

Spinor fields, CPTM symmetry and smallness of cosmological constant in framework of extended manifold

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A model of an extended manifold for the Dirac spinor field is considered. Two Lagrangians related by CPTM (charge-parity-time-mass) symmetry are constructed for a pair of the Dirac spinor fields with each spinor field defined in a separate manifold. An interaction between the matter fields in the manifolds is introduced through gravity. A fermionic effective action of the general system is constructed and a tadpole one-loop spinor diagram and part of the one-loop vacuum diagrams with two external gravitational off-shell fields which contribute to the effective action are calculated. It is demonstrated that among different versions of the second spinor Lagrangian there is a special one for which a cancellation of the mentioned diagrams in the total effective action takes place. As a result, the diagrams do not contribute to the cosmological constant, as well there is a zero contribution of the zero point energies of the spinor fields to the action. The non-zero leading order value of the cosmological constant for each manifold in the framework is proportional to the matter density of each separated manifold or difference of the densities, depending on the chosen model of interaction of gravitational fields with fermions. An appearance of the dark matter in the model is shortly discussed as well as further applications of the approach.

I. INTRODUCTION

In the present paper we continue an investigation of the model of an extended manifold introduced in [1–3]. There we discussed a model of manifold which consists two, at least, parts related by the classical symmetry which reverse charge, parity, time and mass (CPTM symmetry) preserving a form of the metric g at the case of the zero cosmological constant:

$$q \rightarrow -\tilde{q}, r \rightarrow -\tilde{r}, t \rightarrow -\tilde{t}, m_{grav} \rightarrow -\tilde{m}_{grav}; \quad \tilde{q}, \tilde{r}, \tilde{t}, \tilde{m}_{grav} > 0; \quad (1)$$

$$CPTM(g_{\mu\nu}(x)) = \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(\tilde{x}). \quad (2)$$

The general manifold we consider consists of different separate manifolds with the gravitational masses of different signs for matter fields in each one, see details in [1] and discussion in [4] for example. The motivation of the introduction of the negative mass in the different cosmological models is very clear. In any scenario, see [5, 6] for example, the presence of some kind of the repulsive gravitation forces or an additional gravitational field in our Universe helps with an explanation of the existence of dark energy in the models, see also [7–15] and references therein. In the present construction we, nevertheless, consider a different appearance of the negative mass, the way of two metrics introduced is more similarly to the bimetric models perhaps, see [7, 8] for some examples. Namely, there are two different bare actions we consider as well, but, firstly, the only interactions between the manifolds we have are through the weak gravity fields. Secondly, the matter fields of both manifolds are related by the CPTM symmetry as well. These matter fields do not interact each with other directly but only through gravity at least until non-diagonal Green's functions of Schwinger-Keldysh type are not introduced, see about in [16–18]. Additionally, in the case of spinor fields, the whole model's construction depends on the way of an introduction of the gravity field instead vierbeins in the fermion's Lagrangians. There are different variants of this replacement we have and there is a need in some procedure which will fix the possible form of the B-manifold matter Lagrangian with gravity field included.

The gravitation properties of the matter of new B-manifold are also described by Einstein equations in the presented approach, see [5, 12] or [6] and [19] for examples of the application of the discrete symmetries in the case of the quantum and classical systems. In the proposed model, due the CPTM symmetry, we need to account the possible different directions of the time's arrows for the different manifolds, therefore there are variants of the theory with closed time paths for the fields of the general manifold related by the CPTM symmetry. The interaction between these fields leads to the some quantum effective action, the cosmological constant arise then as a part of this effective action. Once arising, the constant's value can be different for both manifolds related as well by the introduced symmetry. Consequently, we will have for the case of different cosmological constants for A and B manifolds:

$$CPTM(\Lambda) = \tilde{\Lambda}, \quad CPTM(g_{\mu\nu}(x, \Lambda)) = \tilde{g}_{\mu\nu}(\tilde{x}, \tilde{\Lambda}). \quad (3)$$

This appearance of the cosmological constant satisfies the naturalness criteria of 't Hooft, see [20]. It's zero value correspondence to the precise symmetry between the metrics Eq. (2) whereas it's small non-zero value reduces the symmetry to Eq. (3) relation. In any extend, the constant in the formalism is not a constant anymore but it is a

functional of the fields in the approach. It, therefore, can acquire different values due to its evolution, the important question we need to clarify is the present smallness of the constant.

As mentioned above, the model assumes a presence of two copies of any matter fields for two manifolds, the case of a scalar fields was discussed in the [2]. In this article we consider two spinor fields, A-spinor and B-spinor fields correspondingly. Therefore, the first question we need to answer is about the construction of the spinor action for the B-manifold by the use of CPTM symmetry, see also [1, 2]. In our case we base the whole construction on the following propositions. First of all, for the matter fields, we begin from the sum of the spinor's actions, their relative sign is fixed by the possible signs of vierbeins in the B-manifold metric and reverse sign of the mass term in the B Lagrangian, see also [21] for a different realization of the similar idea. The vierbein's sign in general can be positive or negative due to the squared appearance of vierbeins in the metric's definition, see also discussions in [22, 23]. Secondly, we construct the quantized one-particle spinor state, due to the M-symmetry present we do not know a priori the commutation properties of the creation and annihilation operators of new state for the B-manifold. Therefore, we fix the commutation relations for the operators by the request that the corresponding vacuum energy-momentum vector for both fields together will be equal to zero, this provides an absence of the zero energy modes in the model for one-particle quantum states. There are two possibilities of the commutation relations for each sign of vierbeins, correspondingly it determines a form of the propagators for the each variant of the action for the B-matter field. On this stage we stay with four different theories of B-manifold spinor action, among them there is an action which similar to the action arises in the Schwinger-Keldysh approach, see [16–18]. So, at the next step we calculate an one-loop effective action which as well provides a large contribution to the cosmological constant value through the corresponding one-loop energy-momentum tensor. In the present manuscript we do not consider the full form of the one-loop spinor action in the external gravitational field, for our purposes it was enough to restrict the calculations only by linear vierbein approximation for the corresponding connection fields. The obtained terms of the effective action are represented by diagrams with one external gravitational field and some diagrams with two external gravitational fields attached to the fermion loop¹. These diagrams are vacuum and particular tadpoles contributions to the energy-momentum tensor. By requiring these contributions to be zero for both fields together A and B, we end up with only one version of the B-manifold spinor action. It is found that this total spinor effective action is formally equivalent to the Schwinger-Keldysh action for a fermionic system. We need to underline, that the cancellation between the different terms of the fermion's effective action takes place when the external gravitational fields attached to the diagrams are the same for the all parts of the action. This is not ordinary assumption in a case of the standard Schwinger-Keldysh formalism, there all fields, including gravitational, must be different in the case of two time's branches, see Discussion section. Therefore, requiring the cancellation of the terms in the effective action, we need to introduce a gravity coupling in the fermion action different from the usual Schwinger-Keldysh formalism set-up.

Having in hand the unique effective spinor's action for the A and B manifolds together, next problem we face is an introduction of the weak gravity field in the actions instead vierbeins fields. In general, there is no clear prescription how to do it when we talk about an extended manifold. Again, we have different possibilities for the A,B spinor actions with gravity included. In a simplest variant, we can follow to the Schwinger-Keldysh approach and introduce two different weak gravity fields for the different manifolds separately. In this case in the each spinor action arise only corresponding weak gravity field, the interaction between the manifolds can arise only due to an interaction similar to the Schwinger-Keldysh interaction mechanism. As mentioned above, this variant does not resolve really the cosmological constant problem, the contributions of the vacuum and tadpole contributions to the energy-momentum tensor can be weaker in this case but do not disappear. Another variant we discuss is an variant based on the assumption that any matter's energy-momentum tensor is a source of the two weak fields appear as a linear combination, a sum in the simplest case. In the corresponding gravity action the weak field, therefore, can appear or separately for the corresponding separate manifold or as a linear combination of two weak fields in each separate Lagrangian, in that particular case the only one weak field for the both manifolds exists. In this case the vierbein field in each spinor's action is equivalent to the sum of the two weak fields and the cancellation of the corresponding terms in the spinor's one-loop effective action is precise. There are additional one-loop contributions which do not cancel when we consider the non-diagonal Green's function appear in the ordinary Schwinger-Keldysh formalism, but their structure and contributions are quite different from the ordinary QFT loops. When the weak field expansion for both manifolds consists of bare metric plus two weak fields from the both manifolds, i.e. only one weak field mutual for the manifolds, some more complicated system of interacting fields in comparison to the ordinary gravity action arises. In this article we do not really explore this possibility, it does not change the main results concern the cosmological constant value. For the both variants, nevertheless, the cosmological constant is turned out to be equal to the difference of the matter densities of the manifolds.

¹ The possible cancellation of different terms in the effective action is due to the fermion's propagators in the loop, the number of the classical gravitational fields attached to the vertex does not matter in this case.

An another important observation is about an appearance of "dark" matter in the framework. In any set-up of the problem when we have two manifolds, we have A and B matter fields which do not interact directly or can interacts through the non-diagonal propagators as in Schwinger-Keldysh formalism. In both cases the structure of the gravitational propagator, aka gravitational potential, is changing due a presence of the interactions with B-manifold and/or with B-spinors. The modification of the propagator in the general case depends on the considered variant of the theory, the simplest variants are when the B-spinor fields can be created in the A-manifold by the gravity fields, these B-spinor field modifies the propagator similarly to a dark matter influence.

Respectively, the article is organized as following. In the Sections II-V we discuss a structure of the bare spinor action for the B-manifold and corresponding spinor propagators. The Section VI is about the standard introduction of the gravity in the spinor's Lagrangian, approximation we made and corresponding spinor's action in an external gravitational field obtained. The next Section VII dedicated to the results of the calculation of the one-loop of effective spinor action in an external gravity fields, the calculations are present in Appendix C. There we explore also in which variant of the effective action the cancellation of corresponding action's terms is happening. The Section VIII is about general structure of the resulting action in the gravitational fields and in the Sections IX-X we discuss a different possibilities of the vierbeins replacement on the weak gravity field in the action. In the Section XI and Section XII the main results of the article and an overview of the formalism are discussed correspondingly.

II. CPTM SYMMETRY FOR DIRAC EQUATION

Consider the expression for the quantized spinor field:

$$\psi(x) = \psi_A(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s (a_{\mathbf{p}}^s u^s(p) e^{-i p x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{i p x}) \quad (4)$$

$$\{a_{\mathbf{p}}^r a_{\mathbf{k}}^{s\dagger}\} = \{b_{\mathbf{p}}^r b_{\mathbf{k}}^{s\dagger}\} = \delta_{p k}^3 \delta^{rs}, \quad (5)$$

with definitions borrowed from [24]. The CPT transformed function has the Eq. (A.6) form

$$\psi_{CPT}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s (b_{\mathbf{p}}^{s\dagger} v^s(p) e^{-i p x} + a_{\mathbf{p}}^s u^s(p) e^{i p x}) . \quad (6)$$

and the last transformation we need is to define the $\mathcal{M}$ transformation for our function obtaining the quantum field solution for the Dirac equation in the B manifold. So, first of all, we put attention now that under the $CPTM$ transform the form of the Dirac equation is changed. Whereas the CPT transformation preserves the form of the Dirac equation, the additional $\mathcal{M}$ transformation clearly change the mass sign:

$$CPTM(\mathcal{L}) = \mathcal{L}_B = \bar{\psi}_B (\imath \gamma^a (\pm E_a^\mu) \partial_\mu + m) \psi_B . \quad (7)$$

In the expression an another interesting observation was made and implemented. Whereas we require the same form of the metric tensor for both manifolds, the sign of the vierbein fields can be changed even at the zero order, see further Section V. Therefore, we separate the possible forms of the Dirac equation in B manifold denoting them as for the positive and negative vierbeins and writing corresponding Lagrangians as $\mathcal{L}_{1B}$ and $\mathcal{L}_{2B}$. We have then:

$$\mathcal{L}_{1B} = \bar{\psi}_B (\imath \gamma^\mu \partial_\mu + m) \psi_B \quad (8)$$

and

$$\mathcal{L}_{2B} = -\bar{\psi}_B (\imath \gamma^\mu \partial_\mu - m) \psi_B . \quad (9)$$

Our next step is to adopt the Eq. (6) solution in the form suitable for the Dirac equation for the B manifold. Consider firstly the Eq. (8) form of the B-manifold Dirac Lagrangian. For the A manifold we have for the spinor classical plane wave solution of Klein-Gordon equation

$$\Psi(x) = u(p) e^{-i x p} + v(p) e^{i x p} , \quad (10)$$

where

$$(\gamma^\mu p_\mu - m) u(p) = 0 , \quad (\gamma^\mu p_\mu + m) v(p) = 0 \quad (11)$$

are the plane wave solutions for the Dirac equation and it's conjugated. We see, that the difference in Eq. (11) solutions is in the change of the mass sign, and, therefore, for the B-manifold we can write:

$$\Psi_{\mathcal{CPTM}}(x) = u(p) e^{i p x} + v(p) e^{-i p x} \quad (12)$$

with the same functional form of spinors preserved of course. The $v(p)$ spinor here is a plane wave solution of the Dirac equation obtained from Eq. (8) Lagrangian and $u(p)$ is it's conjugated. From this point of view, the Eq. (6) function has a correct expansion in terms of the properly defined plane waves of B-manifold, we write then:

$$\psi_{1B}(x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(\left(\hat{\mathcal{M}} b_{\mathbf{p}}^{s\dagger} \right) v^s(p) e^{-i p x} + \left(\hat{\mathcal{M}} a_{\mathbf{p}}^s \right) u^s(p) e^{i p x} \right), \quad (13)$$

with $\hat{\mathcal{M}}$ operator which determines the corresponding change of $b_{\mathbf{p}}^s$ and $a_{\mathbf{p}}^s$ operators under the $\mathcal{M}$ transform.

The Lagrangian Eq. (9) is a replica of the A-manifold Lagrangian, therefore for it we can use the Eq. (4) classical field expression with, of course, redefined operators of creation and annihilation:

$$\psi(x)_{2B} = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(\left(\hat{\mathcal{M}} a_{\mathbf{p}}^s \right) u^s(p) e^{-i p x} + \left(\hat{\mathcal{M}} b_{\mathbf{p}}^{s\dagger} \right) v^s(p) e^{i p x} \right). \quad (14)$$

In both cases, in order to define the anti-commutation relations for the new operators, we need to define some additional conditions to satisfy. This problem we consider in the next Section.

