Dirac equation in exotic space-times

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We find solutions of the Dirac equation in curved space-time. In particular, we consider 1+1 dimensional sections of several exotic metrics: the Alcubierre metric, which describes a scenario that allows faster-than-light (FTL) velocity; the Gödel metric, that describes a universe containing closed timelike curves (CTC); and the Kerr metric, which corresponds to the spacetime of a rotating black hole. Moreover, we also show that the techniques that we use in these cases can be extended to non-static metrics.

I. INTRODUCTION

In 1928, Paul Dirac formulated a special relativistic version of the wave equation in quantum mechanics ($\hbar = 1, c = 1$):

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0,$$

where, in the standard representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This equation was born as an attempt of linearization of the Klein-Gordon equation, that comes from the application of the Einstein relation $E^2 = p^2 + m^2$ to the Schrödinger equation Hamiltonian. The Dirac equation was a turning point in physics. Joining Quantum Mechanics and Relativity, this equation predicts the spin of the electron, the existence of antimatter etc. and it is considered as the natural transition between relativistic quantum mechanics and quantum field theory [1]. Moreover, a renewed interest in the Dirac equation has come from the field of quantum simulations, which has enabled the experimental observation of some of its more interesting features in tabletop experiments [2–8]. Nevertheless, even though the general solutions of this equation are well-known, this is not the case of the Dirac equation in curved space-time. In general, obtaining solutions in curved space is more complicated. Some particular 1+1 D cases can be found, for instance, in [11] and [10].

In this work, the method explained by one of us in [10] will be applied to obtain solutions of the Dirac equation in curved space-time for different exotic metrics in 1+1 dimensional sections, with the help of known solutions in Minkowski space-time. If we take a 1+1 dimensional section of the space-time, the Dirac equations reduces to [11]:

$$i\left(\partial_t + \frac{\dot{\Omega}}{2\Omega}\right)\psi = -i\sigma_x\left(\partial_x + \frac{\Omega'}{2\Omega}\right)\psi + \sigma_z \Omega m \psi$$

where Ω is the conformal factor – note that any metric in 1+1 dimensions allow a coordinate change such that it acquires the form $ds^2 = \Omega^2(-dt^2 + dx^2)$ – and σ_x and σ_z are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Given Ω, we could try to find solutions of this partial differential equation system. However, that procedure could be very involved, depending on the particular form of Ω. The method presented in [10] allows to find the solution of this partial differential equation system through a coordinate change and an existing solution in Minkowski space-time. Thus, the method will be applied to several exotic metrics of interest, such as the Alcubierre metric, which allows FTL velocity; to the Gödel metric, which allows CTC; and the Kerr metric, which describes the space-time geometry generated by a rotating body (rotating black holes). Furthermore, we will see that, although in [10] $\dot{\Omega} = 0$ (static space-time) is an apparent requirement for the validity of the method, in fact the method is also valid for non-static space-times.

Our results might be of interest in the context of quantum simulation of the Dirac equation. Several ideas have been proposed for simulating the Dirac equation in Minkowski space-time for 1+1 dimensional sections: for example, experiments have been already realized with trapped ions [2, 5] and Bose-Einstein condensates [7, 8], while realistic proposals exist with superconducting circuits [9] . Since we obtain solutions in curved space-time through transformations of Minkowski solutions, the results obtained in this work could be helpful for simulations of Dirac equation in non trivial space-times with currently existing setups.

The structure of the paper is the following. We start in Section II by recalling the procedure for obtaining solutions of the Dirac equation in curved spacetimes out of flat-spacetime solutions, first introduced in [10]. Then, we proceed to apply it in Section III to the aforementioned curved spacetimes of interest. Finally, we introduce a generalization of the method to non-static spacetimes in Section IV, which might be useful for further applications. We conclude in Section V with a summary of our results.

