Critique of the use of geodesics in astrophysics and cosmology

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Since particles obey wave equations, in general one is not free to postulate that particles move on the geodesics associated with test particles. Rather, for this to be the case one has to be able to derive such behavior starting from the equations of motion that the particles obey, and to do this for either massless or massive particles one can employ the eikonal approximation. While for massive particles one does obtain standard geodesic behavior this way, for a conformally coupled massless scalar field the eikonal approximation only leads to geodesic behavior if the Ricci scalar is zero. Similarly, for the propagation of the light waves associated with the conformal invariant Maxwell equations geodesic behavior only holds if the Ricci tensor is zero. While for practical purposes such terms might only be of relevance in regions of high curvature, the point of this paper is only to establish their presence in principle. Thus in principle the standard null-geodesic-based gravitational bending formula and the behavior of light rays in cosmology are in need of modification in regions with high enough curvature. We show how to appropriately modify the geodesic equations in such situations. We show that relativistic eikonalization has an intrinsic light-front structure, and show that eikonalization in a theory with local conformal symmetry leads to trajectories that are only globally conformally symmetric. The modifications to geodesics that we find lead to the propagation of massless particles off the light cone. This is a curved space reflection of the fact that when light travels through a refractive medium in flat spacetime its velocity is modified from its free flat spacetime value. In the presence of gravity spacetime itself acts as a medium, and this medium can then take light waves off the light cone.

I. INTRODUCTION

A. Background

As introduced in Riemannian geometry a geodesic is the shortest distance between two points in a general curved space. In physics one takes advantage of this fact by introducing the test particle action

$$I_T = -m\int ds, \tag{1.1}$$

where T denotes test particle, m is the particle mass, and $ds = (-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$ is the proper time (here and throughout we use the notation and conventions in [1] in which g_{00} is negative). Stationary variation of I_T with respect to the particle coordinate x^λ leads to the geodesic equation

$$m\left(\frac{D^2x^\lambda}{Ds^2}\right) = m\left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) = 0, \tag{1.2}$$

where, as introduced here, D^2x^λ/Ds^2 is the total acceleration and $\Gamma^\lambda_{\mu\nu}$ is the connection. As such, only the total D^2x^λ/Ds^2 transforms as a general coordinate vector, with neither of its d^2x^λ/ds^2 or $\Gamma^\lambda_{\mu\nu}(dx^\mu/ds)(dx^\nu/ds)$ components separately doing so, and with it being only their sum with the specifically indicated relative weights that is a general coordinate vector. Moreover, precisely because $\Gamma^\lambda_{\mu\nu}$ is not a general coordinate tensor we are actually able to remove it from (1.2) at any given point via a general coordinate transformation [2], and thus establish the equivalence principle between gravity and acceleration. Since having this particular set of relative weights leads to the equality of inertial and gravitational masses it is generally thought that particles must move on geodesics. However the two terms only appear with this particular set of relative weights because any other combination would not lead to the total D^2x^λ/Ds^2 being a general coordinate vector in the first place. To emphasize the point we consider an illustrative, more general, covariant test particle action

$$I_T = -m\int ds - \kappa\int ds R^\alpha{}_\alpha, \tag{1.3}$$

where $R^{\alpha}{}_{\alpha}$ is the Ricci scalar and κ is a constant. The stationary variation of this I_T with respect to x^{λ} leads to [3]

$$(m+\kappa R^{\alpha}{}_{\alpha})\left(\frac{d^2x^{\lambda}}{ds^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}\right)=-\kappa\left(g^{\lambda\beta}+\frac{dx^{\lambda}}{ds}\frac{dx^{\beta}}{ds}\right)\nabla_{\beta}R^{\alpha}{}_{\alpha}. \tag{1.4}$$

As we see, again it is the total D^2x^{λ}/Ds^2 combination that appears, and its ubiquity is due to the fact that covariance would allow no other possibility. Nonetheless, even though we can still remove the dependence on $\Gamma^{\lambda}_{\mu\nu}$ at any given point, under no coordinate transformation could we also remove the dependence on the Ricci scalar at the same given point precisely because we would simultaneously need both $\Gamma^{\lambda}_{\mu\nu}$ and its derivatives to vanish. Now (1.4) is just as covariant as (1.2), and they both reduce to the flat Cartesian space $d^2x^{\lambda}/ds^2=0$ in the absence of curvature, in exactly the same manner as (1.3) reduces to (1.1) in the same limit. However, only for (1.2) could we simulate the entire effect of a gravitational field at any given point by an acceleration. Since the equivalence principle is commonly understood as the requirement that we can remove all gravitational effects at a given point by a general coordinate transformation, it is commonly assumed that particles move on the purely geodesic (1.2) and not on any other trajectory. However, (1.4) is just as covariant as (1.2), and is allowed by the principle of general coordinate invariance, and in general we should contemplate replacing (1.2) by a more general trajectory such as that given in (1.4). Moreover, in [1] Weinberg also notes that (1.2) could be generalized, and suggested a possible generalization of the form

$$\frac{d^2x^{\lambda}}{ds^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}=-fR^{\lambda}{}_{\mu\nu\kappa}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}S^{k} \tag{1.5}$$

for a particle with spin vector S^k in a geometry with Riemann tensor $R^{\lambda}{}_{\mu\nu\kappa}$, with f being a constant. It is the purpose of this paper to show that rather than considering generalizations of (1.2) to be just a logical possibility, such generalizations, are actually the general rule, with it being (1.2) itself that is the exception. However, since the modifications that we will present from a study of wave equations rather than test particle actions will involve the Ricci tensor, such terms are actually absent in the familiar tests of the equivalence principle that are made in the Ricci flat environment of the solar system. In fact it is the very success of these solar system tests that has led to the use (1.2) in other situations, situations that are not Ricci flat.

Now procedurally there is nothing wrong in varying (1.1) to obtain (1.2). However, our concern here is whether real particles as opposed to test particles can actually be described by the test particle action I_T in the first place. That there might actually be a concern is due to the fact that one cannot use (1.1) for massless particles since even if we drop the factor m and only consider $I_T=\int ds$, for particles that propagate on the light cone $ds=(-g_{\mu\nu}dx^{\mu}dx^{\nu})^{1/2}$ is zero. Nonetheless, regardless of whether or not the general coordinate scalar I_T action without any m factor is to be relevant for massive particles, the variation of I_T does show that the left-hand side of (1.2) is a general coordinate vector. Thus for massless particles one ordinarily replaces (1.2) by the null geodesic equation

$$\frac{d^2x^{\lambda}}{dq^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dq}\frac{dx^{\nu}}{dq}=0, \tag{1.6}$$

where q is a general coordinate scalar affine parameter that measures distance along the trajectory associated with (1.6). As such, with its left-hand side being a true general coordinate vector (dq like ds being a general coordinate scalar), (1.6) does covariantly describe the propagation of massless particles. Now in field theory particles are associated with wave equations rather than with test particle equations, and nowhere in any fundamental Lagrangian description of the fundamental forces does any test particle action that might lead to the massive particle (1.2) or the massless particle (1.6) actually appear. Thus one has to ask whether one could obtain geodesic behavior starting from field equations instead, and whether if in doing so one could instead obtain some modified form of trajectory. This then is the objective of this paper.

In regard to covariantization, we should note that in and of itself the procedure of replacing flat Minkowski space metric and derivatives ($\eta_{\mu\nu}$, ∂_{μ}) by covariant metric and derivatives ($g_{\mu\nu}$ and ∇_{μ}) is actually independent of the presence or absence of curvature. While the condition $d^2x^{\lambda}/ds^2=0$ is the special relativistic flat spacetime ($R_{\lambda\mu\nu\tau}=0$) version of Newton's Law of Motion, as written it is written in Minkowski coordinates, and as such this equation is invariant under transformations with uniform velocity. However, it is not left invariant under transformations to accelerating coordinate systems, transformations that will still keep $R_{\lambda\mu\nu\tau}$ zero. Rather, it is the geodesic equation $d^2x^{\lambda}/ds^2+\Gamma^{\lambda}_{\mu\nu}(dx^{\mu}/ds)(dx^{\nu}/ds)=0$ that is left invariant, and as such it writes Newton's Law of Motion in a form that all flat space observers, accelerating or not accelerating could agree on. However, since all we have done is made a coordinate transformation, if we started in flat spacetime we remain in flat space time. Now, as such, covariantization is generally understood as replacing $d^2x^{\lambda}/ds^2=0$ in a space with $R_{\lambda\mu\nu\tau}=0$ by $d^2x^{\lambda}/ds^2+\Gamma^{\lambda}_{\mu\nu}(dx^{\mu}/ds)(dx^{\nu}/ds)=0$ in a space with $R_{\lambda\mu\nu\tau}\neq 0$. As we see, in general this is an incomplete description for curved space trajectories since we also need to consider the inclusion of curvature-dependent terms, terms that vanish in the flat space limit.

Moreover, as we will see, not only will we get some modification of geodesic motion per se, for massless particles such as photons we will get trajectories that are not even constrained to lie on the curved space light cone (viz. the covariantized flat spacetime light cone in which $\eta_{\mu\nu}$ is replaced by $g_{\mu\nu}$). Such behavior is actually not without precedent as it already occurs in flat spacetime, where the velocity of light in a refractive medium is given not by c itself but by c/n where n is the refractive index. The curving of spacetime by gravity causes spacetime to become a medium, and this not only can but actually is able to take light rays off the covariantized version of the light cone that they would otherwise have travelled on in the absence of any curvature. To see what can happen in the curved space case we begin by identifying some generic features of geometrical optics in flat spacetime.

B. Flat Spacetime Geometrical Optics

Since a study of the eikonal approximation in the relativistic curved spacetime case is the covariant generalization of the standard treatment of flat spacetime geometrical optics, we quickly review how things work in geometrical optics itself. We follow the treatment given in [4] and [5]. We start with the Maxwell equations for electromagnetic fields that oscillate as $e^{-i\omega t}$ in a medium with a spatially-dependent refractive index $n(\mathbf{x}) = (\epsilon(\mathbf{x})/\epsilon_0)^{1/2}$, viz.

$$\boldsymbol{\nabla} \cdot (\epsilon(\mathbf{x})\boldsymbol{E}) = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = i\omega \boldsymbol{B}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{B} = -i\mu_0\omega\epsilon(\mathbf{x})\boldsymbol{E}. \tag{1.7}$$

Manipulation of these equations yields

$$\begin{aligned} &\boldsymbol{\nabla}^2\boldsymbol{E} + \mu_0\omega^2\epsilon(\mathbf{x})\boldsymbol{E} + \boldsymbol{\nabla}\left(\epsilon^{-1}\boldsymbol{E}\cdot\boldsymbol{\nabla}\epsilon\right) = 0, \\ &\boldsymbol{\nabla}^2\boldsymbol{B} + \mu_0\omega^2\epsilon(\mathbf{x})\boldsymbol{B} - i\mu_0\omega\boldsymbol{\nabla}\epsilon \times \boldsymbol{E} = 0. \end{aligned} \tag{1.8}$$

If we assume that the $\boldsymbol{\nabla}\epsilon$ terms are negligible, we can drop the third term in each of the two equations in (1.8) and obtain

$$\left[\boldsymbol{\nabla}^2 + \frac{\omega^2}{c^2}n(\mathbf{x})\right]\alpha = 0, \tag{1.9}$$

where α denotes $\boldsymbol{E}$ or $\boldsymbol{B}$. Setting $\alpha = \exp[i\omega\psi(\mathbf{x})/c)]$ we obtain

$$\frac{\omega^2}{c^2}\left[n^2(\mathbf{x}) - \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi\right] + i\frac{\omega}{c}\boldsymbol{\nabla}^2\psi = 0. \tag{1.10}$$

Then on assuming that $\boldsymbol{\nabla}^2\psi$ is much smaller than $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi$, we obtain

$$\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi = n^2(\mathbf{x}). \tag{1.11}$$

Eq. (1.11) is known as the eikonal equation of geometrical optics, with ψ being known as the eikonal function or eikonal phase.

If we set $\boldsymbol{E} = \boldsymbol{E}_0\exp[i\omega\psi(\mathbf{x})/c]$, $\boldsymbol{B} = \boldsymbol{B}_0\exp[i\omega\psi(\mathbf{x})/c]$, where $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ are constants, we obtain

$$\boldsymbol{\nabla}\psi \times \boldsymbol{E}_0 = c\boldsymbol{B}_0, \quad \boldsymbol{\nabla}\psi\times\boldsymbol{B}_0 = -\frac{n^2(\mathbf{x})}{c}\boldsymbol{E}_0. \tag{1.12}$$

From (1.12) it then follows that $\boldsymbol{\nabla}\psi$ is orthogonal to both $\boldsymbol{E}$ and $\boldsymbol{B}$. It thus points in the same direction as the Poynting vector $\boldsymbol{E}\times\boldsymbol{B}$. With $\boldsymbol{E}$ and $\boldsymbol{B}$ both lying in the wavefront, $\boldsymbol{\nabla}\psi$ is thus in the direction normal to the wavefront, i.e., in the same direction as the light ray. Thus if we introduce a unit vector $\hat{\boldsymbol{k}}$ in the direction parallel to the normal to the wave front we can set

$$\boldsymbol{\nabla}\psi = n(\mathbf{x})\hat{\boldsymbol{k}}. \tag{1.13}$$

Now if the wave travels a distance $d\boldsymbol{r}$ in a distance dq as measured along the ray we can set $d\boldsymbol{r}/dq = \hat{\boldsymbol{k}}$. The trajectory of the ray is thus given by

$$\frac{d\boldsymbol{r}}{dq} = \frac{\boldsymbol{\nabla}\psi}{n(\mathbf{x})}. \tag{1.14}$$

Noting that $d/dq = \hat{\boldsymbol{k}}\cdot\boldsymbol{\nabla}$, differentiation of (1.14) yields

$$n(\mathbf{x})\frac{d^2\boldsymbol{r}}{dq^2} + \frac{d\boldsymbol{r}}{dq}\left(\boldsymbol{\nabla}n(\mathbf{x})\cdot\frac{d\boldsymbol{r}}{dq}\right) = \boldsymbol{\nabla}n(\mathbf{x}). \tag{1.15}$$

For constant $n(\mathbf{x}) = n$ (1.15) reduces to $d^2\boldsymbol{r}/dq^2 = 0$, i.e., light travels on a straight line (with speed c/n), while if $n(\mathbf{x})$ varies in space the trajectory departs from a straight line according to (1.15).

