An introduction to spinors

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(Dated: December 16, 2013)

We introduce spinors, at a level appropriate for an undergraduate or first year graduate course on relativity, astrophysics or particle physics. The treatment assumes very little mathematical knowledge (mainly just vector analysis and some idea of what a group is). The SU(2)–SO(3) homomorphism is presented in detail. Lorentz transformation, chirality, and the spinor Minkowski metric are introduced. Applications to electromagnetism, parity violation, and to Dirac spinors are presented. A classical form of the Dirac equation is obtained, and the (quantum) prediction that $g = 2$ for Dirac particles is presented.

I. INTRODUCING SPINORS

Spinors are mathematical entities somewhat like tensors, that allow a more general treatment of the notion of invariance under rotation and Lorentz boosts[7]. To every tensor of rank k there corresponds a spinor of rank $2k$, and some kinds of tensor can be associated with a spinor of the same rank. For example, a general 4-vector would correspond to a Hermitian spinor of rank 2, which can be represented by a 2×2 Hermitian matrix of complex numbers. A null 4-vector can also be associated with a spinor of rank 1, which can be represented by a complex vector with two components. We shall see why in the following.

Spinors can be used without reference to relativity, but they arise naturally in discussions of the Lorentz group. One could say that a spinor is the most basic sort of mathematical object that can be Lorentz-transformed. The main facts about spinors are given in the box on page 2. The statements in the summary will be explained as we go along.

It appears that Klein originally designed the spinor to simplify the treatment of the classical spinning top in 1897. The more thorough understanding of spinors as mathematical objects is credited to Élie Cartan in 1913. They are closely related to Hamilton's quaternions (about 1845).

Spinors began to find a more extensive role in physics when it was discovered that electrons and other particles have an intrinsic form of angular momentum now called 'spin', and the behaviour of this type of angular momentum is correctly captured by the mathematics discovered by Cartan. Pauli formalized this connection in a non-relativistic (i.e. low velocity) context, modelling the electron spin using a two-component complex vector, and introducing the *Pauli spin matrices*. Then in seeking a quantum mechanical description of the electron that was consistent with the requirements of Lorentz covariance, Paul Dirac had the brilliant insight that an equation of the right form could be found if the electron is described by combining the mathematics of spinors with the existing quantum mechanics of wavefunctions. He introduced a 4-component complex vector, now called a *Dirac spinor*, and by physically interpreting the wave equation thus obtained, he predicted the existence of antimatter.

Here we will discuss spinors in general, concentrating on the simplest case, namely 2-component spinors. These suffice to describe rotations in 3 dimensions, and Lorentz transformations in $3 + 1$ dimensions. We will briefly introduce the spinors of higher rank, which transform like outer products of first rank spinors. We will then introduce Dirac's idea, which can be understood as a pair of coupled equations for a pair of first rank spinors.

Undergraduate students often first meet spinors in the context of non-relativistic quantum mechanics and the treatment of the spin angular momentum. This can give the impression that spinors are essentially about spin, an impression that is fortified by the name 'spinor'. However, you should try to avoid that assumption in the first instance. Think of the word 'spinor' as a generalisation of 'vector' or 'tensor'. We shall meet a spinor that describes an electric 4-current, for example, and a spinor version of the Faraday tensor, and thus write Maxwell's equations in spinor notation.

Just as we can usefully think of a vector as an arrow in space, and a 4-vector as an arrow in spacetime, it is useful to have a geometrical picture of a rank 1 spinor (or just 'spinor' for short); see figure 1. It can be pictured as a vector with two further features: a 'flag' that picks out a plane in space containing the vector, and an overall sign. The crucial property is that under the action of a rotation, the direction of the spinor changes just as a vector would, and the flag is carried along in the same way is if it were rigidly attached to the 'flag pole'. A rotation about the axis picked out by the flagpole would have no effect on a vector pointing in that direction, but it does affect the spinor because it rotates the flag.

The overall sign of the spinor is more subtle. We shall find that when a spinor is rotated through 360°, it is returned to its original direction, as one would expect, but also it picks up an overall sign change. You can think of this as a phase factor ($e^{i\pi} = -1$). This sign has no consequence when spinors are examined one at a time, but it can be relevant when one spinor is compared with another. When we introduce the mathematical description using a pair of complex numbers (a 2-component complex vector) this and all other properties will automatically be taken into account.

