CATEGORICAL GEOMETRY AND THE MATHEMATICAL FOUNDATIONS OF QUANTUM GRAVITY

by Louis Crane, Mathematics Department Kansas State University

ABSTRACT: *We consider two related approaches to quantizing general relativity which involve replacing point set topology with category theory as the foundation for the theory. The ideas of categorical topology are introduced in a way we hope is physicist friendly.*

I. INTRODUCTION

The mathematical structure of a theory is a very abstract collection of assumptions about the nature of the sphere of phenomena the theory studies. Given the great cultural gap which has opened between Mathematics and Physics, it is all too easy for these assumptions to become unconscious.

General Relativity is a classical theory. Its mathematical foundation is a smooth manifold with a pseudometric on it. This entails the following assumptions:

1. Spacetime contains a continuously infinite set of pointlike events which is independent of the observer.

2. Arbitrarily small intervals and durations are well defined quantities. They are either simultaneously measurable or must be treated as existing in principle, even if unmeasurable.

3. At very short distances, special relativity becomes extremely accurate, because spacetime is nearly flat.

4. Physical effects from the infinite set of past events can all affect an event in their future, consequently they must all be integrated over.

The problem of the infinities in quantum general relativity is intimately connected to the consequences of these assumptions.

In my experience, most relativists do not actually believe these assumptions to be reasonable. Nevertheless, any attempt to quantize relativity which begins with a metric on a three or four dimensional manifold, a connection on a manifold, or strings moving in a geometric background metric on a manifold, is in effect making them.

Philosophically, the concept of a continuum of points is an idealization of the principles of classical Physics applied to the spacetime location of events. Observations can localize events into regions. Since classically all observations can be performed simultaneously and with arbitrary accuracy, we can create infinite sequences of contracting regions, which represent points in the limit.

Relativity and quantum mechanics both create obstacles to this process. Determinations of position in spacetime cannot be arbitrarily precise, nor can they be simultaneously well defined.

Unfortunately the classical continuum is thousands of years old and is very deeply rooted in our education. It tends to pass under the radar screen.

I often suspect that quantum physicists are suspicious of Mathematics because so much of it seems wrong to them. I think the solution is more Mathematics rather than less.

One often hears from quantum field theorists that the continuum is the limit of the lattice as the spacing parameter goes to 0. It is not possible to obtain an uncountable infinite point set as a limit of finite sets of vertices, but categorical approaches to topology do allow us to make sense of that statement, in the sense that topoi of categories of simplicial complexes are limits of them.

I have become convinced that the extraordinary difficulty of quantizing gravity is precisely due to the omnipresence of the numerated assumptions. For this reason, this paper will explore the problem of finding the appropriate mathematical concept of spacetime in which a quantum theory of GR could be constructed.

Now, although it is not well known among physicists of any stripe, mathematicians have developed very sophisticated foundations both for topology and the geometry of smooth manifolds, in which an underlying point set is not required.

We will be interested in two related lines of development here; higher category theory and topos theory. Over the last several years, it has become clear that these mathematical approaches have a number of close relationships with interesting new models for quantum gravity, and also with foundational issues in quantum mechanics which will have to be faced in QGR.

In this paper, I hope to introduce these ideas to the relativity community. The most useful approach seems to be to begin with a non-technical introduction to the mathematical structures involved, followed by a survey of actual and potential applications to Physics.

II. SOME MATHEMATICAL APPROACHES TO POINTLESS SPACE AND SPACETIME

A. CATEGORIES IN QUANTUM PHYSICS; FEYNMANOLOGY

Although categorical language is not explicitly familiar in Physics, quantum field theory is in fact dominated by the theory of tensor categories under a different name. A category is a mathematical structure with objects and maps between them called morphisms [1]. A tensor category has the further structure of a product which allows us to combine two objects into a new one.

If we write out a general morphism in a tensor category, we get arrows starting from sets of several objects and ending in different sets of several objects, where we think of maps into and out of the tensor product of objects as maps into and out of their combination. When we compose these, we get exactly Feynman graphs.

The objects in the category are the particles (or more concretely their internal hilbert spaces), and the vertices are the tensor morphisms.

