Spin optics for gravitational waves

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We present geometric optics expansion in the subleading order for the circularly polarized gravitational waves on curved spacetimes. We call "spin optics" to the subleading order geometric optics expansion, which involves modification of the standard eikonal function by including into it a specially chosen helicity dependent correction. We show that, the techniques developed for the propagation of electromagnetic waves could be applied to gravitational waves (difference in their helicity should be accounted), in the limit of spin optics also.

I. INTRODUCTION

Geometric optics approximation can be applied to study high frequency gravitational waves. The result is, in the infinitely large frequency limit, gravitational waves in the fixed curved background behaves as a null ray trajectory, just like electromagnetic waves [6, 8, 9]. Geometric optics expansion reduces the problem of solving linearized Einstein field equations to solving the ray equations and transport equations along these rays. In the limit of geometric optics, the backreaction onto the ray equations from the polarization vector is absent. Because of this, laws of geometric optics are no longer valid at large but finite frequencies.

Backreaction from helicity might cause ray trajectories to deviate from geodesics by an appreciable amount. This effect, in which, the propagation of gravitational waves in the subleading order in curved spacetime (orbital motion) depends on the helicity (spin) is also called gravitational spin Hall effect [2, 10, 35, 36]. Thus spin-orbit interaction modify the gravitational waves propagation from the original ray trajectory of the geometric optics. We here present the covariant formulation of the Wentzel-Kramers-Brillouin (WKB) analysis for the polarized gravitational waves in the subleading order geometric optics expansion in wavelength. WKB analysis, up to the subleading order geometric optics, is called as "spin optics" approximation. The procedure here will be similar to the one developed for electromagnetic waves [1, 10].

Spin Hall effect is the result of the interaction of the polarization/ spin with the orbital motion of the rays [11–13]. The effective ray equations describing this effect for gravitational waves are very similar to those describing the gravitational spin Hall effect of light. Spin Hall effect of light (also called optical Magnus effect [14]) is observed when electromagnetic waves travels in an inhomogeneous medium [15, 16]. The spin-orbital coupling comes from the interaction of the polarization degrees of freedom with the gradient of refractive index of the medium. As a result, transverse deflection of electromagnetic waves occurs in a direction perpendicular to the refractive index gradient. Spin Hall effect of light can be explained in terms of Berry curvature [3, 4] and provides correction to the geometric optics, which scales approximately as the inverse of the frequency in the subleading order. This phenomena of spin Hall effect from condensed matter physics could be imported to the general relativity, where spacetime curvature itself plays the role of an inhomogeneous medium [1, 2, 5, 17–20]. Thus, gravitational spin Hall effect is from the interaction of the polarization with the spacetime curvature itself resulting in spin dependent correction on the particle dynamics.

Higher order geometric optics expansion is not the only approach for calculating spin Hall effect for gravitational waves. The effect of spin on the trajectory of massive particles have been worked out by [26], [27] and [28]. The Mathisson-Papapetrou-Dixon equations have been adapted to the massless case by the work of [25] and [24], and their equations can be used to study the polarization dependent correction on the trajectory of massless particles (see [5] and the reference therein for details) The necessity in calculating the higher order geometric optics correction for gravitational waves arises because of the following reasons. Subleading order correction from the geometric optics for gravitational wave might be necessary while considering the phenomena of gravitational lensing [21, 22]. Spin-orbit coupling plays a role when analyzing the propagation of spinning particles in an inhomogeneous medium when the wavelength of a particle is small but not negligible compared to the inhomogeneity scale of the medium.

This article is organized as follows: In Sec. II, we linearize Einstein field equations whose solution gives the gravitational waves. We also make a couple of initial assumptions on that linearized Einstein equations which will simplify our calculations without losing the generality. Then in Sec. III, we make a WKB approximation to the metric perturbation and substitute it back into linearized Einstein equations to obtain gravitational waves solution in the high frequency limit. Then, in the leading order approximation in $1/\omega$, we reproduce the results of geometric optics. We will show that this gravitational wave solution in the limit of geometric optics can be represented by a set of null tetrad satisfying some orthonormal and completeness conditions. Scalar product of a Fermi transported null tetrad satisfy those conditions everywhere along the null trajectory of gravitational

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waves provided that they are satisfied initially. We also cast the gravitational waves equations in the form of Maxwell equations for electromagnetic waves so that we can describe the circular polarization of the gravitational waves in terms of the self dual and the anti self dual solutions of some bivector/field tensor. After laying the foundations for the generalization of geometric optics, we will write the equations for the trajectory and the polarization of gravitational waves up to the subleading order approximation. In Sec. IV, we calculate the effect of spin-orbit coupling to a couple of simple cases: 1) gravitational lensing of gravitational waves in Schwarzschild spacetime and 2) propagation of gravitational waves in the expanding universe. Finally, we present some concluding remarks and discussions in Sec. V.

In this article, we consider a Lorentzian manifold with the metric $g_{\mu\nu}$ of signature $(-,+,+,+)$. Similarly, we denote the covariant derivative by semicolon (;), write the complex conjugate of z as $\tilde{z}$ and adopt Einstein summation convention. We use the system of units with $G=c=1$. λ denotes the parameter of gravitational waves curves and $\dot{x}=dx/d\lambda$. The sign convention used here is adopted by [6].

II. LINEARIZATION OF EINSTEIN FIELD EQUATIONS

We consider the Einstein field equations with vanishing cosmological constant

$$R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}=0, \tag{1}$$

where $R_{\alpha\beta}$ is the Ricci tensor and R denotes the Ricci scalar. Here, we attempt to describe the propagation of gravitational waves treating it as small metric perturbation around the fixed background solution of vacuum Einstein equations.

Let $g_{\alpha\beta}$ be the solution of Einstein equations in vacuum

$$R_{\alpha\beta}=0. \tag{2}$$

We consider another metric $\tilde{g}_{\alpha\beta}$, which is the result of small perturbation $h_{\alpha\beta}$ on $g_{\alpha\beta}$

$$\tilde{g}_{\alpha\beta}=g_{\alpha\beta}+h_{\alpha\beta}. \tag{3}$$

Substituting this into Eq. (2) gives, up to the first order in $h_{\alpha\beta}$ and its derivatives (see [2])

$$\left(\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}\nabla_{\mu}\nabla^{\mu}-g_{\alpha\beta}g^{\gamma\delta}\nabla_{\mu}\nabla^{\mu}+g^{\gamma\delta}\nabla_{\alpha}\nabla_{\beta}+g_{\alpha\beta}\nabla^{\gamma}\nabla^{\delta}-\delta^{\delta}_{\beta}\nabla^{\gamma}\nabla_{\alpha}-\delta^{\delta}_{\alpha}\nabla^{\gamma}\nabla_{\beta}\right)h_{\gamma\delta}=0. \tag{4}$$

Now, we take the trace of this equation to get

$$\nabla^{\alpha}\nabla^{\beta}h_{\alpha\beta}-\nabla^{\alpha}\nabla_{\alpha}h^{\mu}_{\mu}=0. \tag{5}$$

We can further reduce Einstein field equations to hyperbolic system of equations by using available gauge freedom. We here chose the Lorentz gauge condition such that

$$\nabla^{\alpha}h_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\nabla^{\alpha}h^{\mu}_{\mu}=0. \tag{6}$$

Substituting this back into Eq. (5) gives

$$\nabla^{\alpha}\nabla_{\alpha}h^{\mu}_{\mu}=0. \tag{7}$$

Again substituting Eqs. (6) and (7) into Einstein field Eq. (4) gives

$$\nabla^{\mu}\nabla_{\mu}h_{\alpha\beta}+2R_{\alpha\mu\beta\nu}h^{\mu\nu}=0, \tag{8}$$

where $R_{\alpha\mu\beta\nu}$ is the Riemann curvature tensor.

