Planck scale effects on some low energy quantum phenomena

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Almost all theories of Quantum Gravity predict modifications of the Heisenberg Uncertainty Principle near the Planck scale to a so-called Generalized Uncertainty Principle (GUP). Recently it was shown that the GUP gives rise to corrections to the Schrödinger and Dirac equations, which in turn affect all non-relativistic and relativistic quantum Hamiltonians. In this paper, we apply it to superconductivity and the quantum Hall effect and compute Planck scale corrections. We also show that Planck scale effects may account for a (small) part of the anomalous magnetic moment of the muon. We obtain (weak) empirical bounds on the undetermined GUP parameter from present-day experiments.

Various approaches to quantum gravity, such as String Theory, Doubly Special Relativity (DSR) Theories, Loop Quantum Gravity via so-called Polymer Quantization, as well as black hole physics, predict a minimum measurable length, and a modification of the Heisenberg Uncertainty Principle to a so-called Generalized Uncertainty Principle, or GUP, and a corresponding modification of the commutation relations between position and momenta [1–7]. The only GUP consistent with the symmetries and index structure of the modified commutator bracket between position and momentum from all the above derivations (all of which predict corrections involving at most terms up to second order in the momentum), *and* which ensures $[x_i, x_j] = 0 = [p_i, p_j]$ (via the Jacobi identity)[1] is, to the best of our knowledge [9, 10]

$$[x_i, p_j] = i\hbar\left[\delta_{ij} - a\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + a^2\left(p^2\delta_{ij} + 3p_i p_j\right)\right] \quad (1)$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 - 2a < p > + 4a^2 < p^2 >\right]$$
$$\geq \frac{\hbar}{2}\left[1 + \left(\frac{a}{\sqrt{\langle p^2\rangle}} + 4a^2\right)\Delta p^2 + 4a^2\langle p\rangle^2 - 2a\sqrt{\langle p^2\rangle}\right] \quad (2)$$

where $a = a_0/M_{Pl}c = a_0\ell_{Pl}/\hbar$, M_{Pl} = Planck mass, $\ell_{Pl} \approx 10^{-35}\ m$ = Planck length, and $M_{Pl}c^2$ = Planck energy $\approx 10^{19}\ GeV$. It should be stressed that the GUP-induced terms become important near the Planck scale. It is normally assumed that $a_0 \approx 1$. For phenomenological implications of the above GUP, see [8–12]. Note that although Eqs. (1) and (2) are not Lorentz covariant, they are at least approximately covariant under DSR transformations [6]. We expect the results of our paper to have similar covariance as well. In addition, since DSR transformations preserve not only the speed of light, but also the Planck momentum and the Planck length, it is not surprising that Eqs. (1) and (2) imply the following minimum measurable length *and* maximum measurable momentum

$$\Delta x \ \geq\ (\Delta x)_{min} \approx a_0\ell_{Pl} \quad (3)$$

$$\Delta p \ \leq\ (\Delta p)_{max} \approx \frac{M_{Pl}c}{a_0}\ . \quad (4)$$

It can be shown that the following definitions

$$x_i = x_{0i}\ ,\quad p_i = p_{0i}\left(1 - ap_0 + 2a^2p_0^2\right)\ , \quad (5)$$

(with x_{0i}, p_{0j} satisfying the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar\ \delta_{ij}$, such that $p_{0i} = -i\hbar\partial/\partial x_{0i}$) satisfy Eq.(1). In [8, 9] it was shown using Eq.(5), that any non-relativistic Hamiltonian of the form $H = p^2/2m + V(\vec{r})$ can be written as $H = p_0^2/2m - (a/m)p_0^3 + (5a^2/2m)p_0^4 + V(r) + \mathcal{O}(a^3)$, implying the modified Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{i\alpha\hbar^3}{m}\nabla^3 + \frac{5a^2\hbar^4}{2m}\nabla^4\right]\psi = i\hbar\frac{\partial\psi}{\partial t}\ . \quad (6)$$

We will treat the a and a^2 terms as perturbations, although the higher order Schrödinger equation now has new *non-perturbative* solutions of the form $\psi \sim e^{ix/2a\hbar}$, which may have interesting physical implications [9]. Some phenomenological implications of the above GUP modified Hamiltonian were examined in [11, 12].