C. Implications for Curved Space Eikonalization

From this analysis we identify the main ingredients of the eikonal approximation, viz. excluding terms that are negligible (usually achievable at short wavelengths), and identifying $\boldsymbol{\nabla}\psi$ with the velocity along the ray as in (1.14). Given (1.11), it follows that $d\boldsymbol{r}/dq$ is normalized according to

$$\frac{d\boldsymbol{r}}{dq}=\frac{\boldsymbol{\nabla}\psi}{(\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi)^{1/2}},\quad \frac{d\boldsymbol{r}}{dq}\cdot\frac{d\boldsymbol{r}}{dq}=1. \tag{1.16}$$

Normalizing $\boldsymbol{\nabla}\psi$, which is standard in optics [4], will prove central in the following. In fact not only will it prove central, in Sec. III we will see that it is even required for consistency in the covariant case. In generalizing to the covariant Maxwell case we look for covariant generalizations of (1.16) and (1.15). For (1.16) there are two natural generalizations, either $(dx_\mu/dq)(dx^\mu/dq)=0$ (lightlike) or $(dx_\mu/dq)(dx^\mu/dq)=-1$ (timelike in the $ds^2=-g_{\mu\nu}dx^\mu dx^\nu$ convention with $g_{00}<0$ used in [1]). Similarly, for (1.15) there are also two natural generalizations, either the null geodesics given in (1.6), viz.

$$\frac{d^2x^\lambda}{dq^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}=0, \tag{1.17}$$

or Ricci scalar (or Ricci or Riemann tensor) dependent ones such as the one given in (1.4) with $m=0$, viz.

$$R^\alpha{}_\alpha\left(\frac{d^2x^\lambda}{dq^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}\right)=-\left(g^{\lambda\beta}+\frac{dx^\lambda}{dq}\frac{dx^\beta}{dq}\right)\nabla_\beta R^\alpha{}_\alpha. \tag{1.18}$$

Now initially we would anticipate that the both of these two trajectories would follow from the light cone condition $(dx_\mu/dq)(dx^\mu/dq)=0$ since neither involves any mass term. And by the same token we would anticipate that the $(dx_\mu/dq)(dx^\mu/dq)=-1$ option would only be associated with massive particles. What we will actually find in the curved space Maxwell case that we study in Sec. IV is that there is an explicit dependence on the Ricci tensor in the Maxwell equations themselves, with the curved space vector potential Maxwell equation $\nabla_\mu(\nabla^\mu A^\nu-\nabla^\nu A^\mu)=0$ taking the form

$$\nabla_\mu\nabla^\mu A^\nu-\nabla^\nu\nabla_\mu A^\mu+R^{\nu\alpha}A_\alpha=0. \tag{1.19}$$

As we show below, eikonalization of (1.19) in the $\nabla_\mu A^\mu=0$ gauge leads to a form analogous to (1.18), viz.

$$\frac{d^2x^\lambda}{dq^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}=-\frac{1}{2(-R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta)}\left[g^{\lambda\mu}+\frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right]\frac{\partial(-R^{\gamma\delta}\epsilon_\gamma\epsilon_\delta)}{\partial x^\mu}, \tag{1.20}$$

where ϵ_α is polarization vector. And not only that, (1.20) is also associated with the non light cone $(dx_\mu/dq)(dx^\mu/dq)=-1$ condition, with curvature taking photons off the light cone. There are some situations in which spacetime curvature is too big to be neglected, particularly in the early universe. In the standard Robertson-Walker cosmology with zero spatial three-curvature the expansion radius is given by $a(t)=t^{1/2}$, so that $R_{00}=3\ddot{a}/a=-3/4t^2$. Consequently, $(dR_{00}/dt)/R_{00}=-2/t$ is very big in the early universe. In such a situation photons would not follow null geodesics such as (1.17) but would instead follow (1.20). It could be of interest to ascertain whether this might affect early universe studies.

D. Comparing Eikonalization with a Harmonic Function Analysis

In flat spacetime the wave equation $(\partial_t^2-\boldsymbol{\nabla}^2)S=0$ with scalar field S has general harmonic solution $S=h(t-x)$ for arbitrary function h. Thus if we repeat this analysis in the curved space situation (the procedure discussed in [6]) we would introduce a phase χ, and on setting $S=h(\chi)$ in $\nabla_\mu\nabla^\mu S=0$ would obtain

$$h'\nabla_\mu\nabla^\mu\chi+h''\nabla_\mu\chi\nabla^\mu\chi=0, \tag{1.21}$$

where the prime denotes the derivative with respect to χ. With $h(\chi)$ being arbitrary, the coefficients of both h' and h'' have to vanish separately. This leads to

$$\nabla_\mu\nabla^\mu\chi=0,\quad \nabla_\mu\chi\nabla^\mu\chi=0. \tag{1.22}$$

Then on setting $dx^\mu/dq = \nabla^\mu \chi$ we obtain

$$\nabla_\mu \frac{dx^\mu}{dq} = 0, \quad \frac{dx_\mu}{dq}\frac{dx^\mu}{dq} = 0, \tag{1.23}$$

from which, and as we explicitly show in Sec. II, the null geodesic equation (1.17) follows. As such, this analysis was applied to the Maxwell case in [6], with the presence of the Ricci tensor in (1.19) not affecting the $\nabla_\mu\chi\nabla^\mu\chi = 0$ condition at all, so that the null geodesic (1.17) was recovered.

However, there is another class of solutions to (1.21). Specifically, if $h = e^\chi$ then h' and h'' are equal, and from (1.21) we then only obtain

$$\nabla_\mu\nabla^\mu\chi + \nabla_\mu\chi\nabla^\mu\chi = 0. \tag{1.24}$$

As we show in Sec. II, it is this solution that coincides with eikonalization of $\nabla_\mu\nabla^\mu S = 0$. And as we show in Sec. IV, the Maxwell analog of (1.24) does involve the Ricci tensor directly, leading us not to (1.17) but to (1.20) instead.

II. THE CURVED SPACETIME EIKONAL APPROXIMATION

A. Massless Scalar Fields

If one starts with wave equations one then has both geometrical optics and physical optics, i.e., both ray propagation and diffraction. The ray limit is a short wavelength limit (short on the scale of any system being considered), while the diffraction limit is associated with larger wavelengths that are of the same size as the system under consideration. Now long before the development of general relativity it was known that short wavelength rays travel in straight lines, and as such the use of null geodesics of the form given in (1.6) is just the natural generalization to curved space. However, in curved space one meets a new scale, namely the scale of the curvature rather than the scale of a material system that is being explored. In [7] we explored how such curvature scales can if strong enough cause fluids that are perfect in the absence of gravity to become imperfect in its presence. In this paper we make an analogous analysis for individual light rays. In regard to the covariantization of flat space expressions such as the transition from the flat space $d^2x^\lambda/dq^2 = 0$ to (1.18), we note that the replacing of ordinary derivatives by covariant derivatives is not a sufficient prescription since it ignores the possible presence of intrinsically geometric (i.e., Riemann tensor dependent) terms that have no flat space counterpart. Thus one can go from the curved space (1.18) to the flat space $d^2x^\lambda/dq^2 = 0$, but not vice versa. Thus in studying massless rays in curved space we can anticipate the presence of explicit curvature dependent terms. While for practical purposes such terms might only be of relevance in regions of high curvature, the point of this paper is only to establish their presence in principle. In regions of low curvature our work in this paper shows how to establish geodesic behavior for both massless and massive particles starting from wave equations without ever encountering any test particle action at all. Some of our results have already been prefigured in [3] and they lead to the more general analysis presented here. The key new ingredient presented here is in the normalization of the eikonal function.

In developing a relativistic eikonal approximation one starts with a field such as a scalar field $S(x)$ that obeys a covariant wave equation

$$\nabla_\mu\nabla^\mu S = 0. \tag{2.1}$$

Following the discussion in [3], we introduce an eikonal function $T(x)$ via $S(x) = \exp(iT(x))$, with the scalar phase $T(x)$ then being found to obey the equation

$$\nabla_\mu T\nabla^\mu T - i\nabla_\mu\nabla^\mu T = 0. \tag{2.2}$$

In the short wavelength limit (2.2) reduces to [8]

$$\nabla_\mu T\nabla^\mu T = 0. \tag{2.3}$$

Applying ∇_ν to (2.3) and recalling that since T is a scalar we can set $\nabla_\mu\nabla_\nu T = \nabla_\nu\nabla_\mu T$, we obtain

$$\nabla^\mu T\nabla_\mu\nabla_\nu T = 0. \tag{2.4}$$

Now in the geometrical optics regime rays travel in the direction normal to the wavefront and obey the eikonal relation

$$\nabla^\mu T = \frac{dx^\mu}{dq} = k^\mu, \tag{2.5}$$

where q measures distance along the normal. The vector k^μ is the wave vector of the ray, and according to (2.3) it has to obey $k_\mu k^\mu = 0$. Technically, we should note that given the conventional three-dimensional normalization $d\bar{x}/dq = \bar{\nabla}T/(\bar{\nabla}T\cdot\bar{\nabla}T)^{1/2}$ of the eikonal phase that is used in optics [4], in the covariant case we should set

$$\frac{dx^\mu}{dq} = \frac{\nabla^\mu T}{(-\nabla_\nu T \nabla^\nu T)^{1/2}}, \tag{2.6}$$

a choice that would then allow q to have the dimension of length. However, we cannot use this normalization when $\nabla_\nu T\nabla^\nu T$ is zero. Fortunately, when $\nabla_\nu T\nabla^\nu T$ is zero we do not actually need to introduce any normalization since the desired (2.7) that we seek will follow from (2.3) and (2.5) as is. Below we will meet cases where $\nabla_\nu T\nabla^\nu T$ is not zero, and in those cases we will use the normalization given in (2.6). As we will see, this has substantive consequences, and as we show in Sec. III, would even lead to inconsistencies if not taken into account.

Now on its own an arbitrary general coordinate vector dx^μ/dq would have both longitudinal and transverse components. However, $\nabla^\mu T$ is longitudinal, and thus dx^μ/dq has to be longitudinal too. With dx^μ/dq being directed along the normal to the wavefront, we see that the longitudinal component of the eikonal wave is along the normal (i.e., in the direction of the ray), with the transverse components then being in the wavefront. On noting that $(dx^\mu/dq)(\partial/\partial x^\mu) = d/dq$, from (2.4) we obtain

$$\frac{dx^\mu}{dq}\nabla_\mu \frac{dx^\lambda}{dq} = \frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = 0. \tag{2.7}$$

We recognize (2.7) as being the massless particle geodesic equation, with rays then precisely being found to be geodesic in the eikonal limit.

As such, we have not only derived the massless particle geodesic equation, in deriving it we also derive an expression that obeys the equivalence principle. Thus the equivalence principle is seen to follow from the general coordinate invariance of the field wave equation and does not need to be independently postulated. Now historically the equivalence principle was an important guiding principle in Einstein's original development of general relativity. However, once one has general relativity and general coordinate invariance, then on starting with covariant wave equations the equivalence principle is output rather than input to general relativity.

B. Relativistic Eikonalization and the Light-Front Approach

We should note that with $\nabla^\mu T = dx^\mu/dq = k^\mu$, in general the curved space k^μ will be spacetime dependent. Thus it is immediately suggested to set $T = \int^x k_\mu dx^\mu$ (we could only set $T = k_\mu x^\mu$ if k_μ is constant, which it could only be in flat space). However, if we do set $T = \int^x k_\mu dx^\mu$ we would obtain $T = \int^x (dx_\mu/dq)dx^\mu = \int^x (dx_\mu/dq)(dx^\mu/dq)dq = \int^x k_\mu k^\mu dq$, and with $k_\mu k^\mu = 0$, $(dx_\mu/dq)(dx^\mu/dq) = 0$, such a T would vanish identically. To avoid this we note that if we consider a ray propagating in the z direction so that $k^\mu = (k,0,0,k)$, and then momentarily do set $T = \int^x k_\mu dx^\mu$, we would then obtain $\partial_0 T = k$, $\partial_3 T = -k$, and thus obtain $(\partial_0 + \partial_3)T = 0$. Now this derivative is in a light-front direction. In light-front quantization (see e.g. [9] and references therein) one introduces metric components, coordinates and derivatives of the form [10]

$$g_{\mu\nu}(\text{front}) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.8}$$

$$x^+ = x^0 + x^3, \qquad x^- = x^0 - x^3, \qquad \partial_+ = \frac{1}{2}(\partial_0 + \partial_3), \qquad \partial_- = \frac{1}{2}(\partial_0 - \partial_3), \tag{2.9}$$

so that

$$k_\mu k^\mu = 4k_+k_- - k_1^2 - k_2^2, \qquad x_\mu x^\mu = g_{\mu\nu}x^\mu x^\nu = x^+x^- - (x^1)^2 - (x^2)^2. \tag{2.10}$$

The constraint on T is thus of the form $\partial_+ T = 0$. This constraint can be satisfied by not actually setting $T = \int^x k_\mu dx^\mu$ after all, but by instead setting T equal to the non-vanishing $T = \int^x k_- dx^-$. Then with $k_+ = 0$, $k_1 = 0$, $k_2 = 0$ one still has $k_\mu k^\mu = 0$ even as T is then nonzero (the vanishing of k_+, k_1 and k_2 does not restrict k_- while still keeping $k_\mu k^\mu = 4k_+k_- - k_1^2 - k_2^2$ zero). Thus while non-relativistic eikonalization occurs with the normal to the wavefront being in the x^3 direction so that a non-vanishing eikonal phase T is given by $T = \int^x k_3 dx^3$, $\partial_3 T = k_3 = dx_3/dq$, in relativistic eikonalization the normal is in the longitudinal x^- direction, with a non-vanishing eikonal phase T being given by $T = \int^x k_- dx^-$, $\partial_+ T = 0$, $\partial_- T = k_- = dx_-/dq$.

In the light-front approach, and thus in relativistic eikonalization, one considers propagation in x^+, while in the instant-time approach one considers propagation in x^0. The two approaches thus appear to be different. However, with $g_{\mu\nu}(\text{front})x^\mu x^\nu = x^+ x^- - (x^1)^2 - (x^2)^2$, we see that for either timelike or lightlike events x^+x^- is positive, so x^+ and x^- have the same sign. Now $x^0 = (x^+ + x^-)/2$. Thus if x^0 is positive then for timelike or lightlike events x^+ is positive too. Hence for timelike or lightlike events forward in x^+ is the same as forward in x^0.

C. Massive Scalar Fields

If we now give the scalar field a mass the analysis is analogous. For a massive scalar field that obeys

$$\nabla_\mu \nabla^\mu S - m^2 S = 0, \tag{2.11}$$

we again set $S = \exp(iT)$ and obtain

$$\nabla_\mu T \nabla^\mu T - i\nabla_\mu \nabla^\mu T + m^2 = 0. \tag{2.12}$$

We again drop the $i\nabla_\mu \nabla^\mu T$ term, though in the presence of the mass term we could achieve this by having $\nabla_\mu \nabla^\mu T$ be much smaller in magnitude than m^2 rather than much smaller than $\nabla_\mu T \nabla^\mu T$. On dropping the $i\nabla_\mu \nabla^\mu T$ term for whichever one of these reasons might be relevant we obtain

$$\nabla_\mu T \nabla^\mu T + m^2 = 0. \tag{2.13}$$

Then, since ds is nonzero for massive particles this time we can identify

$$\frac{\nabla^\mu T}{m} = \frac{dx^\mu}{ds} = \frac{P^\mu}{m}, \tag{2.14}$$

where P^μ is the momentum. Since $(-\nabla_\mu T \nabla^\mu T)^{1/2}$ is equal to m, (2.14) satisfies the normalization condition given in (2.6). As introduced, P^μ and dx^μ/ds obey

$$P_\mu P^\mu + m^2 = 0, \quad \frac{dx_\mu}{ds}\frac{dx^\mu}{ds} = -1. \tag{2.15}$$

With $P_\mu = \nabla_\mu T$ we recognize (2.15) as the Hamilton-Jacobi equation, an equation whose solution is precisely $S = \exp(i\int^x P_\mu dx^\mu)$, $T = \int^x P_\mu dx^\mu$. This is just as we would want, with the Hamilton-Jacobi equation describing eikonalization in the massive particle case. Unlike in the massless case, this time there is no difficulty in setting $T = \int^x P_\mu dx^\mu$ since the modes are not propagating on the light cone. Then, with m^2 being a constant we find that on taking the ∇^ν derivative of (2.13) we again obtain (2.4), viz. $\nabla^\mu T \nabla_\mu \nabla^\nu T = 0$. Finally, with $(dx^\mu/ds)(\partial/\partial x^\mu) = d/ds$ (1.2) then follows, viz.

$$m^2\left(\frac{D^2 x^\lambda}{Ds^2}\right) = m^2\left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right) = 0. \tag{2.16}$$

We thus derive the massive particle geodesic equation starting from a massive field wave equation. (As well as satisfy (2.16), on integrating the solution has to obey the timelike constraint $(dx_\mu/ds)(dx^\mu/ds) = -1$ appropriate to a massive particle.) Moreover, as we see, at no point in the calculation did we utilize the test particle action I_T given in (1.1). Now that we have divorced the geodesic equation from the test particle action, we can consider what happens when we use the more complicated, conformal invariant wave equations of relevance to the case of particular concern to us in this paper, namely the propagation of massless particles such as photons.

