that the now generically spacetime-dependent parameters $d\theta_{\alpha\beta}$, figuring in Eqs. (36)-(37), are covariantly constant:

$$0 = \nabla_\nu d\theta_{\alpha\beta}. \tag{52}$$

(This condition does of course not alter any of the previous derivations for flat spacetime in Cartesian coordinates concerning invariance under global 'Lorentz' transformations.) It implies that the connection coefficients, Eq. (48), are globally 'Lorentz' invariant:

$$\begin{aligned}
\delta\Gamma^\rho{}_{\mu\nu} &\equiv \langle \delta s^\rho, \partial_\nu s_\mu \rangle + \langle s^\rho, \partial_\nu \delta s_\mu \rangle \\
&= -\left(d\theta^\rho{}_\sigma\right)\langle s^\sigma, \partial_\nu s_\mu\rangle + \langle s^\rho, \partial_\nu \left[\left(d\theta^\sigma{}_\mu\right) s_\sigma\right]\rangle \\
&= -\left(d\theta^\rho{}_\sigma\right)\langle s^\sigma, \partial_\nu s_\mu\rangle + \left(d\theta^\sigma{}_\mu\right)\langle s^\rho, \partial_\nu s_\sigma\rangle + \left(\partial_\nu d\theta^\sigma{}_\mu\right)\langle s^\rho, s_\sigma\rangle \\
&= \partial_\nu d\theta^\rho{}_\mu + \Gamma^\rho{}_{\sigma\nu} d\theta^\sigma{}_\mu - \Gamma^\sigma{}_{\mu\nu} d\theta^\rho{}_\sigma \\
&\equiv \nabla_\nu d\theta^\rho{}_\mu.
\end{aligned}$$

This in turn implies that $\delta\nabla_\mu = 0$, i.e., that ∇_μ is globally 'Lorentz' invariant. Therefore,

$$\begin{aligned}
\delta\left(\nabla_\mu \psi^\rho\right) &= \nabla_\mu \delta\psi^\rho \\
&= -\frac{1}{2}\nabla_\mu \left[\left(d\theta_{\alpha\beta}\right)\left(\mathbf{S}^{\alpha\beta}\right)^\rho{}_\sigma \psi^\sigma\right] \\
&= -\frac{1}{2}\left(d\theta_{\alpha\beta}\right)\left(\mathbf{S}^{\alpha\beta}\right)^\rho{}_\sigma \nabla_\mu \psi^\sigma,
\end{aligned}$$

using Eqs. (50) and (52), i.e., $\nabla_\mu \psi^\rho$ transforms under global 'Lorentz' transformations as ψ^ρ itself does. From this, it readily follows that the above Lagrangian is itself globally 'Lorentz' invariant. In conjunction with global 'Lorentz' invariance of the metric (note that this holds for arbitrary $d\theta_{\alpha\beta} = -d\theta_{\beta\alpha}$):

$$\begin{aligned}
\delta\langle s_\mu, s_\nu\rangle &\equiv \langle \delta s_\mu, s_\nu\rangle + \langle s_\mu, \delta s_\nu\rangle \\
&= \left(d\theta^\rho{}_\mu\right)\langle s_\rho, s_\nu\rangle + \left(d\theta^\rho{}_\nu\right)\langle s_\mu, s_\rho\rangle \\
&= d\theta_{\nu\mu} + d\theta_{\mu\nu} \\
&\equiv 0,
\end{aligned}$$

this implies that the action $S = \int \mathcal{L}\sqrt{-g}d^4x$ is globally 'Lorentz' invariant. By construction it is, of course, also coordinate invariant. Note that the only conditions posed are the ones of covariant constancy of s_μ and $d\theta_{\alpha\beta}$, Eqs. (47) and (52).