Note that for the earlier versions of the GUP (which did not take into account DSR), the terms in Eqs.(1), (2) and (5) linear in a and the Planck length were effectively absent. In the following sections, we will apply that version to the problems of Superconductivity (Section I) and the Quantum Hall Effect (Section II). In Section III, we will write the Dirac equation that follows from the full GUP, and apply it to the problem of anomalous magnetic moment of the muon.

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[1] (a) In refs. [8–10] α was used in place of a.
(b) The results of this article do not depend on this particular form of GUP chosen, and continue to hold for a a large class of variants, so long as an $\mathcal{O}(a)$ term is present in the right hand side of Eq.(1).

I. SUPERCONDUCTIVITY

The usual Schrödinger current minimally coupled to a Cooper pair of charge $-2e$ and mass $2m$ reads [13]

$$\vec{J} = -\frac{e}{2m}\left[\psi^\star\left\{\left(\frac{\hbar}{i}\vec{\nabla} + \frac{2e}{c}\vec{A}\right)\psi\right\} + \left\{\left(\frac{\hbar}{i}\vec{\nabla} + \frac{2e}{c}\vec{A}\right)\psi\right\}^\star \psi\right] \tag{7}$$

Substituting $\psi = |\psi|e^{i\phi}$, and assuming virtually all spatial dependence of the wavefunction in the phase ϕ, such that $|\psi| \approx$ constant, and $\vec{\nabla}\psi = i\psi\vec{\nabla}\phi$, we get:

$$\vec{J} = -\left[\frac{2e^2}{mc}\vec{A} + \frac{e\hbar}{m}\vec{\nabla}\phi\ .\right]|\psi|^2 \tag{8}$$

Integrating both sides of (8) over a closed loop inside a superconducting material (where $\vec{J} = 0$), we get:

$$0 = \oint \vec{J}\cdot d\vec{l} = \oint\left(\frac{2e^2}{mc}\vec{A} + \frac{e\hbar}{m}\vec{\nabla}\phi\right)\cdot d\vec{l} \tag{9}$$

or, from Stokes theorem:

$$\begin{aligned}\Phi &\equiv \int \vec{B}\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l} = \frac{\hbar c}{2e}\oint \vec{\nabla}\phi\cdot d\vec{l} \\ &= \frac{\hbar c}{2e}\Delta\phi = \frac{\hbar c}{2e}2\pi n \equiv n\Phi_0\ ,\quad \Phi_0 = \frac{hc}{2e}\ , n\in\mathbb{N}\end{aligned} \tag{10}$$

which is the flux quantization in a superconductor.

Next, we would like to estimate GUP effects on the above flux quantum. We see from (6) that because of the ∇^3 operator, the leading Planck scale term of order $O(a)$ is non-local, except in 1-spatial dimension. We do not know of a natural way of circumventing the problem for the non-relativistic case at hand, though such a linearization can modify the Dirac equation [9] (which we shall use in Section III). Thus we will work with the earlier version of the GUP and equivalently the $O(a^2)$ term in Eq.(6). The new conserved current follows (see [11], with $\beta \to 5a^2/2$), again for charge $-2e$ and mass $2m$

$$\begin{aligned}\vec{J} &= \frac{\hbar}{2mi}\left[\psi^\star\vec{\nabla}\psi - \psi\vec{\nabla}\psi^\star\right] \\ &+ \frac{5a^2\hbar^3 e}{2mi}\left[\left(\psi^\star\vec{\nabla}\nabla^2\psi - \psi\vec{\nabla}\nabla^2\psi^\star\right)\right. \\ &\quad \left. + \left(\nabla^2\psi^\star\vec{\nabla}\psi - \nabla^2\psi\vec{\nabla}\psi^\star\right)\right] \qquad (11) \\ &\equiv \vec{J}_0 + \vec{J}_1\ , \qquad (12) \\ \rho &= |\psi|^2 \qquad \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0\ , \qquad (13)\end{aligned}$$

with J_1 being the GUP induced term. Once again, minimal coupling with the Cooper pairs give

$$\begin{aligned}\vec{J}_1 = -\frac{5a^2 e}{2m}\Bigg\{&\psi^\star\left[\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\cdot\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\right]\psi + \left[\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\cdot\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\psi\right]^\star\psi \\ &+ \left[\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\cdot\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\psi\right]^\star\left[\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\psi\right] + \left[\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\cdot\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\psi\right]\left[\left(\frac{\hbar\vec{\nabla}}{i} + \frac{2e}{c}\vec{A}\right)\psi\right]^\star\Bigg\}\end{aligned} \tag{14}$$