II. METHOD

We summarize here the procedure to obtain solutions of the Dirac equation in curved spacetimes, first discussed in [10]. The curved space-time Dirac equation in 1+1 dimensions for a massless particle is given by the following expression:

$$i\left(\partial_t + \frac{\dot{\Omega}}{2\Omega}\right)\psi = -i\sigma_x\left(\partial_x + \frac{\Omega'}{2\Omega}\right)\psi. \quad (1)$$

For the static case, in which the conformal factor is time-independent $\dot{\Omega} = 0$, the equation is:

$$i\partial_t\psi = -i\sigma_x\left(\partial_x + \frac{\Omega'}{2\Omega}\right)\psi. \quad (2)$$

If we make the transformation $\psi = \Omega^{-1/2}\phi$, the equation (2) becomes:

$$i\partial_t\phi = -i\sigma_x\partial_x\phi$$

which corresponds to the Dirac equation for a massless particle in Minkowski space-time, whose solution is well-known (see, for instance, [12]). Thus, using this transformation and the conformal factor of the 1+1 dimensional metric, we can find analytical solutions for the Dirac equation in curved space-time. So, for a given metric, we have to follow this three-step procedure:

1) Find a change of coordinates $(t, x) \to (\bar{t}, \bar{x})$ such that, in the new coordinates, the metric acquires the form $ds^2 = \Omega^2(-d\bar{t}^2 + d\bar{x}^2)$.

2) Using those new coordinates $(\bar{t}, \bar{x})$, apply $\psi(\bar{t}, \bar{x}) = \Omega^{-1/2}(\bar{x})\phi(\bar{t}, \bar{x})$, where $\phi(\bar{t}, \bar{x})$ is the solution of the Minkowski space-time Dirac equation.

3) Once having $\psi(\bar{t}, \bar{x})$, apply the coordinate change $(\bar{t}, \bar{x}) \to (t, x)$ to finally find the solution of the curved space-time Dirac equation in (t, x) coordinates.

Please note that, in spite of its apparent simplicity, in general it might not be necessarily straightforwerd to perform the steps 1) and 3). To write the metric in a conformally-flat form, it might be needed to express the change of coordinates in a differential form, which could generate an equation that might still not be straightforward to solve, as we will see in the (III B) section.

This method was applied in [10] to find solutions of the Dirac equation in a 1+1 dimensional section of a traversable wormhole spacetime. Let us now consider other examples of interest.

III. RESULTS

A. Gödel and Alcubierre metric

Both metrics will be discussed in the same section because of their similarity.

In 1994, Alcubierre proposed a metric that, in principle, allows FTL motion [13]. Note that, in general relativity, FTL speed is only forbidden locally. This is not as exotic as it might seem at first glance: for instance, the expansion of the universe can make that two distant galaxies move at FTL speed between them, while each one is moving locally inside its light-cone. While the oposite might be possible too: if the space-time were contracting fast enough, each galaxy were moving near the speed of light locally (inside its light-cone) in oposite directions, but globally both were getting closer. With these considerations in mind, Alcubierre's idea is simple: to create, in the front of an object, a space-time contraction, and in the back, a space-time dilation. Thus, the contraction will pull the object forward, and the dilation will push the object forward too. Locally the object will be inside its light-cone, but due to this space-time manipulation, it would move FTL– as compared with c, the speed of light in flat-spacetime vacuum.

The Alcubierre metric inside the "bubble" created by the space-temporal contraction/dilation, under the limit $\sigma \to \infty$ [13], and taking a spatial section $y = y_0$ and $z = z_0$ (with constant y_0, z_0), acquires the form [13, 14]:

$$ds^2 = -(1 - v_s^2)dt^2 + dx^2 - 2v_s dx dt. \quad (3)$$

As expected, Eq. (3) becomes the Minkowski metric when $v_s = 0$. Now we make a coordinate change $(t, x) \to (\bar{t}, \bar{x})$, such that:

$$dt = d\bar{t}$$
$$dx = d\bar{x} + v_s d\bar{t}.$$

The Alcubierre metric, using those new coordinates, acquires the form $ds^2 = -d\bar{t}^2 + d\bar{x}^2$, that is simply the Minkowski metric, so the conformal factor will be $\Omega^2 = 1$, and the solution of the curved space-time Dirac equation, $\psi(\bar{t}, \bar{x})$, will be equal to the Minkowski space-time solution, $\phi(\bar{t}, \bar{x})$. Now, due to the big similarity – regarding to the application of our techniques – between the Alcubierre and Gödel metric, we first discuss the latter case and then we compare both results.

