

# THE GRAVITATIONAL SPIN HALL EFFECT OF LIGHT

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ABSTRACT. The propagation of electromagnetic waves in vacuum is often described within the geometrical optics approximation, which predicts that wave rays follow null geodesics. However, this model is valid only in the limit of infinitely high frequencies. At large but finite frequencies, diffraction can still be negligible, but the ray dynamics becomes affected by the evolution of the wave polarization. Hence, rays can deviate from null geodesics, which is known as the gravitational spin Hall effect of light. In the literature, this effect has been calculated *ad hoc* for a number of special cases, but no general description has been proposed. Here, we present a covariant WKB analysis from first principles for the propagation of light in arbitrary curved spacetimes. We obtain polarization-dependent ray equations describing the gravitational spin Hall effect of light. We also present numerical examples of polarization-dependent ray dynamics in the Schwarzschild spacetime, and the magnitude of the effect is briefly discussed. The analysis presented here is analogous to the spin Hall effect of light in inhomogeneous media, which has been experimentally verified.

## 1. INTRODUCTION

The propagation of electromagnetic waves in curved spacetime is often described within the geometrical optics approximation, which applies in the limit of infinitely high frequencies [48, 58]. In geometrical optics, Maxwell's equations are reduced to a set of ray equations, and a set of transport equations along these rays. The ray equations are the null geodesics of the underlying spacetime, and the transport equations govern the evolution of the intensity and the polarization vector. In particular, the geometrical optics approximation predicts that the ray equations determine the evolution of the polarization vector, but there is no backreaction from the polarization vector onto the ray equations. However, this model is valid only in the limit of infinitely high frequencies, and there has been interest in calculating the light propagation more accurately. At large but finite frequencies, diffraction can still be negligible but rays can deviate from geodesics. This is known as the gravitational spin Hall effect of light [50].

The mechanism behind the spin Hall effect is the spin-orbit interaction [13], i.e., the coupling of the wave polarization (spin) with the translational (orbital) motion of the ray as a particle, resulting in polarization (spin)-dependent rays. Related phenomena are found in many areas of physics. In condensed matter physics, electrons travelling in certain materials experience a spin Hall effect, resulting in spin-dependent trajectories, and spin accumulation on the lateral sides of the material [26, 62]. The effect was theoretically predicted by Dyakonov and Perel in 1971 [28, 27], followed by experimental observation in 1984 [6] and 2004 [40]. In optics, the polarization-dependent deflection of light travelling in an inhomogeneous medium is known as the spin Hall effect of light [13, 43]. The effect was predicted by several authors [20, 42, 52, 10, 11, 22, 23], and has recently been verified experimentally by Hosten and Kwiat

[38], and by Bliokh *et al.* [12]. The spin Hall effect of light provides corrections to the geometrical optics limit, which scale roughly with the inverse of frequency. This, and several other effects, can be explained in terms of the Berry curvature [66, 7, 12, 13].

There are several approaches aiming to describe the dynamics of spinning particles or wave packets in general relativity. Using a multipole expansion of the energy-momentum tensor, the dynamics of massive spinning test particles has been extensively studied in the form of the Mathisson–Papapetrou–Dixon equations [47, 53, 68, 15, 16]. A massless limit of these equations was derived by Souriau and Saturnini [63, 57], and particular examples adapted to certain spacetimes have been discussed in [25, 24, 46]. Another commonly used method is the Wentzel–Kramers–Brillouin (WKB) approximation for various field equations on curved spacetimes. For massive fields, this has been done in [5, 54] by considering a WKB approximation for the Dirac equation. For massless fields, using a WKB approximation for Maxwell’s equations on a stationary spacetime, Frolov and Shoom derived polarization-dependent ray equations [30, 31] (see also [71, 19, 18, 36]). With methods less familiar in general relativity, using the Foldy–Wouthuysen transformation for the Bargmann–Wigner equations in a perturbative way, Gosselin *et al.* derived ray equations for photons [35] and electrons [34] travelling in static spacetimes (see also [61, 60, 51]). The gravitational spin Hall effect of gravitational waves was also considered in [71, 70]. However, as discussed in [50], there are inconsistencies between the predictions of these different models, and some of them only work in particular spacetimes.

In this work, we are concerned with describing the propagation of electromagnetic waves in curved spacetime, beyond the traditional geometrical optics approximation. We carry out a covariant WKB analysis of the vacuum Maxwell’s equations, closely following the derivation of the spin Hall effect in optics [12, 55, 56], as well as the work of Littlejohn and Flynn [44]. As a result, we derive ray equations that contain polarization-dependent corrections to those of traditional geometrical optics, and capture the gravitational spin Hall effect of light. As in optics, these corrections can be interpreted in terms of the Berry curvature. To illustrate the effect, we give some numerical examples of the effective ray trajectories in the Schwarzschild spacetime.

Our paper is organized as follows. In Section 2 we start by introducing the variational formulation of the vacuum Maxwell’s equations. Then, we present the specific form of the WKB ansatz to be used, we discuss the role of the Lorenz gauge condition, and we state the assumptions that we are considering on the initial conditions. In Section 3, we present the WKB approximation of the field action, and the corresponding Euler–Lagrange equations. Analyzing these equations at each order in the geometrical optics parameter  $\epsilon$ , we obtain the well-known results of geometrical optics. The dynamics of the polarization vector is expressed in terms of the Berry phase. Finally, we derive an effective Hamilton–Jacobi system that contains  $\mathcal{O}(\epsilon)$  corrections over the standard geometrical optics results. In Section 4, we use the corrected Hamilton–Jacobi equation to derive the ray equations that account for the gravitational spin Hall effect of light. The gauge-invariance of these equations is discussed, and noncanonical coordinates are introduced. In Section 5, we present some basic numerical examples. We consider the effective ray equations on a Schwarzschild background, and compare with the results of Gosselin *et al.* [35]. The magnitude of the effect is estimated numerically. A summary of the result, including the effective Hamiltonian and the effective ray equations, can be found in Section 6.

**Notations and conventions.** We consider an arbitrary smooth Lorentzian manifold  $(M, g_{\mu\nu})$ , where the metric tensor  $g_{\mu\nu}$  has signature  $-+++$ . The absolute value of the metric determinant is denoted as  $g = |\det g_{\mu\nu}|$ . The phase space is defined as the cotangent bundle  $T^*M$ , and phase space points are denoted as  $(x, p)$ . The Einstein summation convention is assumed. Greek indices represent spacetime indices, and run from 0 to 3. Latin indices from the beginning of the alphabet,  $(a, b, c, \dots)$ , represent tetrad indices and run from 0 to 3. Latin indices from the middle of the alphabet,  $(i, j, k, \dots)$ , label the components of 3-vectors and run from 1 to 3.

## 2. MAXWELL'S EQUATIONS AND THE WKB APPROXIMATION

**2.1. Lagrangian formulation of Maxwell's equations.** Electromagnetic waves in vacuum can be described by the electromagnetic tensor  $\mathcal{F}_{\alpha\beta}$ . This is a skew-symmetric, real 2-form, which satisfies the vacuum Maxwell's equations [48]

$$\nabla^\alpha \mathcal{F}_{\alpha\beta} = 0, \quad \nabla_{[\alpha} \mathcal{F}_{\beta\gamma]} = 0. \quad (2.1)$$

Solutions to Maxwell's equations can also be represented by introducing the electromagnetic four-potential  $\mathcal{A}_\alpha$ , which is a real 1-form. Then, the electromagnetic tensor can be expressed as

$$\mathcal{F}_{\alpha\beta} = 2\nabla_{[\alpha} \mathcal{A}_{\beta]}, \quad (2.2)$$

and equations (2.1) become [48]

$$\hat{D}_\alpha{}^\beta \mathcal{A}_\beta = 0, \quad \hat{D}_\alpha{}^\beta = \nabla^\beta \nabla_\alpha - \delta_\alpha^\beta \nabla^\mu \nabla_\mu. \quad (2.3)$$

This equation can be obtained as the Euler–Lagrange equation of the following action:

$$J = \frac{1}{4} \int_M d^4x \sqrt{g} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} = \int_M d^4x \sqrt{g} \mathcal{A}^\alpha \hat{D}_\alpha{}^\beta \mathcal{A}_\beta, \quad (2.4)$$

where the last equality is obtained using integration by parts.

**2.2. WKB Ansatz.** We assume that the vector potential admits a WKB expansion of the form

$$\begin{aligned} \mathcal{A}_\alpha(x) &= \text{Re} \left[ A_\alpha(x, k, \epsilon) e^{iS(x)/\epsilon} \right], \\ A_\alpha(x, k, \epsilon) &= A_{0\alpha}(x, k) + \epsilon A_{1\alpha}(x, k) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (2.5)$$

where  $S$  is a real scalar function,  $A_\alpha$  is a complex amplitude, and  $\epsilon$  is a small expansion parameter. The gradient of  $S$  is denoted as

$$k_\mu(x) = \nabla_\mu S(x). \quad (2.6)$$

Note that we are allowing the amplitude  $A_\alpha$  to depend on  $k_\mu(x)$ . In physical terms, the limit  $\epsilon \ll 1$  indicates that the phase of the vector potential rapidly oscillates, and its variations are much faster than those corresponding to the amplitude  $A_\alpha(x, k, \epsilon)$ .

The role of the expansion parameter  $\epsilon$  becomes clear if we consider a timelike observer, traveling along the worldline  $\tau \mapsto x^\alpha(\tau)$ , with proper time  $\tau$ . This observer measures the frequency

$$\omega = -\frac{t^\alpha k_\alpha}{\epsilon}, \quad (2.7)$$

where  $t^\alpha = dx^\alpha/d\tau$  is the velocity vector field of the observer. The phase function  $S$  and  $\epsilon$  are dimensionless quantities. Working with geometrized units, such that  $c = G = 1$  [69, Appendix F], the velocity  $t^\alpha$  is dimensionless, and  $k_\alpha$  has dimension of inverse length. Hence,

$\omega$  has the dimension of the inverse length, as expected for frequency. Then, an observer sees the frequency going to infinity as  $\epsilon$  goes to 0.

We illustrate the validity condition of the geometrical optics approximation on a Schwarzschild black hole, with Schwarzschild radius  $r_s$ . For a source of light that is falling into the black hole, the gravitational redshift formula implies that the frequency  $\omega_\infty$  measured by an observer at infinity in the rest frame of the central object is smaller than the frequency measured by an observer at finite distance from the black hole. Then, a criterion for the high-frequency limit to hold is

$$\epsilon = (\omega_\infty r_s)^{-1} \ll 1. \quad (2.8)$$

Note that we could have taken any observer at finite distance of the black hole as a criterion. The choice of the observer at infinity provides the simplest expression.

**2.3. Lorenz gauge.** As commonly known, Maxwell's equations in the form (2.3) are not hyperbolic. In particular, they admit pure gauge solutions. We eliminate this problem by introducing a gauge, specifically, the Lorenz gauge

$$\nabla^\beta \mathcal{A}_\beta = 0. \quad (2.9)$$

Using the identity

$$\hat{D}_\alpha{}^\beta \mathcal{A}_\beta + \nabla_\alpha \nabla^\beta \mathcal{A}_\beta = \nabla^\beta \nabla_\beta \mathcal{A}_\alpha + R_{\alpha\beta} \mathcal{A}^\beta, \quad (2.10)$$

one observes that, if Maxwell's equations (2.3) and the Lorenz gauge (2.9) are satisfied, then the following wave equation holds:

$$\nabla^\alpha \nabla_\alpha \mathcal{A}_\beta + R_{\beta\gamma} \mathcal{A}^\gamma = 0. \quad (2.11)$$

It should then be checked that, if the wave equation (2.11) is satisfied, one obtains a solution to Maxwell's equations in the Lorenz gauge. Note that we consider here approximate solutions to Maxwell's equations

$$\hat{D}_\alpha{}^\beta \mathcal{A}_\beta = \mathcal{O}(\epsilon^0). \quad (2.12)$$

Hence, it is sufficient to consider

$$\nabla_\alpha \mathcal{A}^\alpha = \mathcal{O}(\epsilon). \quad (2.13)$$

We reproduce the standard argument recovering Maxwell's equation in the Lorenz gauge from the wave equation (2.11), taking into account that we are considering only approximate solutions. Assume that the wave equation holds:

$$\nabla^\alpha \nabla_\alpha \mathcal{A}_\beta + R_{\beta\alpha} \mathcal{A}^\alpha = \mathcal{O}(\epsilon^0) \Leftrightarrow \begin{cases} k^\alpha k_\alpha = 0 \\ ik^\alpha k_\alpha A_{1\beta} + A_{0\beta} \nabla^\alpha k_\alpha + 2k^\alpha \nabla_\alpha A_{0\beta} = 0 \end{cases}. \quad (2.14)$$