The resemblance is not accidental. The kronecker product which tells us how to combine the hilbert spaces of subsystems is just what mathematicians call the tensor product.

The representations of a lie group form a tensor category, in which the morphisms are the maps which intertwine the group action. This is equivalent to the prescription in feynmanology that we include all vertices not excluded by the symmetries of the theory.

The physicist reader can substitute for the idea of a categorical space the idea that the spacetime is actually a superposition of Feynman graphs, which we can think of as a vacuum fluctuation.

The feynmanological point of view has been developed for the BC model under the name of group field theory [2]. The 4-simplices of the triangulation are treated as vertices in this point of view, and the 3-simplices as particles.

The categorical language is much more developed, and connected to more mathematical examples. I hope I will be forgiven for staying with it.

B. GROTHENDIECK SITES AND TOPOI

The mathematical ideas we shall consider trace back to the work of Alexander Grothendieck, perhaps the deepest mathematical thinker who ever lived. Much of his work was only appreciated after several decades, his deepest ideas are still not fully understood.

Grothendieck made the observation that the open sets of a topological space could be considered as the objects of a category, with a morphism between two objects if the first was contained in the second. He called this the site of the space. This was motivated by the observation that presheaves over the space are the same as functors from the site to the category of whatever type of fiber the sheaf is supposed to have. Since the constructions of topology and geometry can

be reformulated in terms of presheaves, (a bundle, for example can be replaced with the presheaf of its local sections), this opened the way to a far ranging generalization of topology and geometry, in which general categories play the role of spaces.

Grothendieck also realized that rather than the site itself, the central object of study was the category of presheaves over it, (or functors into the category of sets), which he called its topos [3].

Topoi also have an axiomatic definition, which amounts to the idea that they are a category in which all the normal constructions done on sets have analogs. It was then proven that every abstract topos is the topos of some site [3,4].

For this reason, the objects in a topos can be thought of either as abstract sets, or variable or relative sets.

One of the interesting aspects of topos theory is that the objects in a topos can inherit structure from the objects in the category which is its site.

An important example is synthetic differential geometry [5], the study of the topos over the site category of smooth rings, or "analytic spaces" (there are several variants).

Objects in this topos inherit a notion of differential and integral calculus. The object in this category which corresponds to the real numbers has infinitesimal elements. It is much more convenient to treat infinitesimals in a setting where not everything is determined by sets of elements. The result is that the calculus techniques of physicists which mathematicians are forever criticizing suddenly become rigorous.

A topos is a more subtle replacement of the notion of space than a category. It is a category of maps between categories, so it has the character of a relative space. In this paper, we are exploring the possibility that the relativity of objects in a topos could be a model for the relativity of the state of a system to the observer.

C. HIGHER CATEGORIES AS SPACES

The idea that topology and geometry are really about regions and maps between them rather than sets of points, has been a subtle but widespread influence in Mathematics.

A mathematical object with many objects and maps between them is a category [1]. There are many approaches to regarding a category as a kind of space.

Mathematicians have extended the idea of a category to an n-category. A 2-category has objects, maps and maps between maps, known of as homotopies or 2-morphisms. An n-category has 1,2..n morphisms [6].

The simplest situation in which a higher category can be thought of as a kind of space is the case of a simplicial complex.

A simplicial complex is a set of points, intervals, triangles tetrahedra etc referred to as n-simplices, where n is the dimension. The faces of the n-simplices are identified with n-1 simplices, thus giving a discrete set of gluing rules. Faces

are defined combinatorially as subsets of vertices. The whole structure is given by discrete combinatorial data.

A simplicial complex is thus a discrete combinatorial object. It does not contain a sets of internal points. These can be added to form the geometric realization of a simplicial complex, but that is usually not done.

Because the vertices of a simplex are ordered, which fixes an orientation on each of its faces of all dimensions, it is natural to represent it as a higher category. The vertices are the objects, the edges are 1-morphisms, the triangles 2-morphisms etc.

For many purposes, simplicial complexes are just as good as topological spaces or manifolds. Physicists who like to do Physics on a lattice can generalize to curved spacetime by working on a simplicial complex.