6 Making contact with general relativity

Let $s^a \in (\mathbb{C} \otimes \mathbb{H})^+$ be a spacetime-independent basis for $\mathbb{C} \otimes \mathbb{H}$ for which $\langle s^a, s^b \rangle = \eta^{ab}$, and consider the expansion $s^\mu = e^\mu{}_a s^a$, where $e^\mu{}_a = \langle s^\mu, s_a \rangle \in \mathbb{R}$. Inserting this expansion into the expression for δs^ρ, Eq. (37), implies that

$$\begin{aligned} \delta e^\rho{}_c &= -\frac{1}{2}\left(d\theta_{\alpha\beta}\right)\left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\sigma e^\sigma{}_c \\ &= -\frac{1}{2}\left(d\theta_{ab}\right)\left(\mathbf{V}^{ab}\right)^d{}_c e^\rho{}_d, \end{aligned} \tag{53}$$

using the identity $\left(\mathbf{V}^{\alpha\beta}\right)^\rho{}_\sigma = e^\alpha{}_a e^\beta{}_b e^\rho{}_r e^s{}_\sigma \left(\mathbf{V}^{ab}\right)^r{}_s$ with $\left(\mathbf{V}^{ab}\right)^c{}_d \equiv \eta^{ac}\delta^b_d - \eta^{bc}\delta^a_d$ the standard vector representation of the Lorentz algebra, and introducing $d\theta_{ab} \equiv e^\alpha{}_a e^\beta{}_b d\theta_{\alpha\beta}$. This is the standard infinitesimal Lorentz transformation of a vierbein, with transformation parameters $d\theta_{ab}$. The expansion coefficients $e^\mu{}_a$ in conjunction thus seem identifiable with the standard vierbein. In terms of this vierbein, the previously introduced condition of covariant constancy of s_μ, Eq. (47), becomes

$$\nabla_\nu e^\mu{}_a \equiv 0, \tag{54}$$

i.e., the covariant constancy of the vierbein, and the previously introduced connection coefficients, Eq. (48), become

$$\begin{aligned} \Gamma^\rho{}_{\mu\nu} &= \left\langle e^\rho{}_a s^a, \partial_\nu \left(e^b{}_\mu s_b\right)\right\rangle \\ &= e^\rho{}_a \left(\partial_\nu e^b{}_\mu\right) \langle s^a, s_b \rangle \\ &= e^\rho{}_a \partial_\nu e^a{}_\mu, \end{aligned} \tag{55}$$

i.e., the standard Cartan connection, carrying torsion, but no curvature, the setting thus being that of Weitzenböck spacetime and teleparallelism. The spinor Lagrangian, Eq. (46), and the formalism associated with it thus seems to be consistent with the teleparallel formulation of general relativity. A few comments:

- In conjunction with Eq. (52), the covariant constancy of the vierbein, $\nabla_\nu e^\mu{}_a \equiv 0$, immediately implies that $\partial_\nu d\theta_{ab} \equiv \nabla_\nu d\theta_{ab} \equiv 0$, i.e., that the infinitesimal parameters $d\theta_{ab}$ are spacetime-independent, as befits a global Lorentz transformation. Remember that for teleparallelism, only *global* Lorentz transformations are relevant, the local degrees of freedom being frozen out.
- The spinor Lagrangian, Eq. (46), may of course be expanded in terms of the vierbein, simply by inserting $s^\mu = e^\mu{}_a s^a$ into the definitions of $M^{\mu\rho}{}_\sigma$ and $N^\rho{}_\sigma$, Eqs. (9) and (26), but in view of the very aim of this article, that would obviously be counterproductive as it would reintroduce Lorentz indices into the formalism.
- The introduction and use in this section of the vierbein is *not* tantamount to introducing Lorentz indices back into the formalism. Its sole purpose is to be

an auxiliary/calculational device for establishing the consistency between the developed spinor Lagrangian formalism and the teleparallel formulation of general relativity. Note that although the teleparallel formulation of general relativity is build from the vierbein and its first order derivatives, its Lagrangian effectively depends only on the torsion tensor field, $T^{\rho}{}_{\mu\nu}$, compare for instance [8], which does not carry any Lorentz indices.

So, in conclusion, collecting everything, it seems that by combining the above spinor Lagrangian, Eq. (46), with the Lagrangian for the teleparallel formulation of general relativity, a formalism for the coupling of spinor fields to the gravitational field using only world indices is provided, as asserted.

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