Pauli equation and charged spin-1/2 particle in a weak gravitational field

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Abstract

Using the nonrelativistic approximation in the curved-space Dirac equation, the analog of the Pauli equation is derived for a weak, but otherwise arbitrary, gravitational field, in the presence of an electromagnetic field. On top of that, are obtained the equations of motion for the massive spin-1/2 charged particle. In the two particular cases, we consider the previously explored backgrounds of the plane gravitational wave and the homogeneous static gravitational field. Different from the previous works which were employing either the exact or conventional Foldy-Wouthuysen transformations, here we perform calculations in the simplest way, in particular, aiming to clean up the existing discrepancies in the literature.

Keywords. Pauli equation, weak gravitational field, spin-1/2 particle, nonrelativistic approximation

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1 Introduction

General relativity (GR) is a generalization of Newton's laws of classical mechanics, in particular it makes classical law of gravity compatible with the relativistic mechanics at low energies (in the IR). The main symmetry of GR is the general covariance and, in particular, the construction of fields in curved spacetime of GR can be achieved on the basis of this symmetry. E.g., this is the case for the Dirac field describing the spin-1/2 particle. This means, the construction of the action of Dirac field in curved background is essentially based on covariance (see, e.g., [1, 2, 3, 4, 5]). Thus, the nonrelativistic approximation, in particular the Pauli equation for the spin-1/2 particle, in curved space can be constructed only starting from the covariant Dirac equation. When making a nonrelativistic approximation, the covariance is lost, making direct generalization of the Pauli equation to the curved spacetime impossible.

The nonrelativistic approximation has great importance in relativistic quantum mechanics and quantum field theory (QFT). The main reason is that, from the usual QFT perspective, many relevant physical applications correspond to the low-energy domain, where one can meet many high-precision experiments. This statement is valid also for the gravitational experiments, in the sense that the highest precision measurements are done in laboratory and for the motions with the nonrelativistic speeds. Clear examples of this can be found in the kinds of existing and forthcoming experiments (see, e.g., [6, 7]). From the side of the theory, it looks necessary to have well established gravitational contributions to the Pauli equations and to its classical counterpart, i.e., the equations describing the nonrelativistic spin-1/2 charged particle.

Many of the existing literature on relativistic generalizations of the nonrelativistic Schrodinger equation for spinning particle with gravity discusses the gravitational field created by the accelerated reference frames, the constant gravitational acceleration $\vec{g}$, usually obtained by the corresponding approximation in the Schwarzschild metric [8] (see also earlier works [9, 10]), further generalizations, including the rotations [11, 12, 13] and the motion of spin in different configurations of the gravitational field [14] (see also [15, 16]). Another type of gravitational field considered in the literature is the one of a flat gravitational wave [17]. This paper found that this kind of the gravitational field admits an exact Foldy-Wouthuysen transformation (there were also further developments in [18]). However, soon after it was noted that the corresponding calculation had a mistake leading to the discrepancy with the conventional Foldy-Wouthuysen transformation [19].

All of the mentioned works were done using either usual perturbative or exact Foldy-Wouthuysen transformation. Since the equations in different papers sometimes differ, it makes sense to verify the results by a qualitatively different and simpler method, such as the simplest textbook-level one-step derivation of the Pauli equation [20, 21]. In the rest of this paper we shall present, in many details, of such a simple calculation on an arbitrary weak gravitational field. The main restrictions on the background are that the metric can be presented in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

leaving aside the rotational metrics. As it was mentioned above, we assume the weak field restriction, that means $|h_{\mu\nu}| \ll 1$ and the same for its partial derivatives.

In what follows, we obtain the gravitational version of the Pauli equation for the metric (1) and an arbitrary electromagnetic field $\big(\vec{E},\vec{B}\big)$. Furthermore, the canonical quantization of this equation and the subsequent classical limit gives the equations for a spinning particle. These two main results are applied to the particular background of a plane gravitational wave and Newtonian gavity, which were worked out in [11] and [17, 19]. This comparison provides a good portion of control and corrections of coefficients in some cases.

The paper is organized as follows. In Sec. 2 we briefly review the construction of the covariant derivative of a Dirac fermion and of the corresponding action and give necessary expressions for its expansion in the weak gravity limit. Sec. 3 presents the detailed derivation of the Pauli equation. Sec. 4 reports on the calculation of the equations of motions on a background of weak, but otherwise almost arbitrary, gravitational and electromagnetic, fields. The restriction is related to the synchronous gauge fixing condition. Different from cosmology, here this condition makes the results less general,. On the other hand, it greatly reduce the amount of the (already significant) calculations. In Sec. 5, there is a comparison of our results with the results of [17, 19] concerning the gravitational waves background. Sec. 6 makes similar comparison with the Newtonian limit of the Schwarzschild metric [11]. Finally, the Sec. 7, we draw our conclusions.

2 Dirac equation in a gravitational field background

Consider briefly the formulation of the action of Dirac fermion. More details can be found in [4]. The consideration which is based on a similar approach can be found, e.g., in [22, 3] and the treatment based on group theory, e.g., in [5]. The form of the action is

$$S_f = \frac{i}{2}\int d^4x\sqrt{-g}\big(\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\,\gamma^\mu\psi + 2im\big)\,. \tag{2}$$

To make this definition more concrete, we need to construct a covariant version of the gamma-matrices γ^μ and the covariant derivative of a fermion ∇_μ. We start by introducing the tetrads e^μ_a and e^b_ν,

$$e^\mu_a e^{a\nu} = e^\mu_a e^\nu_b \eta^{ab} = g^{\mu\nu}, \qquad e^a_\mu e_{a\nu} = g_{\mu\nu}, \quad e^a_\mu e^\mu_b = \delta^a_b, \quad e^a_\mu e^\alpha_a = \delta^\alpha_\mu. \tag{3}$$

In these and subsequent formulas, Greek letters, e.g., $\mu = 0, 1, 2, 3$ are covariant indices corresponding to the coordinates of the Riemannian space. Latin letters, such as $a = 0, 1, 2, 3$, are Lorentz indices corresponding to the coordinates of the flat Minkowski space which is a tangent space to the manifold in the given point. For an arbitrary vector V^μ, the transformation formulas are $V^\mu = e^\mu_a V^a$ and $V^b = e^b_\nu V^\nu$.