1. Initial conditions

We here assume that the trace of the perturbation tensor $h_{\alpha\beta}$ vanishes initially. Then Eq. (7) ensures that h^{μ}_{μ} vanishes everywhere along the trajectory. Thus, the field equation and the Lorentz condition for gravitational waves reduces to

$$\nabla^{\mu}\nabla_{\mu}h_{\alpha\beta}+2R_{\alpha\mu\beta\nu}h^{\mu\nu}=0, \tag{9}$$

$$\nabla^{\alpha}h_{\alpha\beta}=0, \tag{10}$$

respectively. Note the striking similarity with Maxwell equations and Lorentz condition for electromagnetic radiations, so will be the case for solutions also as we will show below. We also consider circularly polarized gravitational waves for our calculations.

III. FORMULATION OF SPIN OPTICS

Spin optics approximation is valid when the typical wavelength of the waves is very small (but could not be neglected) in comparison to the length scale over which the amplitudes and the wavelength of the waves vary and the radius of curvature of the spacetime on which the waves propagate. In that limit, waves can be approximated locally as a ray propagating on an approximately flat spacetime. Spin optics approximation can be expressed mathematically by using the WKB ansatz

$$h^{\alpha\beta} = a^{\alpha\beta} e^{i\omega\mathcal{S}}, \tag{11}$$

where $a^{\alpha\beta}$ is the slowly varying complex amplitude and $\omega\mathcal{S}$ is the rapidly varying real phase. Here, ω is the is the characteristic frequency of the problem. In later calculations, we write the wave vector $l_\alpha = \mathcal{S}_{;\alpha}$, where semicolon ; denotes the covariant derivative. The wave vector and the polarization tensor can be expanded in terms of ω as

$$l^\alpha = l_0^\alpha + \frac{l_1^\alpha}{\omega} + \frac{l_2^\alpha}{\omega^2} + ..., \tag{12}$$

$$a^{\alpha\beta} = a_0^{\alpha\beta} + \frac{a_1^{\alpha\beta}}{\omega} + \frac{a_2^{\alpha\beta}}{\omega^2} + ..., \tag{13}$$

Below in Sec. III B, we will use the Fermi propagated null tetrad among which two of its components represent the solution of trajectory and polarization of gravitational waves. This Fermi propagation reduces the freedom in the transformation property of the null tetrad $m^\alpha \to e^{i\mathcal{S}_1(\lambda)} m^\alpha$ by the condition

$$\frac{d\mathcal{S}_1(\lambda)}{d\lambda} = 0 \tag{14}$$

This is the reason why we can not absorb higher order phase factors like $\mathcal{S}_1(\lambda)$ into the complex amplitude $a_0^{\alpha\beta}$ by such transformations. We can recover the equations of geometric optics by substituting this perturbation metric onto the source free wave equation (Eq. (9)) and the Lorentz gauge condition. Let us first start from the Lorentz condition, which can be written as

$$l_0^\alpha a_{0\alpha\beta} + \frac{1}{\omega}\left(l_0^\alpha a_{1\alpha\beta} + l_1^\alpha a_{0\alpha\beta} - i a_{0\ \beta;\alpha}^{\alpha}\right) = 0, \tag{15}$$

up to the subleading order terms in ω. We next substitute the perturbation tensor onto the source free wave equation, which again up to the subleading order terms in ω gives

$$t^{\alpha\beta} =: a_0^{\alpha\beta} l_{0\mu} l_0^\mu + \frac{1}{\omega}\left(a_1^{\alpha\beta} l_{0\mu} l_0^\mu + 2a_0^{\alpha\beta} l_{1\mu} l_0^\mu - i\left(a_0^{\alpha\beta} l_{0;\mu}^\mu + 2a_0^{\alpha\beta}{}_{;\mu} l_0^\mu\right)\right) = 0. \tag{16}$$

Let us now calculate the identically vanishing quantity $\tilde{a}_{0\alpha\beta} t^{\alpha\beta} + a_{0\alpha\beta}\tilde{t}^{\alpha\beta}$, and this gives the dispersion relation

$$l_{0\beta} l_0^\beta + \frac{2}{\omega}\left(l_{1\beta} - b_\beta\right) l_0^\beta = 0, \tag{17}$$

where we have used $\tilde{a}_0^{\alpha\beta} a_{0\alpha\beta} = a^2$ and substituted

$$\frac{i}{2a^2}\left(\tilde{a}^{\alpha\beta} a_{\alpha\beta;\mu} - a^{\alpha\beta}\tilde{a}_{\alpha\beta;\mu}\right) := b_\mu, \tag{18}$$

in obtaining this equation.

A. Geometric optics limit

From the above equations Eqs. (15)-(17), we can retrieve the results of geometric optics by substituting $a_1^{\alpha\beta} = 0 = l_1^\beta$. In the leading order approximation in ω, the Lorentz condition Eq. (15) and the wave equation Eq. (16) gives

$$l_0^\alpha a_{0\alpha\beta} = 0 = l_0^\alpha l_{0\alpha}. \tag{19}$$

This equation allows one to write $a_{0\alpha\beta} = a_\alpha\left(c_1 l_{0\beta} + c_2 m_{0\beta}\right)$, where c_1 and c_2 are arbitrary constants, a_α is an arbitrary vector and $m_{0\beta}$ is a complex vector satisfying $m_{0\beta}\tilde{m}_0^\beta = 1$ and $m_{0\beta} l_0^\beta = 0$. As the polarization tensor $a_{0\alpha\beta}$ is symmetric in its indices, we should also have $a_\alpha = c_1 l_{0\alpha} + c_2 m_{0\alpha}$. Now, using the property $\tilde{a}_0^{\alpha\beta} a_{0\alpha\beta} = a^2$, we get $c_2^2 = ae^{i\phi}$. We can further reduce the expression for $a_{0\alpha\beta}$ by considering the gauge transformation

$$h'_{\alpha\beta} = h_{\alpha\beta} - \Xi_{\alpha;\beta} - \Xi_{\beta;\alpha}, \tag{20}$$

under which linearized Einstein field equations presented above are invariant. Here,

$$\Xi_\alpha = g_{\alpha\beta}\Xi^\beta = \frac{1}{\omega}\xi_\alpha e^{i\mathcal{S}}, \tag{21}$$

are small arbitrary functions. This gauge transformation preserves the form of WKB ansatz given in Eq. (11) if

$$a'_{\alpha\beta} = a_{\alpha\beta} - \frac{1}{\omega}\left(\xi_{\alpha;\beta} + \xi_{\beta;\alpha}\right) - i\left(\xi_\beta l_\alpha + \xi_\alpha l_\beta\right), \tag{22}$$

satisfies. The change in amplitude in the leading order approximation can be written as

$$a'_{0\alpha\beta} = a_{0\alpha\beta} - i\left(\xi_{0\beta} l_{0\alpha} + \xi_{0\alpha} l_{0\beta}\right). \tag{23}$$

As $l_{0\alpha} l^{0\alpha} = 0$, the Lorentz condition of Eq. (15) holds for arbitrary $\xi_{0\alpha}$ satisfying $\xi_{0\alpha} l_0^\alpha = 0$. This freedom in the choice of $\xi_{0\alpha}$ can be used to choose $c_1 = 0$. Moreover, an arbitrary constant parameter ϕ from c_2^2 could be absorbed in a redefinition of the phase function $\mathcal{S}$. We finally get

$$a_{0\alpha\beta} = c_2^2 m_{0\alpha} m_{0\beta} = a m_{0\alpha} m_{0\beta}. \tag{24}$$

Note that it is equally legitimate to write $a_{0\alpha\beta} = a_\alpha\left(c_2 m_{0\alpha} + c_3 \tilde{m}_{0\alpha}\right)$, $\tilde{m}_{0\alpha}$ being the complex conjugate of $m_{0\alpha}$ and c_3 being another arbitrary constant. However, as in electromagnetism, if the polarization tensor with $m_{0\alpha}$ represents the right hand circularly polarized waves, then the polarization tensor with $\tilde{m}_{0\alpha}$ represents the left hand circularly polarized waves.