III. CONFORMAL INVARIANT SCALAR FIELD WAVE EQUATION

A. Conformal Scalar Field with no Mass

While we had initially taken the scalar field to be massless, we were subsequently able to add on a mass term since no principle had been invoked that might have prevented us from doing so. However, there is such a principle, namely local conformal invariance, an invariance possessed by massless fermion and massless gauge boson wave equations, an invariance that is thus of relevance to the gravitational bending and lensing of light. We shall explore the gauge boson case below, but first we explore the illustrative conformal scalar field case. Under a local conformal transformation $g_{\mu\nu}(x)$ and $S(x)$ transform as $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$, $S(x) \to \Omega^{-1}(x)S(x)$, where $\Omega(x)$ is a spacetime dependent conformal factor. Since $\nabla_\mu\nabla^\mu S(x)$ would not be left invariant if $\Omega(x)$ is spacetime dependent, one has to introduce a coupling to the geometry, with the conformally coupled wave equation

$$\nabla_\mu\nabla^\mu S + \frac{1}{6}SR^\alpha{}_\alpha = 0 \tag{3.1}$$

then being locally conformal invariant. For this wave equation the substitution $S = \exp(iT)$ leads to

$$\nabla_\mu T\nabla^\mu T - i\nabla_\mu\nabla^\mu T - \frac{1}{6}R^\alpha{}_\alpha = 0, \tag{3.2}$$

so that the curved space eikonal equation then takes the form

$$\nabla_\mu T\nabla^\mu T - \frac{1}{6}R^\alpha{}_\alpha = 0. \tag{3.3}$$

Given (2.6) and (3.3), then as long as $R^\alpha{}_\alpha \neq 0$ we can set

$$\frac{dx^\mu}{dq} = \frac{\nabla^\mu T}{(-R^\alpha{}_\alpha/6)^{1/2}}, \qquad \frac{dx_\mu}{dq}\frac{dx^\mu}{dq} = -1, \tag{3.4}$$

and note that even though we are in a massless theory dx^μ/dq is not a lightlike vector but a unit timelike four-vector instead. (In contrast we should note that for massive particles the constraint given in (2.15) is the timelike $(dx_\mu/ds)(dx^\mu/ds) = -1$, where s is the proper time and not the affine parameter q.)

With (3.3) leading to

$$\nabla^\mu T\nabla_\mu\nabla^\lambda T = \frac{1}{12}\nabla^\lambda R^\alpha{}_\alpha, \tag{3.5}$$

the substitution into (3.5) of the eikonal relation given in (3.4) leads to

$$X^{1/2}\frac{dx^\mu}{dq}\left[X^{1/2}\frac{\partial}{\partial x^\mu}\left(\frac{dx^\lambda}{dq}\right) + \frac{dx^\lambda}{dq}\frac{1}{2X^{1/2}}\frac{\partial X}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu}X^{1/2}\frac{dx^\nu}{dq}\right] = -\frac{1}{2}\nabla^\lambda X, \tag{3.6}$$

where $X = -R^\alpha{}_\alpha/6$. Since $ds^2 = -g_{\mu\nu}dx^\mu dx^\nu$ is nonzero, we could normalize q to s, but will not do so since we would like to keep the normalization of q free, and will actually adjust its normalization in (3.13) below. (I.e., from the start we could have set dx^μ/dq equal to a constant times $\nabla^\mu T/(-R^\alpha{}_\alpha/6)^{1/2}$.) On noting that $(dx^\mu/dq)(\partial/\partial x_\mu) = d/dq$ we obtain

$$\frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = -\frac{1}{2(-R^\alpha{}_\alpha/6)}\left[g^{\lambda\mu} + \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right]\frac{\partial(-R^\alpha{}_\alpha/6)}{\partial x^\mu}. \tag{3.7}$$

Together with its Maxwell analog given in (4.13) below, (3.7) is our main result.

As we see, despite obtaining the geodesic operator on the left-hand side of (3.7), we find that there is a nonzero geometric term on the right-hand side. Moreover, we quite remarkably find that (3.7) is of precisely of the same generic form as (1.18). In fact it would have been identical in form if instead of (1.3) we had varied the test particle action [11]

$$I_T = -\kappa\int ds(-R^\alpha{}_\alpha)^{1/2}. \tag{3.8}$$

Now in [3] we had introduced (1.3) purely for illustrative purposes. Its emergence as being associated with a dynamical wave equation is both surprising and intriguing [12].

To check that there is no obvious error in (3.7), we note that on taking the derivative of $(dx_\mu/dq)(dx^\mu/dq) = -1$ with respect to q we obtain

$$\frac{1}{2}\frac{d}{dq}\left(g_{\lambda\alpha}\frac{dx^\lambda}{dq}\frac{dx^\alpha}{dq}\right) = \frac{dx^\alpha}{dq}\left(g_{\lambda\alpha}\frac{d^2x^\lambda}{dq^2} + \frac{1}{2}\frac{dx^\lambda}{dq}\frac{dx^\nu}{dq}\frac{\partial g_{\lambda\alpha}}{\partial x^\nu}\right) = 0. \tag{3.9}$$

We recognize (3.9) as being of the form

$$g_{\lambda\alpha}\frac{dx^\alpha}{dq}\left(\frac{d^2x^\lambda}{dq^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}\right)=0, \tag{3.10}$$

with the covariant four-velocity being orthogonal to the covariant four-acceleration. Consequently, the right-hand side of (3.7) must be orthogonal to the covariant four-velocity as well, and with it having precisely the form that it does have, one can readily check that this is in fact the case.

It is also of interest to note that had we set $dx^\mu/dq=\nabla^\mu T$ without the normalization factor given in (3.4), we would have obtained [3]

$$\frac{d^2x^\lambda}{dq^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}=\frac{1}{12}g^{\lambda\mu}\frac{\partial R^\alpha{}_\alpha}{\partial x^\mu}. \tag{3.11}$$

Since (3.10) is a mathematical identity, the right-hand side of (3.11) would have to obey

$$\frac{dx^\mu}{dq}\frac{\partial R^\alpha{}_\alpha}{\partial x^\mu}=0, \tag{3.12}$$

a relation that could only hold for specific values of $R^\alpha{}_\alpha$ such as a constant or zero [13]. We thus see that with the normalization of the eikonal that we have used there is no constraint on $R^\alpha{}_\alpha$, with (3.7) holding for any nonzero $R^\alpha{}_\alpha$. (For $R^\alpha{}_\alpha=0$ we would be back to (2.1).) Consequently, we see that the very covariance of the theory requires that we must use the normalized form of the eikonal whenever $\nabla_\mu T\nabla^\mu T$ is nonzero. Moreover, if we do use the normalized form then no matter how small $R^\alpha{}_\alpha$ might be, because of the appearance of $R^\alpha{}_\alpha$ in the denominator of the right-hand side of (3.7) the contribution of $R^\alpha{}_\alpha$ will still be non-trivial.

In addition we note that while we normalized $\nabla_\mu T$ in (3.4), we are still free to rescale q by a numerical constant as it would still measure length along the trajectory and would not affect (3.7). Thus we can introduce a constant E by setting

$$q'=\frac{q}{E^{1/2}}, \tag{3.13}$$

so that

$$\frac{1}{E^{1/2}}\frac{dx^\mu}{dq'}=\frac{\nabla^\mu T}{(-R^\alpha{}_\alpha/6)^{1/2}},\qquad -g_{\mu\nu}\frac{dx^\mu}{dq'}\frac{dx^\nu}{dq'}=E. \tag{3.14}$$

We will see below that this will allow us to make contact with the standard analysis of solar system geodesics in the Schwarzschild de Sitter case, where E serves as an energy.

Moreover, we note that even though we had started with the locally conformal invariant equation of motion given in (3.1), and even though both it and the light cone $ds^2=-g_{\mu\nu}dx^\mu dx^\nu=0$ are left invariant under $g_{\mu\nu}\to\Omega^2(x)g_{\mu\nu}(x)$, nonetheless according to (3.4) $-g_{\mu\nu}dx^\mu dx^\nu$ is not zero [14]. Since it is nonzero it would need to be timelike if it is to describe physical events, and thus E has to be positive, just as needed in (3.13) so as to have q' be real. However, since the standard discussion of propagation of light sets $E=0$, below we will find modifications of the standard discussion of gravitational bending of light once E is nonzero. We comment more on the fact that $-g_{\mu\nu}dx^\mu dx^\nu$ is not zero in our discussion of the implications of conformal invariance and in our discussion the Maxwell field that are given below.

While the left-hand side of (3.7) is the standard term that appears in the geodesic equation, the term on the right-hand side is not. It leads to a departure from strict geodesic behavior and has to be taken into consideration if the derivative of the Ricci scalar is not zero. That this term appears at all is a reflection of our remark in Sec. I that covariantization of a flat space quantity can miss explicit geometric dependent terms. Thus if we start in flat spacetime with $\partial_\mu\partial^\mu\exp(iT)=0$, eikonalization would lead us to $\partial_\mu T\partial^\mu T=0$. Covariantization would then lead us to $\nabla_\mu T\nabla^\mu T=0$, and not to $\nabla_\mu T\nabla^\mu T-R^\alpha{}_\alpha/6=0$. Thus by covariantization we cannot reach $\nabla_\mu T\nabla^\mu T-R^\alpha{}_\alpha/6=0$ starting from $\partial_\mu T\partial^\mu T=0$, even though starting $\nabla_\mu T\nabla^\mu T-R^\alpha{}_\alpha/6=0$ we can reach $\partial_\mu T\partial^\mu T=0$ by taking the flat space limit.

Now while the right-hand side of (3.7) contains the derivative of the Ricci scalar, in the event that the derivative is zero there are actually two possibilities: the Ricci scalar is either zero or equal to a nonzero constant. In either case (3.7) reduces to the standard massless geodesic equation given in (2.7). However, if the Ricci scalar is a nonzero constant, (3.3) does not reduce to (2.3). In this case (3.3) can be considered to be the integral of a (3.5) with a vanishing $\nabla^\lambda R^\alpha{}_\alpha$, with a then constant $\frac{1}{6}R^\alpha{}_\alpha$ term being a nonzero integration constant. In the exterior Schwarzschild situation the Ricci scalar is zero and the standard geodesic discussion of the gravitational bending of massless geodesics by the

Sun remains untouched. However, in the Schwarzschild de Sitter situation the Ricci scalar is a nonzero constant and (3.5) (with $\nabla^\lambda R^\alpha{}_\alpha = 0$) has to be integrated under the constraint provided by (3.3). Finally, we note that in the standard Robertson-Walker cosmology the Ricci scalar is not in general constant and one has to use the full (3.7). We discuss the implications of these features for astrophysics and cosmology in more detail below.

B. Conformal Scalar Field with Dynamical Mass

While conformal invariance would exclude any mass term at the level of the Lagrangian, mass could still be generated through the vacuum by spontaneous symmetry breakdown. For the conformally coupled scalar field S that we are considering here we would need to introduce a second scalar field Q that would provide the symmetry breaking by acquiring a non-vanishing expectation value. Thus for a conformal invariant action of the form

$$I = -\int d^4x(-g)^{1/2}\left[\frac{1}{2}\nabla_\mu S\nabla^\mu S - \frac{1}{12}S^2R^\alpha{}_\alpha + \frac{1}{2}S^2Q^2 + V(Q)\right], \tag{3.15}$$

where $V(Q)$ is to generate a possibly spacetime-dependent expectation value Q_0 for Q, the equation of motion for the scalar field takes the form

$$\nabla_\mu\nabla^\mu S + \frac{1}{6}SR^\alpha{}_\alpha - Q_0^2S = 0, \tag{3.16}$$

while the eikonal phase defined by $S = \exp(iT)$ obeys

$$\nabla_\mu T\nabla^\mu T - i\nabla_\mu\nabla^\mu T - \frac{1}{6}R^\alpha{}_\alpha + Q_0^2 = 0. \tag{3.17}$$

On dropping the $i\nabla_\mu\nabla^\mu T$ term and on setting

$$\frac{dx^\mu}{dq} = \frac{\nabla^\mu T}{(Q_0^2 - R^\alpha{}_\alpha/6)^{1/2}}, \qquad \frac{dx_\mu}{dq}\frac{dx^\mu}{dq} = -1, \tag{3.18}$$

the discussion now follows as above, with (3.7) being replaced by

$$\frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = -\frac{1}{2}\frac{1}{(Q_0^2 - R^\alpha{}_\alpha/6)}\left[g^{\lambda\mu} + \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right]\frac{\partial(Q_0^2 - R^\alpha{}_\alpha/6)}{\partial x^\mu}. \tag{3.19}$$

When Q_0 is a constant m, (3.19) reduces to

$$\frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = -\frac{1}{2}\frac{1}{(m^2 - R^\alpha{}_\alpha/6)}\left[g^{\lambda\mu} + \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right]\frac{\partial(-R^\alpha{}_\alpha/6)}{\partial x^\mu}. \tag{3.20}$$

Interestingly, the scalar field equation based (3.20) is a direct analog of the test particle (1.4), though such an analog of (1.4) is output here not input. Finally, if $Q_0^2 = m^2 \gg R^\alpha{}_\alpha/6$ and $q' = q/E^{1/2}$, (3.18) and (3.20) reduce to

$$\frac{1}{E^{1/2}}\frac{dx^\mu}{dq'} = \frac{\nabla^\mu T}{m}, \qquad \frac{dx_\mu}{dq'}\frac{dx^\mu}{dq'} = -E, \qquad \frac{d^2x^\lambda}{dq'^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq'}\frac{dx^\nu}{dq'} = 0, \tag{3.21}$$

the standard massive particle geodesic equation with energy E. To ascertain whether m might be much larger than any curvature scale, we note that the factor that should appear in the scalar field wave equation is not m but something with the dimension of an inverse length. In quantum field theory the natural length is $mc/\hbar$, the inverse of the Compton wavelength of m (viz. 10^{13} cm^{-1} for a proton). For this choice we would need to compare the square of the inverse of the Compton wavelength of a typical solar mass star (order 10^{140} cm^{-2}) with a typical scale that appears in the Ricci scalar. Since the Ricci scalar vanishes identically in the pure Schwarzschild case, we look instead at a de Sitter geometry where $R^\alpha{}_\alpha = -12K$. The natural expectation for K is that it would be of order the inverse square of the Hubble radius, i.e., of order 10^{-56} cm^{-2}. Alternatively, if the m that appears in the wave equation is to be associated with a classical length scale, the natural length scale is the Schwarzschild radius mG/c^2 (viz. 10^5 cm for a solar mass star). The square of the inverse of this quantity is 10^{-10} cm^{-2}. Thus for either choice the inverse length term dominates over the Ricci scalar, and so for massive particles we are back to standard geodesics.

Effects similar to the ones found here for scalars also occur for fermions, either massless [15] or massive [16], where again there is a dependence on the Ricci scalar. However, while we could generate a mass dynamically in the conformal scalar or conformal fermion field cases, and while such masses could dominate over curvature, below we discuss a case where there is no mass generation, namely the Maxwell field, an equally conformal case where now there is no mass term that is able to dominate over any curvature contribution in the first place. But before doing so, in order to show that the departure from geodesic behavior exhibited in (3.7) is not actually restricted to fields with an underlying conformal structure, we now show that it also occurs with perfect fluids, and in particular for perfect fluid energy-momentum tensors that have no need to be of the traceless form that would be required of conformal perfect fluids.