Assume furthermore that the initial data for the wave equation (2.14) satisfy

$$\begin{aligned} k^\alpha A_{0\alpha} &= 0, \\ \nabla_\alpha A_0{}^\alpha + iA_1{}^\alpha k_\alpha &= 0. \end{aligned} \quad (2.15)$$

Equation (2.14) implies that

$$\nabla_\alpha \nabla^\alpha \left( \nabla^\beta \mathcal{A}_\beta \right) = \mathcal{O}(\epsilon^{-1}). \quad (2.16)$$

The initial data (2.15) for the wave equation (2.14) imply that, initially,

$$\nabla_\alpha \mathcal{A}^\alpha = \mathcal{O}(\epsilon). \quad (2.17)$$

Observe that the condition

$$T^\beta \nabla_\beta (\nabla_\alpha \mathcal{A}^\alpha) = \mathcal{O}(\epsilon^0) \quad (2.18)$$

is automatically satisfied, where  $T$  is a unit future-oriented normal vector to the hypersurface on which initial data are prescribed. Hence, the equation satisfied by the Lorenz gauge source function (2.16) admits initial data as in (2.17) and (2.18) vanishing at the appropriate order in  $\epsilon$  (at  $\mathcal{O}(\epsilon^1)$  and  $\mathcal{O}(\epsilon^0)$ , respectively). This implies that Maxwell's equations

$$\hat{D}_\alpha{}^\beta \mathcal{A}_\beta = \mathcal{O}(\epsilon^0) \Leftrightarrow \begin{cases} k^\alpha A_{0[\alpha} k_{\mu]} = 0 \\ 2k^\alpha \nabla_\alpha A_{0\mu} - (ik^\alpha A_{1\alpha} + \nabla^\alpha A_{0\alpha}) k_\mu - k^\alpha \nabla_\mu A_{0\alpha} \\ -A_{0\alpha} \nabla^\alpha k_\mu + A_{0\mu} \nabla^\alpha k_\alpha + ik^\alpha k_\alpha A_{1\mu} = 0, \end{cases} \quad (2.19)$$

are satisfied in the Lorenz gauge

$$\nabla^\beta \mathcal{A}_\beta = \mathcal{O}(\epsilon^1) \Leftrightarrow \begin{cases} k^\alpha A_{0\alpha} = 0 \\ \nabla_\alpha A_0{}^\alpha + iA_1{}^\alpha k_\alpha = 0 \end{cases} . \quad (2.20)$$

**2.4. Assumption on the initial conditions.** In this paper, we consider solutions of the vacuum Maxwell's equations assuming a WKB ansatz (2.5), with initial conditions satisfying the following properties:

- (1) The Lorenz gauge (2.20) is satisfied initially.
- (2) The initial phase gradient  $k^\alpha$  is a future-oriented null vector. This assumption is in fact a compatibility condition resulting from the dispersion relation (3.8) below, and the Lorenz gauge condition (2.20).
- (3) Initially, the beam has circular polarization (see Section 3.4).

### 3. HIGHER-ORDER GEOMETRICAL OPTICS

**3.1. WKB approximation of the field action.** We compute the WKB approximation for our field theory by inserting the WKB ansatz (2.5) in the field action (2.4):

$$\begin{aligned} J &= \int_M d^4x \sqrt{g} \operatorname{Re} \left( A^\alpha e^{iS/\epsilon} \right) \hat{D}_\alpha{}^\beta \operatorname{Re} \left( A_\beta e^{iS/\epsilon} \right) \\ &= \frac{1}{4} \int_M d^4x \sqrt{g} \left[ A^{*\alpha} e^{-iS/\epsilon} \hat{D}_\alpha{}^\beta \left( A_\beta e^{iS/\epsilon} \right) + \text{c.c.} \right] \\ &\quad + \frac{1}{4} \int_M d^4x \sqrt{g} \left[ A^\alpha e^{iS/\epsilon} \hat{D}_\alpha{}^\beta \left( A_\beta e^{iS/\epsilon} \right) + \text{c.c.} \right]. \end{aligned} \quad (3.1)$$

Since  $e^{iS/\epsilon}$  is a rapidly oscillating function, the Riemann–Lebesgue lemma implies, for sufficiently regular  $f$ ,

$$\int_M d^4x \sqrt{g} e^{\pm i2S(x)/\epsilon} f(x) = o(\epsilon^0). \quad (3.2)$$

Upon expanding the derivative terms in equation (3.1), and keeping only terms of the lowest two orders in  $\epsilon$ , we obtain the following WKB approximation of the field action (for convenience, we are shifting the powers of  $\epsilon$ , such that the lowest-order term is of  $\mathcal{O}(\epsilon^0)$ ):

$$-\epsilon^2 J = \int_M d^4x \sqrt{g} \left[ D_\alpha{}^\beta A^{*\alpha} A_\beta - \frac{i\epsilon}{2} \overset{v}{\nabla}^\mu D_\alpha{}^\beta \left( A^{*\alpha} \nabla_\mu A_\beta - A_\beta \nabla_\mu A^{*\alpha} \right) \right] + \mathcal{O}(\epsilon^2), \quad (3.3)$$

where

$$\begin{aligned} D_\alpha{}^\beta &= \frac{1}{2} k_\mu k^\mu \delta_\alpha^\beta - \frac{1}{2} k_\alpha k^\beta, \\ \overset{v}{\nabla}^\mu D_\alpha{}^\beta &= k^\mu \delta_\alpha^\beta - \delta_\alpha^\mu k^\beta - g^{\mu\beta} k_\alpha. \end{aligned} \quad (3.4)$$

Here,  $D_\alpha^\beta$  represents the symbol [32] of the operator  $\hat{D}_\alpha^\beta$ , evaluated at the phase space point  $(x, p) = (x, k)$ , and we are using the notation  $\overset{v}{\nabla}^\mu D_\alpha^\beta$  for the vertical derivative (Appendix A) of  $D_\alpha^\beta$ , evaluated at the phase space point  $(x, p) = (x, k)$ .

The action depends on the following fields:  $S(x)$ ,  $\nabla_\mu S(x)$ ,  $A_\alpha(x, \nabla S)$ ,  $\nabla_\mu [A_\alpha(x, \nabla S)]$ ,  $A^{*\alpha}(x, \nabla S)$ ,  $\nabla_\mu [A^{*\alpha}(x, \nabla S)]$ . Following the calculations in Appendix B, the Euler–Lagrange equations are

$$D_\alpha^\beta A_\beta - i\epsilon \left( \overset{v}{\nabla}^\mu D_\alpha^\beta \right) \nabla_\mu A_\beta - \frac{i\epsilon}{2} \left( \nabla_\mu \overset{v}{\nabla}^\mu D_\alpha^\beta \right) A_\beta = \mathcal{O}(\epsilon^2), \quad (3.5)$$

$$D_\alpha^\beta A^{*\alpha} + i\epsilon \left( \overset{v}{\nabla}^\mu D_\alpha^\beta \right) \nabla_\mu A^{*\alpha} + \frac{i\epsilon}{2} \left( \nabla_\mu \overset{v}{\nabla}^\mu D_\alpha^\beta \right) A^{*\alpha} = \mathcal{O}(\epsilon^2), \quad (3.6)$$

$$\nabla_\mu \left[ \left( \overset{v}{\nabla}^\mu D_\alpha^\beta \right) A^{*\alpha} A_\beta - \frac{i\epsilon}{2} \left( \overset{v}{\nabla}^\mu \overset{v}{\nabla}^\nu D_\alpha^\beta \right) (A^{*\alpha} \nabla_\nu A_\beta - A_\beta \nabla_\nu A^{*\alpha}) \right] = \mathcal{O}(\epsilon^2). \quad (3.7)$$

In the above equations, the symbol  $D_\alpha^\beta$  and its vertical derivatives are all evaluated at the phase space point  $(x, k)$ .

**3.2. 0th-order geometrical optics.** Starting with equations (3.5)–(3.7), and keeping only terms of  $\mathcal{O}(\epsilon^0)$ , we obtain

$$D_\alpha^\beta A_{0\beta} = 0, \quad (3.8)$$

$$D_\alpha^\beta A_0^{*\alpha} = 0, \quad (3.9)$$

$$\nabla_\mu \left[ \left( \overset{v}{\nabla}^\mu D_\alpha^\beta \right) A_0^{*\alpha} A_{0\beta} \right] = 0. \quad (3.10)$$

Equation (3.8) can also be written as:

$$\frac{1}{2} \left( k_\mu k^\mu \delta_\alpha^\beta - k_\alpha k^\beta \right) A_{0\beta} = 0. \quad (3.11)$$

This equation admits nontrivial solutions if and only if  $A_{0\beta}$  is an eigenvector of  $D_\alpha^\beta$  with zero eigenvalue. Two cases should be discussed:  $k$  is a null vector, or  $k$  is not a null vector.

Assume first that  $k$  is not a null vector,  $k^\mu k_\mu \neq 0$ . Then, equation (3.8) leads to

$$A_{0\alpha} = \frac{k^\beta A_{0\beta}}{k_\mu k^\mu} k_\alpha. \quad (3.12)$$

This entails that

$$A_{0[\alpha} k_{\beta]} = 0 \quad \text{or} \quad \mathcal{F}_{\alpha\beta} = \nabla_{[\alpha} \mathcal{A}_{\beta]} = \mathcal{O}(\epsilon^0). \quad (3.13)$$

In other words, when  $k$  is not a null vector, the corresponding solution is, at the lowest order in  $\epsilon$ , a pure gauge solution. Since the corresponding electromagnetic field vanishes, we do not consider this case further.

If  $k$  is null,  $k^\mu k_\mu = 0$ , equation (3.8) implies

$$k^\beta A_{0\beta} = 0. \quad (3.14)$$

This is consistent with the Lorenz gauge condition (2.20) at the lowest order in  $\epsilon$ . A similar argument can be applied for the complex-conjugate equation (3.9), from which we obtain  $k_\alpha A_0^{*\alpha} = 0$ .

Using equations (3.8)–(3.10), we obtain the well-known system of equations governing the geometrical optics approximation at the lowest order in  $\epsilon$ :

$$k_\mu k^\mu = 0, \quad (3.15)$$

$$k^\alpha A_{0\alpha} = k_\alpha A_0^{*\alpha} = 0, \quad (3.16)$$

$$\nabla_\mu (k^\mu \mathcal{J}_0) = 0, \quad (3.17)$$

where  $\mathcal{J}_0 = A_0^{*\alpha} A_{0\alpha}$  is the lowest-order intensity (more precisely,  $\mathcal{J}_0$  is proportional to the wave action density [67]). Equation (3.17) is obtained from equation (3.10) by using the orthogonality condition (3.16). Using equation (2.6), we have

$$\nabla_\mu k_\alpha = \nabla_\alpha k_\mu, \quad (3.18)$$

and we can use equation (3.15) to derive the geodesic equation for  $k_\mu$ :

$$k^\nu \nabla_\nu k_\mu = 0. \quad (3.19)$$

**3.3. 1st-order geometrical optics.** Here, we examine equations (3.5) and (3.6) at order  $\epsilon^1$  only:

$$D_\alpha^\beta A_{1\beta} - i \left( \overset{v}{\nabla}^\mu D_\alpha^\beta \right) \nabla_\mu A_{0\beta} - \frac{i}{2} \left( \nabla_\mu \overset{v}{\nabla}^\mu D_\alpha^\beta \right) A_{0\beta} = 0, \quad (3.20)$$

$$D_\alpha^\beta A_1^{*\alpha} + i \left( \overset{v}{\nabla}^\mu D_\alpha^\beta \right) \nabla_\mu A_0^{*\alpha} + \frac{i}{2} \left( \nabla_\mu \overset{v}{\nabla}^\mu D_\alpha^\beta \right) A_0^{*\alpha} = 0. \quad (3.21)$$

Using equation (3.4), we can also rewrite equation (3.20) as follows:

$$\begin{aligned} k^\mu \nabla_\mu A_{0\alpha} - \frac{1}{2} k_\alpha \nabla_\mu A_0^\mu - \frac{1}{2} k_\beta \nabla_\alpha A_0^\beta - \frac{i}{2} k_\alpha k^\beta A_{1\beta} \\ + \frac{1}{2} A_{0\alpha} \nabla_\mu k^\mu - \frac{1}{4} A_0^\beta \nabla_\beta k_\alpha - \frac{1}{4} A_0^\beta \nabla_\alpha k_\beta = 0. \end{aligned} \quad (3.22)$$

Using equation (3.18), we can rewrite the last two terms as

$$-\frac{1}{4} A_0^\beta \nabla_\beta k_\alpha - \frac{1}{4} A_0^\beta \nabla_\alpha k_\beta = -\frac{1}{2} A_0^\beta \nabla_\alpha k_\beta. \quad (3.23)$$

With the results obtained at  $\mathcal{O}(\epsilon^0)$ , we also have

$$k_\beta A_0^\beta = 0 \quad \Rightarrow \quad \nabla_\alpha (k_\beta A_0^\beta) = k_\beta \nabla_\alpha A_0^\beta + A_0^\beta \nabla_\alpha k_\beta = 0. \quad (3.24)$$

Then, equation (3.22) becomes

$$k^\mu \nabla_\mu A_{0\alpha} + \frac{1}{2} A_{0\alpha} \nabla_\mu k^\mu - \frac{1}{2} k_\alpha (\nabla_\mu A_0^\mu + i k_\mu A_1^\mu) = 0. \quad (3.25)$$

The last term can be eliminated by using the Lorenz gauge (2.20). The same steps can be applied to the complex-conjugate equation (3.21):

$$\begin{aligned} k^\mu \nabla_\mu A_{0\alpha} + \frac{1}{2} A_{0\alpha} \nabla_\mu k^\mu &= 0, \\ k^\mu \nabla_\mu A_0^{*\beta} + \frac{1}{2} A_0^{*\beta} \nabla_\mu k^\mu &= 0. \end{aligned} \quad (3.26)$$

Furthermore, using the lowest-order intensity  $\mathcal{J}_0$ , we can write the amplitude vectors in the following way:

$$A_{0\alpha} = \sqrt{\mathcal{J}_0} a_{0\alpha}, \quad A_0^{*\alpha} = \sqrt{\mathcal{J}_0} a_0^{*\alpha}, \quad (3.27)$$

where  $a_{0\alpha}$  is a unit complex vector (i.e.  $a_0^{*\alpha}a_{0\alpha} = 1$ ) describing the polarization. Then, from equation (3.26), together with equation (3.17), we obtain

$$k^\mu \nabla_\mu a_{0\alpha} = k^\mu \nabla_\mu a_0^{*\alpha} = 0. \quad (3.28)$$

The parallel propagation of the complex vector  $a_{0\alpha}$  along the integral curve of  $k^\mu$  is another well-known result of the geometrical optics approximation.