There is also a notion of the topology of a simplicial complex including cohomology and homotopy theory. A celebrated theorem states that the categories of homotopy types of simplicial complexes and of topological spaces are equivalent [7].

A naive first approach to quantum spacetime would say that at the Planck scale spacetime is described by a simplicial complex, rather than a continuum. This point of view would nicely accommodate the state sum models for quantum gravity, and the categorical language would allow a very elegant formulation of them, as we shall discuss below. The richness of the connections between category theory and topology allows for more sophisticated versions of this, in which simplicial complexes appear relationally, ie. the information flowing between two regions forms a simplicial complex. We will discuss physical approaches to this below.

Another way to relate categories to simplicial complexes is the construction of the nerve of a category, which is a simplicial complex which expresses the structure of the category. The nerve is constructed by assigning an n-simplex to each chain of n+1 composible morphisms in the category. The n-1 faces are each given by composing one successive pair of morphisms to form an n-chain.

The simplicial complex so formed is a generalization of the classifying space of a group. A group is a category with one object and all morphisms invertible.

There are also constructions which associate a category to a cellular or cubical complex.

The various descriptions of spaces by categories also extend to descriptions of maps between spaces as functors between categories.

Since the setting of a Yang-Mills or Kaluza-Klein theory is a projection map between manifolds, these have categorical generalisations which include more possibilities than the manifold versions.

One very interesting aspect of topos theory is the change in the status of points. A topos does not have an absolute set of points; rather, any topos can have points in any other topos. This was originally discovered by Grothendieck in algebraic geometry [3], where the topoi are called schemes. We shall discuss physical implications of this below.

D. STACKS AND COSMOI

As we shall see in the next section, both higher categorical and topos theoretical notions of space have strong connections to ideas in quantum gravity. For various reasons, it seems desirable to form a fusion of the two; that is, to form relational versions of higher categories.

Interestingly, this was the goal of the final work of Grothendieck on stacks, which he did not complete. Much of this has been worked out more recently by other authors [8].

The maps between two categories form a category, not merely a set. This is because of the existence of natural transformations between the functors. Similarly the morphisms between two 2-categories form a 2-category etc. The analog of sheaves over sites for 2-categories are called stacks. Much as the case of sheaves, these are equivalent to 2-functors. Incidently the word Grothendieck chose for a stack in French is champs, the same as the French word for a physical field.

One can also investigate the 2-categorical analog for a topos, which is a 2-category with an analogous structure to the 2-category of all "small" categories. This has been defined under the name of a cosmos [9].

An interesting class of examples of stacks are the gerbes [10], which have attracted interest in string theory and 2-Yang-Mills theory [11]. Theories with gerbe excitations would generalize naturally into a 2-categoric background spacetime.

III. PHYSICS IN CATEGORICAL SPACETIME

The ideal foundation for a quantum theory of gravity would begin with a description of a quantum mechanical measurement of some part of the geometry of some region; proceed to an analysis of the commutation relations between different observations, and then hypothesize a mathematical structure for spacetime which would contain these relations and give general relativity in a classical limit.

We do not know how to do this at present. However, we do have a number of approaches in which categorical ideas about spacetime fit with aspects of geometry and and quantum theory in interesting ways. We shall present these, and close with some ideas about how to achieve a synthesis.

A. THE BC CATEGORICAL STATE SUM MODEL

The development of the Barrett-Crane model for quantum general relativity [12, 13] begins by substituting a simplicial complex for a manifold. It is possible to adopt the point of view that this is merely a discrete approximation to an underlying continuous geometry located on a triangulation of the manifold. That was never my motivation. Rather considerations of the Planck scale cutoff and the limitations of information transfer in general relativity suggested that discrete geometry was more fundamental.

In any event, the problem of quantizing the geometry on a simplicial complex has proved to be much more tractible than the continuum version.

The bivectors assigned by the geometry to the triangles of the complex can be identified with vectors in the dual of the lorentz algebra, and hence have a very well understood quantization using the Kostant-Kirillov approach [14]. The quantum theory reduces to a careful combination of the unitary representations of the lorentz algebra due to Gelfand [15,16], and of intertwining operators between them.