For the gamma matrices we define $\gamma^\mu = e^\mu_a \gamma^a$ and $\gamma_\nu = e_{\nu b}\gamma^b$. Under the diffeomorphism transformations, γ^μ and γ_ν are contravariant and covariant components of a vector. These objects satisfy the covariant version of Clifford algebra,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \;=\; g_{\mu\nu}. \tag{4}$$

The covariant derivative of a spinor is defined by the relations

$$\nabla_\mu \psi \;=\; \partial_\mu \psi + \frac{i}{2}\,\omega_\mu{}^{ab}_{\cdot\cdot}\,\sigma_{ab}\,\psi\,, \quad \text{where} \quad \sigma_{ab} = \frac{i}{2}\big(\gamma_a\gamma_b - \gamma_b\gamma_a\big)\,. \tag{5}$$

The complex conjugation gives

$$\nabla_\mu \bar{\psi} \;=\; \partial_\mu \bar{\psi} - \frac{i}{2}\,\omega_\mu{}^{ab}_{\cdot\cdot}\,\bar{\psi}\sigma_{ab}\,. \tag{6}$$

In this expressions, $\omega_\mu{}^{ab}_{\cdot\cdot}$ are the real-valued coefficients of the *spinor connection*, which have to be found from the requirement of covariance.

The solution for the spinor connection is

$$\omega_\mu{}^{ab}_{\cdot\cdot} \;=\; \frac{1}{2}\,(e^b_\tau e^{\lambda a}\Gamma^\tau_{\lambda\mu} - e^{\lambda a}\partial_\mu e^b_\lambda) \;=\; -\,\omega_\mu{}^{ba}_{\cdot\cdot}. \tag{7}$$

It is sometimes useful to use the explicitly antisymmetric form,

$$\omega_\mu{}^{ab}_{\cdot\cdot} \;=\; \frac{1}{4}\,(e^b_\tau e^{\lambda a} - e^a_\tau e^{\lambda b})\Gamma^\tau_{\lambda\mu} \;+\; \frac{1}{4}(e^{\lambda b}\partial_\mu e^a_\lambda - e^{\lambda a}\partial_\mu e^b_\lambda)\,. \tag{8}$$

We need to consider the expansion of the action (2) on the flat background (1). For the sake of generality, we can work out a more general expansion

$$g_{\alpha\beta} \;\longrightarrow\; g'_{\alpha\beta} \;=\; g_{\alpha\beta} + h_{\alpha\beta}. \tag{9}$$

The first orders of expansions can be easily found in the form

$$
\begin{aligned}
\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}h\,, \quad \delta g^{\mu\nu} = -h^{\mu\nu}\,, \quad \delta e^c_\mu = \frac{1}{2}\,h^\nu_\mu e^c_\nu\,, \quad \delta e^\rho_b = -\frac{1}{2}\,h^\rho_\lambda e^\lambda_b\,, \\
\delta\Gamma^\lambda_{\alpha\beta} &= \frac{1}{2}\left(\nabla_\alpha h^\lambda_\beta + \nabla_\beta h^\lambda_\alpha - \nabla^\lambda h_{\alpha\beta}\right), \quad \delta\gamma^\mu = -\frac{1}{2}\,h^\mu_\nu\,\gamma^\nu
\end{aligned}
\tag{10}
$$

and

$$
\delta\omega_{\mu\,\cdot\cdot}^{\;\;ab} = \frac{1}{2}\left(e^{a\tau}e^{b\lambda} - e^{b\tau}e^{a\lambda}\right)\nabla_\lambda h_{\mu\tau}
\tag{11}
$$

Here all indices are raised and lowered with the background metrics $g^{\mu\nu}$ and $g_{\mu\nu}$ and the covariant derivatives are constructed with the background metric $g_{\mu\nu}$ and the corresponding connection. It can be shown that the variation of the spinor connection (11) is irrelevant, as its contribution vanishes when substituted into the variation of (2).

Using Eqs. (10), we arrive at the first-order variation of the action (2)

$$
\begin{aligned}
\delta S_f &= \int d^4x\sqrt{-g}\,h_{\alpha\beta}\left\{\frac{i}{4}g^{\alpha\beta}\left(\bar\psi\gamma^\lambda\nabla_\lambda\psi - \nabla_\lambda\bar\psi\gamma^\lambda\psi\right)\right. \\
&\left. - \frac{i}{4}\left(\bar\psi\gamma^\alpha\nabla^\beta\psi - \nabla^\alpha\bar\psi\gamma^\beta\psi\right) - \frac{1}{2}\,g^{\alpha\beta}m\bar\psi\psi\right\},
\end{aligned}
\tag{12}
$$

where $h = h^\mu_\mu$. Taking the flat background metric, the last expression is a good starting point to perform the nonrelativistic expansion in a weak gravitational field.

3 Pauli equation in a weak gravitational field

The standard approach to the nonrelativistic approximation is that it assumes weak external fields because a strong field can accelerate a particle to become relativistic. Indeed, the weak gravitational field condition does not reduce to requiring $|h_{\mu\nu}| \ll 1$ but, as we already mentioned above, demands the same from the partial derivatives of $h_{\mu\nu}$. Following this logic, we come back to the expansion of the metric on a flat background (1) and arrive at the simplified version of (12), which is the starting action of the nonrelativistic approximation

$$
\begin{aligned}
S &= \int d^4x\Big\{\bar\psi\left(i\gamma^\alpha\partial_\alpha - m\right)\psi \\
&+ \frac{i}{4}\left(hg^{\alpha\beta} - h^{\alpha\beta}\right)\left(\bar\psi\,\gamma_\beta\,\partial_\alpha\psi - \partial_\alpha\bar\psi\,\gamma_\beta\,\psi\right) - \frac{mh}{2}\bar\psi\psi\Big\},
\end{aligned}
\tag{13}
$$

where the first term is the basic zeroth-order part. Starting from this point the background metric is $\eta^{\alpha\beta}$, but we use Greek letters for the relativistic indices, e.g., $\alpha = 0, 1, 2, 3$, while the Latin indices are space $3D$, e.g., $i, j, k, l, .. = 0, 1, 2$. The signature is $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$.

In the given approximation, the Dirac equation reads

$$\Big\{ i\gamma^\alpha \partial_\alpha - m + \frac{i}{2}\left(h\eta^{\alpha\beta} - h^{\alpha\beta}\right)\gamma_\beta\,\partial_\alpha + \frac{i}{4}\left(\partial_\beta h - \partial_\alpha h^{\alpha\beta}\right)\gamma_\beta - \frac{mh}{2}\Big\}\,\psi \,=\, 0. \tag{14}$$

Starting from this point, the parenthesis indicate the limit of the action of derivative, i.e., $\partial AB = (\partial A)B + A\partial B$. The next step is to separate time t from the space coordinates x_i and denote $\gamma^0 = \beta$, that gives

$$\begin{aligned}\Big\{ i\beta\,\frac{\partial}{\partial t} &+ i\gamma^k\partial_k - m - \frac{mh}{2} + \frac{i}{2}\big(h - h^{00}\big)\beta\,\partial_t + \frac{i}{2}\big(h\gamma^k\partial_k + h^{jk}\gamma^j\partial_k\big) \\ &+ \frac{i}{2}h^{0k}\gamma^k\partial_t - \frac{i}{2}h^{k0}\beta\partial_k + \frac{i}{4}\dot{h}\beta + \frac{i}{4}\left(\partial_k h\right)\gamma^k - \frac{i}{4}\dot{h}^{00}\beta + \frac{i}{4}\dot{h}^{0k}\gamma^k \\ &- \frac{i}{4}\big(\partial_k h^{k0}\big)\beta + \frac{i}{4}\big(\partial_k h^{kj}\big)\gamma^j\Big\}\,\psi \,=\, 0.\end{aligned} \tag{15}$$