We calculate $\tilde{a}_{0\alpha\beta} t^{\alpha\beta}$ from Eq. (16), while considering only the terms relevant to geometric optics,that is, we take $a_1^{\alpha\beta} = 0$ and $l_{1\mu} = 0$, to get

$$l^\mu_{0;\mu} + \frac{2}{a^2}\tilde{a}_{0\alpha\beta} a_0^{\alpha\beta}{}_{;\mu} l_0^\mu = 0. \tag{25}$$

Adding this equation with its complex conjugate gives

$$l^\mu_{0;\mu} + \frac{1}{a^2}\left(\tilde{a}_{0\alpha\beta} a_0^{\alpha\beta}\right)_{;\mu} l_0^\mu = \left(l_0^\mu \mathcal{I}_0\right)_{;\mu} = 0. \tag{26}$$

where

$$\mathcal{I}_0 =: \tilde{a}_{0\alpha\beta} a_0^{\alpha\beta} - \frac{1}{2}\tilde{a}^\mu_{0\mu} a^\mu_{0\mu} = \tilde{a}_{0\alpha\beta} a_0^{\alpha\beta} \tag{27}$$

is the intensity of the wave (note that by our choice of initial conditions in Sec. II 1, $a^\mu_{0\mu} = 0$). These constitutes the complete set of equations for gravitational waves in curved spacetimes in the limit of geometric optics.

We substitute the value of polarization amplitude from Eq. (24) to Eq. (25), which on further simplification gives

$$l^\beta_{0;\beta} + 2\frac{a_{;\beta}}{a} l_0^\beta + 4\tilde{m}_{0\alpha} m^\alpha_{0;\beta} l_0^\beta = 0. \tag{28}$$

Since the term $\tilde{m}_{0\alpha} m^\alpha_{0;\beta} l_0^\beta$ is purely imaginary and the remaining terms $l^\beta_{0;\beta} + 2\frac{a_{;\beta}}{a} l_0^\beta$ are purely real, they should be separately zero, thereby giving

$$l^\beta_{0;\beta} + 2\frac{a_{;\beta}}{a} l_0^\beta = 0, \qquad m^\alpha_{0;\beta} l_0^\beta = 0. \tag{29}$$

These relations will be of use for further calculations.

B. Introduction of null tetrad

From Eqs. (19) and (29), we can see that the wave vector l_0^α and the polarization vector m_0^α satisfying wave equations in geometric optics limit could be identified with two components of a null tetrad. Remaining two components would be a non-unique auxiliary null vector n_0^α and a complex conjugate $\tilde{m}_0^\alpha$. A set of null tetrad $(l_0^\alpha, n_0^\alpha, m_0^\alpha, \tilde{m}_0^\alpha)$ would satisfy the following orthogonality and the completeness relationships

$$l_0^\alpha m_{0\alpha} = l_0^\alpha l_{0\alpha} = l_0^\alpha \tilde{m}_{0\alpha} = 0, \qquad m_0^\alpha \tilde{m}_{0\alpha} = 1, \tag{30}$$

$$m_0^\alpha m_{0\alpha} = \tilde{m}_0^\alpha \tilde{m}_{0\alpha} = 0, \tag{31}$$

$$n_0^\alpha m_{0\alpha} = n_0^\alpha n_{0\alpha} = n_0^\alpha \tilde{m}_{0\alpha} = 0, \qquad n_0^\alpha l_{0\alpha} = -1, \tag{32}$$

where Eqs. (30) follows from the geometric optics (see Eqs. (19)) and Eqs. (31) and (32) are purely by the choice of $n_{0\alpha}$ and $m_{0\alpha}$. Eqs.(31) do indeed follow from our assumption that the trace of polarization tensor $h_{\alpha\beta}$ vanishes initially and in Sec. III C, we will show that this condition determines the polarization state. In particular, these relations hold for the circularly polarized waves. Moreover, these components of null tetrad would also satisfy

$$l_{0;\beta}^\alpha l_0^\beta = 0, \qquad m_{0;\beta}^\alpha l_0^\beta = 0 = \tilde{m}_{0;\beta}^\alpha l_0^\beta, \tag{33}$$

$$n_{0;\beta}^\alpha l_0^\beta = 0, \tag{34}$$

where Eqs. (33) are again from the geometric optics (see Eq. (29)) and we have used $l_{0\alpha;\beta} = l_{0\beta;\alpha}$ in obtaining the first relation. Eq. (34) is by a choice of n_0^α. To show that this choice is indeed possible, we introduce the Fermi derivative operator $\mathcal{D}_l$ along the ray l^α which gives the following relation when applied to the tensor A^α [1]

$$\mathcal{D}_l A^\alpha = l_0^\beta A_{;\beta}^\alpha - w_\beta A^\beta n^\alpha + A^\beta n_\beta w^\alpha, \tag{35}$$

where $w^\alpha = l_0^\beta l_{;\beta}^\alpha$. w^α is an identically vanishing quantity in geometric optics. As $l^\alpha l_\alpha = 0$, we have $\mathcal{D}_l l^\alpha = 0$. A vector A^α is Fermi propagated if its Fermi derivative $\mathcal{D}_l A^\alpha = 0$, and it is easy to see that the scalar product of any two Fermi propagated vectors is constant. From this, we can conclude that, if tetrad $(l^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$ satisfy the orthogonality and the completeness relations similar to the one given in Eqs. (30)-(32) at some point on the ray and if they are Fermi propagated, then they satisfy those relations everywhere on the ray. So, we Fermi propagate the frame of null tetrad $(l^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$, so that, they satisfy the orthogonality and the completeness relations analogous to the one in Eqs. (30)-(32) along the ray and also obey

$$l_0^\beta n_{;\beta}^\alpha = w^\beta n_\beta n^\alpha, \tag{36}$$

$$l_0^\beta m_{;\beta}^\alpha = w^\beta m_\beta n^\alpha, \tag{37}$$

$$l_0^\beta \tilde{m}_{;\beta}^\alpha = w^\beta \tilde{m}_\beta n^\alpha. \tag{38}$$

We can now use a freedom in the choice of null tetrad

$$l^\alpha \to A l^\alpha, n^\alpha \to A^{-1} n^\alpha, \tag{39}$$

to fix $w^\beta n_\beta = 0$, where A is a real function. This condition fixes the parameter λ along the ray up to its possible rescaling $\lambda \to A^{-1}\lambda$, and we call such choice a canonical parametrization [1]. So, in canonical parametrization, we have

$$l_0^\beta n_{;\beta}^\alpha = 0, \qquad l_0^\beta m_{;\beta}^\alpha = w^\beta m_\beta n^\alpha, \qquad l_0^\beta \tilde{m}_{;\beta}^\alpha = w^\beta \tilde{m}_\beta n^\alpha. \tag{40}$$

These developments in the limit of geometric optics are the relations given in Eqs. (30)-(34). Now, as a generalization, we assume that all these relations hold in the limit of spin optics also for the suitably constructed set of null tetrad.