We tensor together the representations corresponding to the assignments of area variables to the faces, then take the direct sum over all labellings. The resultant expression is what we call a categorical state sum.

The expression obtained for the state sum on any finite simplicial complex has been shown to be finite [17].

In addition, the mathematical form of the state sum is very elegant from the categorical point of view. If we think of the simplicial complex as a higher category, and the representations of the lorentz group as objects in a tensor category (which is really a type of 2-category), then the state sum is a sum over the functors between them.

The BC model is expressed as the category of functors between a spacetime category and a field category. The field category being a suitable subcategory of the unitary representations of the lorentz algebra. This suggests a general procedure for connecting more sophisticated categorical approaches to spacetime to quantum gravity. Namely, we could examine the category of functors from whatever version of spacetime category we are studying to the representation category of the lorentz algebra in order to put in the geometric variables.

It is not necessary for the simplicial complex on which we define the BC model to be equivalent to a triangulation of a manifold. A 4D simplicial complex in general has the topology of a manifold with conical singularities. There has been some work interpreting the behavior of the model near a singular point as a particle, with interesting results [18, 19]. The singularities conic over genus 1 surfaces reproduce, at least in a crude first approximation, the bosonic sector of the standard model, while the higher genus singularities decouple at low energy, with interesting early universe implications. The possibility of investigating singular points would not arise in any theory formulated on a manifold.

The BC model has not yet gained general acceptance as a candidate for quantum general relativity. The fundamental problem is the failure of attempts to find its classical limit.

I want to argue that the work done to date on the classical limit of the model, my own included, has been based on a misconception.

A categorical state sum model is not a path integral, although it resembles one in many aspects. Rather the geometry of each simplex has been quantized separately, and the whole model represented on a constrained tensor product of the local hilbert spaces.

For this reason the terms in the CSS are not classical histories, but rather quantum states. It is not really surprizing, then, that the geometric variables on them do not have simultaneous sharp values, or that they can contain singular configurations. Attempting to interpret them as classical is analogous to confusing the zitterbewegung of the electron with a classical trajectory.

In order to construct the classical limit of the BC model, it is necessary to study the problem of the emergence of a classical world in a quantum system. Fortunately, there has been great progress on this in recent years in the field variously known as consistent histories or decoherence.

The decoherent or consistent histories program has recently been interpreted as indicating that quantum measurements should be considered as occurring in a topos.

In the next sections, we shall briefly review the ideas of consistent histories and decoherence, and explain how they lead to topos theory. Then we shall discuss how to apply these ideas to the BC model.

B. DECOHERENT HISTORIES AND TOPOI

The consistent histories/decoherence approach to the interpretation of quantum mechanics is concerned with the problem of how classical behavior emerges in a suitable approximation in a quantum system [20].

We have to begin by coarse graining the system to be studied by decomposing its hilbert space into a sum of subspaces described as the images of orthogonal projections. A history is a sequence of members of the set of projections at a sequence of times.

Next we need to define the decoherence functional D. It is the trace of the product of the first series of projections time reversed, the density matrix of the original state of the system, and the product of the first series of projections.

$$D(H_1, H_2) = tr(H_1{}^* \rho H_2)$$

Classical behavior occurs if the decoherence functional is concentrated on the diagonal, more precisely if there is a small decoherence parameter η such that

$$D(H_1, H_2) = o(\eta) \text{ if } H_1 \neq H_2.$$

This implies that states described by histories from the chosen set do not interfere significantly. This implies classical behavior.

The next property of D to prove is that it concentrates near histories which correspond to solutions of the equations of motion. This is a way of affirming the correspondence principle for the system.

Since consistency is not perfect, we must think of the classical limit as appearing in the limit of coarse grainings.

Decoherence, the second half of the program, is an extremely robust mechanism causing histories to become consistent. When the variables correspond to typical macroscopic quantities decoherence occurs extremely quickly.

The central observation of the decoherence program is that classical systems can never be effectively decoupled from their environment.

For instance, a piston in a cylinder containing a very dilute gas might experience a negligible force. Nevertheless, the constant collisions with gas molecules

would cause the phase of the piston, treated as a quantum system, to vary randomly and uncontrollably.