**3.4. The polarization vector in a null tetrad.** We observed that the polarization vector satisfies the orthogonality condition

$$k^\alpha a_{0\alpha} = 0. \quad (3.29)$$

Consider the Newman–Penrose tetrad  $\{k_\alpha, n_\alpha, m_\alpha, \bar{m}_\alpha\}$  satisfying

$$\begin{aligned} m_\alpha \bar{m}^\alpha &= 1, & k_\alpha n^\alpha &= -1, \\ k_\alpha k^\alpha &= n_\alpha n^\alpha = m_\alpha m^\alpha = \bar{m}_\alpha \bar{m}^\alpha = 0, \\ k_\alpha m^\alpha &= k_\alpha \bar{m}^\alpha = n_\alpha m^\alpha = n_\alpha \bar{m}^\alpha = 0. \end{aligned} \quad (3.30)$$

Since the Newman–Penrose tetrad is adapted to the vector  $k_\alpha$ , the orthogonality conditions imply that  $m_\alpha$  and  $\bar{m}_\alpha$  are functions of  $k_\alpha$ . Since the polarization vector  $a_{0\alpha}$  is orthogonal to  $k_\alpha$ , we can decompose it as

$$a_{0\alpha}(x, k) = z_1(x)m_\alpha(x, k) + z_2(x)\bar{m}_\alpha(x, k) + z_3(x)k_\alpha(x), \quad (3.31)$$

where  $z_1$ ,  $z_2$ , and  $z_3$  are complex scalar functions. Since  $a_{0\alpha}$  is a unit complex vector, the scalar functions  $z_1$  and  $z_2$  are constrained by

$$z_1^* z_1 + z_2^* z_2 = 1. \quad (3.32)$$

It is important to note that the decomposition (3.31), and more specifically, the choice of  $m_\alpha$ , requires choosing a null vector  $n_\alpha$ . Fixing  $n_\alpha$  is equivalent to choosing a timelike vector field  $t_\alpha$ , that represents an observer. Once  $n_\alpha$  is fixed, the remaining  $SO(2)$  gauge freedom in the choice of  $m_\alpha$  is described by the spin rotation

$$k_\alpha \mapsto k_\alpha, \quad n_\alpha \mapsto n_\alpha, \quad m_\alpha \mapsto e^{i\phi} m_\alpha, \quad \text{for } \phi \in \mathbb{R}. \quad (3.33)$$

Altogether, the gauge freedom in the decomposition (3.31) is described by the little group, that is to say the subgroup of the transformations leaving  $k$  invariant.

Using equations (3.31) and (3.19), the parallel-transport equation for the polarization vector becomes

$$\begin{aligned} 0 = k^\mu \nabla_\mu a_{0\alpha} &= z_1 k^\mu \nabla_\mu m_\alpha + z_2 k^\mu \nabla_\mu \bar{m}_\alpha + m_\alpha k^\mu \nabla_\mu z_1 \\ &+ \bar{m}_\alpha k^\mu \nabla_\mu z_2 + k_\alpha k^\mu \nabla_\mu z_3. \end{aligned} \quad (3.34)$$

Contracting the above equation with  $\bar{m}^\alpha$ ,  $m^\alpha$ , and  $n^\alpha$ , we obtain

$$\begin{aligned} k^\mu \nabla_\mu z_1 &= -z_1 \bar{m}^\alpha k^\mu \nabla_\mu m_\alpha, \\ k^\mu \nabla_\mu z_2 &= -z_2 m^\alpha k^\mu \nabla_\mu \bar{m}_\alpha, \\ k^\mu \nabla_\mu z_3 &= -(z_1 m^\alpha + z_2 \bar{m}^\alpha) k^\mu \nabla_\mu n_\alpha. \end{aligned} \quad (3.35)$$



Recall that in the above equations, the vectors  $m_\alpha$  and  $\bar{m}_\alpha$  are functions of  $x$  and  $k(x)$ . The covariant derivatives are applied as follows:

$$\begin{aligned} k^\mu \nabla_\mu m_\alpha &= k^\mu \nabla_\mu [m_\alpha(x, k)] \\ &= k^\mu \left( \overset{h}{\nabla}_\mu m_\alpha \right) (x, k) + k^\mu (\nabla_\mu k_\nu) \left( \overset{v}{\nabla}^\nu m_\alpha \right) (x, k) \\ &= k^\mu \overset{h}{\nabla}_\mu m_\alpha, \end{aligned} \quad (3.36)$$

where  $\overset{h}{\nabla}_\mu$  is the horizontal derivative (Appendix A). It is convenient to introduce the following 2-dimensional unit complex vector, which is analogous to the Jones vector in optics [29, 9, 55, 56]:

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad (3.37)$$

and we shall also use the hermitian transpose  $z^\dagger$ , defined as follows:

$$z^\dagger = (z_1^* \quad z_2^*). \quad (3.38)$$

Then, the equations for  $z_1$  and  $z_2$  can be written in a more compact form:

$$k^\mu \nabla_\mu z = ik^\mu B_\mu \sigma_3 z, \quad (3.39)$$

where  $\sigma_3$  is the third Pauli matrix,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.40)$$

and  $B_\mu$  is the real 1-form extending to general relativity the Berry connection used in optics [9, 55]

$$B_\mu(x, k) = \frac{i}{2} \left( \bar{m}^\alpha \overset{h}{\nabla}_\mu m_\alpha - m_\alpha \overset{h}{\nabla}_\mu \bar{m}^\alpha \right) = i \bar{m}^\alpha \overset{h}{\nabla}_\mu m_\alpha. \quad (3.41)$$

Furthermore, if we restrict  $z$  to a worldline  $x^\mu(\tau)$ , with  $\dot{x}^\mu = k^\mu$ , we can write

$$\dot{z} = ik^\mu B_\mu \sigma_3 z. \quad (3.42)$$

Integrating along the worldline, we obtain

$$z(\tau) = \begin{pmatrix} e^{i\gamma(\tau)} & 0 \\ 0 & e^{-i\gamma(\tau)} \end{pmatrix} z(0), \quad (3.43)$$

where  $\gamma$  represents the Berry phase [9, 55]

$$\gamma(\tau_1) = \int_{\tau_0}^{\tau_1} d\tau k^\mu B_\mu. \quad (3.44)$$

Using either equation (3.35) or equation (3.42), we see that the evolution of  $z_1$  and  $z_2$  is decoupled in the circular polarization basis, and the following quantities are conserved along  $k^\mu$ :

$$\begin{aligned} 1 &= z_1^* z_1 + z_2^* z_2 = z^\dagger z, \\ s &= z_1^* z_1 - z_2^* z_2 = z^\dagger \sigma_3 z. \end{aligned} \quad (3.45)$$

Based on our assumptions on the initial conditions (Section 2.4), we only consider beams with

$$z(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad z(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.46)$$

Thus, we have  $s = \pm 1$ , depending on the choice of the initial polarization state.

The results described in this section are similar to the description of the polarization of electromagnetic waves travelling in a medium with an inhomogeneous index of refraction [9].

**3.5. Extended geometrical optics.** Now, we take equations (3.5)–(3.7), but without splitting them order by order in  $\epsilon$ . Our aim is to derive an effective Hamilton–Jacobi system that would give us  $\mathcal{O}(\epsilon)$  corrections to the ray equations.

**3.5.1. Effective dispersion relation.** By contracting equation (3.5) with  $A^{*\alpha}$  and equation (3.6) with  $A_\beta$ , and also adding them together, we obtain the following equation:

$$D_\alpha{}^\beta A^{*\alpha} A_\beta - \frac{i\epsilon}{2} \left( \overset{v}{\nabla}{}^\mu D_\alpha{}^\beta \right) (A^{*\alpha} \nabla_\mu A_\beta - A_\beta \nabla_\mu A^{*\alpha}) = \mathcal{O}(\epsilon^2). \quad (3.47)$$

Using equations (3.4), (3.15) and (3.16), we can rewrite this as follows:

$$\begin{aligned} & \frac{1}{2} k_\mu k^\mu (A_0^{*\alpha} A_{0\alpha} + \epsilon A_0^{*\alpha} A_{1\alpha} + \epsilon A_1^{*\alpha} A_{0\alpha}) \\ & - \frac{i\epsilon}{2} k^\mu (A_0^{*\alpha} \nabla_\mu A_{0\alpha} - A_{0\alpha} \nabla_\mu A_0^{*\alpha}) \\ & + \frac{i\epsilon}{4} k_\alpha (A_0^{*\mu} \nabla_\mu A_0^\alpha - A_0^\mu \nabla_\mu A_0^{*\alpha}) = \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.48)$$

Using equation (3.16), we obtain

$$0 = A_0^{*\mu} \nabla_\mu (k_\alpha A_0^\alpha) = k_\alpha A_0^{*\mu} \nabla_\mu A_0^\alpha + A_0^{*\mu} A_0^\alpha \nabla_\mu k_\alpha, \quad (3.49)$$

so we can write:

$$\frac{i\epsilon}{4} k_\alpha (A_0^{*\mu} \nabla_\mu A_0^\alpha - A_0^\mu \nabla_\mu A_0^{*\alpha}) = -\frac{i\epsilon}{2} \nabla_\mu k_\alpha A_0^{*[\mu} A_0^{\alpha]} = 0, \quad (3.50)$$

where the latter equality is due to (3.18). Then, equation (3.47) becomes

$$\begin{aligned} & \frac{1}{2} k_\mu k^\mu (A_0^{*\alpha} A_{0\alpha} + \epsilon A_0^{*\alpha} A_{1\alpha} + \epsilon A_1^{*\alpha} A_{0\alpha}) \\ & - \frac{i\epsilon}{2} k^\mu (A_0^{*\alpha} \nabla_\mu A_{0\alpha} - A_{0\alpha} \nabla_\mu A_0^{*\alpha}) = \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.51)$$

Let us introduce the  $\mathcal{O}(\epsilon^1)$  intensity:

$$\mathcal{J} = \mathcal{A}^{*\alpha} \mathcal{A}_\alpha = A^{*\alpha} A_\alpha = A_0^{*\alpha} A_{0\alpha} + \epsilon A_0^{*\alpha} A_{1\alpha} + \epsilon A_1^{*\alpha} A_{0\alpha} + \mathcal{O}(\epsilon^2). \quad (3.52)$$

Then, we can rewrite the amplitude as

$$A_\alpha = \sqrt{\mathcal{J}} a_\alpha = \sqrt{\mathcal{J}} (a_{0\alpha} + \epsilon a_{1\alpha}) + \mathcal{O}(\epsilon^2), \quad (3.53)$$

where  $a_\alpha$  is a unit complex vector. Then, equation (3.51) can be expressed as follows:

$$\frac{1}{2} k_\mu k^\mu - \frac{i\epsilon}{2} k^\mu (a_0^{*\alpha} \nabla_\mu a_{0\alpha} - a_{0\alpha} \nabla_\mu a_0^{*\alpha}) = \mathcal{O}(\epsilon^2). \quad (3.54)$$

This can be viewed as an effective dispersion relation, containing  $\mathcal{O}(\epsilon)$  corrections to the geometrical optics equation (3.15). Finally, let us assume the notation

$$K_\mu = k_\mu - \frac{i\epsilon}{2} (a_0^{*\alpha} \nabla_\mu a_{0\alpha} - a_{0\alpha} \nabla_\mu a_0^{*\alpha}) \quad (3.55)$$

and rewrite the effective dispersion relation takes the following form:

$$\frac{1}{2} K_\mu K^\mu = \mathcal{O}(\epsilon^2). \quad (3.56)$$

This equation can also be obtained directly from the effective field action (3.3), specifically by varying the latter with respect to  $\mathcal{J}$ .