C. Introduction of polarization basis

In geometric optics limit, circularly polarized gravitational waves has amplitude $a_0^{\alpha\beta} = a m_{0\alpha} m_{0\beta}$ whose evolution is ruled by the equation $m_{0;\beta}^\alpha l_0^\beta = 0$. This means, polarization is parallel propagated to the trajectory in geometric optics. Now, in spin optics approximation, we can keep track of the evolution of gravitational waves polarization by defining an antisymmetric field, in analogy to the electromagnetism

$$F_{\beta\gamma} := A_{\gamma;\beta} - A_{\beta;\gamma}, \tag{41}$$

where the vector A_γ is given as

$$A_\gamma := a m_\gamma e^{i\mathcal{S}/2}, \tag{42}$$

Now, geometric optics approximation up to the subleading order in $1/\omega$ is based on the assumption that typical length scale of variation of amplitude, polarization, and wavelength of gravitational waves is negligible compared to the radius of curvature of the spacetimes through which the waves propagate. In this approximation, the curvature terms appearing in the linearized gravitational wave equation Eq. (9) could be safely neglected, as could be seen from Eq. (16). In such regime, the antisymmetric field $F_{\beta\gamma}$ satisfies

$$F_{\alpha\beta;\gamma} + F_{\gamma\alpha;\beta} + F_{\beta\gamma;\alpha} = 0, \tag{43}$$
$$F^{\beta\gamma}{}_{;\gamma} = 0. \tag{44}$$

Let us define the complex version of the tensor $F^{\alpha\beta}$ as

$$\mathcal{F}^{\pm} = F \pm iF^*, \tag{45}$$

where $F^* = \epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}/2$ is the Hodge dual of $F^{\alpha\beta}$. Here, $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor in four dimension and its components in the tetrad basis is $il \wedge n \wedge m \wedge \tilde{m}$. As $(F^*)^* = -F$, we have $(\mathcal{F}^{\pm})^* = \mp i\mathcal{F}^{\pm}$, a property by the virtue of which we call $\mathcal{F}^{\pm}$ self/anti-self dual antisymmetric field for $+/-$ sign. Now, the field $\mathcal{F}^{+}_{\alpha\beta}$ could be expanded by substituting Eq. (42) into Eq. (41)

$$\mathcal{F}^{+}_{\alpha\beta} = \frac{i\omega}{2}\mathcal{Z}_{\alpha\beta}e^{i\mathcal{S}/2}, \tag{46}$$

where

$$\mathcal{Z}_{\alpha\beta} = a\left(l_\beta m_\gamma - l_\gamma m_\beta - \frac{2i}{\omega}\left(\frac{a_{;\beta}}{a}m_\gamma - \frac{a_{;\gamma}}{a}m_\beta + m_{\gamma;\beta} - m_{\beta;\gamma}\right)\right). \tag{47}$$

For this self dual field $\mathcal{F}^{+}_{\alpha\beta}$, using the property that contraction of a self dual field with an anti-self dual field vanishes, we get

$$\mathcal{Z}_{\alpha\beta}m^\alpha n^\beta = 0, \tag{48}$$
$$\mathcal{Z}_{\alpha\beta}\left(\tilde{m}^\alpha m^\beta - l^\alpha n^\beta\right) = 0, \tag{49}$$
$$\mathcal{Z}_{\alpha\beta}l^\alpha \tilde{m}^\beta = 0, \tag{50}$$

By substituting the value of $\mathcal{Z}_{\alpha\beta}$ from Eq. (47), we can see that Eq. (50) is identically satisfied in the limit of geometric optics. However, Eqs. (48) and (49) gives

$$m_0^\alpha m_{0\alpha} = 0 = l_0^\alpha m_{0\alpha} \tag{51}$$

These are the orthogonality relations given in Eqs. (30) and (31).

D. Equations of spin optics

1. *Defining the Hamiltonian*

In the limit of geometric optics, we have discussed in Sec. III A and III B that light travels in null geodesic

$$l_0^\alpha l_{0\alpha} = 0, \qquad l^\alpha_{0;\beta}l_0^\beta = 0. \tag{52}$$

Let us assume $x^\alpha(\lambda)$ as the integral curve of l^α. Then

$$l_0^\alpha = \frac{dx_0^\alpha}{d\lambda} \equiv \dot{x}_0^\alpha. \tag{53}$$

Now, if we define the Hamiltonian as

$$H = \frac{1}{2}g^{\alpha\beta}l_{0\alpha}l_{0\beta}, \tag{54}$$

Then, the Hamilton's equations of motion gives

$$\frac{dx_0^\alpha}{d\lambda} = \frac{\partial H}{\partial l_{0\alpha}} = g^{\alpha\beta} l_{0\beta}, \tag{55}$$

and

$$\frac{dl_{0\alpha}}{d\lambda} = \frac{\partial H}{\partial x_0^\alpha} = \frac{1}{2}\dot{x}_0^\mu \dot{x}_0^\nu \frac{\partial g_{\mu\nu}}{\partial x_0^\alpha}, \tag{56}$$

where we have used Eq. (55) and the relation

$$\frac{\partial g_{\alpha\beta}}{\partial x_{0\mu}} = -g_{\nu\alpha} g_{\rho\beta} \frac{\partial g^{\nu\rho}}{\partial x_{0\mu}}. \tag{57}$$

in obtaining this. Further simplification of this equation gives

$$\frac{d\left(g_{\alpha\beta}\dot{x}_0^\beta\right)}{d\lambda} - \frac{1}{2}\dot{x}_0^\mu \dot{x}_0^\nu \frac{\partial g_{\mu\nu}}{\partial x_{0\alpha}} = \frac{D^2 x_0^\alpha}{D\lambda^2} = 0, \tag{58}$$

where $D/D\lambda$ denotes the covariant derivative along the curve $x^\alpha(\lambda)$. The two Eqs. (55) and (58) we have obtained by solving the Hamilton's equations of motion are the two equations of Eq. (52). We have thus shown that the Hamiltonian defined in Eq. (54) correctly reproduces the equations of geometric optics. The action corresponding to this Hamiltonian is

$$\mathcal{A}_0 = \frac{1}{2}\int \dot{x}_0^\alpha \dot{x}_{0\alpha} d\lambda. \tag{59}$$

Now as a generalization to the spin optics, we consider the action of the form

$$\mathcal{A} = \frac{1}{2}\int \dot{x}^\alpha \dot{x}_\alpha d\lambda + \frac{1}{\omega}\int b_\alpha \dot{x}^\alpha d\lambda. \tag{60}$$

This action is analogous to the one for electromagnetic waves considered by [34] and [1] to derive the spin Hall effect of light. The variation of the first term, which is the optical optical path length gives

$$\frac{1}{2}\delta\int \dot{x}^\alpha \dot{x}_\alpha d\lambda = \int \dot{x}_\alpha \frac{D\delta x^\alpha}{D\lambda} d\lambda = -\int \frac{D^2 x_\alpha}{D\lambda^2}\delta x^\alpha d\lambda. \tag{61}$$

Similary, variation of the second term, which resembles with the Berry connection of optics give

$$\frac{1}{\omega}\delta\int b_\alpha \dot{x}^\alpha d\lambda = \frac{1}{\omega}\int \delta b_\alpha \dot{x}^\alpha d\lambda + \frac{1}{\omega}\int b_\alpha \frac{D\delta x^\alpha}{D\lambda} d\lambda = \frac{1}{\omega}\int b_{\alpha;\beta}\dot{x}^\alpha \delta x^\beta d\lambda - \frac{1}{\omega}\int b_{\beta;\alpha}\delta x^\beta \dot{x}^\alpha d\lambda. \tag{62}$$

Application of the variational principle $\delta\mathcal{A} = 0$ yields

$$\frac{D^2 x_\beta}{D\lambda^2} + \frac{1}{\omega}\left(b_{\beta;\alpha} - b_{\alpha;\beta}\right)\dot{x}^\alpha = 0. \tag{63}$$