Using $|\vec{A}| \approx |\vec{B}|L$, where L is a typical linear dimension of the sample and $\hbar\vec{\nabla}\psi/i = \hbar\psi\vec{\nabla}\phi \approx \hbar 2\pi\psi/L$, an experiment can be arranged that such that $|\hbar\vec{\nabla}\psi/i| \ll |2e\vec{A}\psi/c|$. For example, for $|\vec{B}| \approx 0.1\ T,\ L \approx 0.1\ m$, $|\hbar\vec{\nabla}\psi/i|/|2e\vec{A}\psi/c| \approx 10^{-3}$. Hence

$$\vec{J}_1 \approx -\frac{80a^2e^4}{mc^3}\vec{A}|\vec{A}|^2|\psi|^2\ . \tag{15}$$

Thus using once again $0 = \oint \vec{J}\cdot d\vec{l} = \oint \vec{J}_0\cdot d\vec{l} + \oint \vec{J}_1\cdot d\vec{l}$, and treating $|\vec{A}|^2 \approx |\vec{B}|^2L^2$ as effectively constant over the domain of integration we now, in lieu of Eq.(10), get the flux $(1 - \frac{40a^2e^2}{c^2}|\vec{A}|^2)\Phi = \frac{hc}{2e}n$ or

$$\Phi = \left(1 + \frac{40a^2e^2}{c^2}|\vec{A}|^2\right)n\Phi_0 \equiv n\left(\Phi_0 + a^2\Phi_1\right) \tag{16}$$

$$\Phi_1 = \frac{40\ e^2|\vec{B}|^2L^2}{c^2}\ \Phi_0\ ,\ n\in\mathbb{N}\ . \tag{17}$$

to leading order in a.

Measurement of the fundamental flux quantum implies $a^2\Phi_1 < \delta\Phi_0/\Phi_0$, where $\delta\Phi_0/\Phi_0$ is the experimental error. Using Eq.(17) above, we obtain an upper bound on a_0,

$$a_0 < \frac{10^{-n/2}}{\sqrt{40}}\frac{M_{Pl}c^2}{eBL} < 10^{19-n/2}\ , \tag{18}$$

assuming experimental precision of 1 part in 10^n, where again we used $|\vec{B}| \approx 0.1\ T,\ L \approx 0.1\ m$. For example, for $n = 4$, $a_0 < 10^{17}$. Conversely, if a significant improvement of precision can be achieved, then small deviations from Φ_0, as predicted above, may be observable! P induced correction to the flux quantum.

II. QUANTUM HALL EFFECT

The modification of the flux quantum has a direct effect on the observable Hall resistance. As is well known, a current density j_x along x in a two dimensional sample in the xy plane subjected to a magnetic field B along z results in a potential difference and an effective electric field $\mathcal{E}_y$ along y. This results in a cancelation of the electric and Lorentz force on the charge carriers (having drift velocity v)

$$e\mathcal{E}_y = evB \ , \tag{19}$$

and sets up a measurable potential difference in that direction. Eq.(19), and the relation $j_x = nev$, where n is the electron density in the sample, imply

$$\mathcal{E}_y = \frac{j_x B}{ne} \ . \tag{20}$$

The Hall resistivity ρ_{xy} is defined by the relation

$$\mathcal{E}_y = \rho_{xy} j_x \ , \tag{21}$$

which combined with Eq.(20) yields

$$\rho_{xy} = \frac{B}{ne} \ . \tag{22}$$

We also know that quantum mechanically, the electrons in the sample subjected to the perpendicular magnetic field give rise to Landau levels, with the energy at level n given by

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right) \ , \tag{23}$$

where $\omega_c = eB/mc$ is the cyclotron frequency. Now, from flux quantization it follows that the density of quanta of magnetic flux is given by

$$n_H = \frac{B}{\Phi_0 + a^2\Phi_1} \tag{24}$$

in terms of the single unit of flux quanta given by (16). This is also the density of states for each Landau level, where we have made the replacement $2e \to e$ in Eq.(10), since now the carriers are electrons as opposed to Cooper pairs.