Multiplying the last equation by β, we change the notations according to $\beta\gamma^k = \alpha^k$, and use the fact that

$$h^{jk} = h_{jk}, \qquad h^k_j = -h_{jk}, \qquad h^{0k} = h^{k0} = -h_{k0}, \qquad h_{0k} = -h^k_0, \tag{16}$$

to arrive at

$$\begin{aligned}\Big\{ i\Big(1 + \frac{1}{2}h - \frac{1}{2}h_{00} - \frac{1}{2}h_{0k}\alpha^k\Big)\,\frac{\partial}{\partial t} &+ i\Big(\alpha^k + \frac{1}{2}h\alpha^k + \frac{1}{2}h_{kj}\alpha^j + \frac{1}{2}h_{0k}\Big)\partial_k \\ &- \beta m\Big(1 + \frac{1}{2}h\Big) + \frac{i}{4}\dot{h} - \frac{i}{4}\dot{h}_{00} + \frac{i}{4}\partial_k h_{k0} + \frac{i}{4}\left(\partial_k h\right)\alpha^k \\ &- \frac{i}{4}\dot{h}_{0k}\,\alpha^k + \frac{i}{4}\left(\partial_j h_{jk}\right)\alpha^k\Big\}\psi \,=\, 0.\end{aligned} \tag{17}$$

Starting from this point, the Latin indices are raised and lowered using the metric of the Euclidean space.

The next operation is to "liberate" the time derivative. For this, we remember that the components of $h_{\mu\nu}$ are taken into account only up to the linear approximation and multiply Eq. (17) by the factor

$$\Big(1 + \frac{1}{2}h - \frac{1}{2}h_{00} - \frac{1}{2}h_{0k}\,\alpha^k\Big)^{-1} \,=\, 1 - \frac{1}{2}h + \frac{1}{2}h_{00} + \frac{1}{2}h_{0k}\,\alpha^k. \tag{18}$$

In this way, after a small algebra, we get

$$\begin{aligned}\Big\{ i\,\frac{\partial}{\partial t} &+ i\Big(\alpha^k + \frac{1}{2}h_{00}\,\alpha^k + \frac{1}{2}h_{0j}\,\alpha^j\alpha^k + \frac{1}{2}h_{kj}\,\alpha^j + \frac{1}{2}h_{0k}\Big)\partial_k \\ &+ \frac{i}{4}\Big(\dot{h} - \dot{h}_{00} + \partial_k h_{k0}\Big) - \beta m\Big(1 + \frac{1}{2}h_{00} + \frac{1}{2}h_{0k}\,\alpha^k\Big) \\ &+ \frac{i}{4}\big(\partial_k h\big)\alpha^k - \frac{i}{4}\dot{h}_{0k}\,\alpha^k + \frac{i}{4}\big(\partial_j h_{jk}\big)\alpha^k\Big\}\psi \,=\, 0.\end{aligned} \tag{19}$$

At this point we recover the powers of c and $\hbar$, change to the momentum representation in the space sector, and introduce the electromagnetic potential $A^\mu = (\phi,\, \vec{A})$, which means trading

$$i\partial_t \to i\hbar\partial_t - e\phi, \qquad -i\partial_k \to -i\hbar\partial_k = p_k, \qquad p_k \to c\Pi_k = c\left(p_k - \frac{e}{c}A_k\right). \tag{20}$$

As a result, we get

$$\begin{aligned}&\bigg\{i\hbar\partial_t - e\phi - c\left(\alpha^k + \frac{1}{2}h_{00}\alpha^k + \frac{1}{2}h_{0j}\alpha^j\alpha^k + \frac{1}{2}h_{kj}\alpha^j + \frac{1}{2}h_{0k}\right)\Pi_k \\ &+ \frac{i\hbar}{4}\left(\dot{h} - \dot{h}_{00}\right) - \beta mc^2\Big(1 + \frac{1}{2}h_{00} + \frac{1}{2}h_{0k}\alpha^k\Big) + \frac{i\hbar c}{4}\left(\partial_k h + \partial_j h_{jk}\right)\alpha^k \\ &+ \frac{i\hbar}{4}\left(c\partial_k h_{k0} + \dot{h}_{0k}\alpha^k\right)\bigg\}\psi = 0.\end{aligned} \tag{21}$$

In the standard representation, one can easily obtain the relation

$$\alpha^j\alpha^k \,=\, -\,\delta^{jk} \,+\, i\epsilon^{jkl}\Sigma^l, \qquad \text{where} \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{22}$$

As a result, Eq. (19) can be written as

$$i\hbar\partial_t\psi \,=\, \hat{H}\psi, \qquad \text{where} \qquad \hat{H} = \hat{H}_\alpha + \hat{H}_\beta \tag{23}$$

and

$$\hat{H}_\alpha = c\left(\Pi_k + \frac{1}{2}h_{00}\Pi_k + \frac{1}{2}h_{jk}\Pi_j\right)\alpha_k - \frac{i\hbar c}{4}(\partial_k h + \partial_j h_{jk})\alpha^k + \frac{i\hbar}{4}\dot{h}_{0k}\alpha_k \tag{24}$$

$$\begin{aligned}\hat{H}_\beta \,&=\, e\phi + \beta mc^2\left(1 + \frac{1}{2}h_{00} + \frac{1}{2}h_{0k}\alpha^k\right) \\ &\quad + \frac{i\hbar}{4}(\dot{h}_{00} - \dot{h}) + \frac{ic}{2}h_{0j}\epsilon^{jkl}\Sigma^l\Pi_k - \frac{i\hbar c}{4}\partial_k h_{k0}\end{aligned} \tag{25}$$

Let us impose the gauge fixing condition, which helps to simplify the expressions. The simplest option is $h_{0k} = 0$, which gives

$$\hat{H}' \,=\, \hat{H}'_\alpha + \hat{H}'_\beta, \tag{26}$$

where

$$\hat{H}'_\beta \,=\, e\phi + \beta mc^2\Big(1 + \frac{1}{2}h_{00}\Big) + \frac{i\hbar}{4}(\dot{h}_{00} - \dot{h}) \tag{27}$$

$$\hat{H}'_\alpha \,=\, c\Big(\Pi_k + \frac{1}{2}h_{00}\Pi_k + \frac{1}{2}h_{jk}\Pi_j\Big)\alpha_k - \frac{i\hbar c}{4}(\partial_k h + \partial_j h_{jk})\alpha^k \,=\, c\alpha^k V_k. \tag{28}$$