We can write the corresponding Hamiltonian as

$$H = \dot{x}^\alpha l_\alpha - \mathcal{L}, \tag{64}$$

where $\mathcal{L} = \dot{x}^\alpha \dot{x}_\alpha/2 + b_\alpha \dot{x}^\alpha/\omega$ is the Lagrangian. We substitute the canonical momentum

$$l_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \dot{x}_\alpha + \frac{b_\alpha}{\omega}, \tag{65}$$

into Eq. (64) to get the Hamiltonian

$$H = \frac{1}{2\omega^2}\left(\omega l_{0\alpha} + l_{1\alpha} - b_\alpha\right)\left(\omega l_0^\alpha + l_1^\alpha - b^\alpha\right). \tag{66}$$

2. Solving Hamilton's equations of motion

Now, for the subleading order calculations in the geometrical optics approximation, we first write the dispersion relation of Eq. (17) as

$$\left(l_{0\beta}+\frac{1}{\omega}(l_{1\beta}-b_\beta)\right)\left(l_0^\beta+\frac{1}{\omega}(l_1^\beta-b^\beta)\right)=\dot{x}_\beta\dot{x}^\beta=0, \tag{67}$$

where Eq. (65) has been used. One can see that in the leading order approximation in $1/\omega$, $\dot{x}^\alpha=l_0^\alpha$ gives the tangent vector. This equation imply that the gravitational waves trajectory in the spin optics approximation is still null. It is, however, not geodesic as can be seen from the fact that

$$\frac{D\dot{x}^\alpha}{D\lambda}=\frac{1}{\omega}\left(l^\alpha_{1;\beta}-b^\alpha_{;\beta}\right)l_0^\beta. \tag{68}$$

To calculate this quantity, we take the Hamiltonian of Eq. (66) for the gravitational waves trajectory we are considering. This Hamiltonian has indeed been solved by [1] to obtain the following equation for right hand circularly polarized ray:

$$\frac{D^2x^\alpha}{D\lambda^2}=\frac{1}{\omega}k^\alpha_\beta l_0^\beta, \tag{69}$$

where $k_{\alpha\beta}=b_{\beta;\alpha}-b_{\alpha;\beta}$ and $b_\alpha=2i\tilde{m}_0^\beta m_{0\beta;\alpha}$. This is our Eq. (63), which has been derived by using the variational principle. Comparing this with Eq. (68), we get $l_{1\alpha;\beta}=b_{\beta;\alpha}$. Further simplifying $k_{\alpha\beta}$, one gets

$$k_{\alpha\beta}=-2iR_{\alpha\beta\mu\nu}m^\mu\tilde{m}^\nu+2i\left(\tilde{m}^\mu_{;\alpha}m_{\mu;\beta}-\tilde{m}^\mu_{;\beta}m_{\mu;\alpha}\right). \tag{70}$$

Substituting this back into Eq. (68) gives

$$\frac{D^2x^\alpha}{D\lambda^2}=-\frac{2i}{\omega}R^\alpha{}_{\beta\mu\nu}m^\mu\tilde{m}^\nu l_0^\beta\approx-\frac{2i}{\omega}R^\alpha{}_{\beta\mu\nu}l_0^\beta m_0^\mu\tilde{m}_0^\nu. \tag{71}$$

We thus get non-geodesic trajectory in the spin optics approximation.

We now can substitute $w^\alpha=l_0^\beta l^\alpha_{;\beta}$ from Eq. (68) into the Eqs. (40), which gives the evolution of the polarization vector

$$l_0^\beta n^\mu_{;\beta}=0, \tag{72}$$

$$l_0^\beta m^\mu_{;\beta}=\frac{1}{\omega}\left(l^\beta_{1;\alpha}-b^\beta_{;\alpha}\right)l_0^\alpha m_\beta n^\mu=\frac{2i}{\omega}R_{\alpha\beta\gamma\delta}l_0^\alpha m_0^\beta m_0^\gamma\tilde{m}_0^\delta n_0^\mu, \tag{73}$$

$$l_0^\beta \tilde{m}^\mu_{;\beta}=\frac{1}{\omega}\left(l^\beta_{1;\alpha}-b^\beta_{;\alpha}\right)l_0^\alpha \tilde{m}_\beta n^\mu=-\frac{2i}{\omega}R_{\alpha\beta\gamma\delta}l_0^\alpha \tilde{m}_0^\beta \tilde{m}_0^\gamma m_0^\delta n_0^\mu. \tag{74}$$

These equations assures that the set of tetrad $(\dot{x}^\alpha,n^\alpha,m^\alpha,\tilde{m}^\alpha)$ satisfy the normalization and orthogonality relations Eqs. (30)-(32) throughout the ray up to the subleading order in $1/\omega$. To see this, let us first simplify Eq. (73) as

$$l_0^\beta m^\mu_{;\beta}\approx\frac{1}{\omega}l_0^\beta m^\mu_{1;\beta}\approx\frac{1}{\omega}l_0^\beta\left(l_1^\alpha-b^\alpha\right)_{;\beta}m_{0\alpha}n_0^\mu. \tag{75}$$

As the covariant derivatives of $m_{0\alpha}$ and n_0^μ are zero, we can write

$$m_1^\mu=\left(l_1^\alpha-b^\alpha\right)m_{0\alpha}n_0^\mu. \tag{76}$$

Simplification of Eqs. (72) and (74) in the similar manner gives

$$n_1^\mu=0,\qquad \tilde{m}_1^\mu=\left(l_1^\alpha-b^\alpha\right)\tilde{m}_{0\alpha}n_0^\mu, \tag{77}$$

respectively. One can also write the subleading order of the component of tetrad $\dot{x}^\alpha$ differently by noting that $w^\alpha l_{0\alpha}=0=w^\alpha n_{0\alpha}$; this is how we have constructed the Fermi propagated tetrad in Sec. III B. So, w^α can be written in the form

$$w^\alpha\equiv\frac{1}{\omega}l_0^\beta\left(l_1^\alpha-b^\alpha\right)_{;\beta}=-\tilde{\kappa}m_0^\alpha-\kappa\tilde{m}_0^\alpha,\qquad \kappa=-\frac{1}{\omega}m_0^\alpha l_0^\beta\left(l_{1\alpha}-b_\alpha\right)_{;\beta}. \tag{78}$$

From this, one is allowed to write

$$l_1^\alpha-b^\alpha=\tilde{m}_0^\beta\left(l_{1\beta}-b_\beta\right)m_0^\alpha+m_0^\beta\left(l_{1\beta}-b_\beta\right)\tilde{m}_0^\alpha. \tag{79}$$

It is easy to see that Eq. (76) in conjunction with the Eqs. (77) and (79) taken as the subleading order correction of the tetrad $(\dot{x}^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$ satisfies the scalar products of Eqs. (30)-(32). The leading order terms of this tetrad is obviously the tetrad $(l_0^\alpha, n_0^\alpha, m_0^\alpha, \tilde{m}_0^\alpha)$ of Sec. III B. Moreover, the subleading order terms of the Lorentz condition of Eq. (15) gives

$$\frac{a_{;\alpha}}{a} m_0^\alpha = -m^\alpha_{0;\alpha} - i b_\alpha m_0^\alpha + m_0^\alpha \tilde{m}_{0\beta;\alpha} m_0^\beta = -m^\alpha_{0;\alpha} - \frac{i}{2} b_\alpha m_0^\alpha. \tag{80}$$