3.5.2. *Effective transport equation.* Using equations (3.4), (3.15) and (3.16), the effective transport equation (3.7) becomes

$$\begin{aligned} \nabla_\mu \left[ k^\mu (A_0^{*\alpha} A_{0\alpha} + \epsilon A_0^{*\alpha} A_{1\alpha} + \epsilon A_1^{*\alpha} A_{0\alpha}) \right. \\ - \frac{\epsilon}{2} k_\alpha (A_0^{*\mu} A_1^\alpha + A_1^{*\alpha} A_0^\mu) + \frac{i\epsilon}{4} (A_0^{*\alpha} \nabla_\alpha A_0^\mu - A_0^\mu \nabla_\alpha A_0^{*\alpha}) \\ + \frac{i\epsilon}{4} (A_0^{*\mu} \nabla_\alpha A_0^\alpha - A_0^\alpha \nabla_\alpha A_0^{*\mu}) \\ \left. - \frac{i\epsilon}{2} g^{\mu\nu} (A_0^{*\alpha} \nabla_\nu A_{0\alpha} - A_{0\alpha} \nabla_\nu A_0^{*\alpha}) \right] = \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.57)$$

We can perform the following replacements in the above equation:

$$\begin{aligned} A_0^{*\alpha} \nabla_\alpha A_0^\mu &= \nabla_\alpha (A_0^{*\alpha} A_0^\mu) - \nabla_\alpha A_0^{*\alpha} A_0^\mu, \\ \nabla_\alpha A_0^{*\mu} A_0^\alpha &= \nabla_\alpha (A_0^{*\mu} A_0^\alpha) - A_0^{*\mu} \nabla_\alpha A_0^\alpha. \end{aligned} \quad (3.58)$$

After rearranging terms, the effective transport equation becomes

$$\begin{aligned} \nabla_\mu \left[ k^\mu (A_0^{*\alpha} A_{0\alpha} + \epsilon A_0^{*\alpha} A_{1\alpha} + \epsilon A_1^{*\alpha} A_{0\alpha}) + \frac{i\epsilon}{4} \nabla_\alpha (A_0^{*[\alpha} A_0^{\mu]}) \right. \\ - \frac{i\epsilon}{2} A_0^\mu (\nabla_\alpha A_0^{*\alpha} - i k_\alpha A_1^{*\alpha}) + \frac{i\epsilon}{2} A_0^{*\mu} (\nabla_\alpha A_0^\alpha + i k_\alpha A_1^\alpha) \\ \left. - \frac{i\epsilon}{2} g^{\mu\nu} (A_0^{*\alpha} \nabla_\nu A_{0\alpha} - A_{0\alpha} \nabla_\nu A_0^{*\alpha}) \right] = \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.59)$$

The following term vanishes due to the symmetry of the Ricci tensor:

$$\begin{aligned} \nabla_\mu \nabla_\alpha (A_0^{*[\alpha} A_0^{\mu]}) &= \nabla_{[\mu} \nabla_{\alpha]} (A_0^{*\alpha} A_0^\mu) \\ &= (R_{\alpha\nu\mu}{}^\nu - R_{\mu\nu\alpha}{}^\nu) A_0^{*\alpha} A_0^\mu \\ &= (R_{\alpha\mu} - R_{\mu\alpha}) A_0^{*\alpha} A_0^\mu \\ &= 0. \end{aligned} \quad (3.60)$$

Furthermore, using the Lorenz gauge condition (2.20), we are left with the following form of the effective transport equation:

$$\begin{aligned} \nabla_\mu \left[ k^\mu (A_0^{*\alpha} A_{0\alpha} + \epsilon A_0^{*\alpha} A_{1\alpha} + \epsilon A_1^{*\alpha} A_{0\alpha}) \right. \\ \left. - \frac{i\epsilon}{2} g^{\mu\nu} (A_0^{*\alpha} \nabla_\nu A_{0\alpha} - A_{0\alpha} \nabla_\nu A_0^{*\alpha}) \right] = \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.61)$$

Introducing the intensity  $\mathcal{J}$ , and the vector  $K_\mu$ , we obtain

$$\nabla_\mu \left\{ \mathcal{J} \left[ k^\mu - \frac{i\epsilon}{2} g^{\mu\nu} (a_0^{*\alpha} \nabla_\nu a_{0\alpha} - a_{0\alpha} \nabla_\nu a_0^{*\alpha}) \right] \right\} = \nabla_\mu (\mathcal{J} K^\mu) = \mathcal{O}(\epsilon^2). \quad (3.62)$$

This is an effective transport equation for the intensity  $\mathcal{J}$ , which includes  $\mathcal{O}(\epsilon)$  corrections to the geometrical optics equation (3.17).

#### 4. EFFECTIVE RAY EQUATIONS

**4.1. Hamilton–Jacobi system at leading order.** The lowest-order geometrical optics equations (3.15) and (3.17) can be viewed as a system of coupled partial differential equations:

$$\frac{1}{2}g^{\mu\nu}k_\mu k_\nu = 0, \quad (4.1)$$

$$\nabla_\mu (\mathcal{J}_0 k^\mu) = 0, \quad (4.2)$$

where  $k_\mu = \nabla_\mu S$ . The first equation is a Hamilton–Jacobi equation for the phase function  $S$ , and the second equation is a transport equation for the intensity  $\mathcal{J}_0$  [49]. The Hamilton–Jacobi equation can be solved using the method of characteristics. This is done by defining a Hamiltonian function on  $T^*M$ , such that

$$H(x, \nabla S) = \frac{1}{2}g^{\mu\nu}k_\mu k_\nu = 0. \quad (4.3)$$

It is obvious that in this case, the Hamiltonian is

$$H(x, p) = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu, \quad (4.4)$$

and Hamilton’s equations take the following form:

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = g^{\mu\nu}p_\nu, \quad (4.5)$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}p_\alpha p_\beta. \quad (4.6)$$

Given a solution  $\{x^\mu(\tau), p_\mu(\tau)\}$  for Hamilton’s equations, we obtain a solution of the Hamilton–Jacobi equation (4.3) by taking [33]:

$$S(x^\mu(\tau_1), p_\mu(\tau_1)) = \int_{\tau_0}^{\tau_1} d\tau [\dot{x}^\mu p_\mu - H(x, p)] + \text{const}. \quad (4.7)$$

Once the Hamilton–Jacobi equation is solved, the transport equation (4.2) can also be solved, at least in principle [49]. However, our main interest is in the ray equations governed by the Hamiltonian (4.4). The corresponding Hamilton’s equations (4.5) and (4.6) describe null geodesics. These equations can easily be rewritten as

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0, \quad (4.8)$$

or in the explicitly covariant form:

$$p^\nu \nabla_\nu p^\mu = \dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0. \quad (4.9)$$

**4.2. Effective Hamilton–Jacobi system.** The effective dispersion relation (3.56), together with the effective transport equation (3.62) introduce  $\mathcal{O}(\epsilon^1)$  corrections over the system discussed above:

$$\frac{1}{2}g^{\mu\nu}k_\mu k_\nu - \frac{i\epsilon}{2}k^\mu (a_0^{*\alpha} \nabla_\mu a_{0\alpha} - a_{0\alpha} \nabla_\mu a_0^{*\alpha}) = \mathcal{O}(\epsilon^2), \quad (4.10)$$

$$\nabla_\mu \left\{ \mathcal{J} \left[ k^\mu - \frac{i\epsilon}{2}g^{\mu\nu} (a_0^{*\alpha} \nabla_\nu a_{0\alpha} - a_{0\alpha} \nabla_\nu a_0^{*\alpha}) \right] \right\} = \mathcal{O}(\epsilon^2). \quad (4.11)$$

Using equation (3.31), the effective dispersion relation becomes

$$\frac{1}{2}g^{\mu\nu}k_\mu k_\nu - \frac{i\epsilon}{2}k^\mu \left( z^\dagger \partial_\mu z - \partial_\mu z^\dagger z \right) - \epsilon s k^\mu B_\mu = \mathcal{O}(\epsilon^2), \quad (4.12)$$

where  $B_\mu = B_\mu(x, k)$  is the Berry connection introduced in equation (3.41), and  $s = \pm 1$ , depending on the initial polarization. Using equation (3.43), together with the assumption on the initial polarization, we can write:

$$-\frac{i\epsilon}{2}k^\mu \left( z^\dagger \partial_\mu z - \partial_\mu z^\dagger z \right) = \epsilon s k^\mu \partial_\mu \gamma. \quad (4.13)$$

Since the value of  $s$  is fixed by initial conditions, the only unknowns are the phase function  $S$  and the Berry phase  $\gamma$ . We can write an effective Hamilton–Jacobi equation for the total phase  $\tilde{S} = S + \epsilon s \gamma$ :

$$H(x, \nabla \tilde{S}) = \frac{1}{2}g^{\mu\nu}k_\mu k_\nu + \epsilon s k^\mu \partial_\mu \gamma - \epsilon s k^\mu B_\mu + \mathcal{O}(\epsilon^2). \quad (4.14)$$

The corresponding Hamiltonian function on  $T^*M$  is

$$H(x, p) = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu - \epsilon s g^{\mu\nu}p_\mu B_\nu(x, p), \quad (4.15)$$

and we have the following Hamilton’s equations:

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = g^{\mu\nu}p_\nu - \epsilon s \left( B^\mu + p^\alpha \nabla^\mu B_\alpha \right), \quad (4.16)$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}p_\alpha p_\beta + \epsilon s p_\alpha \left( \partial_\mu g^{\alpha\beta}B_\beta + g^{\alpha\beta}\partial_\mu B_\beta \right). \quad (4.17)$$

These equations contain polarization-dependent corrections to the null geodesic equations (4.5) and (4.6), representing the gravitational spin Hall effect of light. For  $\epsilon = 0$ , one recovers the standard geodesic equation in canonical coordinates.

We can also write these ray equations in a more compact form

$$\begin{pmatrix} \dot{x}^\mu \\ \dot{p}_\mu \end{pmatrix} = \begin{pmatrix} 0 & \delta_\nu^\mu \\ -\delta_\mu^\nu & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x^\nu} \\ \frac{\partial H}{\partial p_\nu} \end{pmatrix}, \quad (4.18)$$

where the constant matrix on the right-hand side is the inverse of the symplectic 2-form, or the Poisson tensor [45].

**4.2.1. Noncanonical coordinates.** The Hamiltonian (4.15) contains the Berry connection  $B_\mu$ , which is gauge-dependent. The latter means that  $B_\mu$  depends on the choice of  $m_\alpha$  and  $\bar{m}_\alpha$ ; for example, the transformation  $m_\alpha \rightarrow m_\alpha e^{i\phi}$  causes the following transformation of the Berry connection:

$$B_\mu \rightarrow B_\mu - \nabla_\mu \phi. \quad (4.19)$$

This kind of gauge dependence was considered by Littlejohn and Flynn in [44], where they also proposed how to make the Hamiltonian and the equations of motion gauge-invariant. The main idea is to introduce noncanonical coordinates such that the Berry connection is removed from the Hamiltonian and the symplectic form acquires the corresponding Berry curvature, which is gauge-invariant. This is similar to the description of a charged particle in an electromagnetic field in terms of either the canonical or the kinetic momentum of the particle. The Berry connection and Berry curvature play a similar role as the electromagnetic vector potential and the electromagnetic tensor [14].