To specify a spinor state one must furnish 4 real pa-

Spinor summary. A rank 1 spinor can be represented by a two-component complex vector, or by a null 4-vector, angle and sign. The spatial part can be pictured as a flagpole with a rigid flag attached.
The 4-vector is obtained from the 2-component complex vector by

$$\begin{aligned} \mathsf{V}^\mu &= \langle u|\,\sigma^\mu\,|u\rangle \;\; \text{if } \boldsymbol{u} \text{ is a contraspinor (“right-handed”)} \\ \mathsf{V}_\mu &= \langle \tilde{u}|\,\sigma^\mu\,|\tilde{u}\rangle \;\; \text{if } \tilde{\boldsymbol{u}} \text{ is a cospinor (“left handed”).} \end{aligned}$$

Any 2×2 matrix Λ with unit determinant Lorentz-transforms a spinor. Such matrices can be written

$$\Lambda = \exp\left(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2 - \boldsymbol{\sigma}\cdot\boldsymbol{\rho}/2\right)$$

where ρ is rapidity. If Λ is unitary the transformation is a rotation in space; if Λ is Hermitian it is a boost.
If $\boldsymbol{s}' = \Lambda(v)\boldsymbol{s}$ is the Lorentz transform of a right-handed spinor, then under the same change of reference frame a left-handed spinor transforms as $\tilde{\boldsymbol{s}}' = (\Lambda^\dagger)^{-1}\tilde{\boldsymbol{s}} = \Lambda(-v)\tilde{\boldsymbol{s}}$.
The Weyl equations may be obtained by considering $(\mathsf{W}^\alpha \sigma_\alpha)\boldsymbol{w}$. This combination is zero in all frames. Applied to a spinor $\boldsymbol{w}$ representing energy-momentum it reads

$$\begin{aligned} (E/c - \mathbf{p}\cdot\boldsymbol{\sigma})\boldsymbol{w} &= 0 \\ (E/c + \mathbf{p}\cdot\boldsymbol{\sigma})\tilde{\boldsymbol{w}} &= 0. \end{aligned}$$

If we take σ to represent spin angular momentum in these equations, then the equations are not parity-invariant, and they imply that if both the energy-momentum and the spin of a particle can be represented simultaneously by the same spinor, then the particle is massless and the sign of its helicity is fixed.
A Dirac spinor $\Psi = (\phi_R, \phi_L)$ is composed of a pair of spinors, one of each handedness. From the two associated null 4-vectors one can extract two orthogonal non-null 4-vectors

$$\begin{aligned} \mathsf{V}^\mu &= \Psi^\dagger\gamma^0\gamma^\mu\Psi, \\ \mathsf{W}^\mu &= \Psi^\dagger\gamma^0\gamma^\mu\gamma^5\Psi, \end{aligned}$$

where γ^μ, γ^5 are the Dirac matrices. With appropriate normalization factors these 4-vectors can represent the 4-velocity and 4-spin of a particle such as the electron.
Starting from a frame in which $\mathsf{V}^i = 0$ (i.e. the rest frame), the result of a Lorentz boost to a general frame can be written

$$\begin{pmatrix} -m & E + \boldsymbol{\sigma}\cdot\mathbf{p} \\ E - \boldsymbol{\sigma}\cdot\mathbf{p} & -m \end{pmatrix}\begin{pmatrix} \phi_R(\mathbf{p}) \\ \chi_L(\mathbf{p}) \end{pmatrix} = 0.$$

This is the Dirac equation. Under parity inversion the parts of a Dirac spinor swap over and $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$; the Dirac equation is therefore parity-invariant.

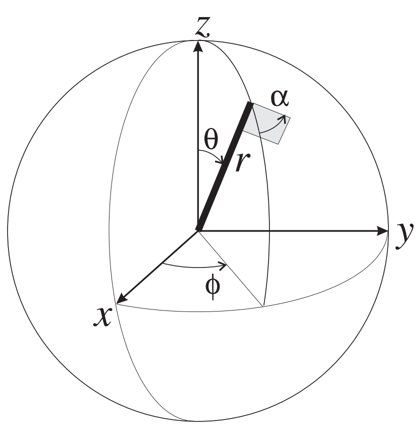

FIG. 1: A spinor. The spinor has a direction in space (‘flagpole’), an orientation about this axis (‘flag’), and an overall sign (not shown). A suitable set of parameters to describe the spinor state, up to a sign, is $(r, \theta, \phi, \alpha)$, as shown. The first three fix the length and direction of the flagpole by using standard spherical coordinates, the last gives the orientation of the flag.

