

Topos Methods in the Foundations of Physics*

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1 Introduction

Over forty years have passed since I first became interested in the problem of quantum gravity. During that time there have been many diversions and, perhaps, some advances. Certainly, the naively-optimistic approaches have long been laid to rest, and the schemes that remain have achieved some degree of stability. The original ‘canonical’ programme evolved into loop quantum gravity, which has become one of the two major approaches. The other, of course, is string theory—a scheme whose roots lie in the old Veneziano model of hadronic interactions, but whose true value became apparent only after it had been re-conceived as a theory of quantum gravity.

However, notwithstanding these hard-won developments, there are certain issues in quantum gravity that transcend any of the current schemes. These involve deep problems of both a mathematical and a philosophical kind, and stem from a fundamental paradigm clash between general relativity—the apotheosis of classical physics—and quantum physics.

In general relativity, space-time ‘itself’ is modelled by a differentiable manifold \mathcal{M} : a set whose elements are interpreted as ‘space-time points’. The curvature tensor of the pseudo-Riemannian metric on \mathcal{M} is then deemed to represent the gravitational field. As a classical theory, the underlying philosophical interpretation is realist: both the space-time and its points truly ‘exist’¹, as does the gravitational field.

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¹At least, that would be the view of unreconstructed, space-time substantialists. However, even purely within the realm of classical physics this position has often been challenged, particularly by those who place emphasis on the relational features that are inherent in general relativity.

On the other hand, standard quantum theory employs a background space-time that is fixed *ab initio* in regard to both its differential structure and its metric/curvature. Furthermore, the conventional interpretation is thoroughly instrumentalist in nature, dealing as it does with counter-factual statements about what would happen (or, to be more precise, the probability of what would happen) *if* a measurement is made of some physical quantity.

In regard to quantum gravity, the immediate question is:

How can such a formalism be applied to space and time themselves?

Specifically, what could it mean to ‘measure’ properties of space and/or time if the very act of measurement requires a spatio-temporal background within which it is made? And how can we meaningfully talk about the ‘probability’ of the results of such measurements? A related question is what meaning, if any, can be ascribed to quantum superpositions of eigenstates of properties of space, time, or space-time. Over the years, this issue has been much discussed by Roger Penrose who concludes that the existing quantum formalism simply cannot be applied to space and time, and that some new starting point is needed.

Another twist is provided by the subject of *quantum cosmology* which aims at applying quantum theory to the entire universe. There is no *prima facie* link between this aspiration and the subject of quantum gravity other than that we have become accustomed to discussing cosmology using various simple solutions to the classical equations of general relativity. However, irrespective of the link with quantum gravity proper, it is still thought provoking to consider in general terms what it means to apply quantum theory to the ‘entire’ universe. Of course, this might simply be a stupid thing to do, but if one does attempt it, then the problem of implementing instrumentalism becomes manifest: for where is the external observer if the entire universe is the system? In contexts like these one can see clearly the attractions of finding a more realist interpretation of quantum theory.

The complexity of such questions is significantly enhanced by the lack of any experimental data that can be unequivocally identified as pertaining to the subject of quantum gravity. In normal theoretical physics there is the (unholy) trinity of (i) real-world data; (ii) mathematical framework; and (iii) conceptual/philosophical framework. These three factors are closely interrelated in any real theory of the natural world: indeed, this tripartite structure underpins all theoretical physics.

However, in quantum gravity the first factor is largely missing, and this raises some curious questions. In particular:

- Would we recognise the/a ‘correct’ theory of quantum gravity even if it was handed to us on a plate? Certainly, the Popperian notion of refutation is hard to apply with such a sparsity of empirical data.
- What makes any particular idea a ‘good’ one as far as the community is concerned? Relatedly, how is it that one particular research programme becomes well established whereas another falls by the wayside or, at best, gains only a relatively small following?

The second question is not just whimsical, especially for our younger colleagues, for it plays a key role in decisions about the award of research grants, post-doc positions, promotion, and the like.

In practice, many of the past and present research programmes in quantum gravity have been developed by analogy with other theories and theoretical structures, particularly standard quantum field theory with gauge groups. And as for why it is that certain ideas survive and other do not, the answer is partly that individual scientists indulge in their own philosophical prejudices, and partly that they like using theoretical tools with which they are already familiar. This, after all, is the fastest route to writing new papers.

At this point it seems appropriate to mention the ubiquitous, oft-maligned ‘Planck length’. This fundamental unit of length comes from combining Planck’s constant \hbar (the ‘quantum’ in quantum gravity) with Newton’s constant G (the ‘gravity’ in quantum gravity) and the speed of light c (which always lurks around) in the form

$$L_P := \left(\frac{G\hbar}{c^3} \right)^{1/2} \quad (1.1)$$

which has a value of around 10^{-35} meters; the corresponding Planck time (defined as $T_P := L_P/c$) has a value or around 10^{-42} seconds.

The general assumption is that something ‘dramatic’ happens to the nature of space and time at these fundamental scales. Precisely what that dramatic change might be has been the source of endless speculation and conjecture. However, there is a fairly wide-spread anticipation that in so far as spatio-temporal concepts have any meaning at all in the ‘deep’ quantum-gravity regime, the appropriate mathematical model will not be based on standard, continuum differential geometry. Indeed, it is not hard to convince oneself that, from a physical perspective, the important ingredient in a space-time model is not the ‘points’ in that space, but rather the ‘regions’ in which physical entities can reside. In the context of a topological space, such

regions are best modelled by open sets: the closed sets may be too ‘thin’² to contain a physical entity, and the only physically-meaningful closed sets are those with a non-empty interior. These reflections lead naturally to the subject of ‘pointless topology’ and the theory of *locales*—a natural step along the road to topos theory.

However, another frequent conjecture is that what we normally call space and time (or space-time) will only ‘emerge’ from the correct quantum gravity formalism in some (classical?) limit. Thus a fundamental theory of quantum gravity may (i) have no intrinsic reference at all to spatio-temporal concepts; and (ii) be such that some of the spatio-temporal concepts that emerge in various limits are *non-standard* and are modelled mathematically with something other than topology and differential geometry. All this leads naturally to the main question that lies behind the work reported in this chapter. Namely:

What is status of (or justification for) using standard quantum theory when trying to construct a theory of quantum gravity?