In the last formula, we introduced the following notations:

$$V_k = \Big(\delta_{jk} + \frac{1}{2}\omega_{jk}\Big)\Pi_j + v'_k, \tag{29}$$

$$v'_k = -\frac{i\hbar}{4}(\partial_k h + \partial_j h_{jk}), \qquad \omega_{jk} = h_{00}\delta_{jk} + h_{jk}. \tag{30}$$

As a first step to take the non-relativistic limit, we change the variable

$$\psi = \psi' \exp\Big(-\frac{imc^2}{\hbar}t\Big). \tag{31}$$

The new spinor ψ' obeys the equation

$$i\hbar\partial_t\psi' = \Big\{e\phi + (\beta - I)mc^2 + \frac{mc^2}{2}\beta h_{00} + \frac{i\hbar}{4}(\dot{h}_{00} - \dot{h}) + H'_\alpha\Big\}\psi'. \tag{32}$$

In the standard representation

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \text{and assuming} \qquad \psi' = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{33}$$

we meet two equations for the two-components spinors φ and χ,

$$i\hbar\frac{\partial\varphi}{\partial t} - e\phi\varphi - \frac{1}{2}mc^2 h_{00}\varphi - \frac{i\hbar}{4}\big(\dot{h}_{00} - \dot{h}\big)\varphi = c\vec{\sigma}\cdot\vec{V}\chi \tag{34}$$

$$i\hbar\frac{\partial\chi}{\partial t} - e\phi\chi + 2mc^2\Big(1 + \frac{1}{4}h_{00}\Big)\chi - \frac{i\hbar}{4}\big(\dot{h}_{00} - \dot{h}\big)\chi = c\vec{\sigma}\cdot\vec{V}\varphi. \tag{35}$$

In the left hand side of (35), the term $2mc^2\big(1+h_{00}/4\big)$ is dominating in the nonrelativistic regime. Thus, we can neglect other terms and arrive at the approximate solution for the "small" component of the Dirac spinor,

$$\chi = \frac{c\vec{\sigma}\cdot\vec{V}}{2mc^2}\Big(1 - \frac{1}{4}h_{00}\Big)\varphi. \tag{36}$$

Substituting this result in Eq. (34), we get

$$i\hbar\frac{\partial\varphi}{\partial t} - e\phi\varphi - \frac{mc^2}{2}h_{00}\varphi - \frac{i\hbar}{4}\big(\dot{h}_{00} - \dot{h}\big)\varphi = \frac{1}{2m}(\vec{\sigma}\cdot\vec{V})^2\Big(1 - \frac{1}{4}h_{00}\Big)\varphi. \tag{37}$$

This is a non-relativistic approximation to the "large" component of the Dirac spinor, in the presence of electromagnetic and weak gravitational fields. The last enters this expression in a rather complicated way, as one can see from the expression for $\vec{V}$, given by (29).

The subsequent transformations are pretty much standard, including

$$(\vec{\sigma}\cdot\vec{V})^2 = \frac{1}{2}(\sigma^j\sigma^k + \sigma^k\sigma^j)V^jV^k + \frac{1}{2}(\sigma^j\sigma^k - \sigma^k\sigma^j)V^jV^k = V_kV_k + i\sigma^l\epsilon^{ljk}V_jV_k, \tag{38}$$

where $V_k V_k = \vec{V}^2$. The *r.h.s.* of Eq. (37) becomes

$$\left\{ \frac{1}{2m} V_k V_k \Big(1 - \frac{1}{4} h_{00}\Big) + \frac{i}{2m} \sigma_k \epsilon^{klj} V_l V_j \Big(1 - \frac{1}{4} h_{00}\Big) \right\} \varphi. \tag{39}$$

Regardless these formulas may be boring, let us elaborate this part in detail. For the first term, we have

$$\frac{1}{2m} V_k V_k \left(1 - \frac{1}{4} h_{00}\right) = \frac{1}{2m} \left(1 - \frac{1}{4} h_{00}\right) \vec{\Pi}^2 + \frac{1}{2m} \omega_{kl} \Pi_k \Pi_l + \frac{1}{m} v_k' \Pi_k \tag{40}$$

and for the second term,

$$\begin{aligned}
\frac{i}{2m}\Big(1 - \frac{1}{4} h_{00}\Big) \vec{\sigma} \cdot [\vec{V} \times \vec{V}] &= \frac{i}{2m} \vec{\sigma} \cdot \big(\vec{\Pi} \times \vec{\Pi}\big)\Big(1 - \frac{1}{4} h_{00}\Big) \\
&\quad + \frac{i}{4m} \sigma_m \epsilon^{mkl} \left(\Pi_k \omega_{lr} \Pi_r + \omega_{rk} \Pi_r \Pi_l\right) \\
&= -\frac{e\hbar}{2mc}\Big(1 - \frac{1}{4} h_{00}\Big) \vec{\sigma} \cdot \vec{B} + \frac{i}{4m} \sigma_m \epsilon^{mkl} (-i\hbar) \left(\partial_k \omega_{lr}\right) \Pi_r \\
&\quad + \frac{i}{4m} \sigma_m \epsilon^{mkl} \omega_{lr} \left(\Pi_k \Pi_r - \Pi_r \Pi_k\right),
\end{aligned} \tag{41}$$

where $\vec{B}$ is a magnetic field. In the last equation, we have

$$\Pi_k \Pi_r - \Pi_r \Pi_k = \frac{ie\hbar}{c} F_{kr} = \frac{ie\hbar}{c} \epsilon_{jrk} B_j. \tag{42}$$

Finally, Eq. (37) becomes

$$\begin{aligned}
i\hbar \frac{\partial \varphi}{\partial t} &= \bigg\{ e\phi + \frac{1}{2} mc^2 h_{00} + \frac{i\hbar}{4}\big(\dot{h}_{00} - \dot{h}\big) + \frac{1}{2m}\Big(1 - \frac{1}{4} h_{00}\Big) \vec{\Pi}^2 \\
&\quad - \frac{e\hbar}{2mc}\Big(1 - \frac{1}{4} h_{00}\Big) \vec{\sigma} \cdot \vec{B} + \frac{1}{2m} \omega_{kl} \Pi_k \Pi_l + \frac{1}{m} v_k' \Pi_k \\
&\quad + \frac{\hbar}{4m} \sigma_m \epsilon^{mkl} \left(\partial_k \omega_{lr}\right) \Pi_r - \frac{e\hbar}{4mc} \sigma_m \epsilon^{mkl} \epsilon_{jrk} \omega_{lr} B_j \bigg\} \varphi.
\end{aligned} \tag{43}$$

This is the extended version of the Pauli equation, taking into account the gravitational field $h_{\mu\nu}$ restricted by the gauge fixing condition $h_{0k} = 0$. Setting $h_{\mu\nu} = 0$, we arrive at the usual version of this equation.