C. Perfect Fluids

For a perfect fluid the energy-momentum tensor is of the form

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu}, \tag{3.22}$$

with energy density ρ, pressure p and a unit timelike fluid four-vector U^μ that is normalized to $U_\mu U^\mu = -1$. We will have no need here to require the perfect fluid to obey the $g_{\mu\nu}T^{\mu\nu} = 3p - \rho = 0$ condition that massless radiation fluids obey. For any general perfect fluid covariant conservation leads to

$$\nabla_\nu \left[(\rho + p)U^\mu U^\nu + p g^{\mu\nu}\right] = U^\mu \nabla_\nu[(\rho + p)U^\nu] + (\rho + p)U^\nu \nabla_\nu U^\mu + \nabla^\mu p = 0, \tag{3.23}$$

and thus to

$$-\nabla_\nu[(\rho + p)U^\nu] + U_\mu \nabla^\mu p = -(\rho + p)U_\mu U^\nu \nabla_\nu U^\mu. \tag{3.24}$$

Then since $\nabla_\nu[U_\mu U^\mu] = 2[U_\mu \nabla_\nu U^\mu] = 0$ we obtain

$$-\nabla_\nu[(\rho + p)U^\nu] + U_\mu \nabla^\mu p = 0. \tag{3.25}$$

The insertion of (3.25) into (3.23) then yields

$$(\rho + p)U^\nu \nabla_\nu U^\mu + [g^{\mu\nu} + U^\mu U^\nu]\nabla_\nu p = 0. \tag{3.26}$$

On setting $U^\mu = dx^\mu/ds$ (we do not need to set $U^\mu = dx^\mu/dq$ since U^μ is timelike, not lightlike), we thus impose the eikonal normalization condition

$$\frac{dx^\mu}{ds} = \frac{U^\mu}{(-U_\nu U^\nu)^{1/2}} = U^\mu \tag{3.27}$$

on using $U_\nu U^\nu = -1$, with the direction of the velocity of the fluid being in the normal, viz. longitudinal, direction. Thus from (3.26) we obtain

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\frac{1}{(\rho + p)}\left[g^{\lambda\nu} + \frac{dx^\lambda}{ds}\frac{dx^\nu}{ds}\right]\nabla_\nu p. \tag{3.28}$$

Thus in general not only does the fluid velocity vector not move on a geodesic, the structure of (3.28) is identical to that of the scalar field (3.7), as it would essentially have to be since with the left-hand side of (3.28) being orthogonal to dx^λ/dq, the right-hand side would have to be orthogonal to dx^λ/dq too.

While the right-hand side of (3.28) would vanish in the pressureless dust situation in which p is zero, it would not vanish otherwise. However, since our derivation of (3.28) holds no matter what the relation between ρ and p might be, it therefore holds for radiation fluids in which $\rho = 3p$. Thus for radiation the right-hand side of (3.28) is necessarily nonzero, and the motion of the fluid velocity is necessarily not geodesic. This behavior parallels that found in (3.7) for conformal scalar fields and in (4.13) for conformal Maxwell fields, where departure from geodesic behavior also occurs

D. Implications for Conformal Invariance

While we started with a locally conformal invariant wave equation in (3.1), the act of dropping the $-i\nabla_\mu\nabla^\mu T$ term in (3.2) leaves us with an eikonal equation in (3.3) that is no longer locally conformal invariant. To see this explicitly we note that under $S \to e^{-\alpha}S$ the eikonal phase changes according to $T \to T + i\alpha$. In consequence, the three terms in (3.2) transform as

$$\begin{aligned} \nabla_\mu T \nabla^\mu T &\to e^{-2\alpha}\left[\nabla_\mu T \nabla^\mu T + 2i\nabla_\mu T\nabla^\mu \alpha - \nabla_\mu \alpha \nabla^\mu \alpha\right], \\ -i\nabla_\mu \nabla^\mu T &\to -ie^{-2\alpha}\left[\nabla_\mu \nabla^\mu T + i\nabla_\mu \nabla^\mu \alpha + 2\nabla_\mu T \nabla^\mu \alpha + 2i\nabla_\mu \alpha \nabla^\mu \alpha\right], \\ -\frac{1}{6}R^\alpha{}_\alpha &\to -e^{-2\alpha}\left[\frac{1}{6}R^\alpha{}_\alpha + \nabla_\mu \nabla^\mu \alpha + \nabla_\mu \alpha \nabla^\mu \alpha\right]. \end{aligned} \tag{3.29}$$

As we see, the $-i\nabla_\mu \nabla^\mu T$ term is needed to maintain the local conformal invariance of $\nabla_\mu T \nabla^\mu T - i\nabla_\mu \nabla^\mu T - (1/6)R^\alpha{}_\alpha$. In its absence we could only maintain local conformal invariance for those particular α that obey $i\nabla_\mu \nabla^\mu \alpha + 2\nabla_\mu T \nabla^\mu \alpha + 2i\nabla_\mu \alpha \nabla^\mu \alpha = 0$. However, as we show momentarily, we should actually satisfy this requirement by taking α to be a constant. Now this does not mean that we have lost all symmetry, only that we have lost local symmetry, as we still have global conformal invariance, an invariance that is often also referred to as global scale invariance [17]. Global conformal invariance is still sufficient to forbid any kinematical mass terms such as the one exhibited in (2.11). Moreover, if we were not to eikonalize, i.e., in the diffraction regime, we would still need to keep the $-i\nabla_\mu \nabla^\mu T$ term and would still have full local conformal symmetry. However, for eikonalization purposes global conformal symmetry will suffice.

That we would want to take α to be a constant when we eikonalize can be seen from the eikonal condition given in (2.6). Specifically, we note that while the fields and the metric transform non-trivially under a local conformal transformation, the coordinates do not. We thus could not enforce the eikonal relation $dx^\mu/dq = \nabla^\mu T/(-\nabla_\nu T \nabla^\nu T)^{1/2}$ if one side of this equation is left untouched while the other side acquires derivatives of α. Thus the only option is to keep α constant, since then $\nabla^\mu T \to \nabla^\mu T + i\nabla^\mu \alpha = \nabla^\mu T$. We should also note that even if we had not normalized the eikonal relation at all and simply set $dx^\mu/dq = \nabla^\mu T$, it still would not be invariant under $T \to T + i\alpha$ unless α is constant.

With the transition from local conformal invariance to global conformal invariance the modified geodesic equation given in (3.7) is only globally conformal invariant. However, we should note that the unmodified geodesic equation does admit of a local conformal invariance, though to maintain it we not only need to set $(dx_\mu/dq)(dx^\mu/dq) = 0$, we, in addition, need to introduce a parameter according to $d/d\bar{q} = \Omega^{-2}d/dq$, and define geodesics with respect to this $\bar{q}$. Thus under a local conformal transformation we obtain

$$\begin{aligned} \frac{d^2x^\lambda}{d\bar{q}^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\bar{q}}\frac{dx^\nu}{d\bar{q}} &\to \frac{1}{\Omega^2}\frac{d}{dq}\left[\frac{1}{\Omega^2}\frac{dx^\lambda}{dq}\right] + \frac{1}{\Omega^4}\left[\Gamma^\lambda_{\mu\nu} + \frac{1}{\Omega}\left(\delta^\lambda_\nu \partial_\mu + \delta^\lambda_\mu \partial_\nu - g_{\mu\nu}\partial^\lambda\right)\Omega\right]\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} \\ &= \frac{1}{\Omega^4}\left[\frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}\right] - \frac{\partial^\lambda \Omega}{\Omega^5}g_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} \\ &= \frac{1}{\Omega^4}\left[\frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}\right]. \end{aligned} \tag{3.30}$$

Now while we could of course do this, our point here is that starting from a wave equation we never could arrive at the massless geodesic equation given in (1.6) in the first place. The intrinsic structure of the eikonalization procedure itself reduces us to global conformal invariance alone.

There is even a second situation in which a local conformal invariance appears to be present. Specifically if we start with the test particle action

$$I_T = -\int ds Q(x), \tag{3.31}$$

where the scalar field $Q(x)$ varies along the trajectory labelled by the proper time, its variation with respect to x^λ leads to [18]

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\frac{1}{Q}\left[g^{\lambda\mu} + \frac{dx^\lambda}{ds}\frac{dx^\mu}{ds}\right]\frac{\partial Q}{\partial x^\mu}. \tag{3.32}$$

Since this particular test particle action is locally conformal invariant ($ds \to e^\alpha ds$, $Q \to e^{-\alpha}Q$) the trajectory given in (3.32) is locally conformal invariant too, something that is shown explicitly in [18]. On identifying a constant expectation value for Q with a mass parameter m, we then reduce (3.32) to

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{3.33}$$

The I_T action given in (3.31) allows us to construct conformal invariant trajectories for massive particles, with the mass parameter transforming under a local conformal transformation since it is actually a scalar field and not a kinematic parameter in a test particle action. Comparing (3.32) with (3.19), we see that if set $dq = ds$ in (3.19), set $Q_0 = Q$, and set $R^{\alpha}{}_{\alpha} = 0$ we precisely obtain (3.32). Thus in this limit (3.19) would become locally conformal invariant. However, all of this is not of significance because we did not derive (3.19) starting from the test particle action I_T given in (3.31) but from the scalar field wave equation given in (3.16), and we had to go through the non-locally conformal invariant eikonal relation given in (3.18) in order to do so. Thus at best (3.32) can only be associated with a globally conformal invariant eikonalization procedure.

IV. CONFORMAL INVARIANT MAXWELL FIELD WAVE EQUATION

In curved space the electromagnetic field is associated with the Maxwell action $I = -\frac{1}{4}\int d^4x(-g)^{1/2}F_{\mu\nu}F^{\mu\nu}$. On setting $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ this action is left invariant under the local conformal transformation $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$, $A_\mu(x) \to A_\mu(x)$ [19]. Consequently, the wave equation $\nabla_\mu F^{\mu\nu} = 0$, viz.

$$\nabla_\mu(\nabla^\mu A^\nu - \nabla^\nu A^\mu) = 0, \tag{4.1}$$

is locally conformal invariant. Recalling the geometric identity

$$\nabla_\kappa\nabla_\nu V_\lambda - \nabla_\nu\nabla_\kappa V_\lambda = V^\sigma R_{\lambda\sigma\nu\kappa} \tag{4.2}$$

that holds for any general coordinate vector V_λ, we can rewrite (4.1) in the convenient form

$$\nabla_\mu\nabla^\mu A^\nu - \nabla^\nu\nabla_\mu A^\mu + R^{\nu\alpha}A_\alpha = 0. \tag{4.3}$$

As we see, just as in the scalar field case discussed above and in fermion field case discussed in [15] and [16], there is an intrinsic dependence on some of the components of the Riemann tensor.

The utility of rewriting (4.1) in this form is that it greatly simplifies in the covariant Lorentz gauge. Thus if set $\nabla_\mu A^\mu = 0$ (4.3) reduces to

$$\nabla_\mu\nabla^\mu A^\nu + R^{\nu\alpha}A_\alpha = 0. \tag{4.4}$$

To eikonalize (4.4) we set $A_\mu(x) = \epsilon_\mu(x)\exp(iT(x))$ where in general the polarization vector $\epsilon_\mu(x)$ is spacetime dependent. With this form the gauge condition takes the form

$$\nabla_\mu\epsilon^\mu + i\epsilon^\mu\nabla_\mu T = 0, \tag{4.5}$$

while (4.4) takes the form

$$-\epsilon^\nu\nabla_\mu T\nabla^\mu T + i\epsilon^\nu\nabla_\mu\nabla^\mu T + 2i\nabla_\mu\epsilon^\nu\nabla^\mu T + \nabla_\mu\nabla^\mu\epsilon^\nu + R^{\nu\alpha}\epsilon_\alpha = 0. \tag{4.6}$$

So far everything is exact. In the eikonal limit we take T to be very large, and in particular we take T to be much larger than any of the components of ϵ_μ or their derivatives. Thus in the short wavelength limit the gauge condition and the wave equation reduce to

$$i\epsilon^\mu\nabla_\mu T = 0, \tag{4.7}$$

$$-\epsilon^\nu\nabla_\mu T\nabla^\mu T + R^{\nu\alpha}\epsilon_\alpha = 0. \tag{4.8}$$

If we set $T = \int^x k_\mu dx^\mu$, $\nabla_\mu T = k_\mu$ we obtain

$$k_\mu\epsilon^\mu = 0, \tag{4.9}$$

$$\epsilon^\nu k_\mu k^\mu - R^{\nu\alpha}\epsilon_\alpha = 0. \tag{4.10}$$

Now as discussed above, in the absence of any explicit curvature term we would have $k_\mu k^\mu = 0$, so that if we were then to set $k^\mu = dx^\mu/dq$ we would find that $\int^x k_\mu dx^\mu$ would be zero. Now initially it looks that this concern could be avoided because of the presence of the $R^{\nu\alpha}\epsilon_\alpha$ term in (4.10). However, since some of the components of $R^{\nu\alpha}\epsilon_\alpha$ could vanish for some specific choice of polarization vector, we shall, as before, instead set $T = \int k_- dx^-$, i.e., in light-front coordinates we set $k_\mu = (0, k_-, 0, 0)$. From (4.9) this leads to $\epsilon^- = 0$. On normalizing the remaining polarization vectors according to $\epsilon_\mu \epsilon^\mu = 1$, in a given polarization mode (4.8) takes the form

$$\nabla_\mu T \nabla^\mu T - R^{\alpha\beta}\epsilon_\alpha \epsilon_\beta = 0. \tag{4.11}$$

We can now proceed as in (3.4) and set

$$\frac{dx^\mu}{dq} = \frac{\nabla^\mu T}{(-R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta)^{1/2}}, \qquad \frac{dx_\mu}{dq}\frac{dx^\mu}{dq} = -1. \tag{4.12}$$

From this point the calculation follows the conformal scalar field case, and leads to

$$\frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = -\frac{1}{2(-R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta)}\left[g^{\lambda\mu} + \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right]\frac{\partial(-R^{\gamma\delta}\epsilon_\gamma\epsilon_\delta)}{\partial x^\mu}. \tag{4.13}$$

This then is our main result, and just as in the conformally coupled scalar field case, in the Maxwell case there again are departures from pure geodesic behavior in the presence of curvature. In general, (4.13) would hold in any geometry, with the right-hand side of (4.13) not vanishing. However in a Schwarzschild de Sitter geometry $R_{\mu\nu}$ is equal to $-3Kg_{\mu\nu}$. Thus if the polarization vectors are normalized so that $\epsilon_\mu\epsilon_\nu = g_{\mu\nu}$, then $R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta$ would be equal to the constant $-3K$. In that case even though the right-hand side of (4.13) would then vanish, nonetheless since K is nonzero the second equation in (4.12) would provide what would then be an integration constant for (4.13). For Schwarzschild de Sitter there thus can be an integration constant, though there is none for pure Schwarzschild itself. Consequently, gravitational bending of light by the Sun is not modified from the standard pure geodesic-based Ricci flat discussion, but when $R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta$ is nonzero there in principle are modifications. We discuss further aspects of Schwarzschild de Sitter modifications below.

It is also of interest to note that covariantizing the flat space Maxwell equations would miss the $R^{\nu\mu}\epsilon_\mu$ term that appears in (4.3). Specifically, in flat space we can write the wave equation in the form

$$\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu + (\partial^\nu\partial^\mu - \partial^\mu\partial^\nu)A_\mu = 0. \tag{4.14}$$

Now while we can drop the $(\partial^\nu\partial^\mu - \partial^\mu\partial^\nu)A_\mu$ term in flat space, in curved space covariant derivatives do not commute, and thus we have to generalize $(\partial^\nu\partial^\mu - \partial^\mu\partial^\nu)A_\mu$ to $(\nabla^\nu\nabla^\mu - \nabla^\mu\nabla^\nu)A_\mu$ and this quantity is not zero, being instead equal to $R^{\nu\mu}A_\mu$. Thus in covariantizing a flat space expression the order in which derivatives are applied cannot be ignored.