It is notable that the main current programmes in quantum gravity *do* all use essentially standard quantum theory. However, around twelve years ago I came to the conclusion that the use of standard quantum theory was fundamentally inconsistent, and I stopped working in quantum gravity proper. Instead, I began studying what, to me, were the central problems in quantum theory itself: a search that lead quickly to the use of topos theory.

Of the various fundamental issues that arise, I will focus here on just two of them. The first is the problem mentioned already: viz, applying the standard instrumentalist interpretation of quantum theory to space and time. The second is what I claim is a category error in the use, *a priori*, of the real and complex numbers in the mathematical formulation of quantum theory when applied in a quantum gravity context. However, to unpack this problem it is first necessary to be more precise about the way that the real (and complex) numbers arise in quantum theory. This is the subject of the next Section.

²In a differentiable manifold, closed sets include points and lines whereas physical entities ‘take up room’.

2 The Problem of Using the Real Numbers in Quantum Gravity

The real numbers arise in theories of physics in three different (but related) ways: (i) as the values of physical quantities; (ii) as the values of probabilities; and (iii) as a fundamental ingredient in models of space and time (especially in those based on differential geometry). All three are of direct concern vis-a-vis our worries about making unjustified, *a priori* assumptions in quantum theory. Let us consider them in turn.

One reason for assuming physical quantities to be real-valued is undoubtedly grounded in the remark that, traditionally (i.e., in the pre-digital age), they are measured with rulers and pointers, or they are defined operationally in terms of such measurements. However, rulers and pointers are taken to be classical objects that exist in the physical space of classical physics, and this space is modelled using the reals. In this sense there is a direct link between the space in which physical quantities take their values (what we call the ‘quantity-value space’) and the nature of physical space or space-time [7].

Thus assuming physical quantities to be real-valued is problematic in any theory in which space, or space-time, is not modelled by a smooth manifold. This is a theoretical-physics analogue of the philosophers’ *category error*.

Of course, real numbers also arise as the value space for probabilities via the relative-frequency interpretation of probability. Thus, in principle, an experiment is to be repeated a large number, N , times, and the probability associated with a particular result is defined to be the ratio N_i/N , where N_i is the number of experiments in which that particular result was obtained. The rational numbers N_i/N necessarily lie between 0 and 1, and if the limit $N \rightarrow \infty$ is taken—as is appropriate for a hypothetical ‘infinite ensemble’—real numbers in the closed interval $[0, 1]$ are obtained.

The relative-frequency interpretation of probability is natural in instrumentalist theories of physics, but it is inapplicable in the absence of any classical spatio-temporal background in which the necessary sequence of measurements can be made (as, for example, is the situation in quantum cosmology).

In the absence of a relative-frequency interpretation, the concept of ‘probability’ must be understood in a different way. One possibility involves the concept of ‘potentiality’, or ‘latency’. In this case there is no compelling

reason why the probability-value space should be a subset of the real numbers. The minimal requirement on this value-space is that it is an ordered set, so that one proposition can be said to be more or less probable than another. However, there is no *prima facie* reason why this set should be *totally* ordered.

In fact, one of our goals is to dispense with probabilities altogether and to replace them with ‘generalised’ truth values for propositions.

It follows from the above that, for us, a key problem is the way in which the formalism of standard quantum theory is firmly grounded in the concepts of Newtonian space and time (or the relativistic extensions of these ideas) which are essentially assumed *a priori*. The big question is how this formalism can be modified/generalised/replaced so as to give a framework for physical theories that is

1. ‘realist’ in some meaning of that word; and
2. not dependent *a priori* on the real and/or complex numbers.

For example, suppose we are told that there is a background space-time \mathcal{C} , but it is modelled on something other than differential geometry; say, a causal set. Then what is the quantum formalism that has the same relation to \mathcal{C} as standard quantum theory does to Newtonian space and time?

This question is very non-trivial as the familiar Hilbert-space formalism is very rigid and does not lend itself to minor ‘fiddling’. What is needed is something that is radically new and yet which can still embody the basic principles³ of the quantum formalism, or beyond. To proceed let us return to first principles and consider what can be said in general about the general structure of mathematical theories of a physical system.

³Of course, the question of what precisely are these ‘basic principles’ is much debatable. I have a fond memory of being in the audience for a seminar by John Wheeler at a conference on quantum gravity in the early 1970s. John was getting well into the swing of his usual enthusiastic lecturing style and made some forceful remark about the importance of the quantum principle. At that point a hand was raised at the back of the lecture room, and a frail voice asked “What *is* the quantum principle?”. John Wheeler paused, looked thoughtfully at his interlocutor, who was Paul Dirac, and answered “Well, to be honest, I don’t know”. He paused again, and then said “Do you?”. “No” replied Dirac.

3 Theories of a Physical System

3.1 The Realism of Classical Physics

“From the range of the basic questions of metaphysics we shall here ask this one question: What is a thing? The question is quite old. What remains ever new about it is merely that it must be asked again and again [8].”

Martin Heidegger

Asking such questions is not a good way for a modern young philosopher to gain tenure. Plato was not ashamed to do so, and neither was Kant, but at some point in the last century the question became ‘Wittgensteined’ and since then it is asked at one’s peril. Fortunately, theoretical physicists are not confined in this way, and I will address the issue front on. Indeed, why should a physicist be ashamed of this margaritiforous question? For is it not what we all strive to answer when we probe the physical world?

Heidegger’s own response makes interesting reading:

“A thing is always something that has such and such properties, always something that is constituted in such and such a way. This something is the bearer of the properties; the something, as it were, that underlies the qualities.”

Let us see how modern physics approaches this fundamental question about the beingness of Being.

In constructing a theory of any ‘normal’⁴ branch of physics, the key ingredients are the mathematical representations of the following:

1. Space, time, or space-time: the framework within which ‘things’ are made manifest to us.
2. ‘States’ of the system: the mathematical entities that determine ‘the way things are’.
3. Physical quantities pertaining to the system.
4. ‘Properties’ of the system: i.e., propositions about the values of physical quantities. A state assigns truth-values to these propositions.

⁴That is, any branch of physics other than quantum gravity!