4 Equations of motion for a non-relativistic particle

The result of the previous section is Eq. (43), which can be rewritten in the form

$$i\hbar \frac{\partial \varphi}{\partial t} = \hat{H} \varphi, \tag{44}$$

where

$$\hat{H} \; = \; A - \vec{\sigma} \cdot \vec{W} + C_{kl} \Pi_k \Pi_l + D_k \Pi_k. \tag{45}$$

In the *r.h.s.* of this formula and below we omit the hats for the sake of brevity. The elements of (45) are defined as follows

$$\begin{aligned} A &= e\phi + \frac{1}{2} mc^2 h_{00} + \frac{1}{4} i\hbar \big(\dot{h}_{00} - \dot{h} \big), \\ \vec{W} &= \frac{e\hbar}{2mc} \Big(1 - \frac{1}{4} h_{00} \Big) \vec{B} + \frac{e\hbar}{4mc} \epsilon^{mkl} \epsilon_{jrk} \omega_{lr} B_j \hat{n}_m, \\ C_{kl} &= \frac{1}{2m} \delta_{kl} \Big(1 - \frac{1}{4} h_{00} \Big) + \frac{1}{2m} \omega_{kl}, \\ D_k &= \frac{1}{m} v'_k + \frac{\hbar}{4m} \sigma_m \epsilon^{mrl} \left(\partial_r \omega_{lk} \right), \end{aligned} \tag{46}$$

where $\{\hat{n}_m\}$ form an orthonormal basis in the Cartesian coordinate system. The four elements (46) depend on time and also on the canonical operators, i.e., coordinates x_i, momenta p_j and spin σ_k. These operators satisfy the canonical commutation relations, which are all zero except

$$[x_i, p_j] \; = \; i\hbar\delta_{ij} \tag{47}$$

$$[\sigma_i, \sigma_j] \; = \; 2i\epsilon_{ijk}\sigma_k \, . \tag{48}$$

By definition, the classical limit is equivalent to the Heisenberg equation for the operator $\mathcal{O}(t)$,

$$\frac{d\mathcal{O}}{dt} \; = \; \frac{1}{i\hbar} \left[\mathcal{O}, H \right] \tag{49}$$

in the limit $\hbar \to 0$. In what follows, we shall use this scheme to derive the equations of motion for classical canonical variables x_i, p_j and σ_k.

Equation of motion for x_i

In the case of x_i, we need to evaluate the commutator

$$[x_i, H] \; = \; [x_i, C_{kl} \Pi_k \Pi_l] + [x_i, D_k \Pi_k] \, . \tag{50}$$

At this point, things start to simplify. Since $D_k = \mathcal{O}(\hbar)$, we have $[x_i, D_k \Pi_k] = \mathcal{O}(\hbar^2)$. As we are interested in the limit $\hbar \to 0$, this term is irrelevant. In the remaining term, also omitting all $\mathcal{O}(\hbar^2)$'s, after certain algebra we get

$$[x_i, C_{kl} \Pi_k \Pi_l] \; = \; \frac{1}{2m} \left[\Big(1 - \frac{1}{4} h_{00} \Big) \delta_{kl} + \omega_{kl} \right] [x_i, \Pi_k \Pi_l] \; = \; \frac{i\hbar}{m} \left[\Big(1 + \frac{3}{4} h_{00} \Big) \Pi_i + h_{il} \Pi_l \right] . \tag{51}$$

In the limit $\hbar \to 0$, the equation of motion for x_i becomes

$$\frac{dx_i}{dt} = \frac{1}{m}\left[\left(1+\frac{3}{4}h_{00}\right)\Pi_i + h_{il}\Pi_l\right]. \tag{52}$$

Equation of motion for p_i

To obtain the equation of motion for p_i, we note that $\vec{W} = \mathcal{O}(\hbar)$ and $D_k = \mathcal{O}(\hbar)$. For this reason, $\left[p_i, \vec{W}\right] = \mathcal{O}(\hbar^2)$ and $[p_i, D_k] = \mathcal{O}(\hbar^2)$, Thus, the corresponding terms can be neglected. as they will not contribute in the limit $\hbar \to 0$. The relevant commutator for p_i is

$$[p_i, H] = [p_i, A] + [p_i, C_{kl}\Pi_k\Pi_l]\,. \tag{53}$$

For the remaining terms, we get

$$[p_i, A] = -ie\hbar\partial_i\phi - \frac{i\hbar}{2}mc^2\left(\partial_i h_{00}\right) + \mathcal{O}(\hbar^2), \tag{54}$$

$$[p_i, C_{kl}\Pi_k\Pi_l] = [p_i, C_{kl}]\,\Pi_k\Pi_l + C_{kl}\,[p_i, \Pi_k]\,\Pi_l + C_{kl}\Pi_k\,[p_i, \Pi_l]\,, \tag{55}$$

where

$$[p_i, C_{kl}] = -\frac{i\hbar}{2m}\left[\frac{3}{4}\delta_{kl}(\partial_i h_{00}) + (\partial_i h_{kl})\right] \qquad \text{and} \qquad [p_i, \Pi_k] = \frac{ie\hbar}{c}(\partial_i A_k). \tag{56}$$

Substituting the equation (56) in the equation (55) we have

$$\begin{aligned}[p_i, C_{kl}\Pi_k\Pi_l] &= -\frac{i\hbar}{2m}\left[\frac{3}{4}\delta_{kl}(\partial_i h_{00}) + (\partial_i h_{kl})\right]\Pi_k\Pi_l \\ &\quad + \frac{ie\hbar}{c}C_{kl}(\partial_i A_k)\Pi_l + \frac{ie\hbar}{c}C_{kl}\Pi_k(\partial_i A_k).\end{aligned} \tag{57}$$

Finally, using (54) and (57) in Eq. (53), we obtain

$$\begin{aligned}[p_i, H] &= -i\hbar\left[e\partial_i\phi + \frac{mc^2}{2}\left(\partial_i h_{00}\right)\right] - \frac{i\hbar}{2m}\left[\frac{3}{4}\delta_{kl}(\partial_i h_{00}) + (\partial_i h_{kl})\right]\Pi_k\Pi_l \\ &\quad + \frac{ie\hbar}{c}\left[C_{kl}(\partial_i A_k)\Pi_l + C_{kl}\Pi_k(\partial_i A_k)\right].\end{aligned} \tag{58}$$