We start by rewriting the Hamiltonian (4.15) as

$$H(x, p) = H_0(x, p) - \epsilon s g^{\mu\nu} p_\mu B_\nu(x, p), \quad (4.20)$$

where  $H_0 = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ . Following [44], the Berry connection can be written in the following way, by using the definition of the horizontal derivative

$$\begin{aligned} p^\mu B_\mu(x, p) &= i p^\mu \bar{m}^\alpha \overset{h}{\nabla}_\mu m_\alpha \\ &= i p^\mu \bar{m}^\alpha \nabla_\mu m_\alpha + i p^\mu p_\sigma \Gamma_{\mu\rho}^\sigma \bar{m}^\alpha \overset{v}{\nabla}^\rho m_\alpha \\ &= i \frac{\partial H_0}{\partial p_\mu} \bar{m}^\alpha \nabla_\mu m_\alpha - i \frac{\partial H_0}{\partial x^\mu} \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha. \end{aligned} \quad (4.21)$$

The Berry connection can be eliminated from the Hamiltonian (4.15) by considering the following change of coordinates on  $T^*M$

$$X^\mu = x^\mu + i \epsilon s \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha, \quad (4.22)$$

$$P_\mu = p_\mu - i \epsilon s \bar{m}^\alpha \nabla_\mu m_\alpha. \quad (4.23)$$

Since  $(X, P)$  are noncanonical coordinates, the symplectic form transforms nontrivially under this change of coordinates. The Hamiltonian (4.15) is as a scalar, so we obtain

$$\begin{aligned} H'(X, P) &= H(x, p) \\ &= H \left( X^\mu - i \epsilon s \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha, P_\mu + i \epsilon s \bar{m}^\alpha \nabla_\mu m_\alpha \right) \\ &= H(X, P) - i \epsilon s \frac{\partial H_0}{\partial x^\mu} \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha + i \epsilon s \frac{\partial H_0}{\partial p_\mu} \bar{m}^\alpha \nabla_\mu m_\alpha \\ &= H_0(X, P). \end{aligned} \quad (4.24)$$

In the new coordinate system  $(X, P)$ , we obtained the following Hamiltonian:

$$H'(X, P) = \frac{1}{2} g^{\mu\nu}(X) P_\mu P_\nu. \quad (4.25)$$

The corresponding Hamilton's equations can be written in a matrix form as:

$$\begin{pmatrix} \dot{X}^\mu \\ \dot{P}_\mu \end{pmatrix} = T' \begin{pmatrix} \frac{\partial H'}{\partial X^\nu} \\ \frac{\partial H'}{\partial P_\nu} \end{pmatrix}, \quad (4.26)$$

where  $T'$  is the Poisson tensor in the new variables. Following Marsden and Ratiu [45, Section 10.4], we obtain

$$T' = \begin{pmatrix} (F_{pp})^{\nu\mu} & \delta_\nu^\mu + (F_{xp})_\nu{}^\mu \\ -\delta_\mu^\nu - (F_{xp})_\mu{}^\nu & -(F_{xx})_{\nu\mu} \end{pmatrix}, \quad (4.27)$$

where we have the following Berry curvature terms

$$\begin{aligned}
(F_{pp})^{\nu\mu} &= i \left( \overset{v}{\nabla}^\mu \bar{m}^\alpha \overset{v}{\nabla}^\nu m_\alpha - \overset{v}{\nabla}^\nu \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha \right. \\
&\quad \left. + \bar{m}^\alpha \overset{v}{\nabla}^{[\mu} \overset{v}{\nabla}^{\nu]} m_\alpha - m_\alpha \overset{v}{\nabla}^{[\mu} \overset{v}{\nabla}^{\nu]} \bar{m}^\alpha \right), \\
(F_{xx})_{\nu\mu} &= i \left( \nabla_\mu \bar{m}^\alpha \nabla_\nu m_\alpha - \nabla_\nu \bar{m}^\alpha \nabla_\mu m_\alpha \right. \\
&\quad \left. + \bar{m}^\alpha \nabla_{[\mu} \nabla_{\nu]} m_\alpha - m_\alpha \nabla_{[\mu} \nabla_{\nu]} \bar{m}^\alpha \right), \\
(F_{px})_{\nu}{}^\mu &= - (F_{xp})^\mu{}_\nu = i \left( \overset{v}{\nabla}^\mu \bar{m}^\alpha \nabla_\nu m_\alpha - \nabla_\nu \bar{m}^\alpha \overset{v}{\nabla}^\mu m_\alpha \right).
\end{aligned} \tag{4.28}$$

Simplified expressions for these terms can be found in appendix C. Now we can write Hamilton's equations in the new variables:

$$\dot{X}^\mu = P^\mu + \epsilon s P^\nu (F_{px})_{\nu}{}^\mu + \epsilon s \Gamma_{\beta\nu}^\alpha P_\alpha P^\beta (F_{pp})^{\nu\mu}, \tag{4.29}$$

$$\dot{P}_\mu = \Gamma_{\beta\mu}^\alpha P_\alpha P^\beta - \epsilon s P^\nu (F_{xx})_{\nu\mu} - \epsilon s \Gamma_{\beta\nu}^\alpha P_\alpha P^\beta (F_{xp})^\nu{}_\mu. \tag{4.30}$$

The last term on the right-hand side of equation (4.29) is the covariant analogue of the spin Hall effect correction obtained in optics,  $(\mathbf{\hat{p}} \times \mathbf{p})/|\mathbf{p}|^3$ , due to the Berry curvature in momentum space [12, 55]. This term is also the source of the gravitational spin Hall effect in the work of Gosselin *et al.* [35]. In equation (4.30), the second term on the right-hand side contains the Riemann tensor, and resembles the curvature term obtained in the Mathisson–Papapetrou–Dixon equations [16].

## 5. NUMERICAL EXAMPLES

To illustrate how the polarization-dependent correction terms modify the ray trajectories, we provide here some numerical examples. For convenience, we use canonical coordinates and treat  $x^0$  as a parameter along the rays. Hence, equations (4.16) and (4.17) become

$$\dot{x}^0 = 1, \tag{5.1}$$

$$\dot{x}^i = \frac{g^{i\nu} p_\nu - \epsilon s \left( B^i + p^\alpha \overset{v}{\nabla}^i B_\alpha \right)}{g^{0\nu} p_\nu - \epsilon s \left( B^0 + p^\alpha \overset{v}{\nabla}^0 B_\alpha \right)}, \tag{5.2}$$

$$\dot{p}_i = \frac{-\frac{1}{2} \partial_i g^{\alpha\beta} p_\alpha p_\beta + \epsilon s p_\alpha (\partial_i g^{\alpha\beta} B_\beta + g^{\alpha\beta} \partial_i B_\beta)}{g^{0\nu} p_\nu - \epsilon s \left( B^0 + p^\alpha \overset{v}{\nabla}^0 B_\alpha \right)}, \tag{5.3}$$

and  $p_0$  is calculated from

$$\frac{1}{2} g^{\mu\nu} p_\mu p_\nu - \epsilon s g^{\mu\nu} p_\mu B_\nu(x, p) = 0. \tag{5.4}$$

This equation can be solved explicitly, using the fact that the velocity  $\dot{x}^\alpha$  is future oriented:

$$\begin{aligned}
p_0 &= (g^{00})^{-1} \left\{ - (g^{0i} p_i - \epsilon s g^{0\mu} B_\mu) \right. \\
&\quad \left. + \left[ (g^{0i} p_i - \epsilon s g^{0\mu} B_\mu)^2 - g^{00} (g^{ij} p_i p_j - 2 \epsilon s p_i g^{i\mu} B_\mu) \right]^{1/2} \right\}.
\end{aligned} \tag{5.5}$$

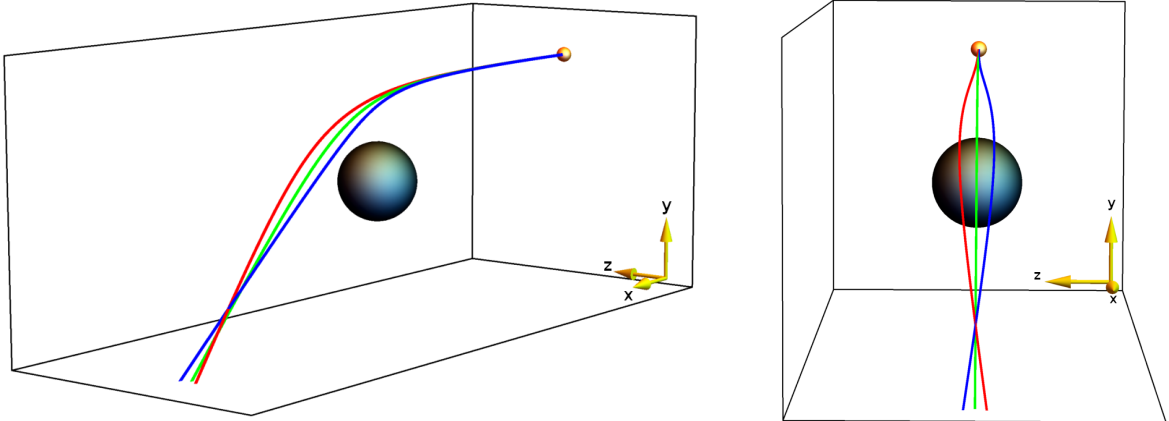


FIGURE 1. Results of numerical simulations illustrating the gravitational spin Hall effect of light around a Schwarzschild black hole. The effect is exaggerated for visualization purposes. The two figures present the same rays from different viewing angles. The central sphere represents the Schwarzschild black hole, and the small orange sphere represents a source of light. The blue and the red trajectories correspond to rays of opposite circular polarizations,  $s = \pm 1$ , while the green trajectory represents a null geodesic. We take  $r_s = 1$ , and we start with the initial position  $x^i(0) = (-50r_s, 15r_s, 0)$ , and initial normalized momentum  $p_i = (1, 0, 0)$ . The wavelength  $\lambda$  is set to a sufficiently large value to make the effect visible on this plot.

Note that in general  $B_\mu$  depends on  $p_0$ . However, since this is an  $\mathcal{O}(\epsilon^1)$  term, we can replace the  $\mathcal{O}(\epsilon^0)$  expression for  $p_0$  in  $B_\mu$ .

In order to compare with the results of Gosselin *et al.* [35], we consider a Schwarzschild spacetime in Cartesian isotropic coordinates  $(t, x, y, z)$ :

$$ds^2 = - \left( \frac{1 - \frac{r_s}{4R}}{1 + \frac{r_s}{4R}} \right)^2 dt^2 + \left( 1 + \frac{r_s}{4R} \right)^4 (dx^2 + dy^2 + dz^2), \quad (5.6)$$

where  $r_s = \frac{2GM}{c^2}$  is the Schwarzschild radius, and  $R = \sqrt{x^2 + y^2 + z^2}$ . We also define the following orthonormal tetrad:

$$e_0 = \frac{1 + \frac{r_s}{4R}}{1 - \frac{r_s}{4R}} \partial_t, \quad e_1 = \left( 1 + \frac{r_s}{4R} \right)^{-2} \partial_x, \quad e_2 = \left( 1 + \frac{r_s}{4R} \right)^{-2} \partial_y, \quad e_3 = \left( 1 + \frac{r_s}{4R} \right)^{-2} \partial_z, \quad (5.7)$$

where  $t^\mu = (e_0)^\mu$  is our choice of observer.

The Berry connection  $B_\mu$  can be explicitly computed by introducing a particular choice of polarization vectors. Using the orthonormal tetrad, we can easily adapt the polarization vectors used in optics [55]. We can write  $p^\mu = P^a (e_a)^\mu$ ,  $v^\mu = V^a (e_a)^\mu$ , and  $w^\mu = W^a (e_a)^\mu$ , where the components of these vectors are given by

$$P = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}, \quad V = \begin{pmatrix} 0 \\ -P^2/P_p \\ P^1/P_p \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 \\ P^1 P^3 / (P_s P_p) \\ P^2 P^3 / (P_s P_p) \\ -P_p / P_s \end{pmatrix}, \quad (5.8)$$



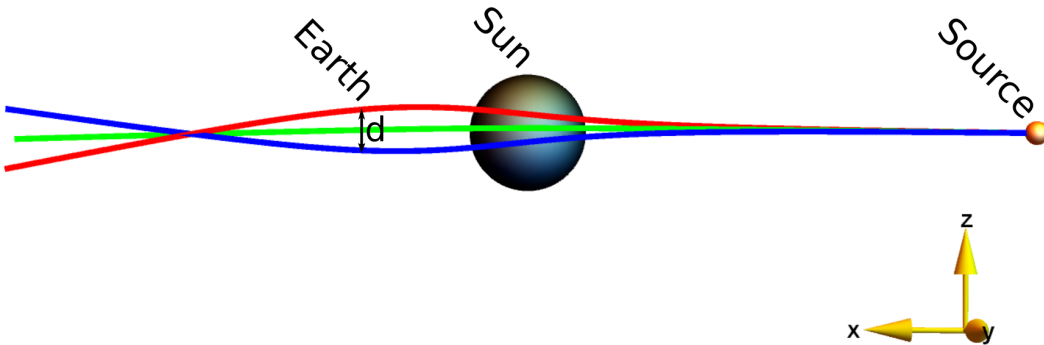


FIGURE 2. Results of numerical simulations illustrating the gravitational spin Hall effect of light around the Sun. The effect is exaggerated for visualization purposes. Viewing angle from above. The separation distance  $d$  is observed from the Earth. The blue and the red trajectories correspond to rays of opposite circular polarization,  $s = \pm 1$ , while the green trajectory represents a null geodesic. We take  $r_s = 3$  km, and we start with the initial position  $x^i(0) = (-10^7 r_s, 3 \times 10^5 r_s, 0)$ , and initial normalized momentum  $p_i = (1, 0, 0)$ .

where

$$P_p = \sqrt{(P^1)^2 + (P^2)^2}, \quad P_s = \sqrt{(P^1)^2 + (P^2)^2 + (P^3)^2}. \quad (5.9)$$

The vectors  $v^\mu$  and  $w^\mu$  are real unit spacelike vectors that represent a linear polarization basis, satisfying (C.2). We now have all the elements required for the numerical integration of equations (5.1)–(5.3). For this purpose, we are using Mathematica [39], with the NDSolve function. The default settings for integration method, precision and accuracy are being used.