K. Gödel, in 1949, found a solution of the Einstein equations [15] corresponding to an homogeneous mass distribution that rotates at each point of the space [16]. That distribution of matter causes unusual effects, such as the existence of CTC. In cylindrical coordinates [16], the metric is given by the following expression:

$$\begin{aligned} ds^2 =& dt^2 - \frac{dr^2}{1+(\frac{r}{2a})^2} - r^2\left(1-\left(\frac{r}{2a}\right)^2\right)d\phi^2 - dz^2 \\ &+ \frac{2r^2}{a\sqrt{2}}dtd\phi \end{aligned} \tag{4}$$

where a is a parameter with units of length, that represents a characteristic distance. In particular, $r_G = 2a$ represents the critical radius from which CTC can exist [16].

Now, taking a radial section with $\phi = \phi_0$ and $z = z_0$ [14], the Gödel metric becomes:

$$ds^2 = dt^2 - \frac{1}{1+\left(\frac{r}{2a}\right)^2}dr^2$$

By making the coordinate change $(t, r) \to (\bar{t}, \bar{r})$, with

$$\begin{aligned} d\bar{r}^2 &= \frac{1}{1+\left(\frac{r}{2a}\right)^2}dr^2 \\ d\bar{t}^2 &= dt^2 \end{aligned} \tag{5}$$

the previous metric transforms into the Minkowski metric $ds^2 = d\bar{t}^2 - d\bar{r}^2$. Furthermore, we can find the relationship between the (t, r) and $(\bar{t}, \bar{r})$ coordinates by performing the following integration:

$$\bar{r}(r) = \int \frac{dr}{\sqrt{1+\left(\frac{r}{2a}\right)^2}}$$

whose solution – setting $\bar{r}(r = 0) = 0$– is:

$$\bar{r}(r) = 2a\sinh^{-1}\left(\frac{r}{2a}\right).$$

With these new coordinates, the conformal factor becomes $\Omega^2 = 1$, and the Minkowski space-time solution will be the same as the curved space-time solution, as in the case of the Alcubierre metric. Thus, the only difference between Gödel and Alcubierre metric is the relationship between the original coordinates and the new coordinates.

Now, if we assume that the wave function has a gaussian initial form:

$$\phi(\bar{x}, 0) = Ne^{-\frac{(\bar{x}-\bar{x}_0)^2}{\sigma^2}}\begin{pmatrix}1\\1\end{pmatrix}$$

where $\bar{x}$ is the conformally flat coordinate and N is a normalization constant, the solution for the Dirac equation in Minkowski's space-time will be [10, 12]:

$$\phi(\bar{x}, \bar{t}) = Ne^{-\frac{(\bar{t}-(\bar{x}-\bar{x}_0))^2}{\sigma^2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{6}$$