III. SPINOR FIELD OF B-MANIFOLD: FIRST VARIANT OF QUANTIZATION FOR THE LAGRANGIAN WITH POSITIVE VIERBEIN

Now we will use the obtained results in order to introduce the commutation relations for the new operators of ψ_{1B} field and consequent calculation of it's propagator. The condition we require is that the energy-momentum tensor for the both manifold will equal to zero at the vacuum state, so we define:

$$\begin{cases} \hat{\mathcal{M}} a_{\mathbf{p}}^s = c_{\mathbf{p}}^{s\dagger} \leftrightarrow a_{\mathbf{p}}^{s\dagger} \\ \hat{\mathcal{M}} b_{\mathbf{p}}^s = d_{\mathbf{p}}^{s\dagger} \leftrightarrow b_{\mathbf{p}}^{s\dagger} \end{cases} \rightarrow \begin{cases} \{ a_{\mathbf{p}}^r a_{\mathbf{k}}^{s\dagger} \} \rightarrow \{ c_{\mathbf{p}}^r c_{\mathbf{k}}^{s\dagger} \} = -\delta_{pk}^3 \delta^{rs}, \quad c_{\mathbf{p}}^r |0\rangle = 0 \\ \{ b_{\mathbf{p}}^r b_{\mathbf{k}}^{s\dagger} \} \rightarrow \{ d_{\mathbf{p}}^r d_{\mathbf{k}}^{s\dagger} \} = -\delta_{pk}^3 \delta^{rs}, \quad d_{\mathbf{p}}^r |0\rangle = 0 \end{cases} \quad (15)$$

with

$$\psi_B(x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(d_{\mathbf{p}}^s v^s(p) e^{-i p x} + c_{\mathbf{p}}^{s\dagger} u^s(p) e^{i p x} \right). \quad (16)$$

Correspondingly considering the mutual vacuum² state for the creation and annihilation operators in both manifolds we have:

$$\begin{cases} a_{\mathbf{p}}^s |0\rangle = b_{\mathbf{p}}^s |0\rangle = 0 \\ \langle 0| a_{\mathbf{p}}^{s\dagger} = \langle 0| b_{\mathbf{p}}^{s\dagger} = 0 \end{cases} \xleftrightarrow{CPTM} \begin{cases} \langle 0| c_{\mathbf{p}}^{s\dagger} = \langle 0| d_{\mathbf{p}}^{s\dagger} = 0 \\ c_{\mathbf{p}}^s |0\rangle = d_{\mathbf{p}}^s |0\rangle = 0. \end{cases} \quad (17)$$

The energy-momentum vector in turn acquires the following form:

$$\begin{aligned} P^\mu &= P_A^\mu + P_B^\mu = \sum_s \int d^3 k k^\mu \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger} + d_{\mathbf{p}}^{s\dagger} d_{\mathbf{p}}^s - c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\dagger} \right) = \\ &= \sum_s \int d^3 k k^\mu \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + d_{\mathbf{p}}^{s\dagger} d_{\mathbf{p}}^s + c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s \right) \end{aligned} \quad (18)$$

that provides with the use of Eq. (15)

$$\langle 0| P^\mu |0\rangle = 0. \quad (19)$$

² We do not discuss here the construction of this state, in feneral it can be also considered as superposition of two vacuum states for the A and B manifolds operatros separately.

From the Eq. (15) formal definition we see that we can define the $\mathcal{M}$ transform as following:

$$\hat{\mathcal{M}}a_{\mathbf{p}}^s = \eta a_{\mathbf{p}}^s, \quad \hat{\mathcal{M}}a_{\mathbf{p}}^{s\dagger} = \eta^* a_{\mathbf{p}}^{s\dagger}, \quad \eta^2 = \eta^{2*} = 1, \quad \eta \eta^* = -1. \quad (20)$$

It provides in turn

$$\{a_{\mathbf{p}}^r a_{\mathbf{k}}^{s\dagger}\} \rightarrow \{c_{\mathbf{p}}^r c_{\mathbf{k}}^{s\dagger}\} = \eta \eta^* \{a_{\mathbf{p}}^r a_{\mathbf{k}}^{s\dagger}\} = -\delta_{p k}^3 \delta^{rs} \quad (21)$$

as defined in Eq. (15).

The next we define it is a Feynman propagator for the B-manifold. We begin from the Feynman propagator of A manifold which is defined as usual:

$$\begin{aligned} S_F(x-y) &= -i \langle 0 | \mathbf{T} (\psi(x) \bar{\psi}(y)) | 0 \rangle = \\ &= -i (\theta(x^0 - y^0) \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle - \theta(y^0 - x^0) \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle) \end{aligned} \quad (22)$$

where the Wightman functions for the fermion fields of the A manifold can be defined through

$$\begin{aligned} D_1(x-y) &= \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = (\hat{\partial}_x + m) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\mathcal{E}_p} e^{-ip(x-y)} = \\ &= (\hat{\partial}_x + m) D(x-y); \end{aligned} \quad (23)$$

$$\begin{aligned} D_2(x-y) &= \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle = -(\hat{\partial}_x + m) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\mathcal{E}_p} e^{ip(x-y)} = \\ &= -(\hat{\partial}_x + m) D(y-x) \end{aligned} \quad (24)$$

with $D(x-y)$ are Wightman function of a scalar field. Consequently, using corresponding definition of the propagator for B manifold with Eq. (16) spinor fields and Eq. (15) anti-commutation relations, we obtain for the Feynman Green's function of the second manifold:

$$\tilde{S}_F(x-y) = CPTM(S_F(x-y)) = -(\hat{\partial}_x - m) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon} \quad (25)$$

similarly to scalar field results obtained in [2, 3], an additional minus here is due the Eq. (15) commutation relations.

IV. SPINOR FIELD OF B-MANIFOLD: SECOND VARIANT OF QUANTIZATION FOR THE LAGRANGIAN WITH POSITIVE VIERBEIN

We again consider Eq. (16) function

$$\psi_B(x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s (d_{\mathbf{p}}^s v^s(p) e^{-ipx} + c_{\mathbf{p}}^{s\dagger} u^s(p) e^{ipx}). \quad (26)$$

but change the properties of the operators, we define this time:

$$\begin{cases} \hat{\mathcal{M}} a_{\mathbf{p}}^s = c_{\mathbf{p}}^{s\dagger} \leftrightarrow a_{\mathbf{p}}^{s\dagger} \\ \hat{\mathcal{M}} b_{\mathbf{p}}^s = d_{\mathbf{p}}^{s\dagger} \leftrightarrow b_{\mathbf{p}}^{s\dagger} \end{cases} \rightarrow \begin{cases} \{a_{\mathbf{p}}^r a_{\mathbf{k}}^{s\dagger}\} \rightarrow \{c_{\mathbf{p}}^r c_{\mathbf{k}}^{s\dagger}\} = \delta_{p k}^3 \delta^{rs}, \quad \langle 0 | c_{\mathbf{p}}^r = 0 \\ \{b_{\mathbf{p}}^r b_{\mathbf{k}}^{s\dagger}\} \rightarrow \{d_{\mathbf{p}}^r d_{\mathbf{k}}^{s\dagger}\} = \delta_{p k}^3 \delta^{rs}, \quad \langle 0 | d_{\mathbf{p}}^r = 0. \end{cases} \quad (27)$$

Similarly to done before, we have for the energy-momentum vector in this case the following expression:

$$\begin{aligned} P^\mu &= P_A^\mu + P_B^\mu = \sum_s \int d^3 k k^\mu (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger} + d_{\mathbf{p}}^{s\dagger} d_{\mathbf{p}}^s - c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\dagger}) = \\ &= \sum_s \int d^3 k k^\mu (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - d_{\mathbf{p}}^s d_{\mathbf{p}}^{s\dagger} - c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\dagger}) \end{aligned} \quad (28)$$

that provides

$$\langle 0 | P^\mu | 0 \rangle = 0 \quad (29)$$

as expected. Now, using again Eq. (22) definition of the propagator together with Eq. (15) definition of the spinor field, we obtain in turn

$$\tilde{S}_F(x-y) = (\hat{\partial}_x - m) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 - i\varepsilon} \quad (30)$$

as a second possibility for the form of the Feynman propagator in the B manifold.

V. SPINOR FIELD OF B-MANIFOLD: QUANTIZATION FOR THE LAGRANGIAN WITH NEGATIVE VIERBEIN

Now we consider the B-manifold fermion as a precise replica of the A-manifold quantum field taking

$$\psi(x)_{2B} = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}p}} \sum_s \left(\left(\hat{\mathcal{M}} a_{\mathbf{p}}^s \right) u^s(p) e^{-i p x} + \left(\hat{\mathcal{M}} b_{\mathbf{p}}^{s\dagger} \right) v^s(p) e^{i p x} \right). \quad (31)$$

For the simplest possible variant

$$\left\{ \begin{array}{l} \hat{\mathcal{M}} a_{\mathbf{p}}^s = c_{\mathbf{p}}^s \leftrightarrow a_{\mathbf{p}}^s \\ \hat{\mathcal{M}} b_{\mathbf{p}}^s = d_{\mathbf{p}}^s \leftrightarrow b_{\mathbf{p}}^s \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \{ a_{\mathbf{p}}^r a_{\mathbf{k}}^{s\dagger} \} \rightarrow \{ c_{\mathbf{p}}^r c_{\mathbf{k}}^{s\dagger} \} = \delta_{p k}^3 \delta^{rs}, \quad c_{\mathbf{p}}^r |0\rangle = 0 \\ \{ b_{\mathbf{p}}^r b_{\mathbf{k}}^{s\dagger} \} \rightarrow \{ d_{\mathbf{p}}^r d_{\mathbf{k}}^{s\dagger} \} = \delta_{p k}^3 \delta^{rs}, \quad d_{\mathbf{p}}^r |0\rangle = 0 \end{array} \right. \quad (32)$$

we, of course will arrive to the

$$\begin{aligned} P^\mu &= P_A^\mu + P_B^\mu = \sum_s \int d^3 k k^\mu (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger} - c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s + d_{\mathbf{p}}^s d_{\mathbf{p}}^{s\dagger}) = \\ &= \sum_s \int d^3 k k^\mu (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s - d_{\mathbf{p}}^{s\dagger} d_{\mathbf{p}}^s) \end{aligned} \quad (33)$$

here the additional minus in front of the Lagrangian of B-manifold fermion, see Eq. (9), is accounted. Correspondingly we have

$$\langle 0 | P^\mu | 0 \rangle = 0 \quad (34)$$

as it must be for the case. The propagator here is the usual, Feynman, one:

$$\tilde{S}_F(x-y) = \left(i\hat{\partial}_x + m \right) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-i k(x-y)}}{k^2 - m^2 + i\varepsilon} \quad (35)$$

see Appendix B.

There is an additional possibilities to define the B-manifold quantization which is similar to the considered previously, this time we define:

$$\left\{ \begin{array}{l} \hat{\mathcal{M}} a_{\mathbf{p}}^s = c_{\mathbf{p}}^s \leftrightarrow a_{\mathbf{p}}^s \\ \hat{\mathcal{M}} b_{\mathbf{p}}^s = d_{\mathbf{p}}^s \leftrightarrow b_{\mathbf{p}}^s \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \{ a_{\mathbf{p}}^r a_{\mathbf{k}}^{s\dagger} \} \rightarrow \{ c_{\mathbf{p}}^r c_{\mathbf{k}}^{s\dagger} \} = -\delta_{p k}^3 \delta^{rs}, \quad \langle 0 | c_{\mathbf{p}}^r = 0 \\ \{ b_{\mathbf{p}}^r b_{\mathbf{k}}^{s\dagger} \} \rightarrow \{ d_{\mathbf{p}}^r d_{\mathbf{k}}^{s\dagger} \} = -\delta_{p k}^3 \delta^{rs}, \quad \langle 0 | d_{\mathbf{p}}^r = 0 \end{array} \right. \quad (36)$$

The energy-momentum vector in this case acquires the following form:

$$\begin{aligned} P^\mu &= P_A^\mu + P_B^\mu = \sum_s \int d^3 k k^\mu (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger} - c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s + d_{\mathbf{p}}^s d_{\mathbf{p}}^{s\dagger}) = \\ &= \sum_s \int d^3 k k^\mu (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\dagger} + d_{\mathbf{p}}^s d_{\mathbf{p}}^{s\dagger}) \end{aligned} \quad (37)$$

that also provides

$$\langle 0 | P^\mu | 0 \rangle = 0. \quad (38)$$

Correspondingly, for the quantum field

$$\psi(x)_{2B} = \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}p}} \sum_s (c_{\mathbf{p}}^s u^s(p) e^{-i p x} + d_{\mathbf{p}}^{s\dagger} v^s(p) e^{i p x}). \quad (39)$$

and Eq. (36) definition of the operators properties, the propagator is the same as in Eq. (30) with only $m \rightarrow -m$ replace, we have:

$$\tilde{S}_F(x-y) = - \left(i\hat{\partial}_x + m \right) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-i k(x-y)}}{k^2 - m^2 - i\varepsilon}. \quad (40)$$

The additional minus here is due the initial minus we accounted in Eq. (9), see the minus also in anti-commutator of Eq. (36). This is Dyson propagator as it defined by Eq. (B.5). We see, that this particular choice of the operators relations determines an analog of the Keldysh approach in the non-equilibrium condensed matter theory.

VI. ACTIONS FOR THE SPINORS IN AN EXTERNAL GRAVITATIONAL FIELD

In this article we do not discuss Einstein-Cartan gravity Lagrangian and action, see [25] for the examples of the Einstein-Cartan gravity formulation. The only assumption we made is that the gravity Lagrangian describes a torsionless relativity theory equivalent to the Einstein general relativity. The torsionless connections is defined then through the vierbein fields:

$$\omega_{cab} = (C_{cab} - C_{abc} + C_{bca}) \quad (41)$$

where

$$C_{abc} = E_a^\mu E_b^\nu (\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) = E_a^\mu E_b^\nu \partial_{[\mu} e_{\nu]c}. \quad (42)$$

The vierbein fields here are defined as usual, we have:

$$e_\mu^a E_b^\mu = \delta_b^a; \quad g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad (43)$$

with η_{ab} as some classical solution for the metric in the case of a non-flat space-time, further for sake of simplicity we will take η_{ab} as the Minkowski metric. Some truncation scheme is requested in this description when a perturbative solution for the vierbein is considered. Namely, in general in such kind of problem we define the vierbein field as some perturbative series:

$$e_\mu^a = \sum_{k=0}^{\infty} e_{k\mu}^a, \quad \eta_{ab} e_{0\mu}^a e_{0\mu}^b = \eta_{\mu\nu}. \quad (44)$$

The $e_{1,\mu}^a$ vierbein in the series can be found solving equations of motion of a corresponding part of the full gravity Lagrangian quadratic with respect to e_1 without or with the energy-momentum tensor in the r.h.s. of the equations, i.e. from the corresponding wave equation with or without source. For our aims it is enough to account only two first orders in Eq. (44), we will take

$$e_\mu^a = e_{0\mu}^a + e_{1\mu}^a \quad (45)$$

with the $e_{1\mu}^a$ vierbein as Einstein-Cartan gravity counterpart of the usual graviton field. For the B-manifold, correspondingly, we require that the metric tensor will remain the same and define

$$e_{B1\mu}^a = \pm (e_{0\mu}^a + e_{1\mu}^a), \quad E_{B1a}^\mu = \pm (E_{0a}^\mu - e_{1a}^\mu). \quad (46)$$

As discussed above, there are two possible vierbein's signs are possible here in general.