While we have shown that there can be departures from geodesic behavior, in a sense more interesting is that (3.4) and (4.12) are both of the form

$$\frac{dx_\mu}{dq}\frac{dx^\mu}{dq} = -1. \tag{4.15}$$

Thus in the massless particle curved space eikonal approximation it is possible for modes to propagate off the light cone in a conformal invariant theory in which there is no intrinsic mass. With $ds^2 = -g_{\mu\nu}dx^\mu dx^\nu$, (4.15) leads to timelike ds^2, and thus modes that obey (4.15) will nicely be causally inside the $ds^2 = 0$ light cone [20]. In addition, we note that the modification to geodesic behavior in the light wave case depends on the polarization of the light wave. As such, (4.13) is somewhat reminiscent of (1.5) as they both involve both spin and geometric factors.

Propagation of light off the light cone is actually a quite general phenomenon, and already occurs in flat space electromagnetism. Specifically, in the presence of a current source the Maxwell equations take the form $\partial_\mu\partial^\mu A^\nu = J^\nu$ in the Lorentz gauge $\partial_\mu A^\mu = 0$, and admit of no solution of the form $A^\nu = \epsilon^\nu \exp(ik\cdot x)$ where $k_\mu k^\mu = 0$. However, if the source is localized one then does get such lightlike solutions far from the source. But if the ray enters a refractive medium with refractive index n, then in the medium the ray travels with a velocity c/n and thus not at velocity c. As discussed earlier, the refractive index is often spatially dependent, and in optics the eikonal approximation with a spatially-dependent eikonal phase was developed for this very purpose [4]. In curved space there is also an effective refractive index, one produced by gravity itself as curvature causes spacetime to become a medium. This gravitational medium will then take a light ray off (the covariantized version of) the light cone that it otherwise would have travelled on in the absence of gravity. The role of the $R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta$ and $R^\alpha{}_\alpha$ terms are thus akin to that of a refractive medium, and in this paper we are exploring what effects such a medium could in principle, even if not necessarily in practice, provide.

V. DEPARTURES FROM GEODESIC BEHAVIOR IN THE SCHWARZSCHILD DE SITTER CASE

We consider the geometry exterior to a static, spherically symmetric system with metric

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{5.1}$$

For the Schwarzschild case where $R_{\mu\nu} = 0$ the exterior metric is given by $B(r) = 1/A(r) = 1 - 2MG/r$, while for the Schwarzschild de Sitter case where $R_{\mu\nu} = -3Kg_{\mu\nu}$ the exterior metric is given by $B(r) = 1/A(r) = 1 - 2MG/r - Kr^2$. In both of these geometries the right-hand side of the massless scalar field (3.7) vanishes, while for an appropriate choice of polarization vectors the right-hand side of the massless vector field (4.13) vanishes too. Thus for all of these cases we can use the massless geodesic equation given in (1.6). However, in the Schwarzschild de Sitter case we would have to integrate the geodesic equation under the constraint given in (4.15).

Starting with the massless geodesic equation given in (1.6), or also with the massive geodesic equation given in (1.2) with q replacing s in it (which is also (3.20) with $m^2 \gg R^\alpha{}_\alpha/6$), we obtain [1]

$$\begin{aligned}
&\frac{d^2r}{dq^2} + \frac{A'}{2A}\left(\frac{dr}{dq}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dq}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\phi}{dq}\right)^2 + \frac{B'}{2A}\left(\frac{dt}{dq}\right)^2 = 0,\\
&\frac{d^2\theta}{dq^2} + \frac{2}{r}\frac{d\theta}{dq}\frac{dr}{dq} - \sin\theta\cos\theta\left(\frac{d\phi}{dq}\right)^2 = 0,\\
&\frac{d^2\phi}{dq^2} + \frac{2}{r}\frac{d\phi}{dq}\frac{dr}{dq} + 2\cot\theta\frac{d\phi}{dq}\frac{d\theta}{dq} = 0,\\
&\frac{d^2t}{dq^2} + \frac{B'}{B}\frac{dt}{dq}\frac{dr}{dq} = 0.
\end{aligned} \tag{5.2}$$

On setting $\theta = \pi/2$ these equations integrate to

$$\frac{dt}{dq} = \frac{1}{B}, \qquad r^2\frac{d\phi}{dq} = J, \qquad \left(\frac{dr}{dq}\right)^2 = \frac{1}{AB} - \frac{J^2}{Ar^2} - \frac{E}{A}, \tag{5.3}$$

where the angular momentum J and the energy E are integration constants. (The integration constant for dt/dq has been set to one.) From these relations we obtain

$$ds^2 = -g_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}dq^2 = Edq^2, \tag{5.4}$$

and on comparing with (3.14) we reinterpret q as q'. Now while for material particles with nonzero mass we take the integration constant E to be nonzero and positive, for massless particles one ordinarily sets $E = 0$ so as to put them on the light cone. Now while one can do this in the Schwarzschild case, one cannot do this in the Schwarzschild de Sitter case since according to (3.4) $-(dx_\mu/dq)(dx^\mu/dq)$ is greater than zero (and analogously for (4.12)). In the de Sitter case then geodesic behavior for massless particles is not compatible with the particles being on the light cone.

For a particle of energy E coming in from an asymptotically Minkowski geometry the gravitational bending of a particle in the solution given in (5.3) is of the form [1]

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{ABJ^2}{r^2(r^2 - J^2B - r^2EB)}, \quad \phi(r) - \phi(\infty) = \int_r^\infty dr\frac{A^{1/2}(r)B^{1/2}(r)r_0(1 - EB(r_0))^{1/2}}{r[r^2B(r_0) - r_0^2B(r) + EB(r)B(r_0)(r_0^2 - r^2)]^{1/2}}, \tag{5.5}$$

where r_0 is the distance of closest approach to the gravitational source located at $r = 0$. When $A = 1/B$ and $E = 0$ (5.5) reduces to

$$\phi(r) - \phi(\infty) = \int_r^\infty dr\frac{r_0}{r[r^2B(r_0) - r_0^2B(r)]^{1/2}}, \tag{5.6}$$

the standard geodesic-based bending formula in a curved space.

As written, (5.6) has a well-known shortcoming. If we set $B(r) = 1 - 2MG/r - Kr^2$ we find that the Kr^2 term drops out identically. However, it is not actually valid to use (5.5) in the de Sitter case [21, 22] since (5.5) was derived on the assumption that far from the source the geometry is asymptotically flat, and for de Sitter this is not the case. Moreover, in [21, 22] Rindler and Ishak provided an alternate prescription for determining gravitational bending that does take the non-asymptotic flatness of the de Sitter geometry into account. While these remarks are valid, to them we should add that for de Sitter one should anyway not be using (5.6) in the first place. Rather, with $R_{\mu\nu} = -3Kg_{\mu\nu}$ and $R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta = -3K\epsilon_\alpha\epsilon^\alpha = -3K$ both being nonzero, for de Sitter one should use (5.5) with $E = 1$ for massless scalar rays and massless vector rays, and now the dependence on K does not drop out. However, even so one still has to incorporate the asymptotic non-flatness considerations discussed in [21, 22] as well. The discussion of gravitational bending is thus instructive since while one does not need to depart from $(dx_\mu/dq)(dx^\mu/dq) = 0$ in the pure Schwarzschild case, one does have to do so in either Schwarzschild de Sitter or pure de Sitter.

For particles with mass we can discuss bound orbits, and for circular bound orbits the analysis is different from the one just given for the gravitational bending of massless light rays, since with dr/dq set to zero in a circular orbit we do not need to integrate the d^2r/dq^2 equation given in (5.2) or address the appropriate value for its integration constant E. Specifically, if we take $\theta = \pi/2$ and set $dr/dq = 0$ in (5.2) we obtain [23]

$$-\frac{r}{A}\left(\frac{d\phi}{dq}\right)^2 + \frac{B'}{2A}\left(\frac{dt}{dq}\right)^2 = 0, \qquad \frac{d^2\phi}{dq^2} = 0, \qquad \frac{d^2t}{dq^2} = 0. \tag{5.7}$$

In an orbit of fixed radius R we have

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{B'(R)}{2R}, \qquad \frac{d\phi}{dq} = C(R), \qquad \frac{dt}{dq} = D(R), \qquad \left(\frac{d\phi}{dt}\right)^2 = \frac{C^2(R)}{D^2(R)} = \frac{B'(R)}{2R}, \tag{5.8}$$

where $C(R)$ and $D(R)$ are integration constants. From (5.8) the orbital velocity v is given by

$$v = R\frac{d\phi}{dt} = \left(\frac{RB'(R)}{2}\right)^{1/2}. \tag{5.9}$$

We recognize (5.9) as the standard expression for massive particle circular orbits with $v^2 < c^2$, i.e., $v < 1$.

Now while we did not need to utilize the condition $(dx_\mu/dq)(dx^\mu/dq) = -1$ given in (3.18) in order to derive the massive particle (5.9), nonetheless (3.18) still holds. Moreover, since for massive particles there is a nonzero integration constant E to begin with, and since we can absorb its magnitude via. (3.13), the discussion is actually the standard one. Thus on absorbing E, for the metric given in (5.1) the $(dx_\mu/dq)(dx^\mu/dq) = -1$ condition takes the form

$$-B\left(\frac{dt}{dq}\right)^2 + A\left(\frac{dr}{dq}\right)^2 + r^2\left(\frac{d\phi}{dq}\right)^2 = -1 \tag{5.10}$$

when $\theta = \pi/2$. In a circular orbit this requires that

$$B(R)D^2(R) - R^2C^2(R) = 1, \tag{5.11}$$

to thus constrain the integration constants given in (5.8) according to

$$D^2(R) = \frac{2}{2B(R) - RB'(R)} = \frac{1}{B(R) - v^2}, \qquad C^2(R) = \frac{B'(R)}{2RB(R) - R^2B'(R)} = \frac{v^2}{R^2(B(R) - v^2)}. \tag{5.12}$$

With both $D^2(R)$ and $C^2(R)$ being positive, v^2 is constrained to be less than $B(R)$. If $B(R)$ is itself less than one then $v^2 < 1$. With $B(R) = 1 - 2MG/R - KR^2$ this can readily be achieved with positive MG and K.

VI. DEPARTURES FROM GEODESIC BEHAVIOR IN THE GENERAL STATIC SPHERICALLY SYMMETRIC MASSLESS CASE

For the conformally coupled massless scalar field and massless vector field the trajectories are given by (3.7) and (4.13) as constrained by (3.4) no matter what the values of the Ricci scalar or tensor. In this general case with the static, spherically symmetric metric given in (5.1), the equations that replace (5.2) and (5.4) are of the form

$$\frac{d^2r}{dq^2} + \frac{A'}{2A}\left(\frac{dr}{dq}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dq}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\phi}{dq}\right)^2 + \frac{B'}{2A}\left(\frac{dt}{dq}\right)^2 = -\frac{1}{2X}\left(\frac{1}{A(r)} + \left(\frac{dr}{dq}\right)^2\right)\frac{dX}{dr},$$
$$\frac{d^2\theta}{dq^2} + \frac{2}{r}\frac{d\theta}{dq}\frac{dr}{dq} - \sin\theta\cos\theta\left(\frac{d\phi}{dq}\right)^2 = -\frac{1}{2X}\frac{d\theta}{dq}\frac{dr}{dq}\frac{dX}{dr},$$

$$\frac{d^2\phi}{dq^2}+\frac{2}{r}\frac{d\phi}{dq}\frac{dr}{dq}+2\cot\theta\frac{d\phi}{dq}\frac{d\theta}{dq}=-\frac{1}{2X}\frac{d\phi}{dq}\frac{dr}{dq}\frac{dX}{dr},$$
$$\frac{d^2t}{dq^2}+\frac{B'}{B}\frac{dt}{dq}\frac{dr}{dq}=-\frac{1}{2X}\frac{dt}{dq}\frac{dr}{dq}\frac{dX}{dr}, \tag{6.1}$$

$$g_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}=-B(r)\left(\frac{dt}{dq}\right)^2+A(r)\left(\frac{dr}{dq}\right)^2+r^2\left(\frac{d\theta}{dq}\right)^2+r^2\sin^2\theta\left(\frac{d\phi}{dq}\right)^2=-1, \tag{6.2}$$

where $X=-R^\alpha{}_\alpha/6$ or $X=-R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta$. With θ being fixed at $\pi/2$, the equations for $d\phi/dq$ and dt/dq can readily be integrated, and with dr/dq then being determinable from (6.2) the general solution is of the form

$$\left(\frac{dt}{dq}\right)^2=\frac{d_1}{B^2X},\qquad \left(\frac{d\phi}{dq}\right)^2=\frac{d_2}{r^4X},\qquad \left(\frac{dr}{dq}\right)^2=\frac{d_1r^2-d_2B-r^2BX}{r^2ABX}, \tag{6.3}$$

where d_1 and d_2 are integration constants. In the solution given in (6.3) gravitational bending is determined from

$$\left(\frac{d\phi}{dr}\right)^2=\frac{d_2AB}{r^2(d_1r^2-d_2B-r^2BX)}. \tag{6.4}$$

With A and B both being positive, the reality of the solution requires that

$$\frac{d_1}{X}>0,\qquad \frac{d_2}{X}>0,\qquad \frac{d_1r^2-d_2B-r^2BX}{X}>0. \tag{6.5}$$

If these conditions are not satisfied (conditions that could restrict the range of allowed values of r), eikonalization is not possible. And when these conditions are satisfied one should in general then determine the dependence of ϕ on r for massless rays from (6.4), and not from the value for $d\phi/dr$ given in (5.5) when evaluated at $E=0$, viz.

$$\left(\frac{d\phi}{dr}\right)^2=\frac{ABJ^2}{r^2(r^2-J^2B)}. \tag{6.6}$$

VII. DEPARTURES FROM GEODESIC BEHAVIOR IN THE CONFORMAL GRAVITY CASE

In static, spherically symmetric geometries there is another situation in which the above analysis is of relevance, namely gravitational bending in the conformal gravity theory. Conformal gravity is based on the action $I_{\rm W}=-\alpha_g\int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}$, where $C_{\lambda\mu\nu\kappa}$ is the Weyl tensor and α_g is a dimensionless gravitational coupling constant. As with the other conformal invariant theories discussed above, $I_{\rm W}$ is invariant under $g_{\mu\nu}\to\Omega^2(x)g_{\mu\nu}(x)$ with arbitrary local $\Omega(x)$. And if we are going to impose local conformal invariance for all fields we must then also use conformal invariant scalar or vector field wave equations for the fields coupled to conformal gravity. In the static, spherically symmetric case we must thus use (6.3) and (6.4), with the gravitational equations of the conformal gravity theory then supplying us with explicit expressions for A, B and X.

To this end the conformal gravity gravitational equations of motion can be written in the compact form [3]

$$4\alpha_g[2\nabla_\kappa\nabla_\lambda C^{\mu\lambda\nu\kappa}-R_{\lambda\kappa}C^{\mu\lambda\nu\kappa}]=T^{\mu\nu}. \tag{7.1}$$

In the static, spherically symmetric situation (7.1) can be solved exactly [24, 25], and since such geometries are not conformal to flat, in them the Weyl tensor does not vanish. For such geometries the imposition of conformal invariance on (5.1) allows us to set $A=1/B$ as a kinematic condition [24], with all nonzero components of the Weyl tensor then being proportional to $B''-2B'/r+2(B-1)/r^2$. With $A=1/B$ the gravitational equations of motion associated with the metric given in (5.1) reduce exactly without any approximation at all to the remarkably compact

$$B''''+\frac{4}{r}B'''\;=\;\nabla^4B(r)=\frac{3}{4\alpha_gB(r)}\left(T^0{}_0-T^r{}_r\right)=f(r),\qquad \frac{1}{3r^4}(1+y^3y'')=\frac{T^r{}_r}{4\alpha_g}. \tag{7.2}$$

Here the first equation in (7.2) serves to define $f(r)$, and in the second we have set $y^2=r^2B'-2rB$.