All of the polarization relations of Eqs. (48)-(50) are not satisfied in the limit of spin optics, thereby implying that the field satisfying Eqs. (30), (31) and (32) is not self-dual in the subleading order in $1/\omega$. To see this we calculate each of these equations separately, starting with Eq. (50)

$$\mathcal{Z}_{\alpha\beta} l^\alpha \tilde{m}^\beta = \frac{2ia}{\omega}\left(-\frac{a_{;\alpha}}{a} l_0^\alpha + m_{0\alpha;\beta} l_0^\alpha \tilde{m}_0^\beta\right) = \frac{2ia}{\omega}\left(\frac{1}{2} l^\alpha_{0;\alpha} - m_0^\alpha l_{0\alpha;\beta}\tilde{m}_0^\beta\right) = 0, \tag{81}$$

where we have used Eq. (29) to arrive at this identity. Similarly, Eq.(49) gives

$$\mathcal{Z}_{\alpha\beta}\left(\tilde{m}^\alpha m^\beta - l^\alpha n^\beta\right) = \frac{2ia}{\omega}\left(\frac{a_{;\alpha}}{a} m_0^\alpha - m_{0\alpha;\beta} l_0^\alpha n_0^\beta\right) = \frac{2ia}{\omega}\left(-m^\alpha_{0;\alpha} - \frac{i}{2} b_\alpha m_0^\alpha - m_{0\alpha;\beta} l_0^\alpha n_0^\beta\right) = 0, \tag{82}$$

where we have used Eq. (80) to obtain this identity. Finally, Eq.(48) gives

$$\mathcal{Z}_{\alpha\beta} m^\alpha n^\beta = \frac{2ia}{\omega}\left(-m_{0\alpha;\beta} m_0^\beta n_0^\alpha\right) = \frac{2ia}{\omega}\left(m_0^\alpha n_{0\alpha;\beta} m_0^\beta\right) \equiv \frac{2ia}{\omega}\tilde{\lambda}, \tag{83}$$

where $\tilde{\lambda}$ is one of the Newman-Penrose scalars. Eq.(48) is thus not satisfied unless $\tilde{\lambda} = 0$.

We have thus shown that the tetrad $(\dot{x}^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$ satisfying the scalar products of Eqs. (30)-(32) does not give the self-dual solution of the field Eqs. (43) and (44). However, there should exist self-dual solution of these field equations in the subleading order geometrical optics approximation. We can find this self dual solution by first writing the Fermi-like derivative operator as follows

$$\mathcal{D}'_l A^\alpha = l_0^\beta A^\alpha_{;\beta} - w_\beta A^\beta n^\alpha + A^\beta n_\beta w^\alpha - \frac{2i}{\omega}\left(\lambda_{;\mu} l^\mu m_\beta A^\beta m^\alpha - \tilde{\lambda}_{;\mu} l^\mu \tilde{m}_\beta A^\beta \tilde{m}^\alpha\right), \tag{84}$$

The vanishing of this derivative $\mathcal{D}'_l A^\alpha = 0$ implies

$$l_0^\beta A^\alpha_{;\beta} = w_\beta A^\beta n^\alpha - A^\beta n_\beta w^\alpha + \frac{2i}{\omega}\left(\lambda_{;\mu} l^\mu m_\beta A^\beta m^\alpha - \tilde{\lambda}_{;\mu} l^\mu \tilde{m}_\beta A^\beta \tilde{m}^\alpha\right), \tag{85}$$

and in that case, the scalar product of any two components of tetrad $(\dot{x}^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$ is constant except that of m^α with itself and that of $\tilde{m}^\alpha$ with itself. This can be seen by calculating

$$\left(a^\alpha b_\alpha\right)_{;\beta} l_0^\beta = a^\alpha b_{\alpha;\beta} l_0^\beta + b^\alpha a_{\alpha;\beta} l_0^\beta = \frac{4i}{\omega}\left(\lambda_{;\mu} l^\mu m_\beta a^\beta m^\alpha b_\alpha - \tilde{\lambda}_{;\mu} l^\mu \tilde{m}_\beta a^\beta \tilde{m}^\alpha b_\alpha\right). \tag{86}$$

This scalar product is non-zero only if $a^\alpha = b^\alpha = m^\alpha$ or $a^\alpha = b^\alpha = \tilde{m}^\alpha$. A tetrad with vanishing Fermi-like derivative satisfy these relations satisfy the following orthogonality and completeness relations everywhere on the ray if they satisfy at some point on the ray:

$$\dot{x}^\alpha m_\alpha = \dot{x}^\alpha \dot{x}_\alpha = \dot{x}^\alpha \tilde{m}_\alpha = 0, \qquad m^\alpha \tilde{m}_\alpha = 1, \tag{87}$$

$$n^\alpha m_\alpha = n^\alpha n_\alpha = n^\alpha \tilde{m}_\alpha = 0, \qquad n^\alpha l_\alpha = -1, \tag{88}$$

However, in general, $m^\alpha m_\alpha \neq 0$ and $\tilde{m}^\alpha \tilde{m}_\alpha \neq 0$ along the circularly polarized ray in the subleading order approximation. That is, the polarization vectors are not null like in the geometrical optics limit. Moreover, the tetrad evolves as

$$l_0^\beta n^\alpha_{;\beta} = 0, \tag{89}$$

$$l_0^\beta m^\alpha_{;\beta} = w^\beta m_\beta n^\alpha - \frac{2i}{\omega}\tilde{\lambda}_{;\mu} l^\mu \tilde{m}^\alpha, \tag{90}$$

$$l_0^\beta \tilde{m}^\alpha_{;\beta} = w^\beta \tilde{m}_\beta n^\alpha + \frac{2i}{\omega}\lambda_{;\mu} l^\mu m^\alpha. \tag{91}$$

Following the simplification procedure used to obtain Eqs. (76) and (77) gives

$$m_1^\mu = \left(l_1^\alpha - b^\alpha\right) m_{0\alpha} n_0^\mu - 2i\tilde{\lambda}\tilde{m}_0^\mu, \qquad n_1^\mu = 0, \tag{92}$$

$$\tilde{m}_1^\mu = \left(l_1^\alpha - b^\alpha\right) \tilde{m}_{0\alpha} n_0^\mu + 2i\lambda m_0^\mu. \tag{93}$$

These components of tetrad constitute the solution of linearized Einstein field equations in Lorentz gauge and they are right hand circularly polarized as they also satisfy Eqs. (48)-(50). So they are the solutions for the propagation of circularly polarized gravitational waves in curved spacetime in the spin optics approximation.

IV. APPLICATIONS

A. Gravitational lensing of gravitational waves

Geometric optics expansion for gravitational waves is valid only when its wavelength is much smaller than the Schwarzschild radius of the lensing body [29, 30]. However, pulsar timing arrays (PTAs) [31] and laser interferometer space antenna (LISA) [32] could collectively detect the gravitational waves in the very wide range, ranging from $10^8 m$ to $10^{17} m$. Thus, the lensing of gravitational waves is likely to give wave effects which is not considered in the limit of geometric optics. Lensing objects in an optically thick regions could still be probed by the lensing of gravitational waves as unlike electromagnetic waves, gravitational waves could travel in such dense regions without much absorption and scattering.

We here calculate the effect of spin-orbital coupling on gravitational waves lensing. In the geometric optics regime, the angle of deflection of gravitational waves by gravitating objects is equal to that of electromagnetic waves and the maximum angle of deflection is given by

$$\Delta\phi \approx \frac{4M}{R_0} \tag{94}$$

where, M is the mass and R_0 is the radius of a gravitating object.