Now if the Fermi energy E_F lies between the energy levels E_k and E_{k+1}, all states $E_i \ , i \le k$ are occupied, resulting in the carrier density

$$n = k n_H, \ k \in \mathbb{N}. \tag{25}$$

Thus from Eqs.(16), (17) and (22), the Hall resistance turns out to be [14]

$$\rho_{xy} = \frac{hc}{ke^2}\left[1 + \frac{10a^2e^2|\vec{B}|^2L^2}{c^2}\right] \ , k \in \mathbb{N} \ . \tag{26}$$

Although the Hall resistance is still quantized, its magnitude has shifted by a small amount. A bound similar to Eq.(18), as well as possibilities of measurement of corrections of the above type can be argued for this case as well.

III. ANOMALOUS MAGNETIC MOMENT OF THE MUON

In this case, we show that the non-relativistic limit of the Dirac equation can be used to extract GUP corrections. First, as in [9] we linearize $p_0 = \sqrt{p_{0x}^2 + p_{0y}^2 + p_{0z}^2}$ by replacing $p_0 \to \vec{\alpha}\cdot\vec{p}$, where α_i ($i = 1,2,3$) and β are the Dirac matrices, for which we use the following representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \ , \ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \ . \tag{27}$$

The GUP-corrected Dirac equation can thus be written to $\mathcal{O}(a)$ as

$$\begin{aligned} H\psi &= \left(c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\psi(\vec{r},t) \\ &= \left(c\,\vec{\alpha}\cdot\vec{p}_0 - c\,a(\vec{\alpha}\cdot\vec{p}_0)(\vec{\alpha}\cdot\vec{p}_0) + \beta mc^2\right)\psi(\vec{r},t) \\ &= i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} \ . \end{aligned} \tag{28}$$

To study the non-relativistic limit, we write the spinor ψ as [15]

$$\psi \ = \ e^{-imt}\begin{pmatrix} \chi_1(\vec{r},t) \\ \chi_2(\vec{r},t) \end{pmatrix} \ , \tag{29}$$

and we include the electromagnetic potential $A^\mu = (\phi, \vec{A})$ in Eq.(28) by the usual minimal coupling prescription [11] $i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} - e\phi$, $\vec{p}_0 \to \vec{\Pi}_0 \equiv \vec{p}_0 - e\vec{A}/c$, obtaining the two component equations

$$\begin{aligned} i\hbar\frac{\partial\chi_1}{\partial t} &= e\phi\chi_1 + c\left(\vec{\sigma}\cdot\vec{\Pi}\right)\chi_2 - ca\left(\vec{\sigma}\cdot\vec{\Pi}\right)^2\chi_1 \\ i\hbar\frac{\partial\chi_2}{\partial t} &= \left(e\phi - 2mc^2\right)\chi_2 + c\left(\vec{\sigma}\cdot\vec{\Pi}\right)\chi_1 - ca\left(\vec{\sigma}\cdot\vec{\Pi}\right)^2\chi_2 \end{aligned} \tag{30}$$

In the non-relativistic limit $mc^2 >> e\phi, |\partial\chi_2/\partial t|$, the second of Eqs.(30) becomes to $\mathcal{O}(a)$

$$\chi_2 = \frac{1}{2mc}\left[1 - \frac{a}{2mc}\left(\vec{\sigma}\cdot\vec{\Pi}\right)^2\right]\left(\vec{\sigma}\cdot\vec{\Pi}\right)\chi_1 \ , \tag{31}$$

which, when substituted into the first of Eqs.(30) yields

$$\begin{aligned} i\hbar\frac{\partial\chi_1}{\partial t} &= e\phi\chi_1 + \frac{1}{2m}\left(\vec{\sigma}\cdot\vec{\Pi}\right)^2\chi_1 \\ &- \frac{a}{(2m)^2c}\left(\vec{\sigma}\cdot\vec{\Pi}\right)^4\chi_1 - ca\left(\vec{\sigma}\cdot\vec{\Pi}\right)^2\chi_1 \ . \end{aligned} \tag{32}$$