The solution to the first equation in (7.2) can be determined in closed form and is given by [25]

$$B(r)\;=\;w-Kr^2-\frac{r}{2}\int_0^r dr'r'^2f(r')-\frac{1}{6r}\int_0^r dr'r'^4f(r')-\frac{1}{2}\int_r^\infty dr'r'^3f(r')-\frac{r^2}{6}\int_r^\infty dr'r'f(r'),$$

$$\begin{aligned}
B'(r) &= -2Kr - \frac{1}{2}\int_0^r dr' r'^2 f(r') + \frac{1}{6r^2}\int_0^r dr' r'^4 f(r') - \frac{r}{3}\int_r^\infty dr' r' f(r'),\\
B''(r) &= -2K - \frac{1}{3r^3}\int_0^r dr' r'^4 f(r') - \frac{1}{3}\int_r^\infty dr' r' f(r'),\\
B'''(r) &= \frac{1}{r^4}\int_0^r dr' r'^4 f(r'),\\
B''''(r) &= -\frac{4}{r^5}\int_0^r dr' r'^4 f(r') + f(r),
\end{aligned} \tag{7.3}$$

from which it follows that (see e.g. [26])

$$\frac{1}{12r^4}\left[4 + \int_0^r dr' r'^2 f(r') \int_0^r dr' r'^4 f(r') - \left(\int_r^\infty dr' r'^3 f(r') - 2w\right)^2\right] = \frac{T^r{}_r}{4\alpha_g}. \tag{7.4}$$

In (7.3) we have given derivatives of $B(r)$ so that it can readily be checked that the first equation in (7.2) is indeed obeyed. Also we have included a $B = w - Kr^2$ contribution as it satisfies $\nabla^4 B(r) = 0$ identically. On evaluating the non-vanishing components of the Weyl tensor for the $B(r)$ given in (7.3) we obtain

$$B'' - \frac{2}{r}B' + \frac{2}{r^2}(B-1) = -\frac{1}{r^3}\int_0^r dr' r'^4 f(r') - \frac{1}{r^2}\int_r^\infty dr' r'^3 f(r') + \frac{2}{r^2}(w-1), \tag{7.5}$$

and confirm that the $-Kr^2$ contribution cancels identically, just as it should since a de Sitter geometry is conformal to flat. We note the absence of any $\int_0^r dr' r'^2 f(r')$ term in (7.5), a point we will return to below.

For the metric given in (5.1) the various components of the Ricci tensor take the form

$$\begin{aligned}
&R_{rr} = \frac{B''}{2B} + \frac{B'}{rB}, \qquad R_{\theta\theta} = -1 + rB' + B,\\
&R_{\phi\phi} = \sin^2\theta R_{\theta\theta}, \qquad R_{tt} = -\frac{B''B}{2} - \frac{B'B}{r},\\
&R^\alpha{}_\alpha = B'' + \frac{4B'}{r} + \frac{2B}{r^2} - \frac{2}{r^2}
\end{aligned} \tag{7.6}$$

when $A = 1/B$. From (7.3) and (7.6) we could determine the form of $R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta$ in any specified polarization state, but we content ourselves here by only determining $R^\alpha{}_\alpha$, and find that it is given by

$$R^\alpha{}_\alpha = -\frac{3}{r}\int_0^r dr' r'^2 f(r') - \frac{1}{r^2}\int_r^\infty dr' r'^3 f(r') - 2\int_r^\infty dr' r' f(r') - 12k + \frac{2}{r^2}(w-1). \tag{7.7}$$

Thus for a given $f(r)$, to determine the gravitational bending of massless rays, in the general (6.4) one should, modulo a modification that is given in (7.15) and (7.16) below, use the relation given in (7.3) for $B(r)$ and $A(r) = -1/B(r)$, and use the $R^\alpha{}_\alpha$ given in (7.7) for $-6X$.

Some simplification of the general $B(r)$ is possible depending on how the matter sources are distributed. For an isolated system such as a star of radius R_0 with $f(r)$ only being nonzero in the range $(0, R_0)$, the metric in $r > R_0$ takes the form

$$B(r > R_0) = w - Kr^2 - \frac{1}{6r}\int_0^{R_0} dr' r'^4 f(r') - \frac{r}{2}\int_0^{R_0} dr' r'^2 f(r') = w - Kr^2 - \frac{2\beta^*}{r} + \gamma^* r, \tag{7.8}$$

$$\beta^* = \frac{1}{12}\int_0^{R_0} dr' r'^4 f(r'), \qquad \gamma^* = -\frac{1}{2}\int_0^{R_0} dr' r'^2 f(r'), \qquad w^2 = 1 - 6\beta^*\gamma^*. \tag{7.9}$$

It was in the form

$$B(r > R_0) = w - \frac{2\beta^*}{r} + \gamma^* r - Kr^2 \tag{7.10}$$

that this solution was introduced in [24, 25]. (For a solar mass star $\beta^* = M_\odot G/c^2$.) However, driven by attempts to fit the systematics of galactic rotation curve data it was gradually realized that this was not the complete solution.

Specifically, since sources are putting out potentials that grow with distance (viz. the γr type term), one cannot ignore the contribution of material outside of any system of interest. There are two forms of such external contributions, one due to the background cosmological Hubble flow and the other due to the inhomogeneities in it. Since the cosmological background is described by a Robertson-Walker geometry and since such a geometry is conformal to flat, in it the Weyl tensor vanishes. However, as seen for instance from (7.5), for the inhomogeneities in the Hubble flow the Weyl tensor does not vanish. In [24, 27] it was realized that when transformed by a coordinate transformation to any static, spherically symmetric rest frame coordinate system, a conformally transformed conformal to flat comoving global Robertson-Walker metric took the form of a universal linear potential term $\gamma_0 r$ contribution to $B(r)$, where γ_0 is fixed by the spatial curvature k of the Universe according to $\gamma_0 = (-4k)^{1/2}$. With the $\int_0^r dr' r'^2 f(r')$ term not contributing to the Weyl tensor in (7.5) and with this same integral being related to γ^* in (7.9), a contribution of the form $B(r) = \gamma_0 r$ would not couple to the Weyl tensor either, just as it should not since it originates from cosmology where the Weyl tensor vanishes. Thus just like its cosmological $-Kr^2$ partner, the $\gamma_0 r$ term can also be added on to the solution to $\nabla^4 B(r) = f(r)$. Thus in [27] the exterior metric

$$B(r > R_0) = w - \frac{2\beta^*}{r} + \gamma^* r + \gamma_0 r - Kr^2 \tag{7.11}$$

and its associated massive particle geodesic equation were used to successfully fit the then available 11 galactic rotation curve bound orbit data for which there were at the time both good radio data (HI) and good photometry (HII), and good fits were found (the contribution of the $-Kr^2$ term was not needed) without the need to introduce any dark matter.

For inhomogeneities, it was noted in [28] that they would contribute to the $r^2 \int_r^\infty dr' r' f(r')$ and $\int_r^\infty dr' r'^3 f(r')$ terms that appear in (7.3), as those integrals continue all the way to infinity, to thus encompass material external to any given system of interest. Taking this exterior matter to begin at some scale R_1, the potential and its derivative in the intermediate $R_0 < r < R_1$ region then take the form

$$\begin{aligned} B(R_0 < r < R_1) &= w - \frac{2\beta^*}{r} + \gamma^* r + \gamma_0 r - \kappa r^2 - \frac{1}{2}\int_{R_1}^\infty dr' r'^3 f(r') - Kr^2, \\ B'(R_0 < r < R_1) &= \frac{2\beta^*}{r^2} + \gamma^* + \gamma_0 - 2\kappa r - 2Kr, \end{aligned} \tag{7.12}$$

where

$$\kappa = \frac{1}{6}\int_{R_1}^\infty dr' r' f(r'). \tag{7.13}$$

We have not merged the $-Kr^2$ and $-\kappa r^2$ terms in (7.12) since while the $-Kr^2$ term applies for all r, the $-\kappa r^2$ term only has the form that it does for $r < R_1$, with the $-Kr^2$ term being due to the cosmological background and the $-\kappa r^2$ term being due to the inhomogeneities in it. With new galactic data having subsequently come on line the data then went out far enough that the κr^2 term was now relevant, and on ignoring the $-Kr^2$ term, through the use of (7.12) very good fitting to the rotation curves of the 138 galaxies was obtained in [28–30] with fixed, universal (i.e., galaxy-independent) parameters

$$\begin{aligned} \beta^* &= 1.48 \times 10^5 \text{cm}, \quad \gamma^* = 5.42 \times 10^{-41} \text{cm}^{-1}, \\ \gamma_0 &= 3.06 \times 10^{-30} \text{cm}^{-1}, \quad \kappa = 9.54 \times 10^{-54} \text{cm}^{-2}, \end{aligned} \tag{7.14}$$

and with there being no need to introduce any dark matter. Since current dark matter fits require two free parameters per galactic halo, the galaxy-dependent 276 free dark matter halo parameters that are needed for the 138 galaxy sample are replaced by just the three universal parameters: γ^*, γ_0 and κ. (The luminous Newtonian β^*-dependent contribution in (7.8) is common to both dark matter and conformal gravity fits and is included in both cases.) With γ_0 being fitted to be of order the inverse of the Hubble radius and with the fitted κ being of order a typical cluster of galaxies scale, the values for γ^0 and κ that are obtained show that they are indeed of the cosmological scales associated with the homogeneous Hubble flow and the inhomogeneities in it. We can thus use stars in galaxies to serve as test particles that measure the global geometry of the universe. From the perspective of a local $1/r$ Newtonian potential the fact that the measured velocities exceed the luminous Newtonian expectation is described as the missing mass problem, with undetected or dark matter within the galaxies themselves being needed in order to be able to account for the shortfall. From the perspective of conformal gravity the shortfall is explained by the rest of the visible mass in the universe. The missing mass is thus not missing at all, it is the rest of the visible universe and it has been hiding in plain sight all along.

Since one cannot ignore the rest of the Universe for bound orbits, one cannot do so for unbound photon trajectories either. As such, the presence of contributions coming from outside of any given galaxy of interest constitute an effect that is foreign to standard Newton-Einstein gravity, namely an external field effect. As well as occurring in conformal gravity, such effects are also present in other alternate gravity theories, and have been identified in a study [31] of Milgrom's Modified Newtonian Dynamics (MOND) and a study [32] of Moffat's Modified Gravity Theory (MOG), and are also discussed in [33]. These external field concerns are of relevance for gravitational bending and lensing, with intent of this paper being to provide a framework for future conformal gravity studies on the topic. In these studies one cannot restrict the range of r as in (7.12) since for a lens composed for instance of a cluster of galaxies the lens is an extended source that light rays both go around and through. Moreover, since light rays are coming in from a non-asymptotically flat background one has to include its cosmological $\gamma_0 r$ contribution to both the full metric and Ricci scalar, viz.

$$B(r) \;=\; w - Kr^2 - \frac{r}{2}\int_0^r dr' r'^2 f(r') - \frac{1}{6r}\int_0^r dr' r'^4 f(r') - \frac{1}{2}\int_r^\infty dr' r'^3 f(r') - \frac{r^2}{6}\int_r^\infty dr' r' f(r') + \gamma_0 r, \tag{7.15}$$

$$R^\alpha{}_\alpha = -\frac{3}{r}\int_0^r dr' r'^2 f(r') - \frac{1}{r^2}\int_r^\infty dr' r'^3 f(r') - 2\int_r^\infty dr' r' f(r') - 12k + \frac{6\gamma_0}{r} + \frac{2}{r^2}(w-1). \tag{7.16}$$

as integrated over all the visible material in the Universe. And in addition one has to adapt the analysis to incorporate the concerns described in [21] and [22].

The issue of gravitational bending in the conformal gravity theory has led to a spirited though not as yet fully resolved discussion in the literature, with some quite varying results being reported. Studies based on the pure null geodesic behavior associated with (5.6) may be found in [34–36]. However, since the linear potential leads to a non-asymptotically flat geometry one cannot actually use (5.6). Studies that took the asymptotic behavior of the conformal gravity metric given in (7.10) into consideration may be found in [37–41]. The present work provides some additional insight into the issue and its follow up could prove to be instructive.

Finally, we should note that while there are potential departures from geodesic behavior for massless light rays there is no such departure for bound orbits of massive particles such as planets in the solar system or stars and gas in a galaxy, since, as noted in Sec. III and [16], the mass term typically dominates over curvature effects. (This is actually quite significant for conformal gravity fits to galactic rotation curves since unlike the $-2MG/r$ term, the linear potential does contribute to the Ricci scalar, and would be relevant if the Ricci scalar were not suppressed by the mass term [42].) For massless particles one has to compare the curvature scale not with a mass scale but with a wavelength scale, and for phenomena such as lensing one has to make the comparison over all radii r including those far from the lens [43]. In regard to curvature effects, we note that these are large in the merger region of two compact sources such as two massive black holes, and in this region curvature effects could impact on the trajectories of gravitational waves. In events such as the merger of two neutron stars high curvature could impact not just on gravitational waves in the merger region but also on the light rays that are emitted in the merger region in these so-called multi-messenger events.

VIII. DEPARTURES FROM GEODESIC BEHAVIOR IN THE ROBERTSON-WALKER CASE

To illustrate what happens in the Robertson-Walker case we consider radial propagation in a spatially flat geometry of the form

$$ds^2 = dt^2 - a^2(t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \tag{8.1}$$

with arbitrary $a(t)$. It is convenient to work in conformal to flat coordinates with $dp = dt/a(t)$, $\Omega(p) = a(t)$, and a metric of the form

$$ds^2 = \Omega^2(p)[dp^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2]. \tag{8.2}$$

With the dot denoting the derivative with respect to p and 0 denoting p, in the $(0,r)$ sector the connection and the Ricci scalar are of the form

$$\Gamma^0_{00} = \Gamma^0_{rr} = \Gamma^r_{0r} = \frac{\dot{\Omega}}{\Omega}, \qquad \Gamma^0_{0r} = \Gamma^r_{00} = \Gamma^r_{rr} = 0, \qquad R^\alpha{}_\alpha = -\frac{6\ddot{\Omega}}{\Omega^3}. \tag{8.3}$$

For the radial modes of a conformally coupled massless scalar field we seek solutions to

$$\frac{d^2x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = -\frac{1}{2(-R^\alpha{}_\alpha)}\left[g^{\lambda\mu} + \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right]\frac{\partial(-R^\alpha{}_\alpha)}{\partial x^\mu}, \qquad \frac{dx_\mu}{dq}\frac{dx^\mu}{dq} = -1, \tag{8.4}$$

for $\lambda = r$, $\lambda = p$. For these components we obtain

$$\begin{aligned}
\frac{d^2r}{dq^2} + 2\frac{\dot{\Omega}}{\Omega}\frac{dr}{dq}\frac{dp}{dq} &= -\frac{1}{2(-R^{\alpha}{}_{\alpha})}\frac{dr}{dq}\frac{dp}{dq}\frac{\partial(-R^{\alpha}{}_{\alpha})}{\partial p}, \\
\frac{d^2p}{dq^2} + \frac{\dot{\Omega}}{\Omega}\frac{dp}{dq}\frac{dp}{dq} + \frac{\dot{\Omega}}{\Omega}\frac{dr}{dq}\frac{dr}{dq} &= -\frac{1}{2(-R^{\alpha}{}_{\alpha})}\left[-\frac{1}{\Omega^2} + \frac{dp}{dq}\frac{dp}{dq}\right]\frac{\partial(-R^{\alpha}{}_{\alpha})}{\partial p}.
\end{aligned} \tag{8.5}$$

With $(dp/dq)d/dp = d/dq$ the equation for dr/dq yields

$$\left(\frac{dr}{dq}\right)^2 = \frac{c_1}{(-R^{\alpha}{}_{\alpha}\Omega^4)}, \tag{8.6}$$

where c_1 is an integration constant, while the equation for dp/dq yields

$$\left(\frac{dp}{dq}\right)^2 = \frac{c_1}{(-R^{\alpha}{}_{\alpha}\Omega^4)} + \frac{1}{\Omega^2} + \frac{c_2}{(-R^{\alpha}{}_{\alpha}\Omega^2)}, \tag{8.7}$$

where c_2 is a second integration constant. In addition we need to satisfy the second equation in (8.4), viz.

$$-\Omega^2\left(\frac{dp}{dq}\right)^2 + \Omega^2\left(\frac{dr}{dq}\right)^2 = -1, \tag{8.8}$$

to thus fix $c_2 = 0$. Hence dp/dq reduces to

$$\left(\frac{dp}{dq}\right)^2 = \frac{c_1}{(-R^{\alpha}{}_{\alpha}\Omega^4)} + \frac{1}{\Omega^2}. \tag{8.9}$$

Finally, from (8.6) and (8.9) we can then determine dr/dp, and find it to be of the form

$$\frac{dr}{dp} = \pm\left(\frac{c_1}{c_1 - \Omega^2 R^{\alpha}{}_{\alpha}}\right)^{1/2}. \tag{8.10}$$

As expected, this signal would be on the light cone and obey $dr/dp = \pm 1$ if were to set $R^{\alpha}{}_{\alpha}$ to zero. For the Maxwell field we find identical results with $R^{\alpha}{}_{\alpha}/6$ replaced by $R^{\alpha\beta}\epsilon_{\alpha}\epsilon_{\beta}$.