We have for the Schwarzschild geometry, the general vector tangent to the congruence of null geodesics could be written as

$$l_{0\mu} = \left(-E, \frac{1}{1-\frac{2M}{r}}\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}, 0, L\sin\theta\right). \tag{95}$$

Now, the three other null fields n^μ, m^μ and $\tilde{m}^\mu$ which along with l^μ forms a set of null tetrad and satisfy the orthogonality and completeness relations are given as

$$\begin{aligned}
n_{0\mu} &= \frac{1-\frac{2M}{r}}{2\left(E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)\right)}\left(-E, -\frac{1}{1-\frac{2M}{r}}\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}, 0, L\sin\theta\right), \\
m_{0\mu} &= \frac{r}{\sqrt{2}}\left(-\frac{iL\left(1-\frac{2M}{r}\right)}{r^2\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}}, 0, 1, \frac{iE\sin\theta}{\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}}\right), \\
\tilde{m}_{0\mu} &= \frac{r}{\sqrt{2}}\left(\frac{iL\left(1-\frac{2M}{r}\right)}{r^2\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}}, 0, 1, -\frac{iE\sin\theta}{\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}}\right).
\end{aligned} \tag{96}$$

We now apply the following transformation relations

$$l_0^\mu \to l_0^\mu, \qquad m_0^\mu \to m_0^\mu + a l_0^\mu, \qquad n_0^\mu \to n_0^\mu + \tilde{a} m_0^\mu + a\tilde{m}_0^\mu + a\tilde{a} l_0^\mu, \tag{97}$$

to the null tetrad such that the resulting tetrad will be F-transported along the rays. Here, the value of a can be evaluated in such a way that the transformed vector $m_0^\mu + a l_0^\mu$ has its covariant derivative zero in the leading order:

$$\left(m_0^\mu + a l_0^\mu\right)_{;\nu} l_0^\nu = m^\mu_{0;\nu} l_0^\nu + a_{,\nu} l_0^\mu l_0^\nu = 0. \tag{98}$$

In spherically symmetric spacetime, a could only be the function of r thereby giving

$$a = \int \frac{m^\mu_{0;\nu} l_0^\nu n_{0\mu}}{l_0^r} dr = \int \frac{iEL(r-3M)}{\sqrt{2} r^3\left(E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)\right)^{3/2}} dr. \tag{99}$$

These transformed null tetrads satisfy Eqs. (30)-(34). These null trajectories l_0^μ and polarization vectors m_0^μ constitute the solutions of linearized Einstein equations in the limit of geometric optics.

Now, to get the first order correction of the null trajectory, we substitute these values into the propagation Eqs. (71) and we get

$$\frac{D^2 x^2}{D\lambda^2} = \frac{6iLM}{\omega}\left(\frac{\sqrt{2}La}{r^6} + \frac{iE}{r^5\sqrt{E^2 - \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)}}\right); \qquad \frac{D^2 x^\mu}{D\lambda^2} = 0 \qquad \text{for} \qquad \mu \neq 2. \tag{100}$$

Integrating this equation gives, up to the second order approximation in M/r

$$\dot{x}^2 = \frac{1}{\omega}\left(\frac{C_1}{r} + \frac{2LM}{Er^4} + \frac{3LM^2}{2Er^5}\right), \tag{101}$$

where, C_1 is an integration constant. For the problem of gravitational lensing, $\dot{x}^2 \to 0$ as $r \to 0$ and for this, we should have, $C_1 = 0$. We can see that $\dot{x}^2 \neq 0$ for $L \neq 0$ and in that case gravitational waves travels in the null non-geodesic trajectory even in Schwarzschild spacetime as a result of gravitational spin Hall effect.

We can now calculate the total angle of deflection of gravitational waves from the lens of M using the relation (here, $x_2 = \theta$ and $x_3 = \phi$)

$$d\Theta = \left(d\theta^2 + \sin^2\theta d\phi^2\right)^{1/2} \approx \left(d\theta^2 + d\phi^2\right)^{1/2} \approx d\phi\left(1 + \frac{1}{2}\left(\frac{d\theta}{d\phi}\right)^2\right), \tag{102}$$

where the last expression is the result of Taylor's expansion based on the assumption $d\theta \ll d\phi$. For the photon starting from infinity and approaching the lens to the closest distance of R_0, we should have $E = 1$ and

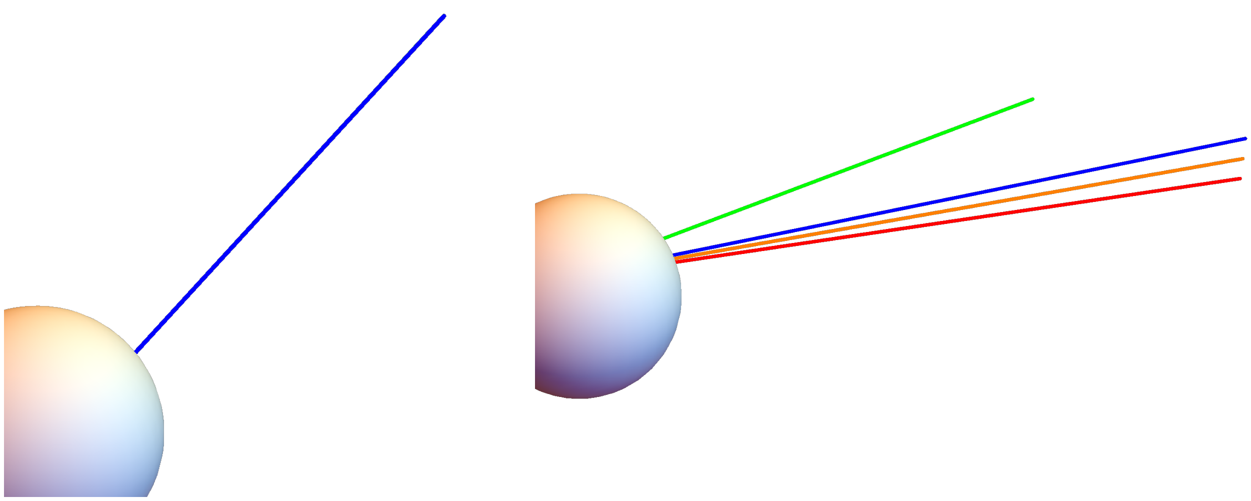
$$L = \frac{R_0}{1 - \frac{2M}{R_0}}. \tag{103}$$

We can now integrate to get the total deflection angle up to the second order expansion in M/r

$$\Theta = 2\left(\Theta_\infty - \Theta_0\right) - \pi = \frac{4M}{R_0} + \frac{7.78097M^2}{R_0^2} + \frac{2.35619M^2}{\omega^2 R_0^4}. \tag{104}$$

The last term gives an additional deflection due to the spin orbit coupling and this quantity might be too small for the lensing object of mass $M \ll R_0$ in the frequency limit detectable by LIGO [33]. This deflection, however, is twice the deflection of electromagnetic waves due to spin orbit correction [29].

We have also solved the trajectory numerically, up to the subleading order correction and plotted the resulting curves in Mathematica using typical value of parameters ($M = 10^{-8}, R_0 = 10, L = 10, \omega = 100$). Light is assumed to be emitted from $r = R_0 = 10$. The resulting curves are shown in Fig. 1.



(a) On Scale

(b) Exaggerated for visual

FIG. 1: Figure showing spin Hall effect of gravitational waves in Schwarzschild spacetime (the sphere on the figure is just inserted for reference and does not represent the actual position of a gravitating object). Red (bottom) curve is for right hand circular polarization, blue (top) curve is for left hand circular polarization; orange (middle) curve is the uncorrected trajectory; green straight line (rear) is the direction of light emission.