In the case of standard quantum theory, the mathematical states are interpreted in a non-realist sense as specifying only what would happen *if* certain measurements are made: so the phrase ‘the way things are’ has to be broadened to include this view. This ontological concept must also apply to the ‘truth values’ that are *intrinsically* probabilistic in nature.

In addition, ‘the way things are’ is normally construed as referring to a specific ‘moment’ of time, but this could be generalised to be compatible with whatever model of time/space-time is being employed. This could even be extended to a ‘history theory’, in which case ‘the way things are’ means the way things are for all moments of time, or whatever is appropriate for the space-time model being employed.

As theoretical physicists with a philosophical bent, a key issue is how such a mathematical framework implies, or encompasses, various possible philosophical positions. In particular, how does it interface with the position of ‘realism’?

This issue is made crystal clear in the case of classical physics, which is represented in the following way.

1. In Newtonian physics (which will suffice for illustrative purposes) space is represented by the three-dimensional Euclidean space, \mathbb{R}^3 , and time by the one-dimensional space, \mathbb{R} .
2. To each system S there is associated a set (actually a symplectic differentiable manifold) \mathcal{S} of states. At each moment of time, $t \in \mathbb{R}$, the system has a unique state $s_t \in \mathcal{S}$.
3. Any physical quantity, A , is represented by a function $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$. The associated interpretation is that if $s \in \mathcal{S}$ is a state, then the value of A in that state is just the real number $\check{A}(s)$. This is the precise sense in which the philosophical position of (naïve) realism is encoded in the framework of classical physics.
4. The basic propositions are of the form “ $A \varepsilon \Delta$ ” which asserts that the value of the physical quantity A lies in the (Borel) subset Δ of the real numbers. This proposition is represented mathematically by the subset $\check{A}^{-1}(\Delta) \subseteq \mathcal{S}$ —i.e., the collection of all states, s , in \mathcal{S} such that $\check{A}(s) \in \Delta$.

This representation of propositions by subsets of the state space \mathcal{S} has a fundamental implication for the *logical* structure of classical physics:

The mathematical structure of set theory implies that, *of necessity*, the propositions in classical physics have the logical structure of a *Boolean algebra*.

We note that classical physics is the paradigmatic implementation of Heidegger’s view of a ‘thing’ as the bearer of properties. However, in quantum theory, the situation is very different. There, the existence of any such realist interpretation is foiled by the famous Kochen-Specker theorem [14]. This asserts that it is impossible to assign values to all physical quantities at once if this assignment is to satisfy the consistency condition that the value of a function of a physical quantity is that function of the value. For example, the value of ‘energy squared’ is the square of the value of energy.

3.2 A Categorical Generalisation of the Representation of Physical Quantities

The Kochen-Specker theorem implies that, from Heidegger’s world-view, there is *no* ‘way things are’. To cope with this, physicists have historically fallen back on the instrumentalist interpretation of quantum theory with which we are all so familiar. However, as I keep emphasising, in the context of quantum gravity there are good reasons for wanting to achieve a more realist view, and the central question is how this can be done.

The problem is the great disparity between the mathematical formalism of classical physics—which is naturally realist—and the formalism of quantum physics—which is not. Let us summarise these different structures as they apply to a physical quantity A in a system S with associated propositions “ $A \in \Delta$ ” where $\Delta \subseteq \mathbb{R}$.

The classical theory of S :

- The state space is a set \mathcal{S} .
- A is represented by a function $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$.
- The propositions “ $A \in \Delta$ ” is represented by the subset $\check{A}^{-1}(\Delta) \subseteq \mathcal{S}$ of \mathcal{S} . The collection of all such subsets forms a Boolean lattice.

The quantum theory of S :

- The state space is a Hilbert space \mathcal{H} .

- A is represented by a self-adjoint operator, \hat{A} , on \mathcal{H} .
- The proposition “ $A \varepsilon \Delta$ ” is represented by the operator $\hat{E}[\hat{A} \in \Delta]$ that projects onto the subset $\Delta \cap \text{sp}(\hat{A})$ of the spectrum, $\text{sp}(\hat{A})$, of \hat{A} . The collection of all projection operators on \mathcal{H} forms a *non-distributive* lattice.

The ‘non-realism’ of quantum theory is reflected in the fact that propositions are represented by elements of a non-distributive lattice, whereas in classical physics a distributive lattice (Boolean algebra) is used.

So how can we find a formalism that goes beyond classical physics and yet which retains some degree of realist interpretation? One possibility is to generalise the axioms of classical physics to a category, τ , *other* than the category of sets. Such a representation of a physical system would have the following ingredients:

The τ -category theory of S :

- There are two special objects, Σ , \mathcal{R} , in τ known respectively as the *state object* and *quantity-value object*.
- A physical quantity, A , is represented by an arrow $\check{A} : \Sigma \rightarrow \mathcal{R}$ in the category τ .
- Propositions about the physical world are represented by sub-objects of the state object Σ . In standard physics there must be some way of embedding into this structure propositions of the form “ $A \varepsilon \Delta$ ” where $\Delta \subseteq \mathbb{R}$.

But does such ‘categorification’ work? In particular, is there some category such that quantum theory can be rewritten in this way?

4 The Use of Topos Theory

4.1 The Nature of a Topos

The simple answer to the question of categorification is ‘no’, not in general. The sub-objects of an object in a general category do not have any logical structure, and I regard possessing such a structure to be a *sine qua non* for the propositions in a physical theory. However, there is a special type of

category, known as a *topos*, in which the sub-objects of any object *do* have a logical structure—a remark that underpins our entire research programme.

Broadly speaking, a topos, τ , is a category that behaves in certain critical respects just like the category, **Sets**, of sets. In particular, these include the following:

1. There is an ‘initial’ object, 0_τ , and a ‘terminal object’, 1_τ . These are the analogues of, respectively, the empty set, \emptyset , and the singleton set $\{*\}$.

A *global element* (or just *element*) of an object X is defined to be an arrow $x : 1_\tau \rightarrow X$. This definition reflects the fact that, in set theory, any element x of a set X can be associated with a unique arrow⁵ $x : \{*\} \rightarrow X$ defined by⁶ $x(*) := x$.

