

Note about the spin connection in general relativity

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In general relativity the fermions are treated from the perspective of the gauged Lorentz group and by introducing the corresponding gauge fields the spin connection. This procedure is intimately related to the so-called "vielbein" formalism and stems from the general structure that can be associated to a manifold, the affine connection. In this work we derive the correct spin connection based only on the general covariance of the theory and on the known space-time properties of fermion bilinears generalized to the curved space. Our result coincides exactly with the spin connection obtain through the tetrad formalism.

In general relativity the ordinary derivative of a tensor, in order to obtain the general behavior of a tensor, is replaced by the covariant derivative written in terms of an affine connection. For example for a vector the covariant derivative is given by:

$$\partial_\mu V_\nu \rightarrow D_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}^\rho V_\rho. \quad (1)$$

Here $\Gamma_{\mu\nu}^\rho$ represents the affine connection which may be independent of a metric. In particular however it is always more amenable to consider an affine connection that satisfies two main requirements [1]:

- a) to be torsion free, i.e. $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$.
- b) to be metric compatible which amount to asking that the covariant derivative of the metric tensor is zero:

$$\Delta_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}^\sigma g_{\sigma\nu} - \Gamma_{\rho\nu}^\sigma g_{\sigma\mu} = 0. \quad (2)$$

Having established how a derivative of a tensor field must be modified in curved space time one needs to consider another type of fields of relevance in a quantum field theory, the fermion fields. In QFT the fermions lie in a four dimensional representation of the Lorentz group $SO(3, 1)$ given by the gamma matrices which span a Clifford algebra with the anti-commutation rule:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad (3)$$

where η^{ab} is the Minkowski metric. The natural approach in the presence of a curved space-time and of a general coordinate transformation would be then to gauge the Lorentz group and to introduce the gauge fields associated to this, the spin connection. Then in a formalism introduced by Cartan and developed further in [2], [3] one defines the gamma matrices in the curved space as:

$$\gamma^\mu(x) = \gamma^a e_a^\mu, \quad (4)$$

where γ^μ depend on the coordinate, μ is the index in the curved space and a is the index in the flat space. The quantities e_a^μ are called a tetrad and satisfy the relation:

$$g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}. \quad (5)$$

It is considered that the gamma matrices in the curved space satisfy a generalized Clifford algebra with the anti commutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (6)$$

If one requires further that the operations of parallel transport and projection on flat and curved indices commute one arrives of the vielbein postulate:

$$D_\rho e_\mu^m = \partial_\rho e_\mu^m(x) - \Gamma_{\rho\mu}^\nu e_\nu^m - \omega_{\rho n}^m e_{m\nu}^n = 0, \quad (7)$$

where $\omega_{\rho n}^m$ is the spin connection which can be extracted from Eq. (7) to be:

$$\omega_\mu^{mn} = e_\nu^m \Gamma_{\sigma\mu}^\nu e^{\sigma n} + e_\nu^m \partial_\mu e^{\nu n} = e_\nu^m \Gamma_{\sigma\mu}^\nu e^{\sigma n} - \partial_\mu e_\nu^m e^{\nu n}. \quad (8)$$

Then the covariant derivative of a Dirac fermion in the curved space time is written as:

$$D_\rho \Psi = \partial_\rho \Psi - \frac{i}{4} \omega_\rho^{ab} \sigma_{ab} \Psi, \quad (9)$$

where $\sigma_{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$.

Various attempts have been made in the literature [4] to introduce fermion covariant derivative without the use of the vielbein formalism in terms of only the curvilinear coordinates. These involved usually new and complicated mathematical structures and an entire formalism of their own.

In the following we will derive the exact expression for the fermion covariant derivative without the use of the vielbein formalism or of the gauged Lorentz group and by only making one natural assumptions.

Consider two Dirac fermions Ψ and $\bar{\Psi}$ and the gamma matrices γ^μ in the curved space time. The main assumption is that the quantity $\bar{\Psi}\gamma^\mu\Psi$ transforms as a vector in the curved space-time. Then the quantity,

$$D_\rho[\bar{\Psi}\gamma^\mu\Psi], \quad (10)$$

where D_ρ was introduced in Eq. (1) should transform as a rank two tensor.

We are interested in writing a covariant derivative such that the quantity $\bar{\Psi}\gamma^\mu\partial_\rho\Psi$ transforms as second rank tensor. Consider that this covariant derivative is expressed as,

$$\bar{\Psi}\gamma^\mu[\partial_\rho + X_\rho]\Psi, \quad (11)$$

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where X_ρ may contain in it gamma matrices in the curved space.

One can expand Eq. (10) which leads to:

$$\begin{aligned} D_\rho[\bar{\Psi}\gamma^\mu\Psi] = & \\ (\partial_\rho\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}(\partial_\rho\gamma^\mu)\Psi + & \\ \bar{\Psi}\gamma^\mu\partial_\rho\Psi + \Gamma_{\rho\sigma}^\mu\bar{\Psi}\gamma^\sigma\Psi. & \end{aligned} \quad (12)$$

We know that the expression in Eq. (12) must behave like a second rank tensor. Similarly to Eq. (11) a corresponding covariant derivative for the field $\bar{\Psi}$ must exist. If one assumes that the main properties of the Dirac fields present in the flat space extend to the curved one one may write:

$$[(\partial_\rho + X_\rho)\Psi]^\dagger = \partial_\rho\Psi^\dagger + \Psi^\dagger X_\rho^\dagger. \quad (13)$$

Further on if one considers that $\bar{\Psi}$ in the curved space is obtained through the same procedure as that in the flat space but this time with gamma matrix γ^0 in the curved space one obtains the covariant derivative for $\bar{\Psi}$ as:

$$D_\rho\bar{\Psi} = \partial_\rho\bar{\Psi} + \bar{\Psi}X_\rho^t, \quad (14)$$

where t signifies transposed.