The main purposes of the approach is an investigation of the model of two manifolds which interact by gravity. Similarly to done in [2, 3], firstly we assume that the $e_{1\mu}^a$ is the same for the both Dirac-Einstein-Cartan systems in A and B manifolds, till the sign of the vierbein. Therefore, we have for the action of the interacting system of fermions:

$$\begin{aligned} S = S_A + S_B = & \int d^4 x_A e_A \bar{\psi}_A (\imath E_c^\mu \gamma^c D_\mu - m) \psi_A \pm \\ & \pm \int d^4 x_B e_B \bar{\psi}_B (\imath E_c^\mu \gamma^c D_\mu^\pm \pm m) \psi_B, \end{aligned} \quad (47)$$

with

$$e_{A,B} = \sqrt{-g(e_{A,B})} \quad (48)$$

and

$$D_\mu^\pm = \partial_\mu + \frac{1}{8} \omega_{\mu ab}^\pm [\gamma^a \gamma^b], \quad (49)$$

the subscripts A and B denote the corresponding manifolds, further we will use these indexes only in the need of special clarification. The subscripts $\pm$, in turn, denotes the terms with positive or negative sign of B-manifold vierbeins accounted. Next we write the positive linearized vierbein fields for A manifold as

$$e_\mu^a = \delta_\mu^a + e_{1\mu}^a, \quad E_a^\mu = \delta_a^\mu - \eta^{\mu\nu} \eta_{ab} e_{1\nu}^b \quad (50)$$

with $\eta^{\mu\nu}$ as Minkowski metric. The spinor field we write in the form of a fluctuation around classical solution taken as background field:

$$\psi_{A,B} \rightarrow \psi_{A,B\,cl} + \chi_{A,B} \quad (51)$$

with $\psi_{A,B\,cl}$ in the r.h.s. as solution of the classical Dirac equation without gravity present for A or B manifold. Further we will omit the $\psi_{A,B\,cl}$ notations writing these fields simply as $\psi_{A,B}$. Preserving in the connections fields only terms linear with respect to the vierbeins, we obtain for the Eq. (41) expression:

$$\omega_{cab} = \partial_c (e_{1ab} - e_{1ba}) - \partial_a (e_{1cb} + e_{1bc}) + \partial_b (e_{1ca} + e_{1ac}) . \quad (52)$$

Further in the calculations the only symmetrical vierbein combination is considered:

$$s_{cb} = \frac{1}{2} (e_{1cb} + e_{1bc}) , \quad s_{cb} = s_{bc} \quad (53)$$

that corresponds to the some chosen form of an external gravitational field of course, see also [26] for example. Now we note, that considering PT transformation of the Eq. (41)-Eq. (42) we obtain that for the case of the positive vierbein's transformation when $(e, E)_A \rightarrow (e, E)_B$, the connection changes sign: due the $\partial \rightarrow -\partial$ PT transform:

$$\omega_{cab}^+ \rightarrow -\omega_{cab} , \quad D_\mu^+ = \partial_\mu - \frac{1}{8} \omega_{\mu ab} [\gamma^a \gamma^b] \quad (54)$$

with ω provided by Eq. (52) of course. Whereas we have the negative vierbein's transformation, $(e, E)_A \rightarrow -(e, E)_B$, the connection does not change the sign

$$\omega_{cab}^- \rightarrow \omega_{cab} , \quad D_\mu^- = \partial_\mu + \frac{1}{8} \omega_{\mu ab} [\gamma^a \gamma^b] . \quad (55)$$

These transformation properties we account in the corresponding Lagrangians. Thereafter we define:

$$\omega_{cab} = \partial_b (e_{1ca} + e_{1ac}) - \partial_a (e_{1cb} + e_{1bc}) = 2 (\partial_b s_{ca} - \partial_a s_{cb}) . \quad (56)$$

For the Eq. (49) covariant derivative part we will have correspondingly:

$$\imath \frac{1}{8} \omega_{\mu ab} \gamma^c [\gamma^a \gamma^b] = \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b \quad (57)$$

Due the simplest perturbative scheme we consider, further we will take:

$$\sqrt{-g} \approx 1 + \frac{1}{2} \eta^{\mu\nu} h_{\mu\nu} = 1 + \eta^{ab} s_{ab} = 1 + s . \quad (58)$$

Collecting all terms in the background expansion of the fields and accounting classical equation of motion, we have for the whole action:

$$S = S_A + S_B^\pm = S_{0A} + S_{1A} + S_{0B}^\pm + S_{1B}^\pm , \quad (59)$$

with S_B^+ and S_B^- as Lagrangians defined for the B-manifold positive and negative vierbeins correspondingly, with all corresponding changes of signs accounted. The Eq. (59) terms, therefore, have the following from:

$$\begin{aligned} S_A = & \int d^4 x_A \bar{\chi}_A (\imath \gamma^\mu \partial_\mu - m) \chi_A + \\ & + \int d^4 x_A \bar{\psi}_A \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b - m s \right) \psi_A + \\ & + \int d^4 x_A \bar{\chi}_A \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b - m s \right) \psi_A + \\ & + \int d^4 x_A \bar{\psi}_A \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b - m s \right) \chi_A + \\ & + \int d^4 x_A \bar{\chi}_A \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b - m s \right) \chi_A . \end{aligned} \quad (60)$$

Correspondingly for the case of the positive B-manifold vierbein we have:

$$\begin{aligned}
S_B^+ = & \int d^4 x_B \bar{\chi}_B (\imath \gamma^\mu \partial_\mu + m) \chi_B + \\
& + \int d^4 x_B \bar{\psi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu - \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b + m s \right) \psi_B + \\
& + \int d^4 x_B \bar{\chi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu - \frac{\imath}{2} (\partial_b e_{B1ca} - \partial_a e_{1cb}) \gamma^c \gamma^a \gamma^b + m s \right) \psi_B + \\
& + \int d^4 x_B \bar{\psi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu - \frac{\imath}{2} (\partial_b e_{B1ca} - \partial_a e_{1cb}) \gamma^c \gamma^a \gamma^b + m s \right) \chi_B + \\
& + \int d^4 x_B \bar{\chi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu - \frac{\imath}{2} (\partial_b e_{B1ca} - \partial_a e_{1cb}) \gamma^c \gamma^a \gamma^b + m s \right) \chi_B, \tag{61}
\end{aligned}$$

whereas for the negative B-manifold vierbein we obtain

$$\begin{aligned}
S_B^- = & - \int d^4 x_B \bar{\chi}_B (\imath \gamma^\mu \partial_\mu - m) \chi_B - \\
& - \int d^4 x_B \bar{\psi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b - m s \right) \psi_B - \\
& - \int d^4 x_B \bar{\chi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b e_{B1ca} - \partial_a e_{1cb}) \gamma^c \gamma^a \gamma^b - m s \right) \psi_B - \\
& - \int d^4 x_B \bar{\psi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b e_{B1ca} - \partial_a e_{1cb}) \gamma^c \gamma^a \gamma^b - m s \right) \chi_B - \\
& - \int d^4 x_B \bar{\chi}_B \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b e_{B1ca} - \partial_a e_{1cb}) \gamma^c \gamma^a \gamma^b - m s \right) \chi_B = \\
& = -S_A, \tag{62}
\end{aligned}$$

In general formulation, the each action has the following form therefore:

$$S = \int d^4 x \bar{\chi} S_0 \chi + \int d^4 x \bar{\chi} M_1 \chi + \int d^4 x \bar{\chi} J(x) + \int d^4 x \bar{J}(x) \chi + \int d^4 x \bar{\psi} M_1(x) \psi, \tag{63}$$

with corresponding redefinition of the terms in the Lagrangian.

Now we have to account the all variants of the propagators for the B-manifold. We begin from the general expression for the whole action in the case of the positive vierbeins, we have:

$$\begin{aligned}
S_{B1,2}^+ = & \int d^4 x_B \bar{\chi}_B (\imath \gamma^\mu \partial_\mu + m) \chi_B + \int d^4 x_B \bar{\chi}_B J_{1B}^+(x) + \int d^4 x_B \bar{J}_{1B}^+(x) \chi_B + \\
& + \int d^4 x_B \bar{\chi}_B M_{1B}^+(x) \chi_B + \int d^4 x_B \bar{\psi}_B M_{1B}^+(x) \psi_B, \tag{64}
\end{aligned}$$

the notations here and further are shortened notations from the Eq. (62). In the cases of the Eq. (25) and Eq. (30) propagators, we have

$$\begin{aligned}
S_{F1}(x-y) = & - \left(\imath \hat{\partial}_x - m \right) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-\imath k(x-y)}}{k^2 - m^2 + \imath \varepsilon}, \\
\left(\imath \hat{\partial}_{\mu x} + m \right) S_{F1}(x-y) = & - \delta^4(x-y) \tag{65}
\end{aligned}$$

and

$$\begin{aligned}
S_{F2}(x-y) = & \left(\imath \hat{\partial}_x - m \right) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-\imath k(x-y)}}{k^2 - m^2 - \imath \varepsilon}, \\
\left(\imath \hat{\partial}_{\mu x} + m \right) S_{F2}(x-y) = & \delta^4(x-y). \tag{66}
\end{aligned}$$

for the the $S_{B1,2}$ actions correspondingly.

For the case of the negative vierbein's sign, we have correspondingly the following action:

$$\begin{aligned}
S_{B3,4}^- = & - \int d^4 x_B \bar{\chi}_B (\imath \gamma^\mu \partial_\mu - m) \chi_B - \int d^4 x_B \bar{\chi}_B J_{1B}^-(x) - \int d^4 x_B \bar{J}_{1B}^-(x) \chi_B - \\
& - \int d^4 x_B \bar{\chi}_B M_{1B}^-(x) \chi_B - \int d^4 x_B \bar{\psi}_B M_{1B}^-(x) \psi_B, \tag{67}
\end{aligned}$$

see Eq. (9) definition. The first variant of the propagator is given by Eq. (B.5)

$$\begin{aligned} S_{F3}(x-y) &= \left(\imath \hat{\partial}_x + m \right) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-\imath k(x-y)}}{k^2 - m^2 + \imath \varepsilon}, \\ \left(\imath \hat{\partial}_x - m \right) S_{F3}(x, y) &= \delta^4(x-y) \end{aligned} \quad (68)$$

whereas the second is

$$\begin{aligned} S_{F4}(x-y) &= - \left(\imath \hat{\partial}_x + m \right) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-\imath k(x-y)}}{k^2 - m^2 - \imath \varepsilon}, \\ \left(\imath \hat{\partial}_x - m \right) S_{F4}(x, y) &= -\delta^4(x-y) \end{aligned} \quad (69)$$

for the two Eq. (67) cases.

VII. ONE LOOP EFFECTIVE ACTION FOR SPINORS: SIMPLEST PERTURBATIVE ANALYSIS

In this Section we calculate one-loop effective action for the A manifold and compare it with the corresponding results for the actions of B manifold. The calculation we perform are restricted by the two simplest diagrams, we calculate a one-loop tadpole diagram and self-energy diagram with one and two external gravitons legs attached correspondingly. This restriction does not change the main consequences of the calculations. The cancellation of the one-loop terms in the effective action is due the loop's internal momenta integral and not affected by the number of the external gravity fields attached to the particular vertex. As mentioned above, of course the cancellation is possible only if we assume that the gravity fields attached to the loop is the same for A and B manifolds, see discussion further. Therefore, the simplest diagrams we considered are enough for the demonstration of the main idea of the proposed framework, more complete perturbative analysis of the full action will be done somewhere else. For the Eq. (60) action, after the integration with respect to χ , we will obtain the following expression for the effective action with one loop precision:

$$Z[\bar{\psi}_A, \psi_A] = Z_0^{-1} e^{\imath \Gamma_A(\bar{\psi}_A, \psi_A)} \quad (70)$$

with

$$\begin{aligned} \Gamma_A(\bar{\psi}_A, \psi_A) &= \int d^4 x \bar{\psi}_A M_1(x) \psi_A - \int d^4 x d^4 y \bar{\psi}_A(x) M_1(x) S_F(x, y) M_1(y) \psi_A(y) + \\ &+ Tr \ln(S_0 + M_1). \end{aligned} \quad (71)$$

In this definition we omit the gravitational field, it is considered as some background field and it's contribution in the generating functional we will account further. For our purposes it is enough to consider the diagrams with only two external gravitons included³. An ordinary expansion of the last term in the Eq. (71) provides to the required precision:

$$\begin{aligned} Tr \ln(S_0 + M_1) &= Tr \ln(S_0) + Tr \ln(I + S_F M_1) \approx Tr \ln(S_0) + Tr(S_F M_1) - \\ &- \frac{1}{2} Tr(S_F M_1 S_F M_1). \end{aligned} \quad (72)$$

We fix the Z_0 normalization constant in the Eq. (70) by condition⁴

$$Z_0^{-1} e^{\imath Tr \ln(S_0)} = 1 \quad (73)$$

leaving us with two contributions we need to calculate. Also, further we will not write explicitly the trace in the expressions if it will not lead to the confusion.

³ We did not consider the vertex of interactions of two vierbeins (gravitons) with a fermion-antifermion pair, therefore the two external gravitons contribution we account is not full one of course.

⁴ In general we can define the constant as $Z_0 = Z[\phi_{cl} = 0]$ with ϕ_{cl} as all possible classical background fields in the problem. In this case the Z_0 defines the contribution of the bubble diagrams in the action, i.e. diagrams without external legs. This definition of the bubble diagrams must be not confused with the bubble diagrams arise in the equation of motion. There it appears in the form of the momentum-energy tensor, i.e. after the variation of the action's diagrams with one external leg with respect to some ϕ_{cl} , see Eq. (80) definition further.

For the second term in the Eq. (72) we have formally:

$$\Gamma_1 = \int d^4x S_F(x, x) M_1(x). \quad (74)$$

This expression can be understood as

$$\Gamma_1 = i S_{FR}^E(0) \int d^4x \left(-i (\delta_c^\mu \partial_\mu s - \eta^{\mu\nu} \partial_\mu s_{c\nu}) \gamma^c + \frac{i}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b - m s \right) \quad (75)$$

with $S_{FR}^E(0)$ as Feynman propagator with some regularization applied in the momentum space after the rotation in an Euclidean space. The only non-zero term we have in this order is the following one:

$$\Gamma_1(p_f) = -4i m^2 G_{FR}^E(0) \int d^4x s(x) \quad (76)$$

where $G_{FR}^E(0)$ is a regularized Feynman propagator of the scalar field.