The set of all global elements of an object X is denoted⁷ ΓX ; i.e., $\Gamma X := \text{Hom}_\tau(1_\tau, X)$.

2. One can form ‘products’ and ‘co-products’ of objects⁸ These are the analogue of the cartesian product and disjoint union in set theory.
3. In set theory, the collection, B^A , of functions $f : A \rightarrow B$ between sets A and B is itself a *set*; i.e., it is an object in the category of sets. In a topos, there is an analogous operation known as ‘exponentiation’. This associates to each pair of objects A, B in τ an object B^A with the characteristic property that

$$\text{Hom}_\tau(C, B^A) \simeq \text{Hom}_\tau(C \times A, B) \quad (4.1)$$

for all objects C . In set theory, the relevant statement is that a parameterised family of functions $c \mapsto f_c : A \rightarrow B$, $c \in C$, is equivalent to a single function $F : C \times A \rightarrow B$ defined by $F(c, a) := f_c(a)$ for all $c \in C$, $a \in A$.

4. Each subset A of a set X is associated with a unique ‘characteristic function’ $\chi_A : X \rightarrow \{0, 1\}$ defined by $\chi_A(x) := 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Here, 0 and 1 can be viewed as standing for ‘false’ and

⁵The singleton, $\{*\}$ is not unique of course. But any two singletons are isomorphic as sets and it does not matter which one we choose.

⁶Note that, hopefully without confusion, we are using the same letter ‘ x ’ for the element in X and the associated function from $\{*\}$ to X .

⁷In general, $\text{Hom}_\tau(X, Y)$ denotes the collection of all arrows in τ from the object X to the object Y .

⁸More generally, there are pull-backs and push-outs.

‘true’ respectively, so that $\chi_A(x) = 1$ corresponds to the mathematical proposition “ $x \in A$ ” being true.

This operation is mirrored in any topos. More precisely, there is a, so-called, ‘sub-object classifier’, Ω_τ , that is the analogue of the set $\{0, 1\}$. Specifically, to each sub-object A of an object X there is associated a ‘characteristic arrow’ $\chi_A : X \rightarrow \Omega_\tau$; conversely, each arrow $\chi : X \rightarrow \Omega_\tau$ determines a unique sub-object of X . Thus

$$\text{Sub}(X) \simeq \text{Hom}_\tau(X, \Omega_\tau) \quad (4.2)$$

where $\text{Sub}(X)$ denotes the collection of all sub-objects of X .

5. The *power set*, PX of any set X is defined to be the set of all subsets of X . Each subset $A \subseteq X$ determines, and is uniquely determined by, its characteristic function $\chi_A : X \mapsto \{0, 1\}$. Thus the set PX is in bijective correspondence with the function space $\{0, 1\}^X$. Analogously, in a general topos, τ , we define the *power object* of an object X , to be $PX := \Omega_\tau^X$. It follows from (4.1) that

$$\begin{aligned} \Gamma(PX) &:= \text{Hom}_\tau(1_\tau, PX) = \text{Hom}_\tau(1_\tau, \Omega_\tau^X) \\ &\simeq \text{Hom}_\tau(1_\tau \times X, \Omega_\tau) \simeq \text{Hom}_\tau(X, \Omega_\tau) \\ &\simeq \text{Sub}(X) \end{aligned} \quad (4.3)$$

Since $\text{Sub}(X)$ is always non-empty (because any object X is always a sub-object of itself) it follows that, for any X , the object PX is non-trivial. As a matter of notation, if A is a sub-object of X the corresponding arrow from 1_τ to PX is called the ‘name’ of A and is denoted $\ulcorner A \urcorner : 1_\tau \rightarrow PX$.

A key result for our purposes is that in any topos, the collection, $\text{Sub}(X)$, of sub-objects of any object X forms a *Heyting algebra*, as does the set $\Gamma\Omega_\tau := \text{Hom}_\tau(1_\tau, \Omega_\tau)$ of global elements of the sub-object classifier Ω_τ . A Heyting algebra is a distributive lattice, \mathfrak{h} , with top and bottom elements 1 and 0, and such that if $\alpha, \beta \in \mathfrak{h}$ there exists an element $\alpha \Rightarrow \beta$ in \mathfrak{h} with the property

$$\gamma \preceq (\alpha \Rightarrow \beta) \text{ if and only if } \gamma \wedge \alpha \preceq \beta \quad (4.4)$$

Then the ‘negation’ of any $\alpha \in \mathfrak{h}$ is defined as

$$\neg\alpha := (\alpha \Rightarrow 0) \quad (4.5)$$

A Heyting algebra has all the properties of a Boolean algebra except that the principle of *excluded middle* may not hold. That is, there may exist $\alpha \in \mathfrak{h}$ such that $\alpha \vee \neg\alpha \prec 1$, i.e., $\alpha \vee \neg\alpha \neq 1$. Equivalently, there may be $\beta \in \mathfrak{h}$ such that $\beta \prec \neg\neg\beta$. Of course, in a Boolean algebra we have $\beta = \neg\neg\beta$ for all β . We shall return to this feature shortly.

4.2 The Mathematics of ‘Neo-realism’

Let us now consider the assignment of truth values to propositions in mathematics. In set theory, let $K \subseteq X$ be a subset of some set X , and let x be an element of X . Then the basic mathematical proposition “ $x \in K$ ” is true if, and only if, x is an element of the subset K . At the risk of seeming pedantic, the truth value, $\llbracket x \in K \rrbracket$, of this proposition can be written as

$$\llbracket x \in K \rrbracket := \begin{cases} 1 & \text{if } x \text{ belongs to } K; \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

In terms of the characteristic function $\chi_K : X \rightarrow \{0, 1\}$, we have

$$\llbracket x \in K \rrbracket = \chi_K \circ x \quad (4.7)$$

where $\chi_K \circ x : \{*\} \rightarrow \{0, 1\}$.⁹

The reason for writing the truth value (4.6) as (4.7) is that this is the form that generalises in an arbitrary topos. More precisely, let $K \in \text{Sub}(X)$ with characteristic arrow $\chi_K : X \rightarrow \Omega_\tau$, and let x be a global element of X so that $x : 1_\tau \rightarrow X$. Then the ‘generalised’ truth value of the mathematical proposition “ $x \in K$ ” is defined to be the arrow

$$\llbracket x \in K \rrbracket := \chi_K \circ x : 1_\tau \rightarrow \Omega_\tau \quad (4.8)$$

Thus $\llbracket x \in K \rrbracket$ belongs to the Heyting algebra $\Gamma\Omega_\tau$. The adjective ‘generalised’ refers to the fact that, in a generic topos, the Heyting algebra contains elements other than just 0 and 1. In this sense, a proposition in a topos can be only *partially* true: a concept that seems ideal for application to the fitful reality of a quantum system.