Next, taking the limit $\hbar \to 0$, we arrive at the classical equation of motion for p_i

$$\frac{dp_i}{dt} = eE_i - \frac{mc^2}{2}\left(\partial_i h_{00}\right) + \frac{2e}{c}C_{kl}(\partial_i A_k)\Pi_l - \frac{1}{2m}\left[\frac{3}{4}\delta_{kl}(\partial_i h_{00}) + (\partial_i h_{kl})\right]\Pi_k\Pi_l, \tag{59}$$

where $E_i = -\partial_i\phi$. Now we remember that $\Pi_i = p_i - \frac{e}{c}A_i$, thus

$$\frac{dp_i}{dt} = \frac{d\Pi_i}{dt} + \frac{e}{c}\frac{\partial A_i}{\partial x_k}\frac{dx_k}{dt} + \frac{e}{c}\frac{\partial A_i}{\partial t}. \tag{60}$$

Substituting Eq. (52) in (60), we get

$$\frac{dp_i}{dt} = \frac{d\Pi_i}{dt} + \frac{e}{mc}\frac{\partial A_i}{\partial x_k}\left[\left(1+\frac{3}{4}h_{00}\right)\Pi_k + h_{kl}\Pi_l\right] + \frac{e}{c}\frac{\partial A_i}{\partial t}. \tag{61}$$

Finally, substituting (61) into (59), we arrive at the result

$$\begin{aligned}\frac{d\Pi_i}{dt} &= eE_i - \frac{mc^2}{2}(\partial_i h_{00}) - \frac{1}{2m}\left[\frac{3}{4}\delta_{kl}(\partial_i h_{00}) + (\partial_i h_{kl})\right]\Pi_k\Pi_l \\ &\quad + \frac{e}{mc}\left(\delta_{kl} + \frac{3}{4}\delta_{kl}h_{00} + h_{kl}\right)\Pi_k \epsilon_{ilj} B_j.\end{aligned} \tag{62}$$

Equation of motion for σ_i

For the spin variable σ_i, the initial commutator has the form

$$[\sigma_i, H] = \left[\sigma_i,\, -\vec{\sigma}\cdot\vec{W} + \frac{\hbar}{4m}\sigma_m \epsilon^{mrl}\left(\partial_r \omega_{ln}\right)\Pi_n\right]. \tag{63}$$

Using the commutation relation (48), for the sigma matrices, we have, after some algebra,

$$\begin{aligned}[\sigma_i, H] &= 2i\epsilon_{ijk}\sigma_k\Big\{-\frac{e\hbar}{2mc}\Big(1-\frac{1}{4}h_{00}\Big)B_j - \frac{e\hbar}{4mc}\epsilon^{jml}\omega_{lr}\epsilon_{nrm}B_n \\ &\quad + \frac{\hbar}{4m}\epsilon^{jml}\left(\partial_m\omega_{ln}\right)\Pi_n\Big\}. \\ &= -\frac{ie\hbar}{mc}\left(1-\frac{1}{4}h_{00}\right)\epsilon_{ijk}B_j\sigma_k + \frac{ie\hbar}{2mc}\epsilon_{ijk}\left(\omega_{rr}B_j - \omega_{lj}B_l\right)\sigma_k \\ &\quad - \frac{i\hbar}{2m}\left[(\partial_i\omega_{kn}) - (\partial_k\omega_{in})\right]\Pi_n\sigma_k.\end{aligned} \tag{64}$$

In the limit $\hbar \to 0$, we get the equation of motion for σ_i,

$$\begin{aligned}\frac{d\sigma_i}{dt} &= -\frac{e}{mc}\Big(1-\frac{1}{4}h_{00}\Big)\epsilon_{ijk}B_j\sigma_k + \frac{e}{2mc}\epsilon_{ijk}\left(\omega_{rr}B_j - \omega_{lj}B_l\right)\sigma_k \\ &\quad - \frac{1}{2m}\left[(\partial_i\omega_{kn}) - (\partial_k\omega_{in})\right]\Pi_n\sigma_k.\end{aligned} \tag{65}$$

5 Gravitational Wave

The equations (52), (62) and (65) obtained in the previous section are not the most general ones because of the synchronous gauge fixing condition $h_{0k} = 0$. It is interesting to discuss the physical origin of this restriction. In gravity, the gauge transformations are related to the coordinate transformations. Thus, fixing the gauge, we choose the reference frame, as it happens in cosmology with the synchronous gauge fixing. However, in the present case, we have no right to perform the transformations of coordinates which violate the form of the expansion (1). In particular, this means that our results do not admit consideration of the gravitational field of rotating body. Anyway, these results are sufficient to reconsider the two important examples, namely the plane gravitational wave and the Schwarzschild metric. Let us start from the first of these two cases.

Consider the metric of the weak plane gravitational wave propagating along the axis OX. The nonzero components of the gravitational perturbations are

$$h_{yy} = -h_{zz} = -2\upsilon, \qquad h_{yz} = h_{zy} = 2u, \tag{66}$$

where υ and u are the functions of the light cone coordinate $ct-x$. Each of these functions describes one possible polarization of the gravitational wave. To compare our results with some of those available in literature, consider only one polarization state, i.e., set $u = 0$. The relevant quantities are

$$\omega_{kl} = 2\upsilon T_{kl} \qquad \text{and} \qquad \upsilon'_k = -\frac{i\hbar}{2} T_{jk}(\partial_j \upsilon), \tag{67}$$

where we introduced notation

$$T_{jk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{68}$$

Pauli equation in the plane gravitational wave background

Using (67) in Eq. (43), after some algebra, we arrive at the equation

$$\begin{aligned} i\hbar \frac{\partial \varphi}{\partial t} &= \Big\{ e\phi + \frac{1}{2m}\Pi_k\Pi_k - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + \frac{\upsilon}{m}T_{kl}\Pi_k\Pi_l \\ &\quad + \frac{\hbar}{2m}\sigma_m \epsilon^{mkl} T_{lr}\Pi_r\,(\partial_k \upsilon) - \frac{e\hbar\upsilon}{2mc}\left(\sigma_m T_{ll} B_m - \sigma_r T_{jr} B_j\right)\Big\}\varphi \\ &= \Big\{ e\phi + \frac{1}{2m}\left(\delta_{kl} + 2\upsilon T_{kl}\right)\Pi_k\Pi_l - \frac{e\hbar}{2mc}\left(\delta_{kl} + \upsilon T_{kl}\right)\sigma_k B_l \\ &\quad + \frac{\hbar}{2m}\sigma_m \epsilon^{mkl} T_{lr}\Pi_r(\partial_k \upsilon)\Big\}\varphi. \end{aligned} \tag{69}$$

This equation can be compared with the one obtained by Quach in [19] using both perturbative and exact Foldy-Wouthuysen transformations. Using the units with $\hbar = c = e = 1$, our Hamiltonian becomes

$$\begin{aligned} H &= m + \frac{1}{2m}\left(\delta_{kl} + 2\upsilon T_{kl}\right)\Pi_k\Pi_l - \frac{1}{2m}\left(\delta_{kl} + \upsilon T_{kl}\right)\sigma_k B_l \\ &\quad + \frac{1}{2m}\sigma_m \epsilon^{mkl} T_{lr}\Pi_r\,(\partial_k \upsilon)\,. \end{aligned} \tag{70}$$