The third term in Eq. (72) has the following form:

$$\Gamma_2 = -\frac{1}{2} \int d^4x \int d^4y M_1(x) S_F(x, y) M_1(y) S_F(y, x) \quad (77)$$

which we write formally as

$$\begin{aligned} \Gamma_2 = & -\frac{1}{2} \int d^4x \int d^4y \left(i \gamma^c (\delta_c^\mu s \partial_\mu - \eta^{\mu\nu} s_{c\nu} \partial_\mu) - \frac{i}{2} \gamma^c [\gamma^a, \gamma^b] s_{ca} \partial_b - m s \right)_x S_F(x, y) \\ & \left(i \gamma^{c_1} (\delta_{c_1}^{\mu_1} s \partial_{\mu_1} - \eta^{\mu_1\nu_1} s_{c_1\nu_1} \partial_{\mu_1}) - \frac{i}{2} \gamma^{c_1} [\gamma^{a_1}, \gamma^{b_1}] s_{c_1 a_1} \partial_{b_1} - m s \right)_y S_F(y, x) \end{aligned} \quad (78)$$

where we performed integration by parts and derivative with respect to x and y act on the both Green's function. In terms of regularized integrals we obtain the following answer for this expression:

$$\begin{aligned} \Gamma_2 = & -\frac{3i}{8} \int \frac{d^4p_i}{(2\pi)^4} \int d^4p_f \tilde{s}(p_i) \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{l_E^2 - 5p_f^2 x(1-x)/2 + 2m^2}{(l_E^2 + \Delta)^2} - \\ & -\frac{3i}{16} \int \frac{d^4p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) \int d^4p_f \tilde{s}^{c\nu}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{l_E^2 + 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} + \\ & + 2i m^2 \int \frac{d^4p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4p_f \tilde{s}(p_f) \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{l_E^2 + p_f^2 x(1-x) - m^2}{(l_E^2 + \Delta)^2}. \end{aligned} \quad (79)$$

see the calculations in Appendix C. An expression for the corresponding energy-momentum tensor for the arbitrary weak field we can obtain from the usual definition of the tensor in the flat space-time:

$$\frac{1}{2} T^{\rho\sigma} = \frac{\delta \Gamma}{\delta h_{\rho\sigma}(z)} - \partial_\mu \frac{\delta \Gamma}{\delta (\partial_\mu h_{\rho\sigma}(z))}. \quad (80)$$

For example, the tensor to the first order approximation has the following form:

$$T_0^{\rho\sigma} = 2 \frac{\delta \Gamma_1}{\delta h_{\rho\sigma}(z)} = \frac{\delta \Gamma_1}{\delta s_{\rho\sigma}(z)} = -4 \int d^4x \delta^4(x - z) m^2 \eta^{\rho\sigma} G_F(x, x) = -4i m^2 G_{FR}^E(z, z) \eta^{\rho\sigma} \quad (81)$$

with $G_{FR}(z, z)$ as a regularized propagator of scalar field. We see, that this term has a form of the cosmological term contribution in the equations of motion, as expected.

A. First variant of the effective action for the B manifold

The first variant of the effective action for the B-manifold is determined by the it's bare action S_{B1} and propagator S_{F1} , Eq. (64) and Eq. (65) correspondingly. For this action the following expansion holds:

$$\begin{aligned} Tr \ln (S_{0B1}^+ + M_{1B}) &= Tr \ln (S_{0B1}^+) + Tr \ln (I - S_{F1} M_{1B}) \approx Tr \ln (S_0^+) - Tr (S_{F1} M_{1B}) - \\ &- \frac{1}{2} Tr (S_{F1} M_{1B} S_{F1} M_{1B}). \end{aligned} \quad (82)$$

Correspondingly, we have for the tadpole contribution:

$$\Gamma_{1B1} = - \int d^4x S_{F1}(x, x) M_{1B}(x) = -4m^2 G_{FR}(0) \int d^4x s(x) = \Gamma_{1A}, \quad (83)$$

i.e. the answer is the same as obtained for the Eq. (76) tadpole contribution in the effective action of A-manifold. For the one-loop contribution expression, first of all, we put attention that the differences between the third terms in Eq. (82) and Eq. (72) is in a sign of the m in the corresponding propagators and in M_1 terms. The second difference between the contributions, is in a sign of the connection field term in the Lagrangians. Because the Eq. (79) answer is invariant with respect to the $m \rightarrow -m$ replace we conclude, therefore, that for this variant of the B-manifold action

$$\Gamma_{B1} + \Gamma_A \neq 0 \quad (84)$$

to the given precision and correspondingly there is no any cancellation of contributions in the momentum-energy tensor.

B. Second variant of the effective action for the B manifold

In the case of Eq. (66) propagator we face the following picture. First of all, we have:

$$\begin{aligned} Tr \ln(S_{0B1}^+ + M_{1B}) &= Tr \ln(S_{0B1}^+) + Tr \ln(I + S_{F2} M_{1B}) \approx Tr \ln(S_{0B1}^+) + Tr(S_{F2} M_{1B}) - \\ &- \frac{1}{2} Tr(S_{F2} M_{1B} S_{F2} M_{1B}). \end{aligned} \quad (85)$$

For the tadpole contribution we write again:

$$\Gamma_{1B2} = - \int d^4x S_{F2}(x, x) M_{1B}(x) = 4\imath m^2 G_{FR}^E(0) \int d^4x s(x) = \Gamma_{1A}. \quad (86)$$

Here the rotation in an Euclidean space is given by $G_{FR} \rightarrow -\imath G_{FR}^E$ expression, there is a Dyson propagator we use. For the one-loop contribution the situation is a bit more complicated. For this term the symmetry in respect to $m \rightarrow -m$ exists as well, but now again we have the Dyson propagator in the expressions instead the Feynman one. Performing the $l_0 \rightarrow -\imath l_{0E}$ continuation into the Euclidean we obtain some part of the terms of Eq. (79) expression with $\imath \rightarrow -\imath$ replace done. Nevertheless, due the different sign of the connection field in the "long" derivative, see Eq. (61) action, not all the terms in the effective action change sign. Therefore, in general we have

$$\Gamma_{2B2} \neq -\Gamma_{2A} \quad (87)$$

and in this variant of the B-manifold we stay with the contributions to the effective action arise from the one-loop diagrams with two external off-shell gravity fields attached, i.e. with some diagrams which formally are of the graviton's mass like type.

C. Third variant of the effective action for the B manifold

In this particular case we have:

$$\begin{aligned} Tr \ln(S_{0B3}^- + M_{1B}^-) &= Tr \ln(S_{0B3}^-) + Tr \ln(I - S_{F3} M_{1B}^-) \approx Tr \ln(S_{0B3}^-) - Tr(S_{F3} M_{1B}^-) - \\ &- \frac{1}{2} Tr(S_{F3} M_{1B}^- S_{F3} M_{1B}^-). \end{aligned} \quad (88)$$

In this particular case the masses in answer for the tadpole contribution comes as

$$\Gamma_{1B3} = -4\imath m^2 G_{FR}^E(0) \int d^4x s(x) = \Gamma_{1A} \quad (89)$$

i.e. it is the same as contribution in the Γ_{1A} , they are doubling and not canceling. The same is correct for the Γ_{2B2} and Γ_{2A} contributions in the general effective action. Correspondingly have a doubling of the contribution into the momentum-energy tensor.

D. Fourth variant of the effective action for the B manifold

The one-loop contributions we consider in this case is the following one:

$$\begin{aligned} \text{Tr} \ln (S_{0B4}^- + M_{1B}^-) &= \text{Tr} \ln(S_{0B4}^-) + \text{Tr} \ln (I + S_{F4} M_{1B}^-) \approx \text{Tr} \ln(S_{0B4}) + \text{Tr} (S_{F4} M_{1B}^-) - \\ &- \frac{1}{2} \text{Tr} (S_{F4} M_{1B}^- S_{F4} M_{1B}^-) . \end{aligned} \quad (90)$$

In this case we have as well

$$\Gamma_{1B4} = - \int d^4x S_{F4}(x, x) M_{1B}(x) = 4m^2 \imath G_{FR}^E(0) \int d^4x s(x) = -\Gamma_{1A} , \quad (91)$$

the change of the sign here is due the use of Dyson propagator of course. So, again, a cancellation of tadpole contributions from S_A and S_{B4}^- actions is present here. More important, that here the identity

$$\Gamma_{2B4} = -\Gamma_{2A} \quad (92)$$

takes place, i.e. these terms are canceling in the total action as well. We obtained, that for this particular variant of the B-manifold action we have a total cancellation of the considered diagrams into the final effective action. Correspondingly, this cancellation occurs also for the the momentum-energy tensor and for the contributions into the cosmological constant value. Only this variant of the effective action, therefore, is considered further and we will denote it as Γ_B .

VIII. EFFECTIVE ACTION WITH GRAVITY INCLUDED

Now we ready to discuss the form of an appearance of gravity field in the model. Firstly consider the matter's fields action we obtained:

$$\begin{aligned} \Gamma(\bar{\psi}, \psi, h) &= \Gamma_A(\bar{\psi}_A, \psi_A, s) + \Gamma_B(\bar{\psi}_B, \psi_B, s) = \\ &= \int d^4x_A \bar{\psi}_A M_1(x_A) \psi_A - \int d^4x_A d^4y_A \bar{\psi}_A(x_A) M_1(x_A) S_F(x_A, y_A) M_1(y_A) \psi_A(y_A) - \\ &- \int d^4x_B \bar{\psi}_B M_1(x_B) \psi_B - \int d^4x_B d^4y_B \bar{\psi}_B(x_B) M_1(x_B) S_D(x_B, y_B) M_1(y_B) \psi_B(y_B) , \end{aligned} \quad (93)$$

here the S_{F4} propagator is denoted as the Dyson's one S_D . The action describes a classical fermion fields in two manifolds which interact with a background gravitational field. Writing precisely the form of M_1 effective vertex, we have:

$$\begin{aligned} M_1(x) &= \left(\imath (s \delta_c^\mu - \eta^{\mu\nu} s_{c\nu}) \gamma^c \partial_\mu + \frac{\imath}{2} (\partial_b s_{ca} - \partial_a s_{cb}) \gamma^c \gamma^a \gamma^b - m s \right) = \\ &= \left(\imath (s \gamma^\mu \partial_\mu - \eta^{\mu\nu} \gamma^c s_{c\nu} \partial_\mu + \gamma^a \partial_a s - \gamma^b \partial_a s_b^a) - m s \right) , \end{aligned} \quad (94)$$

here the Eq. (C.27) identity was used. Taking into account that

$$(\imath \gamma^\mu \partial_\mu - m) \psi_{A,B} = 0, \quad \partial_a (\bar{\psi}_{A,B} \gamma^a \psi_{A,B}) = 0, \quad (95)$$

we stay with

$$M_1(x) = -\imath (\gamma^c s_c^\mu \partial_\mu + \gamma^b \partial_a s_b^a) . \quad (96)$$

Further, in the gravitational field action, we make an usual redefinition of the weak gravity field which in turn leads to

$$s_{ab} = \bar{s}_{ab} - \frac{1}{2} \eta_{ab} \bar{s}, \quad \partial_a \bar{s}_b^a = 0 \quad (97)$$

redefinition and gauge condition for the s field. In terms of redefined s field we obtain for the Eq. (96) expression:

$$M_1(x) = -\imath \gamma^c \bar{s}_c^\mu \partial_\mu + \frac{1}{2} m \bar{s} = \bar{s}_{\mu\nu} \left(J_1^{\mu\nu} + \frac{1}{2} m \eta^{\mu\nu} \right) = \bar{s}_{\mu\nu} J^{\mu\nu} . \quad (98)$$

here again the identities Eq. (95) exact for the classical fermion fields were used.

Our next step is an introduction of the gravity on the stage. In the framework we introduce separately the gravity field limits for the A and B manifolds defining

$$S_{gr} = -m_p^2 \int_{-\infty}^{\infty} dt_A \int d^3x_A \sqrt{-g(x_A)} R(x_A) - m_p^2 \int_{-\infty}^{\infty} dt_B \int d^3x_B \sqrt{-g_B(x_B)} R(x_B) \quad (99)$$

with

$$m_p^2 = \frac{1}{16\pi G} = \frac{2}{\kappa^2}, \quad g_B(x_B) = CPTM(g_A(x_A)), \quad (100)$$

i.e. the B-manifold metric is defined through the metric of A-manifold by corresponding CPTM transform. We also assume that there no any cosmological constant terms in the actions, it will arise dynamically through the interactions between manifolds. In the case of a standard weak field approximation we have:

$$g_{\mu\nu}^A(x) = g_{\mu\nu}^{A0}(x) + h_{\mu\nu}^A(x); \quad (101)$$

$$g_{\mu\nu}^B(y) = g_{\mu\nu}^{B0}(y) + h_{\mu\nu}^B(y); \quad (102)$$

Further we will use the usual redefined h field and gauge condition introduced for it

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}, \quad \partial_\mu \bar{h}^\mu_\nu = 0, \quad (103)$$

see the Eq. (97) in the correspondence. Now, the important question we need to answer on, is about the gravity fields appearance in the Eq. (93) expression. Further we will discuss two different possibilities for the inclusion of the gravity in the Eq. (93). Before that, firstly, we will consider a mostly general expression for the action with gravitational fields and will write the corresponding Green's function and classical solutions adapting them later to the particular cases. In order to avoid a detailed notations in the long expressions, further we will write only the superscripts A and B for the fields without Lorentzian indexes sometimes in the expressions where it will not lead to a confusion.. An another important remarks is about the coordinates notations for different manifolds. In general we have no way to separate the fields by their dependence on the x_A or x_B coordinates. Therefore, we will preserve the A and B indexes only as signs of the different fields and will not distinguish the coordinates of both manifolds in the expressions. The mostly general action we have to the given precision is the following one⁵:

$$\begin{aligned} \Gamma = & \frac{m_p^2}{4} \int d^4x \bar{h}_A^{\mu\nu} G_A^{-1} \bar{h}_{\mu\nu}^A + \frac{m_p^2}{4} \int d^4x \bar{h}_B^{\mu\nu} G_B^{-1} \bar{h}_{\mu\nu}^B + \int d^4x \bar{h}_A^{\mu\nu} T_{\mu\nu A} + \\ & + \int d^4x \bar{h}_B^{\mu\nu} T_{\mu\nu B} + \eta_{\mu\nu} \int d^4x \bar{h}_A^{\mu\nu} \Lambda_{0A} + \eta_{\mu\nu} \int d^4x \bar{h}_B^{\mu\nu} \Lambda_{0B} - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1AA} \bar{h}_A^{\rho\sigma}(y) - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1BB} \bar{h}_B^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1AB} \bar{h}_B^{\rho\sigma}(y) - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1BA} \bar{h}_A^{\rho\sigma}(y) = \\ & = \Gamma_{0gr} + \Gamma_{int}. \end{aligned} \quad (104)$$