Since it is not possible to measure the phases of all the molecules, the determinations an observer could make about the position of the piston would be modelled by projection operators whose images include an ensemble of piston states with random phases, coupled to gas molecule states.

This effect causes pistons (or any macroscopic body) to have diagonal decoherence functionals to a high degree of accuracy, and hence to behave classically.

The definition of a classical system as one which cannot be disentangled is a very useful one. It has enabled experiments to be designed which study systems which are intermediate between classical and quantum behavior [20].

When we observe a system, it is not possible to say exactly what set of consistent histories we are using. It is more natural to think that we are operating in a net of sets of consistent histories simultaneously.

We then expect that the result of an observation will be consistent if we pass from one set of consistent histories to a coarse graining of it.

The idea has been studied that this means that the results of experiments should be thought of as taking values in a topos [21]. The category whose objects are sets of consistent histories and whose morphisms are coarse grainings can be thought of as a site, and the results of experiments take place in presheaves over it.

In my view, the implications of this idea should be studied for physical geometry. Does it mean, for example, that the physical real numbers contain infinitesimals?

C. APPLICATION OF DECOHERENT HISTORIES TO THE BC MODEL

This section is work in progress.

We would like to explore classical histories in the BC model. The goal of this is to show that consistent histories exist for the model which closely approximate the geometry of pseudoriemannian manifolds, and that the decoherence functional concentrates around solutions of Einstein's equation.

The natural choice for macroscopic variables in the BC model would be the overall geometry of regions composing a number of simplices in the underlying complex of the model. It is easier to choose the regions themselves to be simplices which we call large to distinguish them from the fundamental simplices of which they are composed.

The program for showing that the geometric data on the internal small simplices decoheres the overall geometry of the large ones involves two steps.

In the first, we use microlocal analysis to construct a basis of states in which all the geometrical variables of the large simplices are simultaneously sharp to a small inaccuracy. These would combine to give a set of projection operators whose images correspond to pseudoriemannian geometries on the complex, now thought of as a triangulated manifold.

This problem is mathematically similar to finding a wavepacket for a particle. The symplectic space for the tetrahedron turns out to be equivalent to the symplectic structure on the space of euclidean quadrilaterals in the euclidean signature case, and to have an interesting hyperkahler structure in the case of the lorentzian signature. This allows us to use powerful mathematical simplifications, which make me believe the problem is quite solvable.

The second step would be to show the decoherence functional which arises from averaging over the small variables causes the large variables to decohere, and that the decoherence functional concentrates around solutions of Einstein's equation.

This is quite analogous to known results for material systems such as the piston.

The existence of a Brownian motion approximation for the internal variables makes me hopeful that this will work out, similarly to the case of the piston, where an ideal gas approximation is the key to the calculation.

A more challenging problem would be to work out the topos theoretic interpretation of the decoherence program in the case of the BC model.

The site of this topos would be the category whose objects are the "large" triangulations, and whose morphisms are coarse grainings.

One could then apply the ideas about modelling quantum observation in a topos described above to the BC model. This would amount to the construction of a 2-stack, since the BC model itself is 2-categorical.

This would give us a setting to ask the question: "what does one region in a spacetime, treated as classical, observe of the geometry of another part?"

This problem was suggested to me by Chris Isham.

D. CAUSAL SITES

As we explained above, the site of a topological space X is a category whose objects are the open sets of X and whose morphisms are inclusions. The whole construction of a site rests on the relationship of inclusion, which is a partial order on the set of open subsets. This change of starting point has proven enormously productive in Mathematics.

In Physics up to this point, the topological foundations for spacetime have been taken over without alteration from the topological foundation of space. In general relativity, a spacetime is distinguished from a four dimensional space only by the signature of its metric.

Categorical concepts of topology are richer and more flexible than point sets, however, and allow specifically spacetime structures to become part of the topological foundation of the subject.

In particular, regions in spacetime, in addition to the partial order relation of inclusion, have the partial order relation of causal priority, defined when every part of one region can observe every part of the other.

The combination of these two relations satisfy some interesting algebraic rules. These amount to saying that the compact regions of a causal spacetime

are naturally the objects of a two category, in much the same way that open sets form a site.