This brings us to our main contention, which is that it may be profitable to consider constructing theories of physics in a topos other than the familiar topos of sets [2, 3, 4, 5, 6, 9, 10, 11, 12]. The propositions in such a theory admit a ‘neo’-realist interpretation in the sense that there *is* a ‘way things are’, but this is specified by generalised truth values that may not be just true (1) or false (0).

Such a theory has the following ingredients:

1. A physical quantity A is represented by a τ -arrow $\check{A} : \Sigma \rightarrow \mathcal{R}$ from the state object Σ to the quantity-value object \mathcal{R} .

⁹For the notation to be completely consistent, the left hand side of (4.6) should really be written as $\llbracket x \in K \rrbracket(*)$.

2. Propositions about S are represented by elements of the Heyting algebra, $\text{Sub}(\Sigma)$, of sub-objects of Σ . If Q is such a proposition, we denote by $\delta(Q)$ the associated sub-object of Σ ; the ‘name’ of $\delta(Q)$ is the arrow $\ulcorner \delta(Q) \urcorner : 1_\tau \rightarrow P\Sigma$.
3. The topos analogue of a state is a ‘pseudo-state’ which is a particular type of sub-object of Σ . Given this pseudo-state, each proposition can be assigned a truth value in the Heyting algebra $\Gamma\Omega_\tau$. Equivalently, we can use ‘truth objects’: see below.

Conceptually, such a theory is neo-realist in the sense that the propositions and their truth values belong to structures that are ‘almost’ Boolean: in fact they differ from Boolean algebras only in so far as the principle of excluded middle may not apply.

Thus a theory expressed in this way ‘looks like’ classical physics except that classical physics always employs the topos **Sets**, whereas other theories—including, we claim, quantum theory—use a different topos. If the theory requires a background space-time (or functional equivalent thereof) this could be represented by another special object, \mathcal{M} , in the topos. It would even be possible to mimic the actions of differential calculus if the topos is such as to support synthetic differential geometry¹⁰.

The presence of intrinsic logical structures in a topos has another striking implication. Namely, a topos can be used as a *foundation* for mathematics itself, just as set theory is used in the foundations of ‘normal’ (or ‘classical’ mathematics). Thus classical physics is modelled in the topos of sets, and thereby by standard mathematics. But a theory of physics modelled in a topos, τ , other than **Sets** is being represented in an alternative mathematical universe! The absence of excluded middle means that proofs by contradiction cannot be used, but apart from that this, so-called, ‘intuitionistic’ logic can be handled in the same way as classical logic.

A closely-related feature is that each topos has an ‘internal language’ that is functionally similar to the formal language on which set theory is based. It is this internal language that is used in formulating axioms for the mathematical universe associated with the topos. The same language is also used in constructing the neo-realist interpretation of the physical theory.

4.3 The Idea of a Pseudo-state

In discussing the construction of truth values it is important to distinguish clearly between truth values of *mathematical* propositions and truth values

¹⁰This approach to calculus is based on the existence of genuine infinitesimals in certain topoi.

of *physical* propositions. A key step in constructing a physical theory is to translate the latter into the former. For example, in classical physics, the physical proposition “ $A \varepsilon \Delta$ ” is represented by the subset $\check{A}^{-1}(\Delta)$ of the state space, \mathcal{S} ; i.e.,

$$\delta(A \varepsilon \Delta) := \check{A}^{-1}(\Delta). \quad (4.9)$$

Then, for any state $s \in \mathcal{S}$, the truth value of the physical proposition “ $A \varepsilon \Delta$ ” is defined to be the truth value of the *mathematical* proposition “ $s \in \delta(A \varepsilon \Delta)$ ” (or, equivalently, the truth value of “ $\check{A}(s) \in \Delta$ ”).

Given the ideas discussed above, in a topos theory one might expect to represent a physical state by a global element $s : 1_\tau \rightarrow \Sigma$ of the state object Σ . The truth value of a proposition, Q , represented by a sub-object $\delta(Q)$ of Σ , would then be defined as the global element

$$\nu(Q; s) := \llbracket s \in \delta(Q) \rrbracket = \chi_{\delta(Q)} \circ s : 1_\tau \rightarrow \Omega_\tau \quad (4.10)$$

of Ω_τ . However, in the topos version of quantum theory (see Section 5) it transpires that Σ has *no global elements at all*: in fact, this turns out to be precisely equivalent to the Kochen-Specker theorem! This absence of global elements of the state object could well be a generic feature of topos-formulated physics, in which case we cannot use (4.10) and need to proceed in a different way.

In set theory, there are two mathematical statements that are equivalent to “ $s \in K$ ”. These are:

1. $K \in T^s$ (4.11)

where the ‘truth object’ T^s is defined by¹¹

$$T^s := \{J \subseteq \mathcal{S} \mid s \in J\}; \quad (4.13)$$

and

2. $\{s\} \subseteq K$. (4.14)

Thus, in classical physics, “ $A \varepsilon \Delta$ ” is true in a state s if, and only, if (i) $\delta(A \varepsilon \Delta) \in T^s$; and (ii) $\{s\} \subseteq \delta(A \varepsilon \Delta)$.

Let us consider in turn the topos analogue of these two options.