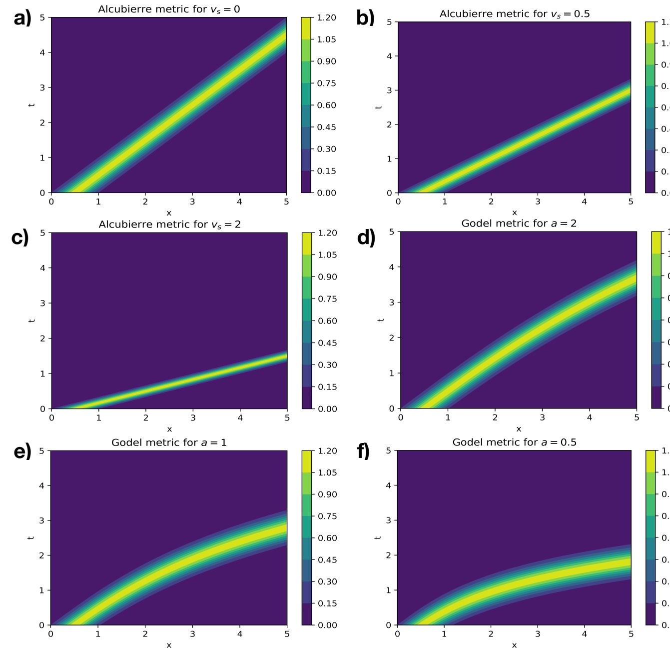

FIG. 1. Space-time diagrams for the probability density of Dirac equation solutions in a-c) Alcubierre metric with a) $v_s = 0$ (equivalent to Minkowski spacetime), b) $v_s = 0.5$ (subluminal space-time bubble), c) $v_s = 2$ (superluminal space-time bubble); d-f) Gödel metric with d) $a = 2$, e) $a = 1$, f) $a = 0.5$. Please note that the wave function is not normalized.

Since $\psi = \phi$, to find the curved space-time solution we only have to apply the coordinate change for each metric. In the Alcubierre case we have:

$$\phi\Big(\bar{x}(x), \bar{t}(t)\Big) = Ne^{-\frac{(t-(x-v_s t-\bar{x}_0))^2}{\sigma^2}}\begin{pmatrix}1\\1\end{pmatrix}, \tag{7}$$

while, for the Gödel metric:

$$\phi\Big(\bar{r}(r), \bar{t}(t)\Big) = Ne^{-\frac{(t-(2a\sinh^{-1}(r/2a)-\bar{x}_0))^2}{\sigma^2}}\begin{pmatrix}1\\1\end{pmatrix}. \tag{8}$$

Both solutions are represented in Figure 1.

For a), b) and c), as the parameter v_s increases, the slope of the gaussian wavepacket's "trajectory" decreases. In other words, the particle's velocity increases as v_s is bigger. In fact, if we take a look to the expression (7), we can see its equivalence to the solution of the Minkowski space-time Dirac equation (6) if we substitute t with $(1+v_s)t$. This means that, physically, the particle moves with a FTL velocity $\bar{c} = 1 + v_s$, as expected for a massless particle since this is precisely the speed of light in this spacetime.

For d), e) and f), we can see that the particle velocity is no longer constant. This variation is inversely proportional to the value of a, as expected, since in the limit $a \to \infty$ the Gödel metric (4) becomes the Minkowski metric in cylindrical coordinates. In addition, the slope decreases with the propagation of the particle, implying

FTL velocity. Of course, in general the existence of CTC –which is the most genuine feature of Gödel spacetime– is intimately linked with the possibility of FTL (see, for instance, [17]).

B. Rotating black hole

The main motivation to study the Kerr metric is the fact that it describes the space-time geometry of a rotating body, so it can be used to analyse the physics of black holes in a more realistic scenario than the one provided by the Schwarzschild metric. Using Boyer-Lindquist coordinates (t, r, θ, ϕ), the Kerr metric is [18][19]:

$$ds^2 = -\Big(1 - \frac{2Mr}{\Sigma}\Big)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\theta + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \Big(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\Big)\sin^2\theta d\phi^2$$

where

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta \qquad \Delta(r) = r^2 + a^2 - 2Mr$$

If we take a radial section ($\theta = \theta_0$, $\phi = \phi_0$), we obtain [14]:

$$ds^2 = -\Big(1 - \frac{2Mr}{\Sigma(r,\theta_0)}\Big)dt^2 + \frac{\Sigma(r,\theta_0)}{\Delta(r)}dr^2,$$

where $\Sigma(r,\theta_0)$ is a function of the coordinate r (for simplicity, $\Sigma(r,\theta_0) = \Sigma(r)$). So, if we make the coordinate change $(t, r) \to (\bar{t}, \bar{x})$ with:

$$\begin{aligned} \frac{\Sigma(r)}{\Delta(r)}dr^2 &= \Big(1 - \frac{2Mr}{\Sigma(r)}\Big)d\bar{x}^2 \\ dt^2 &= d\bar{t}^2 \end{aligned} \tag{9}$$