This suggests the possibility of defining a spacetime directly as a higher categorical object in which topology and causality are unified. A topodynamics to join geometrodynamics.

Recently, Dan Christensen and I implemented this proposal by giving a definition of causal sites and making an investigation of their structure [22].

We began by axiomatizing the properties of inclusion and causal order on compact regions of a strongly causal spacetime, then looked for more general examples not directly related to underlying point sets.

The structure which results is interesting in a number of ways. There is a natural 2-categorical formulation of causal sites. Objects are regions, 1-morphisms are causal chains, defined as sequences of regions each of which is causally prior to the next, and 2-morphisms are inclusions of causal chains, rather technically defined.

We think of causal chains as idealisations of observations, in which information can be retransmitted.

We discovered several interesting families of examples. One family was constructed by including a cutoff minimum spacetime scale. These examples have the interesting property that the set of causal chains between any two regions has a maximal length. This length can be interpreted as the duration of a timelike curve, and can very closely approximate the durations in a classical causal spacetime.

Since the pseudometric of a spacetime can be recovered from its timelike durations, the 2-categorical structure of a causal site can contain not only the topology of a spacetime, but also its geometry.

We also discovered that any two causally related regions have a relational tangent space, which describes the flow of information between them. This space has the structure of a simplicial complex, as opposed to a causal site itself, which has a bisimplicial structure because of the two relations on it. In category theoretic terms, the spacetime is a 2-category, but relationally it is a category.

An interesting feature of causal sites is that regions have relational points, i.e. regions which appear to another region to be indivisible, but perhaps are not absolutely so.

We hope that this feature may make causal sites useful in modelling the theory of observation in general relativity, in which only a finite amount of information can flow from one region to another [23], so that an infinite point set is not observably distinguished.

We also think it an interesting echo of the relational nature of points in topos theory.

If infinite point sets cannot be observed, than according to Einstein's principle, they should not appear in the theory. Causal sites are one possible way to implement this.

E. THE 2-STACK OF QUANTUM GRAVITY? FURTHER DI-RECTIONS

At this point, we have outlined two approaches to categorical spacetime, which include geometric information corresponding to the metric structure in general relativity in two different ways.

In the Barrett-Crane model, the data which expresses the geometry is directly quantum in nature. The geometric variables are given by assigning unitary representations of the lorentz algebra to the 2-faces or triangles of a simplicial complex. These are hilbert spaces on which operators corresponding to elements of the lorentz algebra act, thus directly quantizing the degrees of freedom of the bivector, or directed area element, which would appear on the 2-face if it had a classical geometry, inherited from an embedding into Minkowski space.

This model also has a natural functorial expression, as we mentioned above.

On the other hand, in the causal sites picture spacetime is represented as a family of regions, with two related partial orders on them. Mathematically this can be expressed by regarding the regions as the vertices of a bisimplicial set. Bisimplicial sets are one mathematical approach to 2-categories [24]. This is also an expression of the topological structure of the spacetime, although a more subtle one than a simplicial complex, which could arise from a triangulation of a manifold.

In some interesting examples, the **classical** geometry of a spacetime is naturally included in this bisimplicial complex, measured by the lengths of maximal causal chains. The approximation of the geometry by a causal set [25] can not be as precise, since a causal site has minimal regions which can be adapted to the direction of a path.

Now how could these two picture be synthesized?

One element which has not been included so far in the structure of a causal site is local symmetry. It is clear that this would have to appear in a fully satisfactory development of the theory, since the local symmetries of spacetime are so physically important.

Including local symmetry in the structure of a causal site seems a natural direction to study in linking the causal sites picture to the BC model, since the geometrical variables of the BC model are representations of the lorentz group.

The fundamental variables of a causal site have a yes/no form: region A either is or is not in the causal past of region B.

We could attempt to quantize a causal site by replacing the definite causal relations by causality operators. We can now define a 2-dimensional hilbert space H(A,B) for each pair of regions with a basis representing the yes and no answers to the causal relatedness question. This corresponds to a gravitational experiment in which an observer at B sees or fails to see an event at A. The totality of such experiments should define a quantum geometry on the site in the cases discussed above with bounds on chain length, since the metric can be effectively reconstructed from the classical answers.