Furthermore, we have to elaborate some expressions used in the work [19] as follows:

$$\frac{\beta}{4m}\delta^{ij}\epsilon_{jkl}\sigma^{l}(\partial^{k}A_{i}-\partial_{i}A^{k}) \;=\; -\frac{\beta}{4m}\Sigma^{l}\epsilon_{lkj}\partial^{k}A_{j}-\frac{\beta}{4m}\Sigma^{l}\epsilon_{ljk}\partial_{j}A^{k} \;=\; -\frac{\beta}{2m}\Sigma^{l}B_{l}, \quad (71)$$

where we used $\epsilon_{jkl}\partial^{k}A_{i}=\delta_{il}B_{j}$, and also

$$\frac{\beta}{4m}T^{ij}\epsilon_{jkl}\Sigma^{l}(\partial^{k}A_{i}-\partial_{i}A^{k}) \;=\; -\,\frac{\beta}{2m}\Sigma^{i}T^{ij}B_{j}. \quad (72)$$

Using these relations, taking into account the definition of $\vec{\Sigma}$ in (22) and applying the Hamiltonian of [19] to the wave function ψ' in the form (33), the result is identical to (70), except the factor 2 which multiplies υ of the last term in the first line of Eq. (70). We did not find this factor in our calculation. This difference is likely a misprint, as it does not affect the equations of motion found in [19].

At this point we conclude that the Pauli equation with the gravitational wave, derived in [19], got an independent confirmation.

Equation of motion for x_i

Starting from Eq. (52), we use the metric of the gravitational wave, as explained above. The result is

$$\frac{dx_i}{dt} \;=\; \frac{1}{m}\left(\delta_{il}+2\upsilon T_{il}\right)\Pi_{l}, \quad (73)$$

that is equivalent to the corresponding equation in [17]. Thus, we confirm this result.

Equation of motion for p_i

Using the gravitational wave metric on Eq. (59), we arrive at

$$\frac{dp_i}{dt} \;=\; eE_i-\frac{1}{m}T_{kl}(\partial_i\upsilon)\,\Pi_k\Pi_l+\frac{e}{mc}\big(\delta_{kl}+2\upsilon T_{kl}\big)\Pi_k\big(\partial_i A_l\big). \quad (74)$$

The unique difference with the analogous equation in [17] is the term with electric field, which was not discussed in this reference.

Equation of motion for σ_i

Using the gravitational wave metric in Eq. (65), we get

$$\frac{d\sigma_i}{dt} \;=\; -\,\frac{e}{mc}\,\epsilon_{ijk}B_j\sigma_k-\frac{e\,\upsilon}{mc}\,\epsilon_{ijk}\,T_{lj}\,B_l\sigma_k-\frac{1}{m}\big[\,(T_{kn}\partial_i\upsilon)-(T_{in}\partial_k\upsilon)\,\big]\Pi_n\sigma_k. \quad (75)$$

This equation is different from the one in [17]. On the other hand, using the relation

$$(\partial_l A_n) - (\partial_n A_l) \;=\; F_{ln} = -F_{nl} = -\,\epsilon_{nlr} B_r \tag{76}$$

it can be shown equivalent to the formula obtained by Quach in [19], except the sign of the first, non-gravitational term, which is correct in (75).

6 Newtonian limit of the Schwarzschild's metric

One of the simplest ways to arrive at the constant Newtonian gravitational field is to consider a weak regime of the spherically symmetric case. The Schwarzschild's metric tensor is given by

$$g_{\mu\nu} \;=\; \text{diag}\,\Big(A(r),\; -A^{-1}(r),\; -r^2,\; -r^2 \sin^2\theta\Big), \tag{77}$$

with the notations

$$A(r) \;=\; 1 - \frac{2GM}{rc^2}\,, \qquad \tilde{A}(r) \;=\; 1 + \frac{2GM}{rc^2}\,. \tag{78}$$

In the weak-field (including Newtonian) limit, $A^{-1}(r) = \tilde{A}(r)$ and Eq. (77) becomes

$$g_{\mu\nu} \;=\; \text{diag}\,\Big(A(r),\; -\tilde{A}(r),\; -r^2,\; -r^2 \sin^2\theta\Big), \tag{79}$$

Changing to the isotropic coordinates,

$$r = r'\left(1 + \frac{GM}{2r'c^2}\right)^2 \tag{80}$$

it is easy to verify that Eq. (79) gets the form

$$g_{\mu\nu} \;=\; \text{diag}\,\Big(A(r),\; -\tilde{A}(r),\; -\,r^2\tilde{A}(r),\; -\,r^2 \sin^2\theta \tilde{A}(r)\Big). \tag{81}$$

In this formula, we omitted the prime in the new coordinate, $r' \to r$, for simplicity. Defining the new (quasi-cartesian) coordinates

$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\theta, \tag{82}$$

the metric is cast as

$$g_{\mu\nu} \;=\; \text{diag}\,\Big(A(r),\; -\tilde{A}(r),\; -\,\tilde{A}(r),\; -\,\sin^2\theta \tilde{A}(r)\Big), \tag{83}$$

that is (1), $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with

$$h_{\mu\nu} \;=\; \frac{2\Phi}{c^2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad \text{and} \qquad \Phi \;=\; -\,\frac{GM}{r} \tag{84}$$

is the Newtonian potential. With this metric, we can write the relevant terms in the summands of vector V_k in (30),

$$\omega_{kl} \;=\; \frac{4\Phi}{c^2}\delta_{kl}, \qquad v'_k \;=\; -\,\frac{5}{2c^2}\, i\hbar\, \partial_k \Phi\,. \tag{85}$$

Now we are in a position to consider Pauli equation and the equations for the particle. The comparison will be with the results of Ref. [11].