the metric acquires the form

$$ds^2 = \Omega^2(r)(-d\bar{t}^2 + d\bar{x}^2)$$

with

$$\Omega^2(r) = 1 - \frac{2Mr}{\Sigma(r)}.$$

Due to the conditions (9), r will be a function of $\bar{x}$, whereas $t = \bar{t}$, so the conformal factor $\Omega(r)$ will be a function of $r(\bar{x})$:

$$\Omega^2\big(r(\bar{x})\big) = 1 - \frac{2Mr(\bar{x})}{\Sigma\big(r(\bar{x})\big)} \tag{10}$$

To find the conformal factor $\Omega\big(r(\bar{x})\big)$ we need to obtain $r(\bar{x})$. Using the first equation in (9) and, after doing some algebraic manipulations, we obtain that $\bar{x}(r)$ is given by:

$$\bar{x}(r) = \int dr\Sigma(r)\sqrt{\frac{1}{\Delta(r)\big(\Sigma(r) - 2Mr\big)}} + C \tag{11}$$

In general, solving this integral is not straightforward. Later, we will analyze some particular cases in which the integral can be solved. By now, let us just assume that $r(\bar{x})$ is known. Thus, the conformal factor is be given by the equation (10) and, therefore, the relation between the solution to Dirac equation in curved space-time (ψ) and flat space-time (ϕ) will be:

$$\begin{aligned} |\psi(\bar{x},\bar{t})|^2 &= \Big|\Omega^{-1/2}\big(r(\bar{x})\big)\Big|^2|\phi(\bar{x},\bar{t})|^2 \\ &= \left|\sqrt{\frac{\Sigma\big(r(\bar{x})\big)}{\Sigma\big(r(\bar{x})\big) - 2Mr(\bar{x})}}\right||\phi(\bar{x},\bar{t})|^2 \end{aligned} \tag{12}$$

All the wave function properties caused by the space-time curvature are contained in the factor $\Omega^{-1}\big(r(\bar{x})\big)$, so it is interesting to analyze this function. First, there exist regions in which the probability density of the wave function in curved space-time becomes infinite. It happens when the following condition is fulfilled – we relax the notation and write r instead of $r(\bar{x})$:

$$\Sigma(r) - 2Mr = 0 \tag{13}$$

This condition is satisfied whenever $r = r_\pm$, with:

$$r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta_0}, \tag{14}$$

which are the well-known apparent singularities where the temporal component of the metric changes sign, defining the ergosphere [20]. On the other hand, there exist points in which the wave function is null, independently of the form of the solution in Minkowski's space-time. This will happen whenever $\Omega^{-1}(r) = 0$, thats is, when:

$$\Sigma(r) = 0 \qquad \to \qquad r^2 + a^2\cos^2\theta_0 = 0,$$

which will be satisfied when at least one of those conditions apply:

1) $r = 0$, $a = 0$ (no rotation)
2) $r = 0$, $\theta_0 = \frac{\pi}{2}$.

Finally, there is a region of the space in which the conformal factor is an imaginary number. This occurs if:

$$\Sigma(r) - 2Mr < 0.$$

This second-degree inequality is verified for $r_- < r < r_+$, namely, within the ergosphere.

1. Radial section with $\theta_0 = 0$

Let u consider $\theta_0 = 0$. Then $\Sigma(r) = r^2 + a^2$, $\Delta(r) = \Sigma - 2Mr$, and the integral becomes:

$$\bar{x}(r) = \int dr \frac{\Sigma(r)}{\Delta(r)} + C = \int dr \frac{r^2 + a^2}{r^2 + a^2 - 2Mr} + C$$

whose solution is

$$\begin{aligned}\bar{x}(r) =& \frac{2M^2}{\sqrt{a^2 - M^2}} \tan^{-1}\left(\frac{r - M}{\sqrt{a^2 - M^2}}\right) \\ &+ M \log\left(a^2 - 2Mr + r^2\right) + r + C\end{aligned}$$