¹¹Note that s can be recovered from T^s via

$$\{s\} := \bigcap_{K \in T^s} K \quad (4.12)$$

Option 1 (the truth-object option): Note that T^s is a collection of subsets of \mathcal{S} : i.e., $T^s \in \text{Sub}(P\mathcal{S})$. The interesting thing about (4.11) is that it is of the form “ $x \in X$ ” and therefore has an immediate generalisation to any topos. More precisely, in a general topos τ a truth object, T , would be a sub-object of $P\Sigma$ (equivalently, a global element of $P(P\Sigma)$) with a characteristic arrow $\chi_T : P\Sigma \rightarrow \Omega_\tau$. Then the physical proposition Q has the topos truth value

$$\nu(Q; T) := \llbracket \delta(Q) \in T \rrbracket = \chi_T \circ \ulcorner \delta(Q) \urcorner : 1_\tau \rightarrow \Omega_\tau \quad (4.15)$$

The key remark is that although (4.10) is inapplicable if there are no global elements of Σ , (4.15) *can* be used since global elements of a power object (like $P(P\Sigma)$) always exist. Thus, in option 1, the analogue of a classical state $s \in \mathcal{S}$ is played by the truth object $T \in \text{Sub}(P\Sigma)$.

Option 2 (the pseudo-state option): We cannot use (4.14) in a literal way in the topos theory since, if there are no global elements, s , of Σ , trying to construct an analogue of $\{s\}$ is meaningless. However, although there are no global elements of Σ , there may nevertheless be certain sub-objects, \mathfrak{w} , of Σ (what we call ‘pseudo-states’) that are, in some sense, as ‘close as we can get’ to the (non-existent) analogue of the singleton subsets, $\{s\}$, of \mathcal{S} . We must then consider the mathematical proposition “ $\mathfrak{w} \subseteq \delta(Q)$ ” where \mathfrak{w} and $\delta(Q)$ are both sub-objects of Σ .

As we have seen in (4.8), the truth value of the mathematical proposition “ $x \in K$ ” is a global element of Ω_τ : as such, it may have a value other than 1 (‘true’) or 0 (‘false’). In other words, “ $x \in K$ ” can be only ‘partially true’. What is important for us is that an analogous situation arises if J, K are sub-objects of some object X . Namely, a global element, $\llbracket J \subseteq K \rrbracket$, of Ω_τ can be assigned to the proposition “ $J \subseteq K$ ”: i.e., there is a precise sense in which one sub-object of X can be only ‘partially’ a sub-object of another. In this scenario, a physical proposition Q has the topos truth value

$$\nu(Q; \mathfrak{w}) := \llbracket \mathfrak{w} \subseteq \delta(Q) \rrbracket \quad (4.16)$$

when the pseudo-state is \mathfrak{w} . The general definition of $\llbracket J \subseteq K \rrbracket$ is not important for our present purposes. In the quantum case, the explicit form is discussed in the companion article by my collaborator Andreas Döring [1].

To summarise what has been said so far, the key ingredients in formulating a theory of a physical system in a topos τ are the following:

1. There is a ‘state object’, Σ , in τ .

2. To each physical proposition, Q , there is associated a sub-object, $\delta(Q)$, of Σ .
3. The analogue of a classical state is given by either (i) a truth object, T ; or (ii) a pseudo-state \mathfrak{w} . The topos truth-value of the proposition Q is then $\llbracket \delta(Q) \in T \rrbracket$, or $\llbracket \mathfrak{w} \subseteq \delta(Q) \rrbracket$, respectively.

Note that no mention has been made here of the quantity-value object \mathcal{R} . However, in practice, this also is expected to play a key role, not least in constructing the physical propositions. More precisely, we anticipate that each physical quantity A will be represented by an arrow $\check{A} : \Sigma \rightarrow \mathcal{R}$, and then a typical proposition will be of the form “ $A \varepsilon \Xi$ ” where Ξ is some sub-object of \mathcal{R} .

If any ‘normal’ physics is addressed in this way, physical quantities are expected to be *real*-valued, and the physical propositions are of the form “ $A \varepsilon \Delta$ ” for some $\Delta \subseteq \mathbb{R}$. In this case, it is necessary to decide what sub-object of \mathcal{R} in the topos corresponds to the external quantity $\Delta \subseteq \mathbb{R}$.

There is subtlety here, however, since although (in any topos we are likely to consider) there is a precise topos analogue of the real numbers¹², it would be a mistake to assume that this is necessarily the quantity-value object: indeed, in our topos version of quantum theory this is definitely *not* the case.

The question of relating $\Delta \subseteq \mathbb{R}$ to some sub-object of the quantity-value object \mathcal{R} in τ is just one aspect of the more general issue of distinguishing quantities that are *external* to the topos, and those that are *internal*. Thus, for example, a sub-object Ξ of \mathcal{R} in τ is an internal concept, whereas $\Delta \subseteq \mathbb{R}$ is external, referring as it does to something, \mathbb{R} , that lies outside τ . Any reference to a background space, time or space-time would also be external. Ultimately perhaps—or, at least, certainly in the context of quantum cosmology—one would want to have no external quantities at all. However, in any more ‘normal’ branch of physics it is natural that the propositions about the system refer to the external (to the theory) world to which theoretical physics is meant to apply. And, even in quantum cosmology, the actual collection of what counts as physical quantities is external to the formalism itself.

¹²Albeit defined using the analogue of Dedekind cuts, not Cauchy sequences.

5 The Topos of Quantum Theory

5.1 The Kochen-Specker theorem and contextuality

To motivate our choice of topos for quantum theory let us return again to the Kochen-Specker theorem which asserts the impossibility of assigning values to all physical quantities whilst, at the same time, preserving the functional relations between them [14].

In a quantum theory, a physical quantity A is represented by a self-adjoint operator \hat{A} on the Hilbert space, \mathcal{H} , of the system. A ‘valuation’ is defined to be a real-valued function λ on the set of all bounded, self-adjoint operators, with the properties that: (i) the value $\lambda(\hat{A})$ belongs to the spectrum of \hat{A} ; and (ii) the functional composition principle (or FUNC for short) holds:

$$\lambda(\hat{B}) = h(\lambda(\hat{A})) \tag{5.1}$$

for any pair of self-adjoint operators \hat{A} , \hat{B} such that $\hat{B} = h(\hat{A})$ for some real-valued function h .