rameters and a sign: an illustrative set r, θ, ϕ, α is given in figure 1. One can see that just such a set would be naturally suggested if one wanted to analyse the motion of a spinning top. We shall assume the overall sign is positive unless explicitly stated otherwise. The application to a classical spinning top is such that the spinor could represent the instantaneous positional state of the top. However, we shall not be interested in that application. In this article we will show how a spinor can be used to represent the energy-momentum and the spin of a massless particle, and a pair of spinors can be used to represent the energy-momentum and Pauli-Lubanksi spin 4-vector of a massive particle. Some very interesting properties of spin angular momentum, that otherwise might seem mysterious, emerge naturally when we use spinors.

A spinor, like a vector, can be rotated. Under the action of a rotation, the spinor magnitude is fixed while the angles θ, ϕ, α change. In the flag picture, the flagpole and flag evolve together as a rigid body; this suffices to determine how α changes along with θ and ϕ. In order to write the equations determining the effect of a rotation, it is convenient to gather together the four parameters into two complex numbers defined by

$$\begin{aligned} a &\equiv \sqrt{r}\cos(\theta/2)e^{i(-\alpha-\phi)/2}, \\ b &\equiv \sqrt{r}\sin(\theta/2)e^{i(-\alpha+\phi)/2}. \end{aligned} \quad (1)$$

(The reason for the square root and the factors of 2 will emerge in the discussion). Then the effect of a rotation of the spinor through θ_r about the y axis, for example, is

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \cos(\theta_r/2) & -\sin(\theta_r/2) \\ \sin(\theta_r/2) & \cos(\theta_r/2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \quad (2)$$

We shall prove this when we investigate more general rotations below.

From now on we shall refer to the two-component complex vector

$$\boldsymbol{s} = s e^{-i\alpha/2} \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \quad (3)$$

as a 'spinor'. A spinor of size s has a flagpole of length

$$r = |a|^2 + |b|^2 = s^2. \quad (4)$$

The components (r_x, r_y, r_z) of the flagpole vector are given by

$$r_x = ab^* + ba^*, \quad r_y = i(ab^* - ba^*), \quad r_z = |a|^2 - |b|^2, \quad (5)$$

which may be obtained by inverting (1). You can now see why the square root was required in (1).

The complex number representation will prove to be central to understanding spinors. It gives a second picture of a spinor, as a vector in a 2-dimensional complex vector space. One learns to 'hold' this picture alongside the first one. Most people find themselves thinking pictorially in terms of a flag in a 3-dimensional real space as illustrated in figure 1, but every now and then it is helpful to remind oneself that a pair of opposite flagpole states such as 'straight up along z' and 'straight down along z' are *orthogonal* to one another in the complex vector space (you can see this from eq. (3), which gives $(s, 0)$ and $(0, s)$ for these cases, up to phase factors).

Figure 2 gives some example spinor states with their complex number representation. Note that the two basis vectors $(1, 0)$ and $(0, 1)$ are associated with flagpole directions up and down along z, respectively, as we just mentioned. Considered as complex vectors, these are orthogonal to one another, but they represent directions in 3-space that are opposite to one another. More generally, a rotation through an angle θ_r in the complex 'spin space' corresponds to a rotation through an angle $2\theta_r$ in the 3-dimensional real space. This is called 'angle doubling'; you can see it in eq (3) and we shall explore it further in section II.

The matrix (2) for rotations about the y axis is real, so spinor states obtained by rotation of $(1, 0)$ about the y axis are real. These all have the flag and flagpole in the

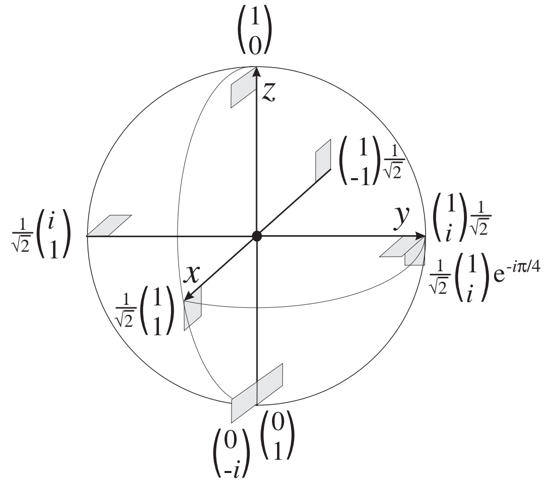

FIG. 2: Some example spinors. In two cases a pair of spinors pointing in the same direction but with flags in different directions are shown, to illustrate the role of the flag angle α. Any given direction and flag angle can also be represented by a spinor of opposite sign to the one shown here.