Pauli equation in the Newtonian limit

Introducing the metric given by Eq. (84) and the terms (85) in Eq. (43), we get, for the Newtonian limit of the Schwarzschild's metric,

$$\begin{aligned} i\hbar\,\frac{\partial\varphi}{\partial t} \;&=\; \Big\{ e\phi + m\Phi + \frac{1}{2m}\Big(1 + \frac{7\Phi}{2c^2}\Big)\vec{\Pi}^2 - \frac{e\hbar}{2mc}\Big(1 - \frac{9\Phi}{2c^2}\Big)\,\vec{\sigma}\cdot\vec{B} \\ &\quad - \frac{5i\hbar}{2mc^2}\,(\partial_k\Phi)\,\Pi_k + \frac{\hbar}{mc^2}\,\sigma_m\epsilon^{mkl}\,(\partial_k\Phi)\,\Pi_l \Big\}\varphi. \end{aligned} \tag{86}$$

To compare with the equation of [11], we have to take zero electromagnetic field, use the natural system of units $\hbar = c = e = 1$ and add the rest energy $mc^2 = m$. With these changes, the Hamiltonian operator in (86) becomes

$$H \;=\; m + m\Phi + \frac{p^2}{2m}\Big(1 + \frac{7}{2}\Phi\Big) + \frac{1}{m}\,\sigma^j\epsilon^{jkl}(\partial_k\Phi)\,p_l - \frac{5i}{2m}\,(\partial_k\Phi)\,p_k, \tag{87}$$

The Hamiltonian obtained in [11] has the form

$$H_{ST} \;=\; \beta\epsilon - \frac{\beta}{2}\left\{\frac{\epsilon^2 + p^2}{\epsilon}, \frac{GM}{r}\right\} \;-\; \frac{\beta\,(2\epsilon + m)}{4\epsilon(\epsilon + m)}\big[2\vec{\Sigma}\cdot(\vec{g}\times\vec{p}) \;+\; \nabla\vec{g}\big], \tag{88}$$

where

$$\{A, B\} = AB + BA, \qquad \epsilon = \sqrt{m^2 + p^2}, \qquad \vec{g} = -\nabla\Phi, \tag{89}$$

and $\vec{\Sigma}$ has been defined in (22). Looking for the correspondence with (87), let us expand

$$\epsilon = m\sqrt{1 + \frac{p^2}{m^2}} \simeq m\left(1 + \frac{p^2}{2m^2}\right). \tag{90}$$

Then, in the leading approximation,

$$\frac{2\epsilon + m}{\epsilon(\epsilon + m)} \simeq \frac{3}{2m} \tag{91}$$

and (88) boils down to

$$H_{ST} \,=\, m + m\Phi + \frac{p^2}{2m}\,(1+4\Phi) + \frac{3}{4m}\,\sigma^l \epsilon^{lkj}\,(\partial_k\Phi)\,p_j + \frac{3}{4m}\nabla^2\Phi, \tag{92}$$

that has different numerical coefficients compared to our result (87).

Equation of motion for x_i

In the same way as before, we derive the equation for x_i in (52) in the case of Newtonian limit of the Schwarzschild metric,

$$\frac{dx_i}{dt} \,=\, \frac{1}{m}\left(1 + \frac{7\Phi}{2c^2}\right)\Pi_i. \tag{93}$$

In this case, there is no equivalent result in the literature to compare.

Equation of motion for p_i

This part may have certain importance related to the experiments devoted to the high-precision measurement of the small ferromagnetic particles caused by the weak gravitational field [7]. Regardless those are macroscopic particles, the model of a massive Dirac particle with spin may be used as a zero-order approximation. Let us note that the motion of small macroscopic particles with magnetic moment was not explored theoretically, until now[4]

The equation (59) with the metric (84), becomes

$$\frac{dp_i}{dt} \,=\, -\,eE_i - m(\partial_i\Phi) - \frac{7}{4mc^2}\,(\partial_i\Phi)\,\vec{\Pi}^2 + \frac{e}{mc}\left(1 + \frac{7\Phi}{c^2}\right)(\partial_i A_l)\Pi_l. \tag{94}$$

Again, to make a comparison with the result of [11], one has to switch off electromagnetic field and use the natural units, that transforms (94) into

$$\frac{dp_i}{dt} \,=\, -\,m(\partial_i\Phi) \,-\, \frac{7}{4m}(\partial_i\Phi)\,p^2. \tag{95}$$

On the other hand, the equation of [11] is

$$\frac{d\vec{p}}{dt} = \frac{\epsilon^2 + p^2}{\epsilon}\vec{g} \tag{96}$$

[4] Authors are grateful to Yu.N. Obukhov for the discussion of this point.

where ϵ an $\vec{g}$ are defined by (89). Using the $\mathcal{O}(1/m)$ approximation, we obtain the equation

$$\frac{dp_i}{dt} = m\partial_i\Phi + \frac{2}{m}(\partial_i\Phi)\,p^2. \tag{97}$$

Once again, there is a numerical difference with Eq. (95). Independent on this difference, we can from both (95) and (97), that an additional acceleration depends on the orientation and magnitude of the velocity of the particle, but not on the spin variable. This means, we do not have too much expectation to meet spin-gravity interactions for small magnetic particles too. Anyway, it would be an interesting issue to check.

Equation of motion for σ_i

Finally, with the metric given by equation (84), the equation for σ_i in (65) becomes

$$\frac{d\sigma_i}{dt} = -\frac{e}{mc}\Big(1+\frac{\Phi}{2c^2}\Big)\epsilon_{ijk}B_j\sigma_k + \frac{4e}{mc^3}\,\Phi\,\epsilon_{ijk}B_j\sigma_k + \frac{2}{mc}\big[\,(\partial_i\Phi)\,\Pi_k\sigma_k - (\partial_k\Phi)\,\Pi_i\sigma_k\big]. \tag{98}$$

Using the natural system of unit with zero electromagnetic field, we arrive at

$$\frac{d\sigma_i}{dt} = -\frac{2}{m}\Big[\,(\partial_i\Phi)\,p_k\sigma_k - (\partial_k\Phi)\,p_i\sigma_k\Big]. \tag{99}$$

Taking the equation obtained in [11] (in our notations)

$$\frac{d\vec{\sigma}}{dt} = \frac{2\epsilon+m}{\epsilon(\epsilon+m)}\,\vec{\sigma}\times(\vec{g}\times\vec{p}). \tag{100}$$

Performing the $\mathcal{O}(1/m)$ expansion, in the first nontrivial order we obtain

$$\frac{d\xi_i}{dt} = -\frac{3}{2m}\Big[(\partial_k\Phi)p_i\sigma_k - (\partial_i\Phi)p_k\sigma_k\Big]. \tag{101}$$

Compared to (99), we meet the same structure, but with a different coefficient.

7 Conclusions

We considered the nonrelativistic approximation for the Dirac equation in a combination of a general weak gravitational field and a general weak electromagnetic field. The calculations are technically more complex and cumbersome that the ones for particular metrics, such as plane gravitational wave or the Newton gravity. On the other hand, our calculations were performed using a more simple and "basic" method, compared to the exact or perturbative Foldy-Wouthuysen transformations. The main result of our efforts are the extended version of the Pauli equation (43) and Eqs. (52), (62) and (65) for a charged spinning particle.

In the particular case of the plane gravitational wave we met a perfect correspondence with the previous equations of [19], which partially corrected the first derivation of [17]. In another particular case, of the Newtonian gravity, we found the same general structure which was previously reported in [11], but in many cases, with different coefficients.

From the physical side, an important result is that the relativistic corrections to the Newton law depends on the velocity of the particles, but does not depend on the spin of the particles. Thus, the *qualitative* situation is somehow similar to the one of MOND [23], which is tested in the experiment [7].

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