Bearing in mind that, for a rotating black hole, $M^2 > a^2$ is satisfied (otherwise, it would not be a black hole, but a highly rotating body [21]), and using the relationship $\tan^{-1}(ix) = i \tanh^{-1}(x)$, we can write the solution as:

$$\begin{aligned}\bar{x}(r) =& \frac{2M^2}{\sqrt{M^2 - a^2}} \tanh^{-1}\left(\frac{M - r}{\sqrt{M^2 - a^2}}\right) \\ &+ M \log\left(a^2 - 2Mr + r^2\right) + r + C\end{aligned} \tag{15}$$

The constant of integration can be chosen in such a way that the constant imaginary part of Eq. (15) cancels out in each of the three spacetime regions defined by the apparent singularities $r_\pm$. Thus:

$$C = \begin{cases} -i\dfrac{\pi M^2}{\sqrt{M^2 - a^2}} & si \quad r < r_- \\ -iM\pi & si \quad r_- < r < r_+ \\ -i\dfrac{\pi M^2}{\sqrt{M^2 - a^2}} & si \quad r_+ < r \end{cases}$$

In Figure 2, we plot $\bar{x}(r)$ for different values of M and a.

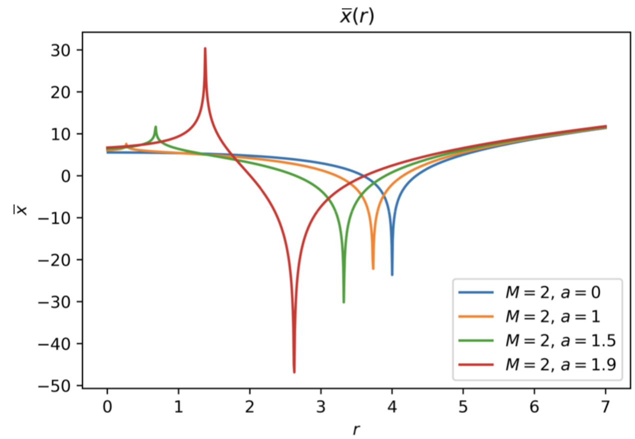

FIG. 2. $\bar{x}(r)$ for different values of the parameter a. In absence of rotation ($a = 0$), the singular points are in $r = 0$ and $r = 2M$ (Schwarzschild's black hole). As the rotation increases, both points ($r = r_\pm$) come closer to $r = M$.

Finally, assuming again that our particle is characterized by a Gaussian wavepacket, the solution in Kerr spacetime is:

$$\begin{aligned}\left|\psi(\bar{x}(r), t)\right|^2 &= \sqrt{\frac{\Sigma(r)}{\Sigma(r) - 2Mr}} \left|\phi(\bar{x}(r), t)\right|^2 \\ &= 2N^2 \sqrt{\frac{\Sigma(r)}{\Sigma(r) - 2Mr}} e^{-\frac{2(t - (\bar{x}(r) - \bar{x}_0))^2}{\sigma^2}}\end{aligned}$$

This solution is represented in Figure (3) for several values of a and M.

Naturally, when we set $M = a = 0$, the metric becomes the Minkowski one, $\Omega^2 = 1$, and the expression (15) reduces to $\bar{x}(r) = r$. In other words, the solution corresponds to a free particle in flat spacetime, as it is observed in Figure (3 a). If we slightly increase the value of M (with $a = 0$, thus we are in Schwarzschild spacetime), we see that the free solution is slightly disturbed, increasing the probability density near the horizon, as can be seen in Figure (3 b). As the mass increases, the probability density accumulates nearby the horizon, placed in $r = 2M$. In addition, we see that the width of the wavepacket is drastically reduced (the particle is more localized in space), and the particle's speed diminishes, vanishing in the horizon. Finally, if we set the mass to a constant value and we change a (Figures (3 e-h), we see that, as a increases, the inner horizon is generated, around which, again, the probability density accumulates in the same way as before. In addition, in the region between both singular points –the ergosphere– we see how the particle tends to be expelled towards the interior or exterior horizon.