Several important results follow from this definition. For example, if \hat{A}_1 and \hat{A}_2 commute, it follows from the spectral theorem that there exists an operator \hat{C} and functions h_1 and h_2 such that $\hat{A}_1 = h_1(\hat{C})$ and $\hat{A}_2 = h_2(\hat{C})$. It then follows from FUNC that

$$\lambda(\hat{A}_1 + \hat{A}_2) = \lambda(\hat{A}_1) + \lambda(\hat{A}_2) \tag{5.2}$$

and

$$\lambda(\hat{A}_1 \hat{A}_2) = \lambda(\hat{A}_1) \lambda(\hat{A}_2). \tag{5.3}$$

The Kochen-Specker theorem says that no valuations exist if $\dim(\mathcal{H}) > 2$. On the other hand, (5.2–5.3) show that, if it existed, a valuation restricted to a commutative sub-algebra of operators would be just an element of the *spectrum* of the algebra, and of course such elements *do* exist. Thus valuations exist on any commutative sub-algebra¹³ of operators, but not on the (non-commutative) algebra, $\mathcal{B}(\mathcal{H})$, of all bounded operators. We shall call such valuations ‘local’.

Within the instrumentalist interpretation of quantum theory, the existence of local valuations is closely related to the possibility of making ‘simultaneous’ measurements on commuting observables. However, the existence of local valuations also plays a key role in the, so-called, ‘modal’ interpretations in which values are given to the physical quantities that belong to

¹³More precisely, since we want to include projection operators, we assume that the commutative algebras are von Neumann algebras. These algebras are defined over the complex numbers, so that non self-adjoint operators are included too.

some specific commuting set. The most famous such interpretation is that of David Bohm where it is the configuration¹⁴ variables in the system that are regarded as always ‘existing’ (in the sense of possessing values).

The topos-implication of these remarks stems from the following observations. First, let V, W be a pair of commutative sub-algebras with $V \subseteq W$. Then any (local) valuation, λ , on W restricts to give a valuation on the sub-algebra V . More formally, if $\underline{\Sigma}_W$ denotes the set of all local valuations on W , there is a ‘restriction map’ $r_{WV} : \underline{\Sigma}_W \rightarrow \underline{\Sigma}_V$ in which $r_{WV}(\lambda) := \lambda|_V$ for all $\lambda \in \underline{\Sigma}_W$. It is clear that if $U \subseteq V \subseteq W$ then

$$r_{VU}(r_{WV}(\lambda)) = r_{WU}(\lambda) \quad (5.4)$$

for all $\lambda \in \underline{\Sigma}_W$ —i.e., restricting from W to U is the same as going from W to V and then from V to U .

Note that, if one existed, a valuation, λ , on the non-commutative algebra of all operators on \mathcal{H} would provide an association of a ‘local’ valuation $\lambda_V := \lambda|_V$ to each commutative algebra V such that, for all pairs V, W with $V \subseteq W$ we have

$$\lambda_W|_V = \lambda_V \quad (5.5)$$

The Kochen-Specker theorem asserts there are no such associations $V \mapsto \lambda_V \in \underline{\Sigma}_V$ if $\dim \mathcal{H} > 2$.

To explore this further consider the situation where we have three commuting algebras V, W_1, W_2 with $V \subseteq W_1$ and $V \subseteq W_2$ and suppose $\lambda_1 \in \underline{\Sigma}_{W_1}$ and $\lambda_2 \in \underline{\Sigma}_{W_2}$ are local valuations. If there is some commuting algebra W so that (i) $W_1 \subseteq W$ and $W_2 \subseteq W$; and (ii) there exists $\lambda \in \underline{\Sigma}_W$ such that $\lambda_1 = \lambda|_{W_1}$ and $\lambda_2 = \lambda|_{W_2}$, then (5.4) implies that

$$\lambda_1|_V = \lambda_2|_V \quad (5.6)$$

However, suppose now the elements of W_1 and W_2 do not all commute with each other: i.e., there is no W such that $W_1 \subseteq W$ and $W_2 \subseteq W$. Then although valuations $\lambda_1 \in \underline{\Sigma}_{W_1}$ and $\lambda_2 \in \underline{\Sigma}_{W_2}$ certainly exist, there is no longer any guarantee that they can be chosen to satisfy the matching condition in (5.6). Indeed, the Kochen-Specker theorem says precisely that it is impossible to construct a collection of local valuations, λ_W , for all commutative sub-algebras W such that all the matching conditions of the form (5.6) are satisfied.

We note that triples V, W_1, W_2 of this type arise when there are non-commuting self-adjoint operators \hat{A}_1, \hat{A}_2 with a third operator \hat{B} and functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that $\hat{B} = f_1(\hat{A}_1) = f_2(\hat{A}_2)$. Of course, $[\hat{B}, \hat{A}_1] = 0$

¹⁴If taken literally, the word ‘configuration’ implies that the state space of the underlying classical system must be a cotangent bundle T^*Q .

and $[\hat{B}, \hat{A}_2] = 0$ so that, in physical parlance, A_1 and B can be given ‘simultaneous values’ (or can be measured simultaneously) as can A_2 and B . However, the implication of the discussion above is that the value ascribed to B (resp. the result of measuring B) depends on whether it is considered together with A_1 , or together with A_2 . In other words the value of the physical quantity B is *contextual*. This is often considered one of the most important implications of the Kochen-Specker theorem.

5.2 The topos of presheaves $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$

To see how all this relates to topos theory let us rewrite the above slightly. Thus, let $\mathcal{V}(\mathcal{H})$ denote the collection of all commutative sub-algebras of operators on the Hilbert space \mathcal{H} . This is a partially-ordered set with respect to sub-algebra inclusion. Hence it is also a category whose objects are just the commutative sub-algebras of $\mathcal{B}(\mathcal{H})$; we shall call it the ‘category of contexts’.

We view each commutative algebra as a context with which to view the quantum system in an essentially classical way in the sense that the physical quantities in any such algebra can be given consistent values, as in classical physics. Thus each context is a ‘classical snapshot’, or ‘world-view’, or ‘window on reality’. In any modal interpretation of quantum theory, only one context at a time is used¹⁵ but our intention is to use the collection of *all* contexts in one mega-structure that will capture the entire quantum theory.