As a first step, we numerically compare our ray equations (5.1)–(5.3) with those predicted by Gosselin *et al.* [35]. Up to numerical errors, we obtain the same ray trajectories with both sets of equations. However, while the equations obtained by Gosselin *et al.* only apply to static spacetimes, equations (5.1)–(5.3) do not have this limitation.

The results of our numerical simulations are shown in Figure 1, which illustrates the general behavior of the gravitational spin Hall effect of light around a Schwarzschild black hole (the actual effect is small, so the figure is obtained by numerical integration of equations (5.1)–(5.3) for unrealistic parameters). Here, we consider rays of opposite circular polarization ( $s = \pm 1$ ) passing close to a Schwarzschild black hole, together with a reference null geodesic ( $s = 0$ ). Except for the value of  $s$ , we are considering the same initial conditions,  $(x^i(0), p_i(0))$ , for these rays. Unlike the null geodesic, for which the motion is planar, the circularly polarized rays are not confined to a plane.

As another example, we used initial conditions  $(x^i(0), p_i(0))$ , such that the rays are initialized as radially ingoing or outgoing. In this case (not represent by any figure, since it is trivial), the gravitational spin Hall effect vanishes, and the circularly polarized rays coincide with the radial null geodesic.

Using these numerical methods, we can also estimate the magnitude of the gravitational spin Hall effect. As a particular example, we consider a similar situation to the one presented in Figure 1, where the black hole is replaced with the Sun. More precisely, we model this

situation by considering a Schwarzschild black hole with  $r_s \approx 3$  km. We consider the deflection of circularly polarized rays coming from a light source far away, passing close to the surface of the Sun, and then observed on the Earth. This situation is illustrated in Figure 2. The numerical results are based on the initial data presented in the caption of Figure 2. When reaching the Earth, the separation distance between the rays of opposite circular polarization depends on the wavelength. For example, taking wavelengths of the order  $\lambda \approx 10^{-9}$  m results in a separation distance of the order  $d \approx 10^{-15}$  m, while for wavelengths of the order of  $\lambda \approx 1$  m we obtain a separation distance of the order  $d \approx 10^{-6}$  m. Although the ray separation is small (about six orders of magnitude smaller than the wavelength), what really matters is that the rays are scattered by a finite angle, so the ray separation grows linearly with distance after the re-intersection point. This means that the effect should be robustly observable if one measures it sufficiently far from the Sun. Furthermore, massive compact astronomical objects, such as black holes or neutron stars, are expected to produce a larger gravitational spin Hall effect.

## 6. CONCLUSIONS

In summary, we have presented a first comprehensive theory of the gravitational spin Hall effect that occurs due to the coupling of the polarization with the translational dynamics of the light rays. The ray dynamics is governed by the corrected Hamiltonian

$$H(x, p) = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu - \epsilon s g^{\mu\nu}p_\mu B_\nu(x, p). \quad (6.1)$$

Here, the first term represents the geometrical optics Hamiltonian, and the second term represents a correction of  $\mathcal{O}(\epsilon^1)$  that is due to the Berry connection, which is given by

$$B_\mu(x, p) = i\bar{m}^\alpha \nabla_\mu^h m_\alpha = i\bar{m}^\alpha \left( \frac{\partial}{\partial x^\mu} m_\alpha - \Gamma_{\alpha\mu}^\sigma m_\sigma + \Gamma_{\mu\rho}^\sigma p_\sigma \frac{\partial}{\partial p_\rho} m_\alpha \right). \quad (6.2)$$

Assuming the noncanonical coordinates (4.22), the corresponding ray equations are

$$\dot{X}^\mu = P^\mu + \epsilon s P^\nu (F_{px})_{\nu}{}^\mu + \frac{2i\epsilon s}{(t^\alpha P_\alpha)^2} \Gamma_{\beta\nu}^\alpha P_\alpha P^\beta m^{[\nu} \bar{m}^{\mu]}, \quad (6.3)$$

$$\dot{P}_\mu = \Gamma_{\beta\mu}^\alpha P_\alpha P^\beta + \epsilon s P^\nu \left[ iR_{\alpha\beta\mu\nu} m^\alpha \bar{m}^\beta + (\tilde{F}_{xx})_{\nu\mu} \right] + \epsilon s \Gamma_{\beta\nu}^\alpha P_\alpha P^\beta (F_{px})_{\mu}{}^\nu, \quad (6.4)$$

where the terms  $F_{px}$  and  $\tilde{F}_{xx}$  and the timelike vector  $t$  are given in appendix C. The term containing  $m^{[\nu} \bar{m}^{\mu]}$  is the covariant analogue of spin Hall correction term usually encountered in optics [12, 35], while the Riemann curvature term is reminiscent of the Mathisson–Papapetrou–Dixon equations [16].

The resulting deviation of the ray trajectories from those predicted by geometrical optics is weak but not unobservable. First of all, even small angular deviations are observable at large enough distances. Second, as shown in [38], weak quantum measurement techniques can be used to detect the spin Hall effect of light, even when the spatial separation between the left-polarized and the right-polarized beams of light is smaller than the wavelength.

Potentially, this work can be naturally extended in two directions. Firstly, the corrected ray equations are yet to be studied more thoroughly, both analytically and numerically. Rigorous numerical investigations are needed to obtain a precise prediction of the effect, in particular for Kerr black holes. Secondly, Maxwell’s equations are a proxy to linearized gravity. It is expected that a similar approach can be carried out to obtain an effective pointwise description of a gravitational wave packet, extending the results of [70].

As discussed in [13], the spin Hall effect of light is directly related to the conservation of total angular momentum. For the discussion presented so far, the considered rays carry extrinsic orbital angular momentum, associated with the ray trajectory, and intrinsic spin angular momentum, associated with the polarization. However, it is well known that light can also carry intrinsic orbital angular momentum [2, 3, 41] (see also [1] and references therein). In principle, the magnitude of the spin Hall effect can be increased by considering optical beams carrying intrinsic orbital angular momentum [8]. The method and ansatz that we have adopted is insufficient to describe this effect. A more realistic and more precise approach involving wave packets, such as Laguerre–Gaussian beams, should be considered. It may be possible to do so using the machinery developed in [17].

A formulation of the special-relativistic dynamics of massless spinning particles and wave packets beyond the geometrical optics limit has been previously reported by Duval and collaborators, cf. [21] for the spin-1/2 case, see also [65]. This analysis relates the modified dynamics to the approach of Souriau [64], making use of so-called spin enslaving. This has been extended to general helicity by Andzejewski *et al.* [4]. We expect that the Hamiltonian formulation presented here corresponds to a general relativistic version of the models considered in the mentioned papers. This will be considered in a future work.

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## APPENDIX A. HORIZONTAL AND VERTICAL DERIVATIVES ON $T^*M$

Let  $(x^\mu, p_\mu)$  be canonical coordinates on  $T^*M$ . Considering fields defined on  $T^*M$ , such as  $u_\alpha(x, p)$  and  $v^\alpha(x, p)$ , the horizontal and vertical derivatives are defined as follows [59, section 3.5]:

$$\overset{v}{\nabla}^\mu u_\alpha = \frac{\partial}{\partial p_\mu} u_\alpha, \quad (\text{A.1a})$$

$$\overset{h}{\nabla}_\mu u_\alpha = \frac{\partial}{\partial x^\mu} u_\alpha - \Gamma_{\alpha\mu}^\sigma u_\sigma + \Gamma_{\mu\rho}^\sigma p_\sigma \frac{\partial}{\partial p_\rho} u_\alpha, \quad (\text{A.1b})$$

$$\overset{v}{\nabla}{}^\mu v^\alpha = \frac{\partial}{\partial p_\mu} v^\alpha, \quad (\text{A.2a})$$

$$\overset{h}{\nabla}{}_\mu v^\alpha = \frac{\partial}{\partial x^a} v^\alpha + \Gamma_{\sigma\mu}^\alpha v^\sigma + \Gamma_{\mu\rho}^\sigma p_\sigma \frac{\partial}{\partial p_\rho} v^\alpha. \quad (\text{A.2b})$$

The extension for general tensor fields on  $T^*M$  is straightforward. Note that, in contrast to [59, section 3.5], we have the opposite sign for the last term in the definition of the horizontal derivative. This is because our fields,  $u_\alpha(x, p)$  and  $v^\alpha(x, p)$ , are defined on  $T^*M$ , and not on  $TM$ , as is the case in the reference mentioned before. We can make use of the following properties:

$$[\overset{h}{\nabla}{}_\mu, \overset{v}{\nabla}{}^\nu] = [\overset{v}{\nabla}{}^\mu, \overset{v}{\nabla}{}^\nu] = 0, \quad \overset{h}{\nabla}{}_\mu p_\alpha = \overset{h}{\nabla}{}_\mu g_{\alpha\beta} = \overset{v}{\nabla}{}^\mu g_{\alpha\beta} = 0. \quad (\text{A.3})$$

## APPENDIX B. VARIATION OF THE ACTION

Here, we derive the Euler–Lagrange equations that correspond to the action

$$J = \int_M d^4x \sqrt{g} \mathcal{L}, \quad (\text{B.1})$$

where the Lagrangian density is of the following form:

$$\mathcal{L} = \mathcal{L} \left\{ S(x), \nabla_\mu S(x), A_\alpha(x, \nabla S(x)), \nabla_\mu [A_\alpha(x, \nabla S(x))], \right. \\ \left. A^{*\alpha}(x, \nabla S(x)), \nabla_\mu [A^{*\alpha}(x, \nabla S(x))] \right\}. \quad (\text{B.2})$$

Here,  $S(x)$  is an independent field, while  $A_\alpha$  and  $A^{*\alpha}$  cannot be considered independent, since they depend on  $\nabla_\mu S$ . Following Hawking and Ellis [37, p. 65], we define the variation of a field  $\Psi_i$  as a one-parameter family of fields  $\Psi_i(u, x)$ , with  $u \in (-\varepsilon, \varepsilon)$  and  $x \in M$ . We use the following notation:

$$\left. \frac{\partial \Psi_i(u, x)}{\partial u} \right|_{u=0} = \Delta \Psi_i. \quad (\text{B.3})$$

Note that the derivative with respect to the parameter  $u$  commutes with the covariant derivative, so we have:

$$\frac{d}{du} \nabla_\mu S(u, x) = \nabla_\mu \left( \frac{\partial S}{\partial u} \right), \quad (\text{B.4})$$

$$\frac{d}{du} A_\alpha(u, x, \nabla S(u, x)) = \frac{\partial A_\alpha}{\partial u} + \frac{\partial A_\alpha}{\partial \nabla_\nu S} \nabla_\nu \left( \frac{\partial S}{\partial u} \right), \quad (\text{B.5})$$

$$\begin{aligned} \frac{d}{du} \nabla_\mu [A_\alpha(u, x, \nabla S(u, x))] &= \nabla_\mu \left[ \frac{d}{du} A_\alpha(u, x, \nabla S(u, x)) \right] \\ &= \nabla_\mu \left[ \frac{\partial A_\alpha}{\partial u} + \frac{\partial A_\alpha}{\partial \nabla_\nu S} \nabla_\nu \left( \frac{\partial S}{\partial u} \right) \right]. \end{aligned} \quad (\text{B.6})$$

We take the variation of the action, taking special care when applying the chain rule:

$$\begin{aligned}
0 = \frac{dJ}{du} \Big|_{u=0} &= \int_M d^4x \sqrt{g} \left\{ \frac{\partial \mathcal{L}}{\partial S} \Delta S + \frac{\partial \mathcal{L}}{\partial \nabla_\mu S} \Delta (\nabla_\mu S) \right. \\
&\quad + \frac{\partial \mathcal{L}}{\partial A_\alpha} \left[ \Delta A_\alpha + \frac{\partial A_\alpha}{\partial \nabla_\mu S} \nabla_\mu (\Delta S) \right] \\
&\quad + \frac{\partial \mathcal{L}}{\partial \nabla_\mu A_\alpha} \nabla_\mu \left[ \Delta A_\alpha + \frac{\partial A_\alpha}{\partial \nabla_\nu S} \nabla_\nu (\Delta S) \right] \\
&\quad + \frac{\partial \mathcal{L}}{\partial A^{*\alpha}} \left[ \Delta A^{*\alpha} + \frac{\partial A^{*\alpha}}{\partial \nabla_\mu S} \nabla_\mu (\Delta S) \right] \\
&\quad \left. + \frac{\partial \mathcal{L}}{\partial \nabla_\mu A^{*\alpha}} \nabla_\mu \left[ \Delta A^{*\alpha} + \frac{\partial A^{*\alpha}}{\partial \nabla_\nu S} \nabla_\nu (\Delta S) \right] \right\}. \tag{B.7}
\end{aligned}$$