xz plane, with the flag pointing in the right handed direction relative to the y axis (i.e. the clockwise direction when the y axis is directed into the page). A rotation about the z axis is represented by a diagonal matrix, so that it leaves spinor states $(1, 0)$ and $(0, 1)$ unchanged in direction. To find the diagonal matrix, consider the spinor $(1, 1)$ which is directed along the positive x axis. A rotation about z should increase ϕ by the rotation angle θ_r. This means the matrix for a rotation of the spinor about the z axis through angle θ_r is

$$\begin{pmatrix} \exp(-i\theta_r/2) & 0 \\ 0 & \exp(i\theta_r/2) \end{pmatrix}. \quad (6)$$

When applied to the spinor $(1, 0)$, the result is $(e^{-i\theta_r/2}, 0)$. This shows that the result is to increase $\alpha + \phi$ from zero to θ_r. Therefore the flag is rotated. In order to be consistent with rotations of spinor directions close to the z axis, it makes sense to interpret this as a change in ϕ while leaving α unchanged.

So far our spinor picture was purely a spatial one. We are used to putting 3-vectors into spacetime by finding a fourth quantity and forming a 4-vector. For the spinor, however, a different approach is used, because it will turn out that the spinor is already a spacetime object that can be Lorentz-transformed. To 'place' the spinor in spacetime we just need to identify the 3-dimensional region or 'hypersurface' on which it lives. We will show in section III A that the 4-vector associated with the flagpole is a null 4-vector. Therefore, the spinor should be regarded as 'pointing along' or 'existing on' the light cone. The word 'cone' suggests a two-dimensional surface, but of course it is 3-dimensional really and therefore can contain a spinor. The event whose light cone is meant will be clear in practice. For example, if a particle has mass or charge then we say the mass or charge is located at each event where the particle is present. In a similar way, if a rank 1 spinor is used to describe a property of a particle, then the spinor can be thought of as 'located at' each event where the particle is present, and lying on the future light cone of the event. (Some spinors of higher rank can also be associated with 4-vectors, not necessarily null ones.) The formula for a null 4-vector, $(\mathsf{X}^0)^2 = (\mathsf{X}^1)^2 + (\mathsf{X}^2)^2 + (\mathsf{X}^3)^2$, leaves open a choice of sign between the time and spatial parts, like the distinction between a contravariant and covariant 4-vector. We shall show in section IV that this choice leads to two types of spinor, called 'left handed' and 'right handed'.

II. THE ROTATION GROUP AND SU(2)

We introduced spinors above by giving a geometrical picture, with flagpole and flag and angles in space. We then gave another definition, a 2-component complex vector. We have an equation relating the definitions, (1). All this makes it self-evident that there must exist a set of transformations of the complex vector that correspond to *rotations* of the flag and flagpole. It is also easy to guess what transformations these are: they have to preserve the length r of the flagpole, so they have to preserve the size $|a|^2 + |b|^2$ of the complex vector. This implies they are *unitary* transformations. If you are happy to accept this, and if you are happy to accept eq. (31) or prove it by others means (such as trigonometry), then you can skip this section and proceed straight to section II B. However, the connection between rotations and unitary 2×2 matrices gives an important example of a very powerful idea in mathematical physics, so in this section we shall take some trouble to explore it.

The basic idea is to show that two groups, which are defined in different ways in the first instance, are in fact the same (they are in one-to-one correspondance with one another, called isomorphic) or else very similar (e.g. each element of one group corresponds to a distinct set of elements of the other, called homomorphic). These are mathematical *groups* defined as in group theory, having associativity, closure, an identity element and inverses. The groups we are concerned with have a continuous range of members, so are called Lie groups. We shall establish one of the most important mappings in Lie group theory (that is, important to physics—mathematicians would regard it as a rather simple example). This is the 'homomorphism'

$$\mathrm{SU}(2) \xrightarrow{2:1} \mathrm{S0}(3) \tag{7}$$

'Homomorphism' means the mapping is not one-to-one; here there are two elements of SU(2) corresponding to each element of SO(3). SU(2) is the *special unitary group of degree 2*. This is the group of two by two unitary[8] matrices with determinant 1. SO(3) is the special orthogonal group of degree 3, isomorphic to the *rotation group*. The former is the group of three by three orthogonal[9] real matrices with determinant 1. The latter is the group of rotations about the origin in Euclidian space in 3 dimensions.