To do this, let us consider the association of the spectrum $\underline{\Sigma}_V$ (the set of local valuations on V) to each commutative sub-algebra V . As explained above, there are restriction maps $r_{WV} : \underline{\Sigma}_W \rightarrow \underline{\Sigma}_V$ for all pairs V, W with $V \subseteq W$, and these maps satisfy the conditions in (5.4). In the language of category theory this means that the operation $V \mapsto \underline{\Sigma}_V$ defines the elements of a *contravariant functor*, $\underline{\Sigma}$, from the category $\mathcal{V}(\mathcal{H})$ to the category of sets; equivalently, it is a covariant functor from the opposite category, $\mathcal{V}(\mathcal{H})^{\text{op}}$ to **Sets**.

Now, one of the basic results in topos theory is that for any category \mathcal{C} , the collection of covariant functors $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Sets}$ is a topos, known as the ‘topos of presheaves’ over \mathcal{C} , and is denoted $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$. In regard to quantum theory, our fundamental claim is that the theory can be reformulated so as to look like classical physics, but in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$. The object $\underline{\Sigma}$ is known as the ‘spectral presheaf’ and plays a fundamental role in our theory. In purely mathematical terms this has considerable interest as the foundation for a type of non-commutative spectral theory; from a physical perspective,

¹⁵In the standard instrumentalist interpretation of quantum theory, a context is selected by choosing to measure a particular set of commuting observables.

we identify it as the state object in the topos.

The terminal object $1_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})}^{\text{op}}}$ in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})}^{\text{op}}$ is the presheaf that associates to each commutative algebra V a singleton set $\{*\}_V$, and the restriction maps are the obvious ones. It is then easy to see that a global element $\lambda : 1_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})}^{\text{op}}} \rightarrow \underline{\Sigma}$ of the spectral presheaf is an association to each V of a spectral element $\lambda_V \in \underline{\Sigma}_V$ such that, for all pairs V, W with $V \subseteq W$ we have (5.5). Thus we have the following basic result:

The Kochen-Specker theorem is equivalent to the statement that the spectral presheaf, $\underline{\Sigma}$, has no global elements.

Of course, identifying the topos and the state object are only the very first steps in constructing a topos formulation of quantum theory. The next key step is to associate a sub-object of $\underline{\Sigma}$ with each physical proposition. In quantum theory, propositions are represented by projection operators, and so what we seek is a map

$$\delta : \mathcal{P}(\mathcal{H}) \rightarrow \text{Sub}(\underline{\Sigma}) \quad (5.7)$$

where $\mathcal{P}(\mathcal{H})$ denotes the lattice of projection operators on \mathcal{H} . Thus δ is a map from a non-distributive quantum logic to the (distributive) Heyting algebra, $\text{Sub}(\underline{\Sigma})$, in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})}^{\text{op}}$.

The precise definition of $\delta(\hat{P})$, $\hat{P} \in \mathcal{P}(\mathcal{H})$, is described in the companion article by my collaborator Andreas Döring [1]. Suffice it to say that, at each context V , it involves approximating \hat{P} with the ‘closest’ projector to \hat{P} that lies in V : an operator that we call ‘daseinisation’ in honour of Heidegger’s memorable existentialist perspective on ontology.

The next step is to construct the quantity-value object, $\underline{\mathcal{R}}$. We do this by applying the Gel’fand spectral transformation in each context V . The most striking remark about the result is that $\underline{\mathcal{R}}$ is *not* the real-number object, $\underline{\mathbb{R}}$, in the topos, although the latter is a sub-object of the former. One result of this construction is that we are able to associate an arrow $\check{A} : \underline{\Sigma} \rightarrow \underline{\mathcal{R}}$ with each physical quantity A ; i.e., with each bounded, self-adjoint operator \hat{A} . This is a type of non-commutative spectral theory.

The final, key, ingredient is to construct the truth objects, or pseudo-states; in particular, we wish to do this for each vector $|\psi\rangle$ in the Hilbert space \mathcal{H} . Again, the details can be found in the chapter by Andreas Döring, but suffice it to say that the pseudo-state, $\underline{\mathbf{w}}^{|\psi\rangle}$ associated with $|\psi\rangle \in \mathcal{H}$ can be written in the simple form

$$\underline{\mathbf{w}}^{|\psi\rangle} = \delta(|\psi\rangle\langle\psi|) \quad (5.8)$$

In other words, the pseudo-state corresponding to the unit vector $|\psi\rangle$ is just the daseinisation of the projection operator onto $|\psi\rangle$.

6 Conclusions

We have revisited the oft-repeated statement about fundamental incompatibilities between quantum theory and general relativity. In particular, we have argued that the conventional quantum formalism is inadequate to the task of quantum gravity both in regard to (i) the non-realist, instrumentalist interpretation; and (ii) the *a priori* use of the real and complex numbers.

Our suggestion is to employ a mathematical formalism that ‘looks like’ classical physics (so as to gain some degree of ‘realism’) but in a topos, τ , other than the topos of sets. The ingredients in such a theory are:

1. A state-object, Σ , and a quantity-value object, \mathcal{R} .
2. A map $\delta : \mathcal{P} \rightarrow \text{Sub}(\Sigma)$ from the set of propositions, \mathcal{P} , to the Heyting algebra, $\text{Sub}(\Sigma)$, of sub-objects of Σ .
3. A set of pseudo-states, \mathfrak{w} , or truth objects, T . Then the truth value of the physical proposition Q in the pseudo-state \mathfrak{w} (resp. the truth object T) is $\llbracket \mathfrak{w} \subseteq \delta(Q) \rrbracket$ (resp. $\llbracket \delta(Q) \in T \rrbracket$) which is a global element of the sub-object classifier, Ω_τ , in τ . The set, $\Gamma\Omega_\tau$ of all such global elements is also a Heyting algebra.

In the case of quantum theory, in our published papers we have shown in great detail how this programme goes through with the topos being the category, $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$, of presheaves over the category/poset $\mathcal{V}(\mathcal{H})$ of commutative sub-algebras of the algebra of all bounded operators on the Hilbert space \mathcal{H} .

The next step in this programme is to experiment with ‘generalisations’ of quantum theory in which the context category is not $\mathcal{V}(\mathcal{H})$ but some category, \mathcal{C} , which has no fundamental link with a Hilbert space, and therefore with the real and/or complex numbers. In a scheme of this type the intrinsic contextuality of quantum theory is kept, but its domain of applicability could include spatio-temporal concepts that are radically different from those currently in use.

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