The notations used are simple, the $M_{\mu\nu\rho\sigma}^{1IJ}$ here are vertices of effective interaction between $I = A, B$ and $J = A, B$ gravity fields, the interaction can take place between the fields in the same or in the different manifolds, all other quantities in the expression are clear. The cosmological constant term in the expression must be understood as appearing from the corresponding terms in the matter's action and not as introduced by "hands". The Green's functions equation has the following form in turn:

$$\frac{m_p^2}{2} \begin{pmatrix} G_A^{-1} - \frac{2}{m_p^2} M_1^{AA} & -\frac{2}{m_p^2} M_1^{AB} \\ -\frac{2}{m_p^2} M_1^{BA} & G_B^{-1} - \frac{2}{m_p^2} M_1^{BB} \end{pmatrix} \begin{pmatrix} G_{AA} & G_{AB} \\ G_{BA} & G_{BB} \end{pmatrix} = \begin{pmatrix} \delta_{AA} & 0 \\ 0 & \delta_{BB} \end{pmatrix}.$$

The S_{AB} and S_{BA} propagators here are analog of the Wightman propagators $S_>$ and $S_<$ which connect the points of the different time's contours in the Schwinger-Keldysh technique, see for examples and definitions [18]. In order not

⁵ We adjusted the numerical coefficients and notations here for the simplification of the further calculations.

to write overweighted expressions, we presents here the Green's functions for the A-manifold only, expressions for the second pair of the Green's functions can be obtained by replace of A and B indexes there:

$$G_A^{-1} G_{0AA} = \frac{2}{m_p^2} \delta_{AA}, \quad G_B^{-1} G_{0BB} = \frac{2}{m_p^2} \delta_{BB}; \quad (105)$$

$$G_B^{-1} G_{0BA} = 0, \quad G_A^{-1} G_{0AB} = 0; \quad (106)$$

$$G_{AA} = G_{0AA} + \int G_{0AA} M^{1AA} G_{AA} + \int G_{0AA} M^{1AB} G_{BA} + \\ + \int G_{0AB} M^{1BA} G_{AA} + \int G_{0AB} M^{1BB} G_{BA}; \quad (107)$$

$$G_{AB} = G_{0AB} + \int G_{0AA} M^{1AB} G_{BB} + \int G_{0AA} M^{1AA} G_{AB} + \\ + \int G_{0AB} M^{1BA} G_{AB} + \int G_{0AB} M^{1BB} G_{BB}; \quad (108)$$

the repeating indexes mean also a corresponding integration in the expression, of course. The most general expression for the gravitational field we have is the following therefore:

$$h_I = h_{0I} - \frac{2}{m_p^2} \int G_{IJ} \frac{\delta \Gamma_{int}}{\delta h_J} + \xi_I; \quad G_I^{-1} h_{0I} = 0; \quad I, J = A, B, \quad (109)$$

with ξ_I as a fluctuation around a classical solution represented by the first two terms in the expression.

IX. APPEARANCE OF COSMOLOGICAL CONSTANT IN THE MODEL: FIRST VARIANT

The terms of the M^{1AB} and M^{1BA} type arise in the corresponding effective action in the Schwinger-Keldysh approach. We did not consider the full analog of the Schwinger-Keldysh action in our framework, it will be done in an separate publication. Nevertheless, in general the formal difference between the present and Schwinger-Keldysh frameworks⁶ it is an absence in the Eq. (93) effective action of additional vertices of the following form⁷

$$M_{1AB}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{2} \bar{\psi}_A(x) J_x^{\mu\nu} S_>(x, y) J_y^{\rho\sigma} \psi_B(y); \quad (110)$$

$$M_{1BA}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{2} \bar{\psi}_B(x) J_x^{\mu\nu} S_<(x, y) J_y^{\rho\sigma} \psi_A(y) \quad (111)$$

attached to the s graviton field similarly to written in Eq. (93), see also Eq. (98). Formally speaking, these vertices generate an additional one loop contributions to the fermionic effective action through the $S_>(x, y) S_<(y, x)$ type of expressions. We did not discuss and explore these types of contributions postponing that for a separate publication, so further by one loop contributions we will mean only ordinary contributions discussed above.

Now, the main question we ask is about a connection between the s and $h_{A,B}$ fields. In the case of the Schwinger-Keldysh like formulation of the approach we assume that there are two different fields

$$s_A = \frac{h_A}{2}, \quad s_B = \frac{h_B}{2} \quad (112)$$

for the A and B manifolds separately exist. We note immediately, therefore, that then the cancellation between the effective action parts does not take place, see further. Making the substitutions in Eq. (93) action, we will obtain an precise analog of Eq. (104):

$$\Gamma = \frac{m_p^2}{4} \int d^4x \bar{h}_A^{\mu\nu} G_A^{-1} \bar{h}_{\mu\nu}^A + \frac{m_p^2}{4} \int d^4x \bar{h}_B^{\mu\nu} G_B^{-1} \bar{h}_{\mu\nu}^B + \int d^4x \bar{h}_A^{\mu\nu} T_{\mu\nu A} - \\ - \int d^4x \bar{h}_B^{\mu\nu} T_{\mu\nu B} + \eta_{\mu\nu} \int d^4x \bar{h}_A^{\mu\nu} \Lambda_{0A} - \eta_{\mu\nu} \int d^4x \bar{h}_B^{\mu\nu} \Lambda_{0B} - \\ - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1AA} \bar{h}_A^{\rho\sigma}(y) - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1BB} \bar{h}_B^{\rho\sigma}(y) - \\ - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1AB} \bar{h}_B^{\rho\sigma}(y) - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1BA} \bar{h}_A^{\rho\sigma}(y) \quad (113)$$

⁶ See also Discussion section

⁷ The signs of the terms and numerical coefficients in the front of them are not important for the further discussion, we simply write them similarly to the other vertices and introduce in the action by "hand".

with

$$M_{1AA}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{2} \bar{\psi}_A(x) J_x^{\mu\nu} S_F(x, y) J_y^{\rho\sigma} \psi_A(y); \quad (114)$$

$$M_{1BB}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{2} \bar{\psi}_B(x) J_x^{\mu\nu} S_D(x, y) J_y^{\rho\sigma} \psi_B(y); \quad (115)$$

$$T_{A,B}^{\mu\nu} = \frac{1}{2} \bar{\psi}_{A,B} J_1^{\mu\nu} \psi_{A,B}, \quad \Lambda_{0A,B} = \frac{1}{4} m \bar{\psi}_{A,B} \psi_{A,B}; \quad (116)$$

see Eq. (98) definition again. Now we can determine what is the cosmological constant in the framework, to the leading order we define it as

$$\begin{aligned} \Lambda_A &= \frac{1}{m_p^2} \left. \frac{\delta \Gamma}{\delta \bar{h}_A} \right|_{\bar{h}_A=0} = \\ &= \frac{m}{4m_p^2} \bar{\psi}_A \psi_A - \frac{m^2}{16m_p^2} \int d^4y (\bar{\psi}_A(x) S_>(x, y) \psi_B(y) + \bar{\psi}_B(y) S_<(y, x) \psi_A(x)) \bar{h}_B(y). \end{aligned} \quad (117)$$

and correspondingly

$$\begin{aligned} \Lambda_B &= \frac{1}{m_p^2} \left. \frac{\delta \Gamma}{\delta \bar{h}_B} \right|_{\bar{h}_B=0} = \\ &= -\frac{m}{4m_p^2} \bar{\psi}_B \psi_B - \frac{m^2}{16m_p^2} \int d^4y (\bar{\psi}_A(y) S_>(y, x) \psi_B(x) + \bar{\psi}_B(x) S_<(x, y) \psi_A(y)) \bar{h}_A(y). \end{aligned} \quad (118)$$

We see that to leading order the constant is proportional to the following values:

$$\Lambda_{0A} = \frac{1}{4m_p^2} m \bar{\psi}_A \psi_A; \quad (119)$$

$$\Lambda_{0B} = -\frac{1}{4m_p^2} m \bar{\psi}_B \psi_B; \quad (120)$$

with plus or minus sign, the next-order corrections are due the interactions between the fermion fields from different manifolds.

Now we can comment on the above results, not in depth, but clarifying interesting points. First of all we note that

$$m \bar{\psi}_{A,B} \psi_{A,B} = (T_{0\mu}^\mu)_{A,B}, \quad (121)$$

i.e. the cosmological constant to that order is proportional to the leading order of trace of energy-momentum tensor. This result is expected, the constant occurs in the effective action after the following decomposition of the on-shell energy-momentum tensor:

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \eta_{\mu\nu} T_\rho^\rho \quad (122)$$

see Eq. (98) above. Therefore, it is quite natural to assume that the cosmological constant term must emerge in the Eq. (113) action as

$$\Gamma_\Lambda = \eta_{\mu\nu} \int d^4x \bar{h}_A^{\mu\nu} T_{A\rho}^\rho - \eta_{\mu\nu} \int d^4x \bar{h}_B^{\mu\nu} T_{B\rho}^\rho. \quad (123)$$

Having in mind that we in general can write the trace of the tensor as

$$T_\mu^\mu = T_{cl\mu}^\mu + T_{a\mu}^\mu = (\epsilon - 3p) + T_{a\mu}^\mu, \quad (124)$$

where we approximated the classical part of the trace by usual expression and as $T_{a\mu}^\mu$ we understand a trace of the conformal anomaly of the tensor, see [27], we obtain:

$$\Lambda_{0A} = \frac{1}{4m_p^2} (\epsilon - 3p + T_{a\mu}^\mu). \quad (125)$$

With precise inequality

$$p \leq \frac{\epsilon}{3} \quad (126)$$

and small conformal anomaly contribution, the Eq. (125) determines the sign of the constant. So, the second comment we have is about the issue of the constant's sign. In our notations the positive value of the constant (if it positive after all of course) appears at the r.h.s. of the Einstein equations

$$G_{\mu\nu} = \frac{2}{m_p^2} (T_{\mu\nu} + \eta_{\mu\nu} T^\rho_\rho) \quad (127)$$

thereby with the positive value of Eq. (125) constant, the constant is negative in usual definition of the term (and vice versa of course). Therefore, the conclusion that Eq. (125) and direct application of the Schwinger-Keldysh formalism do provide the correct answer unless Eq. (125) value is positive.

As we mentioned above, in an analog of the Schwinger-Keldysh formulation of the problem there is no cancellation between the different contributions into the effective spinor's action. The spinor's action terms under consideration are depend on s field, see Eq. (76) or/and Eq. (78), and when the fields on both manifolds are different, see Eq. (112), these terms are not canceled. Indeed, we have for the classical fields:

$$s_A = \frac{h_A}{2} = \frac{1}{2} h_{0A} - \int G_{AA} \Lambda_{0A} + \dots \quad (128)$$

$$s_B = \frac{h_B}{2} = \frac{1}{2} h_{0B} - \int G_{BB} \Lambda_{0B} + \dots, \quad (129)$$

and even assuming that

$$h_{0A} = h_{0B} \quad (130)$$

we have that the cancellation of the contributions occurs only to leading order. In general there is some perturbative equation for the cosmological constant due the not-full cancellation of the one-loop contribution to the constant. We can achieve the full cancellation of the mentioned terms assuming that $h_A = h_B = h$. In this case the CPTM symmetry can be, perhaps, understood in the sense of "clone" model of 't Hooft, see [28]. We have only one gravity field overall in this case, with corresponding change in Eq. (113) action we correspondingly will obtain to leading order:

$$\Lambda_A = \frac{1}{m_p^2} \frac{\delta \Gamma}{\delta \bar{h}} \Big|_{\bar{h}=0} = \frac{m}{4m_p^2} (\bar{\psi}_A \psi_A - \bar{\psi}_B \psi_B) \quad (131)$$

or for general situation we can assume that

$$\Lambda_A = \frac{1}{m_p^2} \frac{\delta \Gamma}{\delta \bar{h}} \Big|_{\bar{h}=0} = \frac{1}{4m_p^2} (\mathcal{T}_{A\mu}^\mu - \mathcal{T}_{B\mu}^\mu) = -\frac{1}{4m_p^2} (\mathcal{T}_{B\mu}^\mu - \mathcal{T}_{A\mu}^\mu) \quad (132)$$

The correct value of the constant is provided in this case by

$$\mathcal{T}_{B\mu}^\mu > \mathcal{T}_{A\mu}^\mu \quad (133)$$

condition. Therefore, the particular formulation can be defined as a "dark" matter model, more similar to the [5, 6] construction than to the bimetric models [7].

We return now to the Green's functions, Eq. (107) expression, and consider a change of the function's structure due the B-manifold presence in comparison to the usual gravity Green's function. We have:

$$G_{AA} = G_{AA}^1 + G_{AA}^2 \quad (134)$$

with

$$G_{AA}^1 = G_{0AA} + \int G_{0AA} M^{1AA} G_{AA} \quad (135)$$

as bare A-manifold graviton's Green's function and

$$G_{AA}^2 = \int G_{0AA} M^{1AB} G_{BA} + \int G_{0AB} M^{1BA} G_{AA} + \int G_{0AB} M^{1BB} G_{BA} + \int G_{0AA} G_{0AB} G_{0BA} + \dots \quad (136)$$

as corrections to the G_{AA}^1 propagator arise in the formalism due the B-manifold presence. Therefore, the first part of the full propagator is an usual propagator of a graviton, i.e. it represents an usual gravity potential, whereas the

second term is a correction to the potential due an interactions of matter and gravity between A and B-manifolds. In that extend, the B-manifold plays a role of the dark matter in the approach of the extended manifold. Subsequently, when we consider only one gravity field, see discussion above, then the corrections to the usual propagator will acquire the following form:

$$G_{AA}^2 = \int G_{0AA} M^{1AB} M^{1BA} G_{AA} + \dots \quad (137)$$

These corrections are due the Schwinger-Keldysh like mechanism of the interactions applied to the different fermions fields only, see Eq. (110)-Eq. (111) definitions. In the case when we have no any connection between A and B manifolds, i.e. in the case of an absence of the Schwinger-Keldysh type of interactions for all fields, the gravitational propagators are not changing but the one-loop cancellation of the effective action terms occurs as well in the case of the same gravitational field of course.

X. APPEARANCE OF COSMOLOGICAL CONSTANT IN THE MODEL: SECOND VARIANT

As an another possibility for the introduction of h fields in the action with s vierbein present, we consider the following definition of both A,B vierbein fields in the effective fermion's action:

$$\bar{s}_{\mu\nu} = \frac{1}{2} \bar{h}_{\mu\nu} = \frac{1}{4} (\bar{h}_{\mu\nu}^A + \bar{h}_{\mu\nu}^B) \quad (138)$$

The field is the same for the A,B manifolds therefore the precise cancellation of the one loop Γ_2 terms exists in this case. The assumption made here is that any matter is a source for the both $h_{A,B}$ fields and this is a main difference between the Eq. (138) and Eq. (112) expressions. Namely, in the previous section we considered the interaction between the h_A and h_B fields only through the Schwinger-Keldysh type of interaction, each field was present only on it's own manifold. Now, instead, we discuss the interaction which are initially non-linear because of the momentum-energy tensor which interacts with both fields. Consequently, we also can consider different definitions of the full metric through the weak fields for the separated manifolds, it can be defined similarly to done in Eq. (101)-Eq. (102) or as

$$g_{\mu\nu}^A(x) = g_{\mu\nu}^{A0}(x) + \frac{1}{2} (h_{\mu\nu}^A(x) + h_{\mu\nu}^B(x)) ; \quad (139)$$

$$g_{\mu\nu}^B(y) = g_{\mu\nu}^{B0}(y) + \frac{1}{2} (h_{\mu\nu}^A(y) + h_{\mu\nu}^B(y)) , \quad (140)$$

further we consider the both cases.