Using the identities $\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$ and $\left(\vec{\sigma}\cdot\vec{\Pi}\right)^2 = |\vec{\Pi}|^2 - e\hbar\vec{\sigma}\cdot\vec{B}/c$ and the identification of the spin operator $\vec{S} = \vec{\sigma}/2$, Eq.(32) becomes

$$\begin{aligned} i\hbar\frac{\partial\chi_1}{\partial t} &= \left[\left(\frac{1}{2m} - ca\right)|\vec{\Pi}|^2 - \frac{a}{(2m)^2c}\Pi^4 + e\phi\chi_1 \right. \\ &-2\frac{e\hbar}{2mc}\left(1 - 2acm - \frac{a}{mc}\Pi^2\right)\vec{S}\cdot\vec{B} - \frac{ae^2\hbar^2}{(2m)^2c^3}|\vec{B}|^2 \\ &\left. -\frac{ie\hbar a}{2(mc)^2}\left(\vec{\nabla}(\vec{\sigma}\cdot\vec{B})\cdot\vec{\Pi} - \frac{e}{c}\vec{A}\cdot\vec{\nabla}(\vec{\sigma}\cdot\vec{B}) + i\hbar\nabla^2(\vec{\sigma}\cdot\vec{B})\right)\right]\chi_1 \end{aligned} \tag{33}$$

where the terms in the first line correspond to the GUP corrected kinetic terms (including the Π^4 term) and the potential energy term, while those on the second and third lines pertain to the interaction with the electron with an external magnetic field. The ones in the third line are also new terms which depend on derivatives of the magnetic field[2] . Since $e/2mc$ is the Bohr magneton, one gets $g = 2(1 - 2acm - \frac{a}{mc}\Pi^2)$, or

$$\left(\frac{g-2}{2}\right)_{GUP} = -\left[2acm + \frac{a\Pi^2}{mc}\right] . \tag{34}$$

Note that GUP predicts a slight decrease in the value of g, and for a measurement accuracy of 1 part in 10^n, one obtains the bound

$$\begin{aligned} a_0 &< 10^{-n}\frac{m_{Pl}}{m_\mu} \\ &< 10^{20-n} , \end{aligned} \tag{35}$$

where we have used $m_{Pl} = 1.2 \times 10^{19}\ GeV/c^2$ and $m_\mu = 105.7\ MeV/c^2$. Thus for the present-day precision level of $n = 12$, $a_0 < 10^8$, which gives a much tighter bound. Conversely, further increase of accuracies may enable one to observe the above deviation.

IV. CONCLUSIONS

In this paper we have explored Planck scale effects on some low energy systems via the Generalized Uncertainty principle, which appears to be a robust prediction of most theories of Quantum Gravity. We found that small but non-zero effect are present for the fundamental flux quantum of superconductivity, for the integer quantum Hall effect, and for the anomalous magnetic moment of the muon. Since these effects have not been observed so far, one can obtain important upper bounds on the GUP parameter, which turns out to be $a_0 < 10^{17}$ and $a_0 < 10^8$ from current experiments in Superconductivity and muon experiments respectively. These can be compared with the bounds obtained in [8], between 10^{18} and 10^{25}, and in [10], between 10^{10} and 10^{23}. Although the above bounds appear to be rather weak, future experiments of greater precision will either provide better bounds on the GUP parameter or in an optimistic scenario, may be able to detect some of these effects. In a broader context this approach may open up a low energy window to quantum gravity phenomenology. This strongly suggests that more work needs to be done in this direction.

[2] We used the following identities in their derivation: $[\vec{\Pi}, \vec{\sigma}\cdot\vec{B}] = \vec{p}_0(\vec{\sigma}\cdot\vec{B})$, $[\Pi^2, \vec{\sigma}\cdot\vec{B}] = \vec{\Pi}\cdot\vec{p}_0(\vec{\sigma}\cdot\vec{B}) + \{\vec{p}_0(\vec{\sigma}\cdot\vec{B})\}\cdot\vec{\Pi}$ and $(\vec{\sigma}\cdot\vec{B})\Pi^2 = \Pi^2(\vec{\sigma}\cdot\vec{B}) - [\Pi^2, \vec{\sigma}\cdot\vec{B}]$

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Note added

After completion of this work, we became aware of paper [16] (we thank the referee for pointing this out to us), in which the authors obtain a lower bound on the fundamental (higher dimensional) Planck mass in theories with extra dimensions, from muon $(g-2)$ measurements, by directly using the modified Dirac equation. In this paper on the other hand, we use the non-relativistic limit of the GUP modified Dirac equation to obtain upper bounds on a_0.

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