B. Propagation of binary's waves through an expanding universe

If we model the large scale structure of the expanding universe by a line element

$$ds^2 = R_0 t^2 \left(-dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{105}$$

then no spin Hall effect of gravitational waves from binary is observed during propagation. The reason behind this is on this spacetime, null rays along which gravitational waves propagate in the geometric optics regime are the curves of constant θ, ϕ and $t-r$. A set of null tetrad for such spacetime could be written as

$$\begin{aligned}
l_{0\mu} &= R_0^2 \left(-1,1,0,0\right), \\
n_{0\mu} &= \frac{t^4}{2}\left(-1,-1,0,0\right), \\
m_{0\mu} &= \frac{R_0 t^2 r}{\sqrt{2}}\left(0,0,1,i\sin\theta\right), \\
\tilde{m}_{0\mu} &= \frac{R_0 t^2 r}{\sqrt{2}}\left(0,0,1,-\sin\theta\right).
\end{aligned} \tag{106}$$

This is indeed F-transported null tetrad along the rays that satisfies Eqs. (30)-(34). So, these null trajectories l_0^μ and polarization vectors m_0^μ constitute the gravitational waves solutions of linearized Einstein equations in the limit of geometric optics.

Now, using the propagation Eqs. (71), we get no first order correction of the null trajectory thereby implying that there is no spin-orbit coupling for the propagation of gravitational waves in such spacetime. Thus, the propagation of gravitational waves from binary is not deviated from the null geodesic trajectories in such spacetime. Absence of terms with subleading order correction in geometric optics approximation in such spacetime could be attributed to the fact that waves propagate in the curves of constant θ and ϕ. The absence of orbital motion/angular momentum on gravitational waves propagating in such spacetime causes spin-orbit coupling to vanish.

V. CONCLUSION AND DISCUSSIONS

We have presented the calculations that account for the spin Hall effect of gravitational waves on curved background, and applied it to some simple spacetimes. [2] have also developed the covariant formulation that accounts for this effect. However, our results differ slightly from theirs, and this is primarily due to the difference in our eikonal function Eq. (11) with theirs. In their approach

- Amplitude $h_{\alpha\beta}$ is assumed to be the function of the phase gradient l^α, that is, $h_{\alpha\beta} = h_{\alpha\beta}\left(\lambda, l(\lambda)\right)$.

WKB analysis with this form of eikonal, gives the following equation for the ray trajectory:

$$\dot{x}^\alpha = \frac{1}{\omega}\left(l^\alpha - B^\alpha - l_\mu \frac{\partial B^\mu}{\partial l_\alpha}\right), \tag{107}$$

where $B_\beta(\lambda, l(\lambda))$ is identified with the Berry connection of optics

$$B_\beta(\lambda, l(\lambda)) = \frac{i}{2}\left(\tilde{m}^\alpha \overset{h}{\nabla}_\beta m_\alpha - m^\alpha \overset{h}{\nabla}_\beta \tilde{m}_\alpha\right) = i\tilde{m}^\alpha \overset{h}{\nabla}_\beta m_\alpha, \qquad \overset{h}{\nabla}_\beta m_\alpha = \nabla_\beta m_\alpha + \Gamma^\mu_{\beta\nu} l_\mu \frac{\partial m_\alpha}{\partial l_\nu}, \tag{108}$$

and $\overset{h}{\nabla}_\beta$ is called as horizontal derivative. These equations were obtained for circularly polarized waves whose amplitude has the form

$$h_{\alpha\beta}(\lambda, l(\lambda)) = a m_\alpha m_\beta. \tag{109}$$

As null tetrad $(l^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$ is supposed to be a linearly independent basis of 4-dimensional spacetimes onto which all the tensors/vector can be projected, the partial derivative of m_α with respect to l_α should be zero, that is, $\partial m_\alpha / \partial l_\nu = 0$ (see appendix A). By projecting amplitude $h_{\alpha\beta}(\lambda, l(\lambda))$ into the null tetrad basis, its form in Eq. (109) was indeed obtained by [2]. In that case,

$$B_\beta(\lambda, l(\lambda)) = B_\beta(\lambda) = b_\beta(\lambda)/2, \tag{110}$$

where b_β is defined in Eq. (18) and the ray trajectory given by [2] would be same as that of ours in Eq. (71).

All the equations of spin optics developed here for gravitational waves have the exact same form as for their electromagnetic counterparts. The only difference here is the factor of two which correctly accounts for the difference in helicity between gravitational and electromagnetic waves. This factor of two causes twice a deviation of gravitational waves from the null geodesic trajectory than that of electromagnetic waves due to spin-orbit coupling. It should also be noted that this deviation is frequency dependent.

In the leading order geometric optics, the trajectory of light/gravitational waves is geodesic. However, this is no longer true in the subleading order correction. As gravitons/photons still travel at the speed of light, it takes longer time for them to reach the observer from the source [29]. It takes even longer time for gravitons than for photons owing to their difference in helicity. This frequency dependent time delay for circularly polarized waves might be observed in cosmological events like neutron star merging [7]. We can also notice from Sec. III C that electromagnetic and gravitational waves in weak field limit (up to the subleading order geometric optics approximation) follow exactly same set of equations except for difference in the factor of two, that accounts for their helicity difference. So, the subleading order correction procedure described here could be applied to the massless particles of arbitrary spin (obviously, by taking account of the difference in spin of these particles).

There are different approaches for studying the motion of massless particles with spin in curved spacetimes [5]. The similarity of our geometric optics approach to the quantum mechanical approach used by [23] to get gravitational spin Hall effect is evident from the similarity of our Hamiltonian Eq. (66) with his. This results in the similar equation for the null trajectory deviating from the geodesic in the subleading order in $1/\omega$, where the term proportional to b_β occurring in our Hamiltonian should be interpreted as the Berry connection. Similarly, the equivalence between the quantum mechanical approach, for example of Gosselin, and the approach that uses Souriau-Saturnini equations [24, 25] at least for Schwarzschild spacetime is demonstrated by [5]. The equations of the trajectory from all these approaches resemble closely, for Schwarzschild spacetime, to the method used in optics to derive the spin Hall effect, where spacetime is treated as effective medium with perfect impedance matching. This resemblance is particularly encouraging as spin Hall effect has been verified experimentally in optics. We are also looking to establish the equivalence of our spin optics approach with these different approaches rigorously in the near future.

Appendix A: Partial derivatives of linearly independent vectors

Consider two linearly independent vectors $\mathbf{l}$ and $\mathbf{m}$. Together they constitute two-space, whose elements can be written in the form

$$\mathbf{v}(\mathbf{l},\mathbf{m}) = c_1\mathbf{l} + c_2\mathbf{m}. \tag{A1}$$

Let us consider a vector $\mathbf{w}(\mathbf{l},\mathbf{m}) = c\mathbf{m}$ and calculate its partial derivative with respect to $\mathbf{l}$

$$\frac{\partial w^\alpha}{\partial l^\beta} = \lim_{\mathbf{h}\to 0} \frac{w^\alpha\left(\mathbf{l}+\mathbf{h},\mathbf{m}\right) - w^\alpha\left(\mathbf{l},\mathbf{m}\right)}{h^\beta} = \lim_{\mathbf{h}\to 0} \frac{cm^\alpha - cm^\alpha}{h^\beta} = 0. \tag{A2}$$

Example: We take functions x and x^2. They generate function space, whose elements are of the form

$$v(x,x^2) = c_1 x + c_2 x^2. \tag{A3}$$

Now, we define a function $w(x,x^2) = cx^2$ and calculate its partial derivative with respect to x

$$\frac{\partial w}{\partial x} = \lim_{h\to 0} \frac{w\left(x+h,x^2\right) - w\left(x,x^2\right)}{h} = \lim_{h\to 0} \frac{cx^2 - cx^2}{h} = 0. \tag{A4}$$

Thus, the partial derivatives of two linearly independent vectors are zero irrespective of their functional dependence. However, the total derivative might not be zero.

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