So, for the Eq. (101)-Eq. (102) weak fields definition, the mostly general Lagrangian we have is the following one:

$$\begin{aligned} \Gamma = & \frac{m_p^2}{4} \int d^4x \bar{h}_A^{\mu\nu} G_{AA}^{-1} \bar{h}_{\mu\nu}^A + \frac{m_p^2}{4} \int d^4x \bar{h}_B^{\mu\nu} G_{BB}^{-1} \bar{h}_{\mu\nu}^B + \int d^4x \bar{h}_A^{\mu\nu} T_{\mu\nu A} + \\ & + \int d^4x \bar{h}_B^{\mu\nu} T_{\mu\nu B} + \eta_{\mu\nu} \int d^4x \bar{h}_A^{\mu\nu} \Lambda_{0A} + \eta_{\mu\nu} \int d^4x \bar{h}_B^{\mu\nu} \Lambda_{0B} - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1AA} + M_{\mu\nu\rho\sigma}^{1BB})_{xy} \bar{h}_A^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1AA} + M_{\mu\nu\rho\sigma}^{1BB})_{xy} \bar{h}_B^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1AA} + M_{\mu\nu\rho\sigma}^{1BB})_{xy} \bar{h}_B^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1AA} + M_{\mu\nu\rho\sigma}^{1BB})_{xy} \bar{h}_A^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1AB} \bar{h}_B^{\rho\sigma}(y) - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1BA} \bar{h}_A^{\rho\sigma}(y) \end{aligned} \quad (141)$$

with

$$M_{1AA}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{8} \bar{\psi}_A(x) J_x^{\mu\nu} S_F(x, y) J_y^{\rho\sigma} \psi_A(y); \quad (142)$$

$$M_{1BB}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{8} \bar{\psi}_B(x) J_x^{\mu\nu} S_D(x, y) J_y^{\rho\sigma} \psi_B(y); \quad (143)$$

$$M_{1AB}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{8} \bar{\psi}_A(x) J_x^{\mu\nu} S_{>}(x, y) J_y^{\rho\sigma} \psi_B(y); \quad (144)$$

$$M_{1BA}^{\mu\nu\rho\sigma}(x, y) = \frac{1}{8} \bar{\psi}_B(x) J_x^{\mu\nu} S_{<}(x, y) J_y^{\rho\sigma} \psi_A(y). \quad (145)$$

The $T_{A,B}^{\mu\nu}$ and $\Lambda_{0A,B}$ in the expression are defined by the Eq. (93) effective action, we have for the A-manifold to leading order precision:

$$\begin{aligned} & \left. \frac{\delta\Gamma(\bar{\psi}, \psi, h)}{\delta\bar{h}_A^{\mu\nu}(z)} \right|_{\bar{h}_A=0} = \\ &= \frac{1}{4} \frac{\delta}{\delta\bar{h}_A^{\mu\nu}(z)} \left(\int d^4x \bar{\psi}_A(x) \left(\bar{h}_A^{\mu\nu}(x) + \bar{h}_B^{\mu\nu}(x) \right) \left(J_{1\mu\nu} + \frac{1}{2} \eta_{\mu\nu} m \right) \psi_A(x) - \right. \\ & \quad \left. - \int d^4x \bar{\psi}_B(x) \left(\bar{h}_A^{\mu\nu}(x) + \bar{h}_B^{\mu\nu}(x) \right) \left(J_{1\mu\nu} + \frac{1}{2} \eta_{\mu\nu} m \right) \psi_B(x) \right) = \\ &= \frac{1}{4} \left[\bar{\psi}_A(z) J_{1\mu\nu} \psi_A(z) - \bar{\psi}_B(z) J_{1\mu\nu} \psi_B(z) + \right. \\ & \quad \left. + \frac{1}{2} m \eta_{\mu\nu} (\bar{\psi}_A(z) \psi_A(z) - \bar{\psi}_B(z) \psi_B(z)) \right] = \\ &= T_{\mu\nu A} + m_p^2 \eta_{\mu\nu} \Lambda_A. \end{aligned} \quad (146)$$

So, the cosmological constants we have here are

$$\Lambda_{0A} = \Lambda_{0B} = \frac{1}{8} \frac{m}{m_p^2} (\bar{\psi}_A(z) \psi_A(z) - \bar{\psi}_B(z) \psi_B(z)) \quad (147)$$

or

$$\Lambda_{0A} = \Lambda_{0B} = -\frac{1}{8m_p^2} (\mathcal{T}_{B\mu}^\mu - \mathcal{T}_{A\mu}^\mu); \quad (148)$$

the answer is different from the previous Eq. (119)-Eq. (120) expressions by absolute values and by sign of the Λ_B . The cosmological constants in the expression depend on the difference of the densities of the manifolds at the same moment of time but for the functions which can live in a different time's scales. Again, additionally to the trivial

$$\mathcal{T}_{B\mu}^\mu = \mathcal{T}_{A\mu}^\mu \quad (149)$$

identity⁸, we can consider a case when the time variable of the functions is

$$t_{B,A} = \frac{T}{2} \pm t; \quad \zeta = t/T \quad (150)$$

obtaining

$$\Lambda_{0A} = -\frac{1}{8m_p^2} \left(\mathcal{T}_{B\mu}^\mu \left(\frac{T}{2} + t \right) - \mathcal{T}_{A\mu}^\mu \left(\frac{T}{2} - t \right) \right) \approx -\frac{1}{4Tm_p^2} \left(\frac{\partial \mathcal{T}_{0B\mu}^\mu}{\partial \zeta} + \frac{\partial \mathcal{T}_{0A\mu}^\mu}{\partial \zeta} \right) t. \quad (151)$$

Here the $T/2$ we can consider roughly as a half of Universe time age, the cosmological constant is changing with time and rate of the the change is defined by the change of the trace of the energy-momentum tensor, which, in turn, is changing due a new matter creation and/or annihilation in our Universe. The model, in fact describes this effect.

⁸ This identity can not be absolutely correct due the non-zero value of Λ_A of course.

Considering the action structure, we see from Eq. (93) expression, that there are terms which describe a creation of ψ_B fermions through propagation of the $\bar{h}_A^{\mu\nu}$ gravitational wave. This field does not interact directly with an ordinary matter and affects on the propagation of the gravitational fields similarly to the A-spinor fields. Indeed, the Green's functions for this variant of the action can be obtained from Eq. (107) result by the following replacement:

$$M_{1AA}, M_{1BB} \rightarrow M_{1AA} + M_{1BB}. \quad (152)$$

Therefore, in comparison to Eq. (135) answer, we obtain that in this particular case there is a modification of the propagator not only through interactions with the B manifold but also through the creation and/or interaction of the gravitational wave with the B-spinors present in the A manifold. In this extend these spinors are dark matter, of course an additional change of the gravitational potential through the interaction with B manifold can be presented here as well.

For the Eq. (139)-Eq. (140) weak fields we have the following effective action:

$$\begin{aligned} \Gamma = & \frac{m_p^2}{16} \int d^4x \bar{h}_A^{\mu\nu} (G_{AA}^{-1} + G_{BB}^{-1}) \bar{h}_{\mu\nu}^A + \frac{m_p^2}{16} \int d^4x \bar{h}_B^{\mu\nu} (G_{BB}^{-1} + G_{AA}^{-1}) \bar{h}_{\mu\nu}^B + \\ & + \frac{m_p^2}{8} \int d^4x \bar{h}_A^{\mu\nu} (G_{AA}^{-1} + G_{BB}^{-1}) \bar{h}_{\mu\nu}^B + \int d^4x \bar{h}_A^{\mu\nu} T_{\mu\nu A} + \\ & + \int d^4x \bar{h}_B^{\mu\nu} T_{\mu\nu B} + \eta_{\mu\nu} \int d^4x \bar{h}_A^{\mu\nu} \Lambda_{0A} + \eta_{\mu\nu} \int d^4x \bar{h}_B^{\mu\nu} \Lambda_{0B} - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1AA} + M_{\mu\nu\rho\sigma}^{1BB})_{xy} \bar{h}_A^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1BB} + M_{\mu\nu\rho\sigma}^{1AA})_{xy} \bar{h}_B^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1AA} + M_{\mu\nu\rho\sigma}^{1BB})_{xy} \bar{h}_B^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) (M_{\mu\nu\rho\sigma}^{1AA} + M_{\mu\nu\rho\sigma}^{1BB})_{xy} \bar{h}_A^{\rho\sigma}(y) - \\ & - \frac{1}{2} \int d^4x \int d^4y \bar{h}_A^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1AB} \bar{h}_B^{\rho\sigma}(y) - \frac{1}{2} \int d^4x \int d^4y \bar{h}_B^{\mu\nu}(x) M_{\mu\nu\rho\sigma}^{1BA} \bar{h}_A^{\rho\sigma}(y). \end{aligned} \quad (153)$$

The difference between this action and the expression Eq. (141) in this order of precision is only in the propagators, the G_{AA} and G_{BB} propagators can be different in general. For example, if they are Feynman and Dyson propagators correspondingly, then in the expression we obtain the sum or difference of them, depending on the overall sign of the G_{BB}^{-1} operator. We note also, that this issue is definitely important for the gravitational wave propagation and dark matter subjects discussion but is not will change drastically overall results obtained by the use of Eq. (141) action. Therefore, we do not discuss the Eq. (153) action further preserving the task of it's detailed exploration for the future investigation.

XI. MAIN RESULTS

The main result of the proposed approach is a construction of an effective action where a cancellation of the one loop contributions of the matter spinor field to the cosmological constant occurs. The result is based on the CPTM symmetry used for the construction of the the B-manifold actions, see Eq. (8)-Eq. (9) expressions. Consequently, for the full effective bare action, the general energy-momentum vector of a quantum one-particle state is zero, see Eq. (18)-Eq. (19), Eq. (28)-Eq. (29) and Eq. (34)-Eq. (38). On this stage we have a four variants of the B-manifold Lagrangians with different forms of the spinors propagators found. At the next step we construct an one loop effective action in an external gravitational field that allowed, requiring the cancellation of unwanted terms, to obtain a final variant of the B-manifold action, see Eq. (67) and Eq. (69).

In the proposed scheme we do not construct and consider an precise analog of the Schwinger-Keldysh mechanism of interactions between A and B spinor and gravitational fields, i.e. non-diagonal terms in the corresponding Green's functions matrix, see for example Eq. (104)-Eq. (108) expressions. In general these contributions can be important when we discuss the one-loop contributions cancellation, see discussion further in the next Section. Without these contributions the structure of the action is simpler and cancellation of the ordinary one-loops contributions occurs if we assume the same gravitational field attached to the A,B spinors through corresponding vierbeins. If the fields are different, see Eq. (112), then the cancellation has no place, the external gravity "legs" attached to spinor's loop

are different simply, see Eq. (128)-Eq. (130). In the later case, the cosmological constant value is given by Eq. (117) expression, to the leading order it is proportional to the trace of the on-shell classical value of the corresponding energy-momentum tensor, but the problem of the vacuum loop contributions will remain. In the former case, see Eq. (130), the cancellation is here and the cosmological constant to the leading order is defined by the difference of traces of the A and B energy-momentum tensors, see Eq. (131). The Eq. (130) condition required for the weak gravitational fields in this case hence must be explained somehow in the framework of the CPTM symmetry and/or Schwinger-Keldysh description of the interacting A,B systems.

The symmetry between the weak gravity fields of A and B manifolds, requested for the one-loops cancellation, can be introduced from the very beginning if we assume that any matter field is interacting with both fields equivalently or that both metrics have a sum of the A,B weak gravitational fields as a weak field for each separate manifold (i.e. there is only one weak field for both Lagrangians exists), see Eq. (138) or Eq. (139)-Eq. (140) correspondingly. In the both cases the cosmological constant is proportional to the difference of the traces of the energy-momentum tensors, see Eq. (147) and Eq. (151). There are also vertices M_{1AA} and M_{1BB} which allows an interaction between the A, B weak gravitational fields. These vertices, therefore, allows a creation of the B matter in the A manifold and vice versa creation of the A fields in the B manifold without direct interactions between the manifolds introduced. The model, through this mechanism of the A and B fields interactions through gravity only, describes some dark matter appearance in the A manifold therefore. Namely, there is the B spinor field which do not interact with A field directly but only through gravity and if we consider the dark matter manifestation as some corrections to the graviton propagator aka gravity potential, we see that the corrections appear inevitably. The form and structure the corrections depend on the form of interactions between A and B manifolds and/or on the interaction and creation of of B spinor fields with and by the gravity in the A manifold. Anyway, these corrections are in the model simply by construction, see Eq. (105)-Eq. (108) and Eq. (134)-Eq. (136). We do not really calculated these corrections in the present article, this task requires an additional exploration. Another important remark about the "dark" matter appearance defined here as B-spinor fields, is that for the correct positive constant the Eq. (133) condition must be satisfied that, to first sight, is in accordance with the statement about a dark matter dominance.

XII. DISCUSSION

The present article is devoted to an exploration of a model intended to explain the smallness of the cosmological constant and, at the same time, the absence of unwanted contributions from the one-loop effective action of spinor field. The framework is based on an application of the CPTM discrete symmetry, introduced in [1–3], to fermion fields. As an additional result, it is also demonstrated that dark matter appears in the approach in the form of a modified graviton propagator inevitably, the propagator's modification appears due to the second portion of the matter introduced in the form of B-spinors. Different possibilities of this construction are discussed through the article.

Throughout the article we underlined that the approach is similar to the Schwinger-Keldysh approach in the non-equilibrium condensed matter physics. The similarities are clear of course. There is a request of the CPT symmetry in both approaches, doubling of degrees of freedom and two arrows of time. Therefore it is important to underline the differences between the frameworks. The Schwinger-Keldysh approach can be considered as a variant of the CPT invariant QFT intended describe non-equilibrium states of matter at nonzero temperature in general, in any case there is only one field which we define on the time contour defined in a complex time's plane. In the present approach the fields are different, there are A and B clones of the matter field related by the CPTM symmetry and the additional time direction arises due of the presence of an additional B manifold. The new time arrow can be in parallel to the usual, "our", time direction or can be directed in an opposite direction, see [2]. Moreover, we fix the form of the final action by the introduction of the external gravitational field and by consequent request of the cancellation of the undesirable terms in the mutual effective action. It is not clear whether this can be done in the Schwinger-Keldysh approach with gravity included. Indeed, the Eq. (9) Lagrangian is an initial point of the consideration in the Schwinger-Keldysh approach whereas in the present framework we obtained this form of the Lagrangian only after mass sign inverse and negative vierbein introduction. There is an another problem will arise if we will follow the canonical Schwinger-Keldysh formalism and this a problem of the form of the gravity field. By construction there are two different gravity fields belong to the two different actions in the Schwinger-Keldysh approach and then again there is no full one-loop cancellation can happen.