IV. NON-STATIC SPACE-TIMES

So far, we have only analyzed static metrics, that is, $\dot{\Omega} = 0$. We now show that our techniques are also valid for the case of non-static metrics ($\dot{\Omega} \neq 0$). Performing the substitution $\psi = \Omega^{-1/2}\phi$ in Equation (1) and with a little algebra, we find:

$$i\left(\partial_t + \frac{\dot{\Omega}}{2\Omega}\right)(\Omega^{-1/2}\phi) = -i\sigma_x\left(\partial_x + \frac{\Omega'}{2\Omega}\right)(\Omega^{-1/2}\phi)$$

$$\partial_t(\Omega^{-1/2}\phi) + \frac{\dot{\Omega}}{2\Omega}\Omega^{-1/2}\phi = -\sigma_x\left[\partial_x(\Omega^{-1/2}\phi) + \frac{\Omega'}{2\Omega}\Omega^{-1/2}\phi\right]$$

Notice that

$$\partial_t(\Omega^{-1/2}\phi) = -\frac{1}{2}\Omega^{-3/2}\dot{\Omega}\phi + \Omega^{-1/2}\partial_t\phi. \tag{16}$$

Therefore the terms not containing $\partial_t\phi$ cancel out. The same thing happens with the spatial partial derivative:

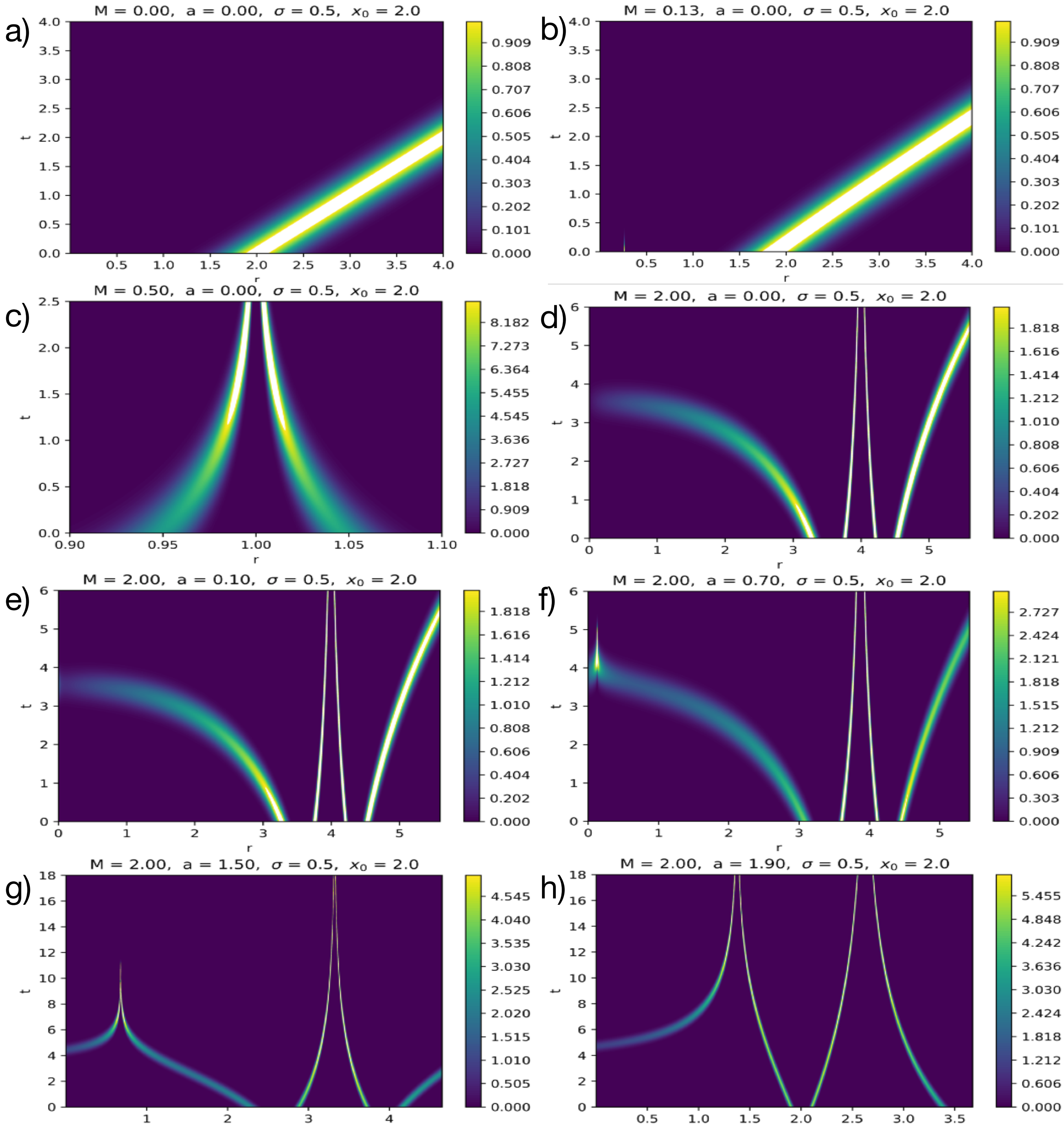

FIG. 3. Dirac equation solution for Kerr metric and different values of the parameters a and M. a)-d), $a = 0$ (no rotation, Schwarzschild metric) and M ranging from 0 to 2. The horizon is at $r = 2M$. e)-h) $M = 2$ and a ranging from 0 to 2, two singularities appear according to Eq. (14) with $\theta_0 = 0$ giving rise to an ergosphere which splits the full spacetime into three separate regions . Notice that as a increases r_+, r_- move from $2M$ and 0, respectively, to M, where they would merge in the limit $M = a$. Please note that the wave function is not normalized.

the terms without $\partial_x \phi$ cancel out. Thus, putting everything together we recover the equation:

$$i\partial_t \phi = -i\sigma_x \partial_x \phi$$

that is the Dirac equation for Minkowski space-time. Therefore, our techniques could be also applied to non-static spacetimes, such as FRLW spacetime, as in [11].

V. CONCLUSIONS

Following [10], we have shown that is possible to obtain solutions of the Dirac equation in curved spacetime by means of a transformation $\psi(\bar{x}, \bar{t}) = \Omega^{-1/2}(\bar{x})\phi(\bar{x}, \bar{t})$ where $\psi(\bar{x}, \bar{t})$ is the curved space-time solution, $\phi(\bar{x}, \bar{t})$ is the Minkowski space-time solution and $\Omega(\bar{x})$ is the conformal factor. Then, in order to be able to apply this transformation, it is necessary to perform first a coordinate change $(x, t) \rightarrow (\bar{x}, \bar{t})$, such that in the new coordinates, the metric is conformally flat. We have applied this technique to three different metrics: 1+1 dimensional sections of Alcubierre, Gödel and Kerr metrics. For Alcubierre and Gödel, the conformal factor turns out to be $\Omega^2 = 1$, so the only difference between both solutions is the coordinate transformation employed in each case. Setting a gaussian solution in Minkowski space-time, and after undoing the coordinate transformation, we obtain the solutions (7) and (8), for Alcubierre and Gödel spacetimes, respectively. In this way, we have been able to analyse the FTL behaviour of massless particles in these exotic spacetimes. In the case of the Kerr metric, the conformal factor is non-trivial $\Omega^2(r) = 1 - \frac{2Mr}{\Sigma(r)}$. We find an analytical solution in a spatial section with $\theta_0 = 0$. We have discussed the dependence of this solution on the black-hole mass and rotation, as well as the behaviour of the particle near the singularities and within the ergosphere. Finally, we have shown that our technique is valid also for non-static space-times.

Our results might, in principle, be useful in the context of quantum simulations of the Dirac equation. In particular, following the spirit of [10], notice that our technique entails that curved spacetimes can be encoded into a transformation realized onto a flat-spacetime Dirac wavepacket. Therefore, in principle, our results could be tested in already existing quantum simulators of the Dirac equation in flat spacetime.

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