Integrating by parts, and assuming the boundary terms vanish, we obtain:

$$\begin{aligned}
0 = \frac{dJ}{du} \Big|_{u=0} &= \int_M d^4x \sqrt{g} \left\{ \left( \frac{\partial \mathcal{L}}{\partial A_\alpha} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu A_\alpha} \right) \Delta A_\alpha \right. \\
&\quad + \left( \frac{\partial \mathcal{L}}{\partial A^{*\alpha}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu A^{*\alpha}} \right) \Delta A^{*\alpha} \\
&\quad + \frac{\partial \mathcal{L}}{\partial S} \Delta S - \nabla_\mu \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\mu S} + \frac{\partial A_\alpha}{\partial \nabla_\mu S} \left( \frac{\partial \mathcal{L}}{\partial A_\alpha} - \nabla_\nu \frac{\partial \mathcal{L}}{\partial \nabla_\nu A_\alpha} \right) \right. \\
&\quad \left. + \frac{\partial A^{*\alpha}}{\partial \nabla_\mu S} \left( \frac{\partial \mathcal{L}}{\partial A^{*\alpha}} - \nabla_\nu \frac{\partial \mathcal{L}}{\partial \nabla_\nu A^{*\alpha}} \right) \right] \Delta S \left. \right\}. \tag{B.8}
\end{aligned}$$

Since the above equation must be satisfied for all variations  $\Delta S$ ,  $\Delta A_\alpha$ , and  $\Delta A^{*\alpha}$ , we obtain the following Euler–Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A^{*\alpha}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu A^{*\alpha}} = \mathcal{O}(\epsilon^2), \tag{B.9}$$

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu A_\alpha} = \mathcal{O}(\epsilon^2), \tag{B.10}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial S} - \nabla_\mu \left[ \frac{\partial \mathcal{L}}{\partial \nabla_\mu S} + \frac{\partial A_\alpha}{\partial \nabla_\mu S} \left( \frac{\partial \mathcal{L}}{\partial A_\alpha} - \nabla_\nu \frac{\partial \mathcal{L}}{\partial \nabla_\nu A_\alpha} \right) \right. \\
\left. + \frac{\partial A^{*\alpha}}{\partial \nabla_\mu S} \left( \frac{\partial \mathcal{L}}{\partial A^{*\alpha}} - \nabla_\nu \frac{\partial \mathcal{L}}{\partial \nabla_\nu A^{*\alpha}} \right) \right] = \mathcal{O}(\epsilon^2). \tag{B.11}
\end{aligned}$$

Furthermore, equation (B.11) can be simplified by using equations (B.9) and (B.10). Thus, as a final result, we have the following set of Euler–Lagrange equations:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A^{*\alpha}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu A^{*\alpha}} &= \mathcal{O}(\epsilon^2), \\ \frac{\partial \mathcal{L}}{\partial A_\alpha} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu A_\alpha} &= \mathcal{O}(\epsilon^2), \\ \frac{\partial \mathcal{L}}{\partial S} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu S} &= \mathcal{O}(\epsilon^2).\end{aligned}\tag{B.12}$$

### APPENDIX C. BERRY CURVATURE

In order to calculate the Berry curvature terms (4.28), it is enough to use a tetrad  $\{t^\alpha, p^\alpha, v^\alpha, w^\alpha\}$ , where  $t^\alpha$  is a future-oriented timelike vector field representing a family of observers and  $p^\alpha$  is a generic vector, not necessarily null, representing the momentum of a point particle (ray). The vectors  $v^\alpha$  and  $w^\alpha$  are real spacelike vectors related to  $m^\alpha$  and  $\bar{m}^\alpha$  by the following relations:

$$m^\alpha = \frac{1}{\sqrt{2}}(v^\alpha + iw^\alpha), \quad \bar{m}^\alpha = \frac{1}{\sqrt{2}}(v^\alpha - iw^\alpha).\tag{C.1}$$

The elements of the tetrad  $\{t^\alpha, p^\alpha, v^\alpha, w^\alpha\}$  satisfy the following relations:

$$\begin{aligned}t_\alpha t^\alpha &= -1, & p_\alpha p^\alpha &= \kappa, & t_\alpha p^\alpha &= -\epsilon\omega, & v_\alpha v^\alpha &= w_\alpha w^\alpha = 1, \\ t_\alpha v^\alpha &= t_\alpha w_\alpha = p_\alpha v^\alpha = p_\alpha w^\alpha = v_\alpha w^\alpha & &= 0.\end{aligned}\tag{C.2}$$

Note that the vectors  $v^\alpha$  and  $w^\alpha$  depend of  $p^\mu$  through the orthogonality condition, while  $t^\alpha$  is independent of  $p^\mu$ . We start by computing the vertical derivatives of the vectors  $v^\alpha$  and  $w^\alpha$ . Using the tetrad, we can write:

$$\overset{v}{\nabla}^\mu v^\alpha = \frac{\partial v^\alpha}{\partial p_\mu} = c_1^\mu t^\alpha + c_2^\mu p^\alpha + c_3^\mu v^\alpha + c_4^\mu w^\alpha,\tag{C.3}$$

$$\overset{v}{\nabla}^\mu w^\alpha = \frac{\partial w^\alpha}{\partial p_\mu} = d_1^\mu t^\alpha + d_2^\mu p^\alpha + d_3^\mu v^\alpha + d_4^\mu w^\alpha,\tag{C.4}$$

where  $c_i^\mu$  and  $d_i^\mu$  are unknown vector fields that need to be determined. Using the properties from equation (C.2), we obtain

$$\begin{aligned}\overset{v}{\nabla}^\mu v^\alpha &= \frac{\epsilon\omega}{\epsilon^2\omega^2 + \kappa} v^\mu t^\alpha - \frac{1}{\epsilon^2\omega^2 + \kappa} v^\mu p^\alpha + c_4^\mu w^\alpha, \\ \overset{v}{\nabla}^\mu w^\alpha &= \frac{\epsilon\omega}{\epsilon^2\omega^2 + \kappa} w^\mu t^\alpha - \frac{1}{\epsilon^2\omega^2 + \kappa} w^\mu p^\alpha + d_3^\mu v^\alpha.\end{aligned}\tag{C.5}$$

Applying the same arguments to the terms  $\nabla_\mu v_\alpha$  and  $\nabla_\mu w_\alpha$ , we also obtain

$$\begin{aligned}\nabla_\mu v_\alpha &= -\frac{1}{\epsilon^2\omega^2 + \kappa} (\epsilon\omega p_\sigma \nabla_\mu v^\sigma + \kappa t_\sigma \nabla_\mu v^\sigma) t_\alpha \\ &\quad + \frac{1}{\epsilon^2\omega^2 + \kappa} (p_\sigma \nabla_\mu v^\sigma - \epsilon\omega t_\sigma \nabla_\mu v^\sigma) p_\alpha + f_{4\mu} w_\alpha, \\ \nabla_\mu w_\alpha &= -\frac{1}{\epsilon^2\omega^2 + \kappa} (\epsilon\omega p_\sigma \nabla_\mu w^\sigma + \kappa t_\sigma \nabla_\mu w^\sigma) t_\alpha \\ &\quad + \frac{1}{\epsilon^2\omega^2 + \kappa} (p_\sigma \nabla_\mu w^\sigma - \epsilon\omega t_\sigma \nabla_\mu w^\sigma) p_\alpha + g_{3\mu} v_\alpha.\end{aligned}\tag{C.6}$$

Note that the fields  $c_{4\mu}$ ,  $d_{3\mu}$ ,  $f_{4\mu}$ , and  $g_{3\mu}$  are undetermined within this approach, but this is not a problem, because they do not affect the Berry curvature.

**C.1.  $F_{pp}$ .** We compute  $(F_{pp})^{\nu\mu}$  by using equation (C.5) and setting  $\kappa = 0$ . Since vertical derivatives commute, we can write

$$\begin{aligned}
(F_{pp})^{\nu\mu} &= i \left( \overset{v}{\nabla}{}^\mu \bar{m}^\alpha \overset{v}{\nabla}{}^\nu m_\alpha - \overset{v}{\nabla}{}^\nu \bar{m}^\alpha \overset{v}{\nabla}{}^\mu m_\alpha \right) \\
&= \overset{v}{\nabla}{}^\nu v^\alpha \overset{v}{\nabla}{}^\mu w_\alpha - \overset{v}{\nabla}{}^\mu v^\alpha \overset{v}{\nabla}{}^\nu w_\alpha \\
&= \frac{2}{\epsilon^2 \omega^2} v^{[\nu} w^{\mu]} \\
&= \frac{2i}{\epsilon^2 \omega^2} m^{[\nu} \bar{m}^{\mu]}.
\end{aligned} \tag{C.7}$$

**C.2.  $F_{xx}$ .** We have

$$\begin{aligned}
(F_{xx})_{\nu\mu} &= i \left( \nabla_\mu \bar{m}^\alpha \nabla_\nu m_\alpha - \nabla_\nu \bar{m}^\alpha \nabla_\mu m_\alpha \right. \\
&\quad \left. + \bar{m}^\alpha \nabla_{[\mu} \nabla_{\nu]} m_\alpha - m_\alpha \nabla_{[\mu} \nabla_{\nu]} \bar{m}^\alpha \right).
\end{aligned} \tag{C.8}$$

The last two terms can be expressed in terms of the Riemann tensor:

$$i \left( \bar{m}^\alpha \nabla_{[\mu} \nabla_{\nu]} m_\alpha - m_\alpha \nabla_{[\mu} \nabla_{\nu]} \bar{m}^\alpha \right) = -i R_{\alpha\beta\mu\nu} m^\alpha \bar{m}^\beta. \tag{C.9}$$

The first two terms can be computed using equation (C.6) and  $\kappa = 0$ :

$$\begin{aligned}
(\tilde{F}_{xx})_{\nu\mu} &= i \left( \nabla_\mu \bar{m}^\alpha \nabla_\nu m_\alpha - \nabla_\nu \bar{m}^\alpha \nabla_\mu m_\alpha \right) = \nabla_\nu v^\alpha \nabla_\mu w_\alpha - \nabla_\mu v^\alpha \nabla_\nu w_\alpha \\
&= \frac{1}{\epsilon^2 \omega^2} \left( p_\sigma \nabla_\mu v^\sigma p_\rho \nabla_\nu w^\rho - p_\sigma \nabla_\nu v^\sigma p_\rho \nabla_\mu w^\rho \right. \\
&\quad \left. - \epsilon \omega p_\sigma \nabla_\mu v^\sigma t_\rho \nabla_\nu w^\rho + \epsilon \omega p_\sigma \nabla_\nu v^\sigma t_\rho \nabla_\mu w^\rho \right. \\
&\quad \left. - \epsilon \omega t_\sigma \nabla_\mu v^\sigma p_\rho \nabla_\nu w^\rho + \epsilon \omega t_\sigma \nabla_\nu v^\sigma p_\rho \nabla_\mu w^\rho \right) \\
&= \frac{1}{\epsilon^2 \omega^2} \left( p_\sigma \nabla_\mu m^\sigma p_\rho \nabla_\nu \bar{m}^\rho - p_\sigma \nabla_\nu m^\sigma p_\rho \nabla_\mu \bar{m}^\rho \right. \\
&\quad \left. - \epsilon \omega p_\sigma \nabla_\mu m^\sigma t_\rho \nabla_\nu \bar{m}^\rho + \epsilon \omega p_\sigma \nabla_\nu m^\sigma t_\rho \nabla_\mu \bar{m}^\rho \right. \\
&\quad \left. - \epsilon \omega t_\sigma \nabla_\mu m^\sigma p_\rho \nabla_\nu \bar{m}^\rho + \epsilon \omega t_\sigma \nabla_\nu m^\sigma p_\rho \nabla_\mu \bar{m}^\rho \right)
\end{aligned} \tag{C.10}$$