However, an important lesson we learned from the Schwinger-Keldysh approach is that there are non-diagonal Green's functions constructed from the correlators of the fields belong to the different time's branches of the contour. These correlators describe a direct interaction between the fields with different complex time arguments in the Schwinger-Keldysh approach and can describe an interaction between the fields belong to different manifolds in the present formalism. Correspondingly, there are a plenty additional spinor one-loop contributions into the cosmological

constant arises when we account the loops with $S_>(x, y)$ and $S_<(y, x)$ non-diagonal propagators included. We do not consider these contributions in the article but formally in the present framework we have off-diagonal Green's functions as well. Important to note, that in the Schwinger-Keldysh formalism these functions in momentum representation are on-shell delta functions, see [18] for examples, and their contribution to the constant can be quite different from the ordinary matter loops. The calculations of these type of contributions it is definitely very interesting task which we postpone for an additional publication. In the case when these non-diagonal interactions are absent, we stay with a dark matter model where the B-fields (dark matter) are created by the gravity and both kinds of the matter interact with the same gravity field.

The problem of the cosmological constant, consequently, is resolved in the approach as following. Without the non-diagonal contributions, the one-loop contributions to the constant must be canceled independently on the number of the external gravitons attached⁹. Indeed, in this case there is only one internal momenta integration occurs and there are always a pair of the regularized diagrams which are continued to the Euclidean space differently with different signs in the front of them. The situation of course is more complicated with non-diagonal contribution introduced, then there are different contributions to the effective action constructed from the non-diagonal Green's functions, as mentioned above this case will be explored separately. Assuming the cancellation of the large one-loop contributions, therefore, to the leading order the constant is proportional to the trace of the energy-momentum tensor if gravity fields are different, or to the difference of A and B tensors traces if we operate with the same gravity field. In the later case we have two possibilities: it can be precisely zero if the values of the different spinor fields are the same at the same value of their arguments or it can be non-zero value if the values are different, see discussion in [3]. In the case of the first possibility, the value of the constant is suppressed even more in comparison to the represented result. In the second case we can obtain an equation of the Eq. (151) type, there the cosmological constant is a rate of a change of the trace of the manifold's energy-momentum tensor during some large period of time which can be taken as the lifetime of the Universe. Another interesting observation is the following one. The trace consists of two parts, classical and quantum on-shell contributions. Both of them are small in general, and if we talk about the difference of the traces it could be that one parts of the trace is zero whereas the second provides the current small value of the constant.

In this article we did not discuss much a dark matter issue. Nevertheless, similarly to the modification of the propagators in the Schwinger-Keldysh model, here we obtain the same result as well. Any additional matter introduced in the model inevitably leads to the additional terms in the graviton's propagator and modifies it. There are different scenario of the modifications depends on the matter properties. In the simplest variant, without non-diagonal Green's functions, the matter creation and/or absorbtion is due a gravity only whereas in the Schwinger-Keldysh like variant there are direct interactions between the A and B spinor and gravity fields. In is interesting to note, that in some extend the additional terms in the A-manifold effective action, which leads to the new graviton propagator, are similar to the additional terms in the gravity action of [30] model where a creation of the matter (C-field) in our branch of Universe was proposed. Also, the models can describe the MOND, [31], modification of the gravity at large distances in principal, see for example [32] with a model of negative mass distribution introduced for the MOND modification. Of course, some detailed calculations are required in order to demonstrate whether and in which form the modified propagator will be able to mimic the main properties of the MOND model.

One from the important properties of the framework is its fundamentally possible check for a falsification. Namely, each from the proposed and discussed or mentioned possibilities for the cosmological constant appearance and value is different from an another. Therefore, any possible description of the experimental data based on the proposed model will depend on the particularly chosen scheme and can be verified in principal. Additionally to this type of a correctness check, there is an another interesting property we can to explore: models's renormalization properties will depend on the particular scheme of the approach. Therefore, in general, it is interesting to check the behavior of the divergent diagrams, see for example [26], in the case when we have a doubling of the degrees of freedom in any from the proposed ways.

There is an additional task which we can consider as a further application of the proposed ideas. it is very interesting to consider an application of the CPTM symmetry and/or Schwinger-Keldysh like approach to the problem of the pure gravitational interactions between two manifolds without matter fields. Namely, the important question is if it possible to construct an extended effective action manifold with a similar cancellation of "bad" contributions at the level of the energy-momentum vector for a one-particle state and at the level of an one-loop effective action. This task could clarify the form and structure of the weak gravitational fields of A and B manifolds in the model that, in turn, will allow to complete the construction of the full one-loop effective action with both spinor and gravity fields included. Of course, it is interesting and important as well to calculate the effective action of the model including

⁹ There is no cancellation, of course, if we talk about the diagrams with external legs attached to some given sources of the gravity field, see discussion in [29] for example.

fermion and graviton degrees of freedom to two-loops precision. At this precision a non-trivial mechanism of the possible cancellation of "bad" loop contributions with different Green's functions mixing could be revealed clearer. An another important direction of the model's developing it is an application of it's results to the experimental data description including a direct calculation of the cosmological constant value. Hopefully, these task will clarify yet non clear details of the approach and will help to understand it's correctness.

Appendix A: CPT transform of solution of Dirac equation

We take the Eq. (4) quantum fields and perform the CPT transform for this field following, mainly, to definitions of [24]. We have for the $\mathcal{P}$ (parity) transform:

$$\begin{aligned}\hat{\mathcal{P}}\psi(x) &= \psi_P(x) = \gamma^0 \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(a_{\mathbf{p}}^s u^s(p) e^{-i(p_0 x^0 + x^i p^i)} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{i(p_0 x^0 + x^i p^i)} \right) = \\ &= \gamma^0 \psi(t, -\mathbf{x}).\end{aligned}\tag{A.1}$$

The next is $\mathcal{T}$ (time reversal) transform we account:

$$\begin{aligned}\hat{\mathcal{T}}\psi_P(x) &= \psi_{PT}(x) = \\ &= \gamma^0 \gamma^1 \gamma^3 \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(a_{\mathbf{p}}^{-s} u^{-s}(p) e^{i(p_0 x^0 - x^i p^i)} + b_{\mathbf{p}}^{-s\dagger} v^{-s}(p) e^{-i(p_0 x^0 - x^i p^i)} \right) = \\ &= \gamma^0 \gamma^1 \gamma^3 \psi(-t, -\mathbf{x}).\end{aligned}\tag{A.2}$$

An additional transform we account is the charge conjugation $\mathcal{C}$ which defined through

$$\mathcal{C} a_{\mathbf{p}}^s \mathcal{C} = b_{\mathbf{p}}^s, \quad \mathcal{C} b_{\mathbf{p}}^s \mathcal{C} = a_{\mathbf{p}}^s\tag{A.3}$$

that provides:

$$\begin{aligned}\hat{\mathcal{C}}\psi_{PT}(x) &= \psi_{CPT}(x) = \\ &= \gamma^0 \gamma^1 \gamma^3 \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(b_{\mathbf{p}}^s u^s(p) e^{i(p_0 x^0 - x^i p^i)} + a_{\mathbf{p}}^{s\dagger} v^s(p) e^{-i(p_0 x^0 - x^i p^i)} \right).\end{aligned}\tag{A.4}$$

Using the following identities

$$u^s(p) = -i \gamma^2 v^{s*}(p), \quad v^s(p) = -i \gamma^2 u^{s*}(p)\tag{A.5}$$

we obtain instead Eq. (A.4):

$$\begin{aligned}\psi_{CPT}(x) &= \gamma^5 \psi^*(x) = (\bar{\psi} \gamma^0 \gamma^5)^T = \\ &= \gamma^5 \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(b_{\mathbf{p}}^s v^{s*}(p) e^{i p x} + a_{\mathbf{p}}^{s\dagger} u^{s*}(p) e^{-i p x} \right)\end{aligned}\tag{A.6}$$

and correspondingly finally we obtain for the classical solution Eq. (4) after the $\mathcal{CPT}$ transform:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mathcal{E}_p}} \sum_s \left(b_{\mathbf{p}}^{s\dagger} v^s(p) e^{-i p x} + a_{\mathbf{p}}^s u^s(p) e^{i p x} \right).\tag{A.7}$$

As it seems, the function can be obtained by the $x \rightarrow -x$ transformation of the initial Eq. (4).

Appendix B: Propagators of fermion fields

We use the Eq. (22) definition of the Feynman propagator and write is as

$$S_F(x-y) = -i \frac{i}{2\pi} \int d\omega \frac{e^{-i\omega(x^0-y^0)}}{\omega + i\varepsilon} \sum_s \int \frac{d^3k}{(2\pi)^3 2\omega_k} u^s(k) \bar{u}^s(k) e^{-i\omega_k(x^0-y^0)+i\vec{k}(\vec{x}-\vec{y})} + \\ + i \frac{i}{2\pi} \int d\omega \frac{e^{-i\omega(y^0-x^0)}}{\omega + i\varepsilon} \sum_s \int \frac{d^3k}{(2\pi)^3 2\omega_k} v^s(k) \bar{v}^s(k) e^{-i\omega_k(y^0-x^0)+i\vec{k}(\vec{y}-\vec{x})} \quad (\text{B.1})$$

with

$$\omega_k = \sqrt{k^2 + m^2}. \quad (\text{B.2})$$

After the variable change $\omega + \omega_k \rightarrow \tilde{\omega}$ in the both terms we arrive to

$$S_F(x-y) = \int \frac{d\tilde{\omega} d^3k}{(2\pi)^4 2\omega_k} (\hat{k} + m) \frac{e^{-i\tilde{\omega}(x-y)}}{\tilde{\omega} - \omega_k + i\varepsilon} - \int \frac{d\tilde{\omega} d^3k}{(2\pi)^4 2\omega_k} (\hat{k} - m) \frac{e^{-i\tilde{\omega}(y-x)}}{\tilde{\omega} - \omega_k + i\varepsilon} \quad (\text{B.3})$$

and replacing

$$k \rightarrow -k \quad (\text{B.4})$$

in the second term we obtain finally:

$$S_F(x-y) = (i\hat{\partial}_x + m) \int \frac{d\tilde{\omega} d^3k}{(2\pi)^4 2\omega_k} \frac{e^{-i\tilde{\omega}(x-y)}}{\tilde{\omega} - \omega_k + i\varepsilon} - (i\hat{\partial}_x + m) \int \frac{d\tilde{\omega} d^3k}{(2\pi)^4 2\omega_k} \frac{e^{-i\tilde{\omega}(x-y)}}{\tilde{\omega} + \omega_k - i\varepsilon} = \\ = (i\hat{\partial}_x + m) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i\tilde{\omega}(x-y)}}{k^2 - m^2 + i\varepsilon} = (i\hat{\partial}_x + m) G_F \quad (\text{B.5})$$

with G_F as Feynman propagator of scalar field and with

$$(i\hat{\partial}_x - m) S_F(x, y) = \delta^4(x-y) \quad (\text{B.6})$$

satisfied. Defining the Dyson propagator as

$$S_D(x-y) = -i (\theta(x^0 - y^0) < 0 | \bar{\psi}(y) \psi(x) - \theta(y^0 - x^0) < 0 | \psi(x) \bar{\psi}(y) | 0 > | 0 >) \quad (\text{B.7})$$

we obtain after similar calculations

$$S_D(x-y) = - (i\hat{\partial}_x + m) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i\tilde{\omega}(x-y)}}{k^2 - m^2 - i\varepsilon} = (i\hat{\partial}_x + m) G_D \quad (\text{B.8})$$

with

$$- (i\hat{\partial}_x - m) S_D(x, y) = \delta^4(x-y) \quad (\text{B.9})$$

satisfied.

Appendix C: One loop integrals

First term

The first term contributes in Eq. (78) has the following form:

$$Int_{2a} = \int d^4x s(x) \int d^4y s(y) (\gamma^\mu (\partial_{\mu x} \partial_{\mu_1 y} S_F(x, y)) \gamma^{\mu_1} S_F(y, x) + \gamma^\mu (\partial_{\mu x} S_F(x, y)) \gamma^{\mu_1} (\partial_{\mu_1 y} S_F(y, x))) . \quad (C.1)$$

After Fourier transform we have:

$$\begin{aligned} Int_{2a} &= \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) \delta^4(p_i + p_f) \int \frac{d^4 p}{(2\pi)^4} \left[\gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{\mu_1} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_\mu p_{\mu_1} - \right. \\ &\quad \left. - \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{\mu_1} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_\mu (p - p_f)_{\mu_1} \right] = \\ &= \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) \delta^4(p_i + p_f) \int \frac{d^4 p}{(2\pi)^4} \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{\mu_1} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_\mu p_{f\mu_1} \end{aligned} \quad (C.2)$$

Introducing Feynman parameters and using a symmetry of the integral with respect to the $x \leftrightarrow (1 - x)$ replace and also $\rho \leftrightarrow \rho_1$ symmetry in the expressions with gamma matrices trace, we obtain:

$$\begin{aligned} Int_{2a} &= 2i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 x dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1 - x) + 2m^2}{(l_E^2 + \Delta)^2} = \\ &= i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1 - x) + 2m^2}{(l_E^2 + \Delta)^2} . \end{aligned} \quad (C.3)$$

here $\Delta = m^2 - x(1 - x)p_f^2$, $l = p - xp_f$ and Wick rotation $l_0 \rightarrow il_{0E}$ was performed. If the on-shell $p_f^2 = 0$ condition is satisfied for the $s_{\mu\nu}(x)$ gravitational field, after a renormalization the integral is zero of course.

Second term

Now we calculate the contribution of the following two (there is a symmetry with respect to x and y interchange in Eq. (78)) terms in Eq. (78):

$$\begin{aligned} Int_{2b} &= - \int d^4x s(x) \int d^4y s_{c_1\nu_1}(y) \eta^{\mu_1\nu_1} \gamma^\mu (\partial_{\mu x} \partial_{\mu_1 y} S_F(x, y)) \gamma^{c_1} S_F(y, x) - \\ &\quad - \int d^4x s(x) \int d^4y s_{c_1\nu_1}(y) \eta^{\mu_1\nu_1} \gamma^\mu (\partial_{\mu x} S_F(x, y)) \gamma^{c_1} (\partial_{\mu_1 y} S_F(y, x)) - \\ &\quad - \int d^4x s(x) \int d^4y s_{c_1\nu_1}(y) \eta^{\mu_1\nu_1} \gamma^\mu (\partial_{\mu_1 y} S_F(x, y)) \gamma^{c_1} (\partial_{\mu x} S_F(y, x)) - \\ &\quad - \int d^4x s(x) \int d^4y s_{c_1\nu_1}(y) \eta^{\mu_1\nu_1} \gamma^\mu S_F(x, y) \gamma^{c_1} (\partial_{\mu x} \partial_{\mu_1 y} S_F(x, y)) . \end{aligned} \quad (C.4)$$

Performing Fourier transform we obtain:

$$\begin{aligned} Int_{2b} &= - \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}_{c_1\nu_1}(p_f) \delta^4(p_i + p_f) \eta^{\mu_1\nu_1} \\ &\quad \int \frac{d^4 p}{(2\pi)^4} \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{c_1} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_{f\mu} p_{f\mu_1} \end{aligned} \quad (C.5)$$

that provides the final answer:

$$Int_{2b} = -2i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}_{\mu\nu}(p_f) p_f^\mu p_f^\nu \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1 - x) + 2m^2}{(l_E^2 + \Delta)^2} . \quad (C.6)$$

Because we do not really use the results for the one-loop action, the next simplification we can make is to simplify the integrals on $p_{i,f}$ variables obtaining in this particular case:

$$Int_{2b} = -\frac{i}{2} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2}. \quad (C.7)$$

We note, that here for the case of polarization of outgoing (or incoming) classical gravitation field $s_{\mu\nu} p_f^\mu = 0$ the integral is zero, see further also the next term contribution.