For comparison, we note that had we simply looked at the massless geodesic given in (1.6), we would instead have had to satisfy

$$\frac{d^2r}{dq^2} + 2\frac{\dot{\Omega}}{\Omega}\frac{dr}{dq}\frac{dp}{dq} = 0, \qquad \frac{d^2p}{dq^2} + \frac{\dot{\Omega}}{\Omega}\frac{dp}{dq}\frac{dp}{dq} + \frac{\dot{\Omega}}{\Omega}\frac{dr}{dq}\frac{dr}{dq} = 0, \tag{8.11}$$

and on integrating would have obtained

$$\left(\frac{dr}{dq}\right)^2 = \frac{d_1}{\Omega^4}, \quad \left(\frac{dp}{dq}\right)^2 = \frac{d_1}{\Omega^4} + \frac{d_2}{\Omega^2}, \tag{8.12}$$

where d_1 and d_2 are integration constants. On requiring that the solutions to massless geodesics be on the light cone yields

$$-\Omega^2\left(\frac{dp}{dq}\right)^2 + \Omega^2\left(\frac{dr}{dq}\right)^2 = 0, \tag{8.13}$$

to thus lead to $d_2 = 0$, and thereby give

$$\left(\frac{dr}{dq}\right)^2 = \frac{d_1}{\Omega^4}, \quad \left(\frac{dp}{dq}\right)^2 = \frac{d_1}{\Omega^4}. \tag{8.14}$$

The radial trajectories are thus of the form

$$\left(\frac{dr}{dp}\right)^2 = 1, \tag{8.15}$$

just as we would have directly obtained if we had set $ds^2 = 0$ in (8.2). As we see, radial trajectories for conformally coupled scalar fields differ quite substantially from standard radial geodesics. And this continues to be the case for the Maxwell field as well, since like the conformal scalar field the Maxwell field is also conformally coupled.

For nonzero spatial curvature we set $r = \sinh\chi$ when $k = -1$ and $r = \sin\chi$ when $k = +1$. This then leads to conformal to flat metrics of the form

$$ds^2(k=-1) = \Omega^2(p)[dp^2 - d\chi^2 - \sinh^2\chi d\theta^2 - \sinh^2\chi \sin^2\theta d\phi^2],$$
$$ds^2(k=+1) = \Omega^2(p)[dp^2 - d\chi^2 - \sin^2\chi d\theta^2 - \sin^2\chi \sin^2\theta d\phi^2]. \tag{8.16}$$

The discussion in the (p,χ) sector just duplicates the discussion in the (p,r) sector. And with

$$\Gamma^0_{00} = \Gamma^0_{\chi\chi} = \Gamma^\chi_{0\chi} = \frac{\dot{\Omega}}{\Omega}, \qquad \Gamma^0_{0\chi} = \Gamma^\chi_{00} = \Gamma^\chi_{\chi\chi} = 0, \qquad R^\alpha{}_\alpha = -\frac{6\ddot{\Omega}}{\Omega^3} - \frac{6k}{\Omega^2}, \tag{8.17}$$

we obtain radial trajectories in the conformally coupled case of the form

$$\left(\frac{d\chi}{dq}\right)^2 = \frac{c_2}{(-R^\alpha{}_\alpha\Omega^4)}, \qquad \left(\frac{dp}{dq}\right)^2 = \frac{c_2}{(-R^\alpha{}_\alpha\Omega^4)} + \frac{1}{\Omega^2}, \qquad \frac{d\chi}{dp} = \pm\left(\frac{c_2}{c_2 - \Omega^2 R^\alpha{}_\alpha}\right)^{1/2}, \tag{8.18}$$

where c_2 is an integration constant. These $k \neq 0$ radial trajectories differ from their $k = 0$ counterparts only in the form required of $R^\alpha{}_\alpha$.

Since for a de Sitter geometry $R^\alpha{}_\alpha$ is a constant and $\nabla^\lambda R^\alpha{}_\alpha$ is zero we solve de Sitter in and of itself. For a de Sitter geometry the conformally coupled (8.4) takes the form

$$\frac{d^2r}{dq^2} + 2\frac{\dot{\Omega}}{\Omega}\frac{dr}{dq}\frac{dp}{dq} = 0, \qquad \frac{d^2p}{dq^2} + \frac{\dot{\Omega}}{\Omega}\frac{dp}{dq}\frac{dp}{dq} + \frac{\dot{\Omega}}{\Omega}\frac{dr}{dq}\frac{dr}{dq} = 0, \tag{8.19}$$

with solution (for r or χ) of the form

$$\left(\frac{dr}{dq}\right)^2 = \frac{c_3}{\Omega^4}, \qquad \left(\frac{dp}{dq}\right)^2 = \frac{c_3}{\Omega^4} + \frac{c_4}{\Omega^2}, \tag{8.20}$$

where c_3 and c_4 are integration constants. If we now impose

$$dx_\mu dx^\mu = -\Omega^2\left(\frac{dp}{dq}\right)^2 + \Omega^2\left(\frac{dr}{dq}\right)^2 = -1, \tag{8.21}$$

we obtain $c_4 = 1$, to thus give

$$\left(\frac{dr}{dq}\right)^2 = \frac{c_3}{\Omega^4}, \qquad \left(\frac{dp}{dq}\right)^2 = \frac{c_3}{\Omega^4} + \frac{1}{\Omega^2}, \qquad \frac{dr}{dp} = \pm\left(\frac{c_3}{c_3 + \Omega^2}\right)^{1/2}. \tag{8.22}$$

Comparing with (8.10) we identify $c_1 = -c_3 R^\alpha{}_\alpha$. Also we note that if we were to impose $(dx_\mu/dq)(dx^\mu/dq) = 0$ we would obtain $c_4 = 0$ and be back to $dr/dp = \pm 1$.

With all of these results equally applying in the Maxwell case, and recalling that since in cosmological perturbation theory photons are taken to travel on the perturbed light cone, we conclude this paper by briefly indicating what our analysis here might entail for fluctuations in the cosmological microwave background should this not be the case.

IX. COSMOLOGICAL FLUCTUATIONS

For fluctuations around a Robertson-Walker background it is very convenient to use the scalar, vector, tensor decomposition of the fluctuations that was introduced in [44, 45]. Some recent reviews may be found in [46–51] and [52–56]. With such a decomposition the metric is written as

$$ds^2 = -(g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = \Omega^2(p)\left[dp^2 - \frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2\right]$$
$$+ \Omega^2(p)\left[2\phi dp^2 - 2(\tilde{\nabla}_i B + B_i)dp dx^i - [-2\psi\tilde{\gamma}_{ij} + 2\tilde{\nabla}_i\tilde{\nabla}_j E + \tilde{\nabla}_i E_j + \tilde{\nabla}_j E_i + 2E_{ij}]dx^i dx^j\right]. \tag{9.1}$$

In (9.1) $\tilde{\nabla}_i=\partial/\partial x^i$ and $\tilde{\nabla}^i=\tilde{\gamma}^{ij}\tilde{\nabla}_j$ (with Latin indices) are defined with respect to the background three-space metric $\tilde{\gamma}_{ij}$, and $(1,2,3)=(r,\theta,\phi)$. And with

$$\tilde{\gamma}^{ij}\tilde{\nabla}_jV_i=\tilde{\gamma}^{ij}[\partial_jV_i-\tilde{\Gamma}^k_{ij}V_k] \tag{9.2}$$

for any three-vector V_i in a three-space with three-space connection $\tilde{\Gamma}^k_{ij}$, the elements of (9.1) are required to obey

$$\tilde{\gamma}^{ij}\tilde{\nabla}_jB_i=0,\quad \tilde{\gamma}^{ij}\tilde{\nabla}_jE_i=0,\quad E_{ij}=E_{ji},\quad \tilde{\gamma}^{jk}\tilde{\nabla}_kE_{ij}=0,\quad \tilde{\gamma}^{ij}E_{ij}=0. \tag{9.3}$$

With the three-space sector of the background geometry being maximally three-symmetric, it is described by a Riemann tensor of the form

$$\tilde{R}_{ijk\ell}=k[\tilde{\gamma}_{jk}\tilde{\gamma}_{i\ell}-\tilde{\gamma}_{ik}\tilde{\gamma}_{j\ell}]. \tag{9.4}$$

To get a sense of what is possible, we could perturb either (8.4) or (8.10). However, for illustrative purposes we shall only consider a perturbation to the dr/dp solution to (8.4) as given in (8.10) (or analogously $d\chi/dp$ as given in (8.18)). (For $k=-1$ we have $d\chi/dp=(1+r^2)^{-1/2}dr/dp$, and for $k=1$ we have $d\chi/dp=(1-r^2)^{-1/2}dr/dp$.) Perturbing (the square of) (8.10) we obtain

$$\delta\left(\frac{dr}{dp}\right)=\frac{\Omega^2}{2c_1}\left(\frac{dr}{dp}\right)^3\delta R^{\alpha}{}_{\alpha}. \tag{9.5}$$

With $R^{\alpha}{}_{\alpha}=g^{\alpha\beta}R_{\alpha\beta}$, the fluctuation in the Ricci scalar is given by

$$\delta R^{\alpha}{}_{\alpha}=-h^{\alpha\beta}R_{\alpha\beta}+g^{\alpha\beta}\delta R_{\alpha\beta}. \tag{9.6}$$

With the Einstein tensor being of the form $G_{\mu\nu}=R_{\mu\nu}-(1/2)g_{\mu\nu}R^{\alpha}{}_{\alpha}$, we can rewrite (9.6) in the form

$$\delta R^{\alpha}{}_{\alpha}=h^{\alpha\beta}G_{\alpha\beta}-g^{\alpha\beta}\delta G_{\alpha\beta}. \tag{9.7}$$

This form is more convenient since for the metric given in (9.1) $G_{\alpha\beta}$ and $\delta G_{\alpha\beta}$ have been tabulated in many studies. Here we follow the presentation given in [57–60], where it was shown that

$$G_{00}=-3k-3\dot{\Omega}^2\Omega^{-2},\quad G_{0i}=0,\quad G_{ij}=\tilde{\gamma}_{ij}(k-\dot{\Omega}^2\Omega^{-2}+2\ddot{\Omega}\Omega^{-1}),\quad R^{\alpha}{}_{\alpha}=-6\Omega^{-2}k-6\ddot{\Omega}\Omega^{-3}, \tag{9.8}$$

$$\begin{aligned}
\delta G_{00} &= -6k\phi-6k\psi+6\dot{\psi}\dot{\Omega}\Omega^{-1}+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^aB-2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}-2\tilde{\nabla}_a\tilde{\nabla}^a\psi,\\
\delta G_{0i} &= 3k\tilde{\nabla}_iB-\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_iB+2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_iB-2k\tilde{\nabla}_i\dot{E}-2\tilde{\nabla}_i\dot{\psi}-2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\phi+2kB_i-k\dot{E}_i\\
&\quad -B_i\dot{\Omega}^2\Omega^{-2}+2B_i\ddot{\Omega}\Omega^{-1}+\frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^aB_i-\frac{1}{2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}_i,\\
\delta G_{ij} &= -2\ddot{\psi}\tilde{\gamma}_{ij}+2\dot{\Omega}^2\tilde{\gamma}_{ij}\phi\Omega^{-2}+2\dot{\Omega}^2\tilde{\gamma}_{ij}\psi\Omega^{-2}-2\dot{\phi}\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1}-4\dot{\psi}\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1}-4\ddot{\Omega}\tilde{\gamma}_{ij}\phi\Omega^{-1}\\
&\quad -4\ddot{\Omega}\tilde{\gamma}_{ij}\psi\Omega^{-1}-2\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^aB-\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B}+\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E}+2\dot{\Omega}\tilde{\gamma}_{ij}\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}\\
&\quad -\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\phi+\tilde{\gamma}_{ij}\tilde{\nabla}_a\tilde{\nabla}^a\psi+2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_iB+\tilde{\nabla}_j\tilde{\nabla}_i\dot{B}-\tilde{\nabla}_j\tilde{\nabla}_i\ddot{E}-2\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_i\dot{E}\\
&\quad +2k\tilde{\nabla}_j\tilde{\nabla}_iE-2\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_j\tilde{\nabla}_iE+4\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_j\tilde{\nabla}_iE+\tilde{\nabla}_j\tilde{\nabla}_i\phi-\tilde{\nabla}_j\tilde{\nabla}_i\psi+\dot{\Omega}\Omega^{-1}\tilde{\nabla}_iB_j+\frac{1}{2}\tilde{\nabla}_i\dot{B}_j\\
&\quad -\frac{1}{2}\tilde{\nabla}_i\ddot{E}_j-\dot{\Omega}\Omega^{-1}\tilde{\nabla}_i\dot{E}_j+k\tilde{\nabla}_iE_j-\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_iE_j+2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_iE_j+\dot{\Omega}\Omega^{-1}\tilde{\nabla}_jB_i+\frac{1}{2}\tilde{\nabla}_j\dot{B}_i\\
&\quad -\frac{1}{2}\tilde{\nabla}_j\ddot{E}_i-\dot{\Omega}\Omega^{-1}\tilde{\nabla}_j\dot{E}_i+k\tilde{\nabla}_jE_i-\dot{\Omega}^2\Omega^{-2}\tilde{\nabla}_jE_i+2\ddot{\Omega}\Omega^{-1}\tilde{\nabla}_jE_i-\ddot{E}_{ij}-2\dot{\Omega}^2E_{ij}\Omega^{-2}\\
&\quad -2\dot{E}_{ij}\dot{\Omega}\Omega^{-1}+4\ddot{\Omega}E_{ij}\Omega^{-1}+\tilde{\nabla}_a\tilde{\nabla}^aE_{ij},\\
g^{\mu\nu}\delta G_{\mu\nu} &= 6\dot{\Omega}^2\phi\Omega^{-4}+6\dot{\Omega}^2\psi\Omega^{-4}-6\dot{\phi}\dot{\Omega}\Omega^{-3}-18\dot{\psi}\dot{\Omega}\Omega^{-3}-12\ddot{\Omega}\phi\Omega^{-3}-12\ddot{\Omega}\psi\Omega^{-3}-6\ddot{\psi}\Omega^{-2}+6k\phi\Omega^{-2}\\
&\quad +6k\psi\Omega^{-2}-6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^aB-2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\dot{B}+2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\ddot{E}+6\dot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^a\dot{E}\\
&\quad -2\dot{\Omega}^2\Omega^{-4}\tilde{\nabla}_a\tilde{\nabla}^aE+4\ddot{\Omega}\Omega^{-3}\tilde{\nabla}_a\tilde{\nabla}^aE+2k\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^aE-2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\phi+4\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a\psi.
\end{aligned} \tag{9.9}$$

In [57–60] it was shown that for any Robertson-Walker background with any k or $\Omega(p)$ the particular combinations

$$\alpha=\phi+\psi+\dot{B}-\ddot{E},\quad \gamma=-\dot{\Omega}^{-1}\Omega\psi+B-\dot{E},\quad B_i-\dot{E}_i,\quad E_{ij} \tag{9.10}$$

of the fluctuations are gauge invariant. Interestingly, these combinations are independent of k even if k is nonzero.