In the presence of an action of local symmetry on the regions of the site, the tensor product of the spaces H(A,B) would decompose into representations of the local symmetry group.

If this led to the reappearance of the BC model on the relative tangent space between two regions in a site, it would create a setting in which the idea of the BC model as describing the geometry of one region as observed by another could be realized.

The physical thought is that since only a finite amount of information can pass from A to B in general relativity, the set of vertex points in a relative BC model could include all the topology of A which B could detect.

The idea of construcing a topos version of the BC model using decoherent histories also points to a BC model which varies depending on which classical observer the model is observed by.

Both of these ideas (neither implementsd yet, and neither easy) seem to hint at a simultaneously higher categorical and topos theoretical description of quantum spacetime which would fulfill the physical idea of a relational spacetime.

Perhaps there is an as yet unguessed construction of a 2-stack which will provide a synthesis of these ideas. Einsteins relational ideas may find their final form in the mathematical ideas of Grothendieck.

BIBLIOGRAPHY

1. S. Maclane, Categories for the working mathematician Springer Verlag, N.Y. 1971.

2. M. P. Reisenberger and C. Rovelli, Space time as feynman diagrams, the connection formulation, CQG 18 (2001) 121-140.

3. M. Artin, Theorie des topos et cohomologie etale des schemas, Springer, Berlin, N.Y. 1972.

4. S. Maclane and I. Moerdijk, Sheaves in geometry and logic, a first introduction to topos theory, Springer N. Y. 1992

5. A. Kock Synthetic differential geometry, Cambridge University Press, 1981.

6. J. Benabou, Introduction to bicategories in Reports of the midwest category theory seminar LNM 47 Springer, 1967 p1-77.

7. P. Goerss and R. Jardine, Simplicial homotopy theory, Birkhauser, 1999

8. L. Breem, On the classification of 2-Gerbes and 2-Stacks, Societe Mathematique de France, Providence RI, 1994

9. R. Street, Cosmoi of internal categories, AMS Transactions, 1980

10. R. Picken A cohomological description of abelian bundles and gerbes proceedings XXth workshop on geometric methods in physics, Bielowieza, July 1-7 2001.

11. J. Baez and J. Dolan, Higher Yang-Mills theory, hep-th 0206130

12. J. Barrett and L. Crane, Relativistic spin networks and quantum gravity, J. Math. Phys. 39 (1998) 3296-3302

13. J. Barrett and L. Crane, A lorentzian signature model for quantum general relativity, CQG 17(2000) 3101-3118

14. J. C. Baez and J. Barrett, The quantum tetrahedron in 3 and 4 dimensions, Adv. Theor. Math. Phys. 3, 815-850, 1999

15. G. Gelfand et. al. Generalized functions vol. 5 Integral geometry and representation theory, Academic Press 1966

16. I. M. Gelfand and M. A. Naimark, Unitary representations of the proper lorentz group, Izv. Akad. Nauk. SSSR 11 411 (1947)

17. L. Crane, A. Perez and C. Rovelli, A finiteness proof for the lorentzian state sum model for quantum general relativity, Phys. Rev. Lett. 87(2001) 181301

18. L. Crane A new approach to the geometrization of matter, gr-qc 0110060.

19. S. Alexander, L. Crane, M. D. Sheppeard, The geometrization of matter proposal in the Barrett Crane model and resolution of cosmological problems gr-qc 0306079.

20. R. Omnes, Understanding quantum mechanics, Princeton University Press, 1999.

21. C. J. Isham, J. Butterfield, Some possible roles for topos theory in quantum theory and quantum gravity, Found. Phys. 30 (2000) 1707-1735.

22. J. D. Christensen and L. Crane, Causal sites and quantum gravity, to appear JMP.

23. M. A. Perlath, R. M. Wald, Comment on entropy bounds and the generalized second law, Phys. Rev. D. 60 (1999) 104009.

24. Z. Tamsamani Sur les notions de ∞ categorie et ∞ groupoide non-stricte via des ensemble multi-simpliciaux, alg-geom 9512006

25. R. Sorkin, Causal sets, Discrete gravity, gr-qc 0309009