Third term

The third term (two terms together in Eq. (78)) we calculate is the following one:

$$\begin{aligned} Int_{2c} = & -\frac{1}{2} \int d^4 x s(x) \int d^4 y s_{c_1 a_1}(y) \gamma^\mu (\partial_{\mu x} \partial_{b_1 y} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] S_F(y, x) - \\ & -\frac{1}{2} \int d^4 x s(x) \int d^4 y s_{c_1 a_1}(y) \gamma^\mu (\partial_{\mu x} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] (\partial_{b_1 y} S_F(y, x)) - \\ & -\frac{1}{2} \int d^4 x s(x) \int d^4 y s_{c_1 a_1}(y) \gamma^\mu (\partial_{b_1 y} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] (\partial_{\mu x} S_F(y, x)) - \\ & -\frac{1}{2} \int d^4 x s(x) \int d^4 y s_{c_1 a_1}(y) \gamma^\mu S_F(x, y) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] (\partial_{\mu x} \partial_{b_1 y} S_F(x, y)). \end{aligned} \quad (C.8)$$

Similarly to done before we have:

$$\begin{aligned} Int_{2c} = & -\frac{1}{2} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}_{c_1 a_1}(p_f) \delta^4(p_i + p_f) \\ & \int \frac{d^4 p}{(2\pi)^4} \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_{f\mu} p_{fb_1}. \end{aligned} \quad (C.9)$$

Using the usual identities

$$\gamma^\mu \gamma^\nu \gamma^\rho = \eta^{\mu\nu} \gamma^\rho + \eta^{\nu\rho} \gamma^\mu - \eta^{\mu\rho} \gamma^\nu - i \epsilon^{\mu\nu\rho\sigma} \eta_{\sigma\sigma_1} \gamma^{\sigma_1} \gamma^5 \quad (C.10)$$

we have

$$\gamma^\mu [\gamma^\nu \gamma^\rho] = 2 (\eta^{\mu\nu} \gamma^\rho - \eta^{\mu\rho} \gamma^\nu - i \epsilon^{\mu\nu\rho\sigma} \eta_{\sigma\sigma_1} \gamma^{\sigma_1} \gamma^5). \quad (C.11)$$

and obtain

$$\begin{aligned} Int_{2c} = & - \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}_{c_1 a_1}(p_f) \delta^4(p_i + p_f) \\ & \int \frac{d^4 p}{(2\pi)^4} \left[\eta^{c_1 a_1} \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{b_1} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} - \right. \\ & - \eta^{c_1 b_1} \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{a_1} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} - \\ & \left. - i \epsilon^{c_1 a_1 b_1 d} \eta_{dd_1} \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{d_1} \gamma^5 \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} \right] p_{f\mu} p_{fb_1}. \end{aligned} \quad (C.12)$$

The second term in this expression is canceled with the previous one, the Eq. (C.5) answer; the third term is zero due product of symmetric s and antisymmetric tensors. So, we stay here therefore with only non-zero first term, which is

$$Int_{2c} = -2i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2}. \quad (C.13)$$

Fourth term

The next term we have is the following one:

$$Int_{2d} = i m \int d^4 x s(x) \int d^4 y s(y) \gamma^\mu ((\partial_{\mu x} S_F(x, y)) S_F(y, x) + S_F(x, y) (\partial_{\mu x} S_F(y, x))) . \quad (C.14)$$

Performing Fourier transform we obtain that this contribution is zero

$$\begin{aligned} Int_{2d} &= m \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) \delta^4(p_i + p_f) \int \frac{d^4 p}{(2\pi)^4} \gamma^\mu \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_{f\mu} = \\ &= m^2 \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) \delta^4(p_i + p_f) \int \frac{d^4 p}{(2\pi)^4} \frac{\gamma^\mu (\gamma^\rho p_\rho + \gamma^{\rho_1} (p - p_f)_{\rho_1})}{(p^2 - m^2 + i\varepsilon) ((p - p_f)^2 - m^2 + i\varepsilon)} p_{f\mu} = 0 . \end{aligned} \quad (C.15)$$

due the linear dependence of $l = p - x p_f$ in the numerator of the integral.

Fifth term

The fifth term is the new one, we have:

$$\begin{aligned} Int_{2e} &= \int d^4 x s_{c\nu}(x) \int d^4 y s_{c_1\nu_1}(y) \eta^{\mu\nu} \eta^{\mu_1\nu_1} \\ &\quad (\gamma^c (\partial_{\mu x} \partial_{\mu_1 y} S_F(x, y)) \gamma^{c_1} S_F(y, x) + \gamma^c (\partial_{\mu x} S_F(x, y)) \gamma^{c_1} (\partial_{\mu_1 y} S_F(y, x))) . \end{aligned} \quad (C.16)$$

After the Fourier transform we have:

$$\begin{aligned} Int_{2e} &= \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) \int d^4 p_f \tilde{s}_{c_1\nu_1}(p_f) \delta^4(p_i + p_f) \eta^{\mu\nu} \eta^{\mu_1\nu_1} \\ &\quad \int \frac{d^4 p}{(2\pi)^4} \gamma^c \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{c_1} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_\mu p_{f\mu_1} . \end{aligned} \quad (C.17)$$

The calculation gives for this term:

$$\begin{aligned} Int_{2e} &= i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) \int d^4 p_f \tilde{s}_{c\nu_1}^c(p_f) p_f^\nu p_f^{\nu_1} \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ &- 4i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) p_f^c p_f^\nu \int d^4 p_f \tilde{s}_{c_1\nu_1}(p_f) p_f^{c_1} p_f^{\nu_1} \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{x(1-x)}{(l_E^2 + \Delta)^2} . \end{aligned} \quad (C.18)$$

or

$$\begin{aligned} Int_{2e} &= \frac{i}{4} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) \int d^4 p_f \tilde{s}^{c\nu}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ &- \frac{i}{4} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^4 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{x(1-x)}{(l_E^2 + \Delta)^2} . \end{aligned} \quad (C.19)$$

Sixth term

For the sixth term we write:

$$\begin{aligned} Int_{2f} &= \frac{1}{2} \int d^4 x s_{c\nu}(x) \int d^4 y s_{c_1 a_1}(y) \eta^{\mu\nu} \\ &\quad (\gamma^c (\partial_{\mu x} \partial_{b_1 y} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] S_F(y, x) + \gamma^c (\partial_{\mu x} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] (\partial_{b_1 y} S_F(y, x)) \\ &\quad + \gamma^c (\partial_{b_1 y} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] (\partial_{\mu x} S_F(y, x)) + \gamma^c S_F(x, y) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] (\partial_{\mu x} \partial_{b_1 y} S_F(y, x))) . \end{aligned} \quad (C.20)$$

After the Fourier transform we obtain:

$$Int_{2f} = \frac{1}{2} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) \int d^4 p_f \tilde{s}_{c_1 a_1}(p_f) \delta^4(p_i + p_f) \eta^{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \gamma^c \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_{f\mu} p_{fb_1}. \quad (C.21)$$

After some calculation we get:

$$Int_{2f} = 2i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^c p_f^\nu \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ - 2i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) p_f^\nu \int d^4 p_f \tilde{s}_{c_1}^c(p_f) p_f^{c_1} \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} + \\ + 8i \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) p_f^c p_f^\nu \int d^4 p_f \tilde{s}_{c_1 \nu_1}(p_f) p_f^{c_1} p_f^{\nu_1} \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{x(1-x)}{(l_E^2 + \Delta)^2} \quad (C.22)$$

or

$$Int_{2f} = \frac{i}{2} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ - \frac{i}{2} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) \int d^4 p_f \tilde{s}^{c\nu}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} + \\ + \frac{i}{2} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^4 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{x(1-x)}{(l_E^2 + \Delta)^2}. \quad (C.23)$$

Seventh term

The next term we consider is the following one:

$$Int_{2g} = -i m \int d^4 x s(x) \int d^4 y s_{c\nu}(y) \eta^{\mu\nu} \gamma^c ((\partial_{\mu x} S_F(x, y)) S_F(y, x) + S_F(x, y) (\partial_{\mu x} S_F(y, x))) = 0, \quad (C.24)$$

similarly to fourth term.

Eighth term

The eighth term is

$$Int_{2h} = \frac{1}{4} \int d^4 x s_{ca}(x) \int d^4 y s_{c_1 a_1}(y) (\gamma^c [\gamma^a \gamma^b] (\partial_{bx} \partial_{b_1 y} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] S_F(y, x) + \gamma^c [\gamma^a \gamma^b] (\partial_{bx} S_F(x, y)) \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] (\partial_{b_1 y} S_F(y, x))) . \quad (C.25)$$

After the Fourier transform we have:

$$Int_{2h} = \frac{1}{4} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{ca}(p_i) \int d^4 p_f \tilde{s}_{c_1 a_1}(p_f) \delta^4(p_i + p_f) \int \frac{d^4 p}{(2\pi)^4} \gamma^c [\gamma^a \gamma^b] \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + i\varepsilon} \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + i\varepsilon} p_{fb} p_{fb_1}. \quad (C.26)$$

With the help of

$$\gamma^\mu \gamma^\nu \gamma^\rho = \eta^{\mu\nu} \gamma^\rho + \eta^{\nu\rho} \gamma^\mu - \eta^{\mu\rho} \gamma^\nu - i \epsilon^{\mu\nu\rho\sigma} \eta_{\sigma\sigma_1} \gamma^{\sigma_1} \gamma^5 \quad (C.27)$$

and

$$\gamma^\mu [\gamma^\nu \gamma^\rho] = 2 (\eta^{\mu\nu} \gamma^\rho - \eta^{\mu\rho} \gamma^\nu - i \epsilon^{\mu\nu\rho\sigma} \eta_{\sigma\sigma_1} \gamma^{\sigma_1} \gamma^5) \quad (C.28)$$

we have for the trace we need:

$$\begin{aligned} Tr_1 = Tr [\gamma^c [\gamma^a \gamma^b] \gamma^\rho \gamma^{c_1} [\gamma^{a_1} \gamma^{b_1}] \gamma^{\rho_1}] = Tr [(\eta^{ca} \gamma^b - \eta^{cb} \gamma^a - \imath \epsilon^{cab\sigma} \eta_{\sigma\sigma_1} \gamma^{\sigma_1} \gamma^5) \gamma^\rho \\ (\eta^{c_1 a_1} \gamma^{b_1} - \eta^{c_1 b_1} \gamma^{a_1} - \imath \epsilon^{c_1 a_1 b_1 \gamma} \eta_{\gamma\gamma_1} \gamma^{\gamma_1} \gamma^5) \gamma^{\rho_1}] . \end{aligned} \quad (C.29)$$

For the symmetrical $s_{\mu\nu}$ we stay therefore with

$$Tr_1 = Tr [(\eta^{ca} \gamma^b - \eta^{cb} \gamma^a) \gamma^\rho (\eta^{c_1 a_1} \gamma^{b_1} - \eta^{c_1 b_1} \gamma^{a_1}) \gamma^{\rho_1}] . \quad (C.30)$$

Now, after some algebra, we obtain finally:

$$\begin{aligned} Int_{2h} = \frac{\imath}{4} \int \frac{d^4 p_i}{(2\pi)^4} \int d^4 p_f \tilde{s}(p_i) \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ - \frac{\imath}{2} \int \frac{d^4 p_i}{(2\pi)^4} \int d^4 p_f \tilde{s}(p_i) \tilde{s}_{c\nu}(p_f) p_f^c p_f^\nu \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} + \\ + \frac{\imath}{4} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) p_f^\nu \int d^4 p_f \tilde{s}_{\nu_1}^c(p_f) p_f^{\nu_1} \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ - \imath \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{ac}(p_i) p_f^a p_f^c \int d^4 p_f \tilde{s}_{a_1 c_1}(p_f) p_f^{a_1} p_f^{c_1} \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{x(1-x)}{(l_E^2 + \Delta)^2} . \end{aligned} \quad (C.31)$$

and

$$\begin{aligned} Int_{2h} = \frac{\imath}{4} \int \frac{d^4 p_i}{(2\pi)^4} \int d^4 p_f \tilde{s}(p_i) \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ - \frac{\imath}{8} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 - 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} + \\ + \frac{\imath}{16} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}_{c\nu}(p_i) \int d^4 p_f \tilde{s}^{c\nu}(p_f) p_f^2 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + 2p_f^2 x(1-x) + 2m^2}{(l_E^2 + \Delta)^2} - \\ - \frac{\imath}{16} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) p_f^4 \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{x(1-x)}{(l_E^2 + \Delta)^2} . \end{aligned} \quad (C.32)$$

Ninth term

Similarly to the previous similar terms, the term

$$Int_{2I} = -\frac{\imath m}{2} \int d^4 x s(x) \int d^4 y s_{ca}(y) \gamma^c [\gamma^a \gamma^b] ((\partial_{bx} S_F(x, y)) S_F(y, x) + S_F(x, y) (\partial_{bx} S_F(y, x))) = 0 , \quad (C.33)$$

is zero as well.

Tenth term

The last term we need is

$$Int_{2j} = -\frac{m^2}{2} \int d^4 x s(x) \int d^4 y s(y) S_F(x, y) S_F(y, x) . \quad (C.34)$$

Fourier transform provides:

$$Int_{2j} = -\frac{m^2}{2} \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) \delta^4(p_i + p_f) \int \frac{d^4 p}{(2\pi)^4} \frac{\gamma^\rho p_\rho + m}{p^2 - m^2 + \imath \varepsilon} \frac{\gamma^{\rho_1} (p - p_f)_{\rho_1} + m}{(p - p_f)^2 - m^2 + \imath \varepsilon} \quad (C.35)$$

and after a calculation we obtain the following answer

$$Int_{2j} = 2 \imath m^2 \int \frac{d^4 p_i}{(2\pi)^4} \tilde{s}(p_i) \int d^4 p_f \tilde{s}(p_f) \delta^4(p_i + p_f) \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{l_E^2 + p_f^2 x(1-x) - m^2}{(l_E^2 + \Delta)^2} . \quad (C.36)$$

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