These Lie groups SU(2) and SO(3) have the same 'dimension', where the dimension counts the number of real parameters needed to specify a member of the group. This 'dimension' is the number of dimensions of the abstract 'space' (or *manifold*) of group members (do not confuse this with dimensions in physical space and time). The rotation group is three dimensional because three parameters are needed to specify a rotation (two to pick an axis, one to give the rotation angle); the matrix group SO(3) is three dimensional because a general 3×3 matrix has nine parameters, but the orthogonality and unit determinant conditions together set six constraints; the matrix group SU(2) is three dimensional because a general 2×2 unitary matrix can be described by 4 real parameters (see below) and the determinant condition gives one constraint.

The strict definition of an isomorphism between groups is as follows. If $\{M_i\}$ are elements of one group and $\{N_i\}$ are elements of the other, the groups are isomorphic if there exists a mapping $M_i \leftrightarrow N_i$ such that if $M_i M_j = M_k$ then $N_i N_j = N_k$. For a homomorphism the same condition applies but now the mapping need not be one-to-one.

The SU(2), SO(3) mapping may be established as follows. First introduce the *Pauli spin matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that these are all Hermitian and unitary. It follows that they square to one:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I. \tag{8}$$

They also have zero trace. It is very useful to know their *commutation relations*:

$$[\sigma_x, \sigma_y] \equiv \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z \tag{9}$$

and similarly for cyclic permutation of x, y, z. You can also notice that

$$\sigma_x\sigma_y = i\sigma_z, \;\; \sigma_y\sigma_x = -i\sigma_z$$

and therefore any pair anti-commutes:

$$\sigma_x\sigma_y = -\sigma_y\sigma_x \tag{10}$$

or in terms of the 'anticommutator'

$$\{\sigma_x, \sigma_y\} \equiv \sigma_x\sigma_y + \sigma_y\sigma_x = 0. \tag{11}$$

Now, for any given spinor $\boldsymbol{s}$, the components of the flagpole vector, as given by eqn (5), can be written

$$r_x = \boldsymbol{s}^\dagger \sigma_x \boldsymbol{s}, \; r_y = \boldsymbol{s}^\dagger \sigma_y \boldsymbol{s}, \; r_z = \boldsymbol{s}^\dagger \sigma_z \boldsymbol{s} \tag{12}$$

which can be written more succinctly,

$$\mathbf{r} = \boldsymbol{s}^\dagger \boldsymbol{\sigma} \boldsymbol{s} = \langle s | \, \boldsymbol{\sigma} \, | s \rangle \tag{13}$$

where the second version is in Dirac notation[10].

Consider (exercise 1)

$$e^{i(\theta/2)\sigma_j} = \cos(\theta/2)I + i\sin(\theta/2)\sigma_j \tag{14}$$

hence

$$e^{i(\theta/2)\sigma_x} = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{15}$$

$$e^{i(\theta/2)\sigma_y} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{16}$$

$$e^{i(\theta/2)\sigma_z} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}. \tag{17}$$

We shall call these the 'spin rotation matrices.' We will now show that when the spinor is acted on by the matrix $\exp(i\theta\sigma_x/2)$, the flagpole is rotated through the angle θ about the x-axis. This can be shown directly from eq. (3) by trigonometry, but it will be more instructive to prove it using matrix methods, as follows. Let

$$\boldsymbol{s}' = e^{i(\theta/2)\sigma_x}\,\boldsymbol{s}$$

then

$$\mathbf{r}' = \langle s'|\,\boldsymbol{\sigma}\,|s'\rangle = \langle s|\,e^{-i(\theta/2)\sigma_x}\,\boldsymbol{\sigma}\,e^{i(\theta/2)\sigma_x}\,|s\rangle \tag{18}$$

where we used that σ_x is Hermitian. Consider first the x-component of this expression:

$$\begin{aligned} x' &= \langle s|\,e^{-i(\theta/2)\sigma_x}\sigma_x e^{i(\theta/2)\sigma_x}\,|s\rangle \\ &= \langle s|\,\sigma_x e^{-i(\theta/2)\sigma_x} e^{i(\theta/2)\sigma_x}\,|s\rangle \\ &= \langle s|\,\sigma_x\,|s\rangle = x, \end{aligned} \tag{19}$$