Without loss of generality one may write:

$$X_\rho = -iA_{\rho\alpha\beta}\sigma^{\alpha\beta}, \quad (15)$$

where all the indices are considered in the curved space time. Then one has:

$$\begin{aligned} D_\rho\Psi &= \partial_\rho\Psi - iA_{\rho\alpha\beta}\sigma^{\alpha\beta}\Psi \\ D_\rho\bar{\Psi} &= \partial_\rho\bar{\Psi} - iA_{\rho\alpha\beta}\sigma^{\beta\alpha}\bar{\Psi}. \end{aligned} \quad (16)$$

If we introduce the expression in Eq. (16) into Eq. (12) one obtains that the quantity,

$$\begin{aligned} D_\rho\bar{\Psi}\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu D_\rho\Psi + & \\ +\bar{\Psi}i\gamma^\mu A_{\rho\alpha\beta}\sigma^{\alpha\beta}\Psi - \bar{\Psi}iA_{\rho\alpha\beta}\sigma^{\alpha\beta}\gamma^\mu\Psi + & \\ \bar{\Psi}(\partial_\rho\gamma^\mu)\Psi + \Gamma_{\rho\sigma}^\mu\bar{\Psi}\gamma^\sigma\Psi, & \end{aligned} \quad (17)$$

behaves as a second rank tensor. Since the term in the first line of Eq. (17) behave like a tensor then also the

terms on the second plus the third line must behave as a second rank tensor. Then,

$$\begin{aligned} \bar{\Psi}i\gamma^\mu A_{\rho\alpha\beta}\sigma^{\alpha\beta}\Psi - \bar{\Psi}iA_{\rho\alpha\beta}\sigma^{\alpha\beta}\gamma^\mu\Psi + & \\ \bar{\Psi}(\partial_\rho\gamma^\mu)\Psi + \Gamma_{\rho\sigma}^\mu\bar{\Psi}\gamma^\sigma\Psi = T_\rho^\mu, & \end{aligned} \quad (18)$$

where T_ρ^μ is an arbitrary tensor expressed in terms of the fermion fields. Since there is not such tensor besides those introduced at this point with the correct mass dimension one can consider this tensor zero.

One may rewrite Eq. (17) as:

$$iA_{\rho\alpha\beta}[\gamma^\mu, \sigma^{\alpha\beta}] = -\Gamma_{\rho\sigma}^\mu\gamma^\sigma - \partial_\rho\gamma^\mu. \quad (19)$$

In the flat space we know that:

$$\frac{1}{2}[\gamma^a, \sigma^{bc}] = i(\eta^{ab}\gamma^c - \eta^{ac}\gamma^b). \quad (20)$$

Since we consider in the curved space a similar Clifford algebra this time with the gamma matrices space time dependent the same relation should work if the flat indices would be replaced by the curved indices. Then Eq. (19) becomes:

$$-4A_{\rho\beta}^\mu\gamma^\beta = -\Gamma_{\rho\sigma}^\mu - \partial_\rho\gamma^\mu. \quad (21)$$

We multiply Eq. (21) by γ^λ and take the trace to obtain:

$$A_{\rho\lambda}^\mu = \frac{1}{4}\Gamma_{\rho\lambda}^\mu + \frac{1}{16}\text{Tr}[\gamma^\lambda\partial_\rho\gamma^\mu]. \quad (22)$$

Finally the covariant derivative for the fermion fields is written in terms of only quantities in the curved space as:

$$D_\rho\Psi = \partial_\rho\Psi - iA_{\rho\beta}^\alpha\sigma_\alpha^\beta\Psi, \quad (23)$$

where $A_{\rho\beta}^\alpha$ is given in Eq. (22).

Next we will show that the spin connection introduced in Eq. (23) is identical to that in Eq. (9). For that we write:

$$\begin{aligned} -iA_{\rho\lambda}^\mu\sigma_\mu^\lambda &= \\ -i\frac{1}{4}[\Gamma_{\rho\lambda}^\mu + \frac{1}{4}\text{Tr}[\gamma^\lambda\partial_\rho\gamma^\mu]]\sigma_\mu^\lambda &= \\ -i\frac{1}{4}[\Gamma_{\rho\lambda}^\mu + \partial_\rho e_a^\mu e_b^\lambda \eta^{ab}]e_{\mu c}e_d^\lambda\sigma^{cd} &= \\ -i\frac{1}{4}[\Gamma_{\rho\lambda}^\mu e_{\mu c}e_d^\lambda + \partial_\rho e_d^\mu e_{\mu c}] \sigma^{cd}. & \end{aligned} \quad (24)$$

Eq. (24) shows that the exact expression of the spin connection obtained through the vielbein formalism can be obtained by using only quantities defined in the curved space with the gamma matrices in the curved space satisfying a similar Clifford algebra.

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