Given (9.10), we thus look to reexpress (9.7) in terms of gauge invariant combinations, and following some algebra, we obtain

$$\begin{aligned} g^{\alpha\beta}\delta G_{\alpha\beta} = & - 12\ddot{\Omega}\Omega^{-3}(\alpha-\dot{\gamma}) - 6\dot{\Omega}\Omega^{-3}(\dot{\alpha}-\ddot{\gamma}) - 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a(\alpha+3\dot{\Omega}\Omega^{-1}\gamma) + (2k\Omega^{-2}+4\ddot{\Omega}\Omega^{-3}-2\dot{\Omega}^2\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^a E \\ & + 6\dot{\Omega}^2\Omega^{-4}\phi + 6\dot{\Omega}^2\Omega^{-4}\psi + 6\ddot{\Omega}\Omega^{-3}\psi + 6k\Omega^{-2}\phi + 6k\Omega^{-2}\psi - 6\dddot{\Omega}\Omega^{-2}\dot{\Omega}^{-1}\psi, \end{aligned} \tag{9.11}$$

$$\begin{aligned} -h^{\alpha\beta}G_{\alpha\beta} = & - (2k\Omega^{-2}+4\ddot{\Omega}\Omega^{-3}-2\dot{\Omega}^2\Omega^{-4})\tilde{\nabla}_a\tilde{\nabla}^a E \\ & - 6k\Omega^{-2}\phi - 6\dot{\Omega}^2\Omega^{-4}\phi + 6k\Omega^{-2}\psi - 6\dot{\Omega}^2\Omega^{-4}\psi + 12\ddot{\Omega}\Omega^{-3}\psi. \end{aligned} \tag{9.12}$$

Combining these relations we obtain

$$\delta R^{\alpha}{}_{\alpha} = 12\ddot{\Omega}\Omega^{-3}(\alpha-\dot{\gamma}) + 6\dot{\Omega}\Omega^{-3}(\dot{\alpha}-\ddot{\gamma}) + 2\Omega^{-2}\tilde{\nabla}_a\tilde{\nabla}^a(\alpha+3\dot{\Omega}\Omega^{-1}\gamma) + 6\Omega^{-3}\dot{\Omega}^{-1}\dot{A}\psi, \tag{9.13}$$

where

$$A = \Omega\ddot{\Omega} - 2\dot{\Omega}^2 - k\Omega^2, \quad \dot{A} = \Omega\dddot{\Omega} - 3\dot{\Omega}\ddot{\Omega} - 2k\Omega\dot{\Omega}. \tag{9.14}$$

We should thus use (9.13) in (9.5). Consequently we identify the gauge invariant quantity not as $\delta(dr/dp)$ itself but as

$$\bar{\delta}\left(\frac{dr}{dp}\right) = \delta\left(\frac{dr}{dp}\right) - \frac{3}{c_1\Omega\dot{\Omega}}\left(\frac{dr}{dp}\right)^3\left(\frac{dA}{dp}\right)\psi \tag{9.15}$$

instead [61]. Since $\delta R^{\alpha}{}_{\alpha}$ is associated with fluctuations in the spin zero scalar field sector, only the scalar fluctuations given in (9.1) appear in (9.13). The other gauge invariant combinations given in (9.10) can appear in fluctuations involving vector modes or even those involving the propagation of gravitational modes themselves. For spin one and spin two modes we would replace $\delta R^{\alpha}{}_{\alpha}$ by $\delta R^{\alpha\beta}\epsilon_\alpha\epsilon_\beta$ and $\delta R^{\alpha\beta}\epsilon_{\alpha\beta}$, where ϵ_α and $\epsilon_{\alpha\beta}$ are polarization vectors and tensors. These polarization vectors and tensors would project onto the various components of $\delta G_{\mu\nu}$ as listed in (9.9) and not just onto its trace.

While in general we would have to deal with the function $A(p)$ in any application of (9.5), it was noted in [58, 60] that if the background perfect fluid is just a cosmological constant term (such as in the inflationary universe), then for standard or conformal gravity fluctuations $A(p)$ just happens to vanish identically. For $k=0$ this entails that $A(p)=1/p$, for $k=-1$ we find that $\Omega(p)=1/\sinh p$, while for $k=+1$ we find that $\Omega(p)=1/\sin p$.

As noted in [54], the standard use of the Boltzmann equation in cosmological fluctuation theory can be recast as a study of the propagation of light on a perturbed Robertson-Walker light cone. Since we are arguing in this paper that gravity actually takes massless particles off the light cone we need to ask what then happens to the Boltzmann equation. In equilibrium the distribution function $f(q,x^\alpha,U^\beta)$ schematically obeys

$$\left[\frac{\partial}{\partial q} + U^\lambda\frac{\partial}{\partial x^\lambda} + \frac{\partial U^\lambda}{\partial q}\frac{\partial}{\partial U^\lambda}\right] f(q,x^\alpha,U^\beta) = 0, \tag{9.16}$$

where $U^\lambda = dx^\lambda/dq$. For particles that obey the standard massless particle geodesic equation (1.6) we can rewrite (9.16) as

$$\left[\frac{\partial}{\partial q} + U^\lambda\frac{\partial}{\partial x^\lambda} - \Gamma^\lambda_{\mu\nu}U^\mu U^\nu\frac{\partial}{\partial U^\lambda}\right] f(q,x^\alpha,U^\beta) = 0. \tag{9.17}$$

However, given (8.4) we must replace (9.17) by

$$\left[\frac{\partial}{\partial q} + U^\lambda\frac{\partial}{\partial x^\lambda} - \Gamma^\lambda_{\mu\nu}U^\mu U^\nu\frac{\partial}{\partial U^\lambda}\right] f(q,x^\alpha,U^\beta) = \frac{1}{2(-R^\alpha{}_\alpha)}\left[g^{\lambda\mu} + \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right]\frac{\partial(-R^\alpha{}_\alpha)}{\partial x^\mu}\frac{\partial f(q,x^\alpha,U^\beta)}{\partial U^\lambda}. \tag{9.18}$$

In the language of transport theory the right-hand side of (9.18) may be thought of as being a gravitationally-induced viscosity.

X. FINAL COMMENTS

In this paper we have shown that departures from both test particle geodesics and the massless particle light cone are generally to be expected in an eikonalization of field theoretic wave equations. In particular, we have shown that such departures are to be expected in some of the most prominent geometries that are used in astrophysics and cosmology, namely Schwarzschild de Sitter, pure de Sitter, and Robertson-Walker geometries. As such, our results are points of principle, and their phenomenological relevance will depend on how big the curvature contributions that we have identified might actually be in specific cases. In this paper we have integrated some specific modified geodesic equations because they were simple enough for us to be able to do so. However, this may not be so straightforward to do in more complicated cases. In those cases an alternate procedure is possible, one that bypasses the modified geodesics altogether. Specifically, we first solve the wave equation directly in order to determine the eikonal function T in the large T limit. And only then do we set $dx^\mu/dq = \nabla^\mu T/(-\nabla_\nu T\nabla^\nu T)^{1/2}$ in order to get the trajectories.

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[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[2] Under the transformation $x'^\lambda = x^\lambda + \frac{1}{2}x^\mu x^\nu(\Gamma^\lambda_{\mu\nu})_P$, the primed coordinate $(\Gamma'^\lambda_{\mu\nu})_P$ will vanish at the point P, regardless in fact of how large the Riemann tensor in the neighborhood of the point P might actually be.

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[7] P. D. Mannheim, J. G. O'Brien and D. E. Cox, Gen. Rel. Gravit. **42**, 2561 (2010).

[8] For T possessing a term of the form k_0x^0, $\nabla_\mu T\nabla^\mu T$ behaves like k_0^2 for large k_0 while $\nabla_\mu\nabla^\mu T$ is of only order k_0.

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[10] As is common in field theory this section on light-front systematics is written so that in (x^0, x^1, x^2, x^3) Minkowski coordinates the metric has signature $[1, -1, -1, -1]$. The transformation to light-front coordinates then yields the metric given in (2.8). In the rest of the paper we follow [1] and use metric signature $[-1, 1, 1, 1]$.

[11] Choosing to take the integrand to be $(-R^\alpha{}_\alpha)^{1/2}$ in the action $I_T = -\kappa\int ds(-R^\alpha{}_\alpha)^{1/2}$ renders the constant κ dimensionless.

[12] While beyond the scope of the present paper, this result does raise the question of whether any eikonalized trajectory that can be obtained form a wave equation could also correspond to one obtained from some appropriate test particle action.

[13] This point had not been noted in [3].

[14] If in (3.2) we set $T = \int^x k_\mu dx^\mu$ we would obtain $k_\mu k^\mu - i\nabla_\mu k^\mu - R^\alpha{}_\alpha/6 = 0$. Thus unless $-i\nabla_\mu k^\mu - R^\alpha{}_\alpha/6 = 0$, $k_\mu k^\mu$ could not be zero. Heuristically, we could regard $k_\mu k^\mu - i\nabla_\mu k^\mu - R^\alpha{}_\alpha/6 = 0$ as being the "light cone" in this case as conformal modes would satisfy this condition.

[15] The curved space massless fermion Dirac action $I_D = -\int d^4x(-g)^{1/2}i\bar{\psi}\gamma^c V^\mu_c(x)[\partial_\mu + \Gamma_\mu(x)]\psi$ is locally conformal invariant under $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$, $V^a_\mu(x) \to \Omega(x)V^a_\mu(x)$, $\psi(x) \to \Omega^{-3/2}(x)\psi(x)$, where $V^\mu_c(x)$ is a vierbein and $\Gamma_\mu = -(1/8)[\gamma_a, \gamma_b](V^b_\nu\partial_\mu V^{a\nu} + V^b_\lambda\Gamma^\lambda{}_{\nu\mu}V^{a\nu})$ is the spin connection. On "squaring" the Dirac equation $i\gamma^c V^\mu_c(x)[\partial_\mu + \Gamma_\mu(x)]\psi(x) = 0$ we obtain [3] $[\partial_\mu + \Gamma_\mu][\partial^\mu + \Gamma^\mu]\psi(x) + \frac{1}{4}R^\alpha{}_\alpha\psi(x) = 0$, to thus generate a dependence on the Ricci scalar similar to that found in the scalar field case. On defining $D_\mu = \partial_\mu + \Gamma_\mu$ and setting $\psi = \psi_0 e^{iT}$ where ψ_0 is a Dirac spinor, eikonalization for fermions is of the form $dx_\mu/dq = \nabla_\mu T$. In passing we note that both the scalar and the fermion wave equation operators can generically be written in the form $\nabla_\mu\nabla^\mu + (d/6)R^\alpha{}_\alpha$, $D_\mu D^\mu + (d/6)R^\alpha{}_\alpha$, where d is the dimension of the field ($d = 1$ for scalars and $d = 3/2$ for fermions).

[16] For fermions we would take the action to be of the form $I_D = -\int d^4x(-g)^{1/2}\bar{\psi}[i\gamma^c V^\mu_c(x)[\partial_\mu + \Gamma_\mu(x)] + Q]\psi$, and on setting $Q = Q_0 = m$ would obtain $[\partial_\mu + \Gamma_\mu][\partial^\mu + \Gamma^\mu]\psi(x) + [\frac{1}{4}R^\alpha{}_\alpha - m^2]\psi(x) = 0$. On setting $D_\mu = \partial_\mu + \Gamma_\mu$ as per [15], eikonalization in a form analogous to (3.20) would follow if we set $dx^\mu/dq = D^\mu T/(m^2 - R^\alpha{}_\alpha/4)^{1/2}$. While this would in principle actually lead to a departure from the standard massive particle geodesics given in (1.2), in practice this would not be the case since we had noted in Sec. III that m^2 is typically much greater than curvature scales such as $R^\alpha{}_\alpha$.

[17] To appreciate the distinction, we note that each of the $\int d^4x(-g)^{1/2}R_{\mu\nu}R^{\mu\nu}$ and $\int d^4x(-g)^{1/2}(R^\alpha{}_\alpha)^2$ actions is separately globally conformal invariant under $g_{\mu\nu} \to e^{2\alpha}g_{\mu\nu}$ when α is as constant, but only the particular linear combination $\int d^4x(-g)^{1/2}[R_{\mu\nu}R^{\mu\nu} - (1/3)(R^\alpha{}_\alpha)^2]$ is invariant when α is an arbitrary function of the spacetime coordinates.

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[19] Even though $g_{\mu\nu}(x)$ does transform non-trivially under a local conformal transformation, and even though $A_\mu(x)$ has dimension one, under a local conformal transformation $A_\mu(x)$ is left invariant. This behavior is complementary to the behavior found in a gauge theory, since there it is $A_\mu(x)$ that transforms non-trivially under a local gauge transformation (viz. $A_\mu(x) \to A_\mu(x) - \nabla_\mu\chi$) while it is $g_{\mu\nu}(x)$ that is left invariant.
[20] Obtaining contributions inside the light cone is actually quite familiar in conformal theories. In the fourth-order derivative conformal gravity theory for instance the perturbative Green's function associated with fluctuations around flat space obeys the fourth-order derivative $(-\partial_t^2+\nabla^2)^2G(x)=\delta^4(x)$ and is of the form $G(x)=\theta(t-r)/8\pi$. It thus takes support both on and inside the $t^2-r^2=0$ light cone. Being inside the light cone is also familiar from flat space optics since $c/n<c$.
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[42] In a typical 10^{11} solar mass galaxy the square of the inverse of the Schwarzschild radius is $10^{-32}\ cm^{-2}$. In the kiloparsec distance range of relevance to galactic rotation curves the γ_0/r term in (7.16) is of order the altogether smaller $10^{-51}\ cm^{-2}$.
[43] These asymptotic considerations are not of concern for the gravitational bending of light by the Sun, since what is relevant is not a single measurement but a difference between two, one when the Sun is along a line of sight to the target stars and the other half a year later when it is not. In this difference the background contribution drops out. Moreover, since only contributions at radii close to the Sun remain, its $\gamma^* r$ contribution is not significant. However, on the much larger distances associated with the lensing of quasar light by an intervening cluster of galaxies one can only make one measurement, and then the asymptotic behavior of the background and the linear potentials of the galaxies in the cluster cannot be ignored.
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[60] A. Amarasinghe, T. Liu, D. A. Norman and P. D. Mannheim, Phys. Rev. D **103**, 104022 (2021).
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