**C.3.  $F_{px}$  and  $F_{xp}$ .** Since  $(F_{px})_\nu{}^\mu = -(F_{xp})^\mu{}_\nu$ , it is enough to compute only one term. Using equations (C.5) and (C.6), and setting  $\kappa = 0$ , we obtain

$$\begin{aligned}
(F_{px})_\nu{}^\mu &= i \left( \overset{v}{\nabla}{}^\mu \bar{m}^\alpha \nabla_\nu m_\alpha - \nabla_\nu \bar{m}^\alpha \overset{v}{\nabla}{}^\mu m_\alpha \right) \\
&= \nabla_\nu v^\alpha \overset{v}{\nabla}{}^\mu w_\alpha - \overset{v}{\nabla}{}^\mu v^\alpha \nabla_\nu w_\alpha \\
&= \frac{1}{\epsilon^2 \omega^2} [(p_\sigma \nabla_\nu w^\sigma - \epsilon \omega t_\sigma \nabla_\nu w^\sigma) v^\mu - (p_\sigma \nabla_\nu v^\sigma - \epsilon \omega t_\sigma \nabla_\nu v^\sigma) w^\mu] \\
&= \frac{i}{\epsilon^2 \omega^2} [(p_\sigma \nabla_\nu \bar{m}^\sigma - \epsilon \omega t_\sigma \nabla_\nu \bar{m}^\sigma) m^\mu - (p_\sigma \nabla_\nu m^\sigma - \epsilon \omega t_\sigma \nabla_\nu m^\sigma) \bar{m}^\mu].
\end{aligned} \tag{C.11}$$

## REFERENCES

- [1] S. Aghapour, L. Andersson, and R. Bhattacharyya. Helicity and spin conservation in Maxwell theory and Linearized Gravity. *arXiv preprint arXiv:1812.03292*, 2018.
- [2] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman. Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes. *Physical Review A*, 45:8185–8189, Jun 1992.
- [3] D. L. Andrews and M. Babiker. *The Angular Momentum of Light*. Cambridge University Press, 2012.
- [4] K. Andrzejewski, A. Kijanka-Dec, P. Kosiński, and P. Maślanka. Chiral fermions, massless particles and Poincare covariance. *Physics Letters B*, 746:417–423, June 2015.
- [5] J. Audretsch. Trajectories and spin motion of massive spin- $\frac{1}{2}$  particles in gravitational fields. *Journal of Physics A: Mathematical and General*, 14:411–422, 1981.
- [6] A. A. Bakun, B. P. Zakharchenya, A. A. Rogachev, M. N. Tkachuk, and V. G. Fleisher. Observation of a surface photocurrent caused by optical orientation of electrons in a semiconductor. *Soviet Journal of Experimental and Theoretical Physics Letters*, 40:1293, Dec. 1984.
- [7] A. Bérard and H. Mohrbach. Spin Hall effect and Berry phase of spinning particles. *Physics Letters A*, 352(3):190–195, 2006.
- [8] K. Y. Bliokh. Geometrical optics of beams with vortices: Berry phase and orbital angular momentum Hall effect. *Physical Review Letters*, 97:043901, Jul 2006.
- [9] K. Y. Bliokh. Geometrodynamics of polarized light: Berry phase and spin Hall effect in a gradient-index medium. *Journal of Optics A: Pure and Applied Optics*, 11(9):094009, aug 2009.
- [10] K. Y. Bliokh and Y. P. Bliokh. Modified geometrical optics of a smoothly inhomogeneous isotropic medium: The anisotropy, Berry phase, and the optical Magnus effect. *Physical Review E*, 70:026605, Aug 2004.
- [11] K. Y. Bliokh and Y. P. Bliokh. Topological spin transport of photons: The optical Magnus effect and Berry phase. *Physics Letters A*, 333(3):181–186, 2004.
- [12] K. Y. Bliokh, A. Niv, V. Kleiner, and E. Hasman. Geometrodynamics of spinning light. *Nature Photonics*, 2:748, Nov 2008.
- [13] K. Y. Bliokh, F. J. Rodríguez-Fortuño, F. Nori, and A. V. Zayats. Spin-orbit interactions of light. *Nature Photonics*, 9(12):796–808, 2015.
- [14] D. Chruściński and A. Jamiołkowski. *Geometric phases in classical and quantum mechanics*, volume 36. Springer Science & Business Media, 2012.
- [15] W. G. Dixon. A covariant multipole formalism for extended test bodies in general relativity. *Il Nuovo Cimento (1955-1965)*, 34(2):317–339, 1964.
- [16] W. G. Dixon. The new mechanics of Myron Mathisson and its subsequent development. In *Equations of Motion in Relativistic Gravity*, pages 1–66. Springer, 2015.
- [17] I. Y. Dodin, D. E. Ruiz, K. Yanagihara, Y. Zhou, and S. Kubo. Quasioptical modeling of wave beams with and without mode conversion. I. Basic theory. *Physics of Plasmas*, 26(7):072110, 2019.
- [18] S. R. Dolan. Geometrical optics for scalar, electromagnetic and gravitational waves on curved spacetime. *International Journal of Modern Physics D*, 27:1843010, 2018.
- [19] S. R. Dolan. Higher-order geometrical optics for circularly-polarized electromagnetic waves. *arXiv preprint arXiv:1801.02273*, 2018.
- [20] A. V. Dooghin, N. D. Kundikova, V. S. Liberman, and B. Y. Zel’dovich. Optical Magnus effect. *Physical Review A*, 45:8204–8208, Jun 1992.
- [21] C. Duval, M. Elbistan, P. A. Horváthy, and P. M. Zhang. Wigner-Souriau translations and Lorentz symmetry of chiral fermions. *Physics Letters B*, 742:322–326, Mar. 2015.
- [22] C. Duval, Z. Horváth, and P. A. Horváthy. Fermat principle for spinning light. *Physical Review D*, 74:021701, Jul 2006.
- [23] C. Duval, Z. Horváth, and P. A. Horváthy. Geometrical spinoptics and the optical Hall effect. *Journal of Geometry and Physics*, 57(3):925–941, 2007.
- [24] C. Duval, L. Marsot, and T. Schücker. Gravitational birefringence of light in Schwarzschild spacetime. *Physical Review D*, 99:124037, Jun 2019.
- [25] C. Duval and T. Schücker. Gravitational birefringence of light in Robertson-Walker cosmologies. *Physical Review D*, 96:043517, Aug 2017.
- [26] M. I. Dyakonov and A. V. Khaetskii. *Spin Hall Effect*, pages 211–243. Springer Berlin Heidelberg, 2008.



- [27] M. I. Dyakonov and V. I. Perel. Current-induced spin orientation of electrons in semiconductors. *Physics Letters A*, 35(6):459–460, 1971.
- [28] M. I. Dyakonov and V. I. Perel. Possibility of orienting electron spins with current. *Soviet Journal of Experimental and Theoretical Physics Letters*, 13:467, June 1971.
- [29] G. R. Fowles. *Introduction to Modern Optics*. Courier Corporation, 1989.
- [30] V. P. Frolov and A. A. Shoom. Spinoptics in a stationary spacetime. *Physical Review D*, 84:044026, Aug 2011.
- [31] V. P. Frolov and A. A. Shoom. Scattering of circularly polarized light by a rotating black hole. *Physical Review D*, 86:024010, Jul 2012.
- [32] S. A. Fulling. Pseudodifferential operators, covariant quantization, the inescapable van Vleck–Morette determinant, and the  $\frac{R}{6}$  controversy. *International Journal of Modern Physics D*, 05(06):597–608, 1996.
- [33] H. Goldstein, C. P. Poole, and J. L. Safko. *Classical Mechanics*. Addison Wesley, 2002.
- [34] P. Gosselin, A. Bérard, and H. Mohrbach. Semiclassical dynamics of Dirac particles interacting with a static gravitational field. *Physics Letters A*, 368(5):356–361, 2007.
- [35] P. Gosselin, A. Bérard, and H. Mohrbach. Spin Hall effect of photons in a static gravitational field. *Physical Review D*, 75:084035, Apr 2007.
- [36] A. I. Harte. Gravitational lensing beyond geometric optics: I. Formalism and observables. *General Relativity and Gravitation*, 51(1):14, Jan 2019.
- [37] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, 1973.
- [38] O. Hosten and P. Kwiat. Observation of the spin Hall effect of light via weak measurements. *Science*, 319(5864):787–790, 2008.
- [39] W. R. Inc. Mathematica, Version 12.0. Champaign, IL, 2019.
- [40] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom. Observation of the spin Hall effect in semiconductors. *Science*, 306(5703):1910–1913, 2004.
- [41] M. Krenn and A. Zeilinger. On small beams with large topological charge: II. Photons, electrons and gravitational waves. *New Journal of Physics*, 20(6):063006, 2018.
- [42] V. S. Liberman and B. Y. Zel’dovich. Spin-orbit interaction of a photon in an inhomogeneous medium. *Physical Review A*, 46:5199–5207, Oct 1992.
- [43] X. Ling, X. Zhou, K. Huang, Y. Liu, C.-W. Qiu, H. Luo, and S. Wen. Recent advances in the spin Hall effect of light. *Reports on Progress in Physics*, 80(6):066401, 2017.
- [44] R. G. Littlejohn and W. G. Flynn. Geometric phases in the asymptotic theory of coupled wave equations. *Physical Review A*, 44:5239–5256, Oct 1991.
- [45] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry: A basic exposition of classical mechanical systems*, volume 17. Springer Science & Business Media, 2013.
- [46] L. Marsot. How does the photon’s spin affect gravitational wave measurements? *Physical Review D*, 100:064050, Sep 2019.
- [47] M. Mathisson. Republication of: New mechanics of material systems. *General Relativity and Gravitation*, 42(4):1011–1048, 2010.
- [48] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman San Francisco, 1973.
- [49] B. B. Moussa and G. T. Kossioris. On the System of Hamilton-Jacobi and Transport Equations Arising in Geometrical Optics. *Communications in Partial Differential Equations*, 28(5-6):1085–1111, 2003.
- [50] M. A. Oancea, C. F. Paganini, J. Joudioux, and L. Andersson. An overview of the gravitational spin Hall effect. *arXiv preprint arXiv:1904.09963*, 2019.
- [51] Y. N. Obukhov, A. J. Silenko, and O. V. Teryaev. General treatment of quantum and classical spinning particles in external fields. *Physical Review D*, 96(10):105005, 2017.
- [52] M. Onoda, S. Murakami, and N. Nagaosa. Hall effect of light. *Physical Review Letters*, 93:083901, Aug 2004.
- [53] A. Papapetrou. Spinning test-particles in general relativity. I. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 209(1097):248–258, 1951.
- [54] R. Rüdiger. The Dirac equation and spinning particles in general relativity. *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences*, 377:417–424, 1981.
- [55] D. E. Ruiz and I. Y. Dodin. First-principles variational formulation of polarization effects in geometrical optics. *Physical Review A*, 92:043805, Oct 2015.

- [56] D. E. Ruiz and I. Y. Dodin. Extending geometrical optics: A Lagrangian theory for vector waves. *Physics of Plasmas*, 24(5):055704, 2017.
- [57] P. Saturnini. *Un modle de particule à spin de masse nulle dans le champ de gravitation*. Thesis, Université de Provence, 1976.
- [58] P. Schneider, J. Ehlers, and E. E. Falco. *Gravitational Lenses*. 1992.
- [59] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.
- [60] A. J. Silenko. Foldy-Wouthyusen transformation and semiclassical limit for relativistic particles in strong external fields. *Physical Review A*, 77(1):012116, 2008.
- [61] A. J. Silenko and O. V. Teryaev. Semiclassical limit for Dirac particles interacting with a gravitational field. *Physical Review D*, 71:064016, Mar 2005.
- [62] J. Sinova, S. O. Valenzuela, J. Wunderlich, C. H. Back, and T. Jungwirth. Spin Hall effects. *Reviews of Modern Physics*, 87:1213–1260, Oct 2015.
- [63] J.-M. Souriau. Modele de particulea spin dans le champ électromagnétique et gravitationnel. *Annales de l'Institut Henri Poincaré A*, 20:315–364, 1974.
- [64] J.-M. Souriau. *Structure of dynamical systems*, volume 149 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1997. A symplectic view of physics, Translated from the French by C. H. Cushman-de Vries, Translation edited and with a preface by R. H. Cushman and G. M. Tuynman.
- [65] M. Stone. Berry phase and anomalous velocity of Weyl fermions and Maxwell photons. *International Journal of Modern Physics B*, 30(2):1550249, Dec. 2016.
- [66] G. Sundaram and Q. Niu. Wave-packet dynamics in slowly perturbed crystals: Gradient corrections and Berry-phase effects. *Physical Review B*, 59:14915–14925, Jun 1999.
- [67] E. R. Tracy, A. J. Brizard, A. S. Richardson, and A. N. Kaufman. *Ray Tracing and Beyond: Phase Space Methods in Plasma Wave Theory*. Cambridge University Press, 2014.
- [68] W. Tulczyjew. Motion of multipole particles in general relativity theory. *Acta Physica Polonica*, 18:393, 1959.
- [69] R. M. Wald. *General relativity*. University of Chicago Press, Chicago, IL, 1984.
- [70] N. Yamamoto. Spin Hall effect of gravitational waves. *Physical Review D*, 98:061701, Sep 2018.
- [71] C.-M. Yoo. Notes on spinoptics in a stationary spacetime. *Physical Review D*, 86:084005, Oct 2012.

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