where we used that σ_x commutes with I and itself, and you should confirm that

$$e^{-i(\theta/2)\sigma_x} e^{i(\theta/2)\sigma_x} = I \tag{20}$$

(more generally, if a matrix H is Hermitian then $\exp(iH)$ is unitary). Now consider the y-component of (18):

$$y' = \langle s|\,e^{-i(\theta/2)\sigma_x}\sigma_y e^{i(\theta/2)\sigma_x}\,|s\rangle \tag{21}$$

To reduce clutter in the following, introduce $\alpha = \theta/2$. Then, using (14), the operator in the middle of (21) is

$$\begin{aligned} &(\cos\alpha - i\sin\alpha\sigma_x)\sigma_y(\cos\alpha + i\sin\alpha\sigma_x) \\ &\quad = \sigma_y(\cos\alpha + i\sin\alpha\sigma_x)(\cos\alpha + i\sin\alpha\sigma_x) \\ &\quad = \sigma_y(\cos^2\alpha - \sin^2\alpha + 2i\sin\alpha\cos\alpha\sigma_x) \\ &\quad = \sigma_y(\cos\theta + i\sin\theta\sigma_x) \end{aligned} \tag{22}$$

where in the first step we brought σ_y to the front by using that σ_x and σ_y anti-commute (eqn (10)), and in the second step we used that $\sigma_x^2 = I$. Upon subsituting the result (22) into (21) we have

$$y' = \cos\theta\,\langle s|\,\sigma_y\,|s\rangle + i\sin\theta\,\langle s|\,\sigma_y\sigma_x\,|s\rangle \tag{23}$$

but $\sigma_y\sigma_x = -i\sigma_z$, so this is

$$y' = \cos(\theta)y + \sin(\theta)z. \tag{24}$$

The analysis for z' goes the same, except that we have $\sigma_z\sigma_x = +i\sigma_y$ in the final step, so

$$z' = \cos(\theta)z - \sin(\theta)y. \tag{25}$$

The overall result is $\mathbf{r}' = R_x\mathbf{r}$, where R_x is the matrix representing a rotation through θ about the x axis. Owing to the fact that the commutation relations (9) are obeyed by cyclic permutations of x, y, z, the corresponding results for σ_y and σ_z immediately follow. Therefore, we have shown that multiplying a spinor by each of the spin rotation matrices (15)-(17) results in a rotation of the flagpole by the corresponding matrix for a rotation in three dimensions:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{26}$$

$$R_y = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{27}$$

$$R_z = \begin{pmatrix} \cos\theta & +\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

These are rotations about the x, y and z axes respectively, but note the *angle doubling*: the rotation angle θ is twice the angle $\theta/2$ which appears in the 2×2 'spin rotation' matrices. The sense of rotation is such that R represents a change of reference frame, that is to say, a rotation of the coordinate axes in a right-handed sense[11].

We have now almost established the homomorphism between the groups SU(2) and SO(3), because we have explicitly stated which member of SU(2) corresponds to which member of SO(3). It only remains to note that *any* member of SU(2) can be written (exercise 2)

$$U = e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}} \tag{29}$$

and *any* member of SO(3) can be written

$$R = e^{i\mathbf{J}\cdot\boldsymbol{\theta}}$$

where $\mathbf{J}$ are the generators of rotations in three dimensions:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

Note also that to obtain a given rotation R, we can use either U or $-U$. We have now fully established the mapping between the groups:

spinor $\boldsymbol{s}$	$\leftrightarrow$ vector $\mathbf{r} = \langle s \vert \boldsymbol{\sigma} \vert s \rangle$
Members U and $-U$ of SU(2)	$\leftrightarrow$ member R of SO(3)
$U = e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2}$	$R = e^{i\mathbf{J}\cdot\boldsymbol{\theta}}$

(31)

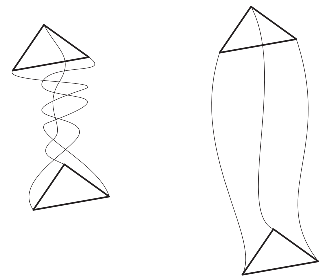

FIG. 3: 'Tangloids' is a game invented by Piet Hein to explore the effect of rotations of connected objects. Two small wooden poles or triangular blocks are joined by three parallel strings. Each player holds one of the blocks. The first player holds one block still, while the other player *rotates the other wooden block for two full revolutions about any fixed axis.* After this, the strings appear to be tangled. The first player now has to untangle them *without* rotating either piece of wood. He must use a parallel transport, that is, a translation of his block (in 3 dimensions) without rotating it or the other block. The fact that it can be done (for a 720° initial rotation, but not for a 360° initial rotation) illustrates a subtle property of rotations. After swapping roles, the winner is the one who untangled the fastest.

Let us note also the effect of an inversion of the coordinate system through the origin (called parity inversion). Vectors such as displacement change sign under such an inversion and are called *polar vectors*. Vectors such as angular momentum do not change sign under such an inversion, and are called *axial vectors* or *pseudovectors*. Suppose polar vectors $\mathbf{a}$ and $\mathbf{b}$ are related by $\mathbf{b} = R\mathbf{a}$. Under parity inversion these vectors transform as $\mathbf{a}' = -\mathbf{a}$ and $\mathbf{b}' = -\mathbf{b}$, so one finds $\mathbf{b}' = R\mathbf{a}'$, hence the rotation matrix is unaffected by parity inversion: $R' = R$. It follows that, in the expression $R = \exp(i\mathbf{J}\cdot\boldsymbol{\sigma})$, we must take $\mathbf{J}$ and $\boldsymbol{\sigma}$ as either both polar or both axial. The choice, whether we consider $\boldsymbol{\sigma}$ to be polar or axial, depends on the context in which it is being used.

With the benefit of hindsight, or else with a good knowledge of group theory, one could 'spot' the SU(2)–SO(3) homomorphism, including the angle doubling, simply by noticing that the commutation relations (9) are the same as those for the rotation matrices J_i, apart from the factor 2. This is because, if you look back through the argument, you can see that it would apply to any set of quantities obeying those relations. More generally, therefore, we say that the Pauli matrices are defined to be a set of entities that obey the commutation relations, and their standard expressions using 2×2 matrices are one *representation* of them.

The angle doubling leads to the curious feature that when $\theta = 2\pi$ (a single full rotation) the spin rotation matrices all give $-I$. It is *not* that the flagpole reverses direction—it does not, and neither does the flag—but rather, the spinor picks up an overall sign that has no ready representation in the flagpole picture.

It is worth considering for a moment. We usually consider that a 360° leaves everything unchanged. This is true for a global rotation of the whole universe, or for a rotation of an isolated object not interacting with anything else. However, when one object is rotated while interacting with another that is not rotated, more possibilities arise. The fact that a spinor rotation through 360° does not give the identity operation captures a valid property of rotations that is simply not modelled by the behaviour of vectors. Place a fragile object such as a china plate on the palm of your hand, and then rotate your palm through 360° (say, anticlockwise if you use your right hand) while keeping your palm horizontal, with the plate balanced on it. It can be done but you will now be standing somewhat awkwardly with a twist in your arm. But now *continue* to rotate your palm *in the same direction* (still anticlockwise). It can be done: most of us find ourselves bringing our hand up over our shoulder, but note: the palm and plate remain horizontal and continue to rotate. After thus completing two full revolutions, 720°, you should find yourself standing comfortably, with no twist in your arm! This simple experiment illustrates the fact that there is more to rotations than is captured by the simple notion of a direction in space. Mathematically, it is noticed in a subtle property of the Lie group SO(3): the associated smooth space is not 'simply connected' (in a topological sense). The group SU(2) exhibits it more clearly: the result of one full rotation is a sign change; a second full rotation is required to get a total effect equal to the identity matrix. Figure 3 gives a further comment on this property.

A. Rotations of rank 2 spinors

The mapping between SU(2) and SO(3) can also be established by examining a class of second rank spinors. This serves to introduce some further useful ideas.

For any real vector $\mathbf{r} = (x, y, z)$ one can construct the traceless Hermitian matrix

$$X = \mathbf{r}\cdot\boldsymbol{\sigma} = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}. \quad (32)$$

It has determinant

$$|X| = -(x^2 + y^2 + z^2).$$

Now consider the matrix product

$$UXU^\dagger = X' \quad (33)$$

where U is unitary and of unit determinant. For *any* unitary U, if X is Hermitian then the result X' is (i) Hermitian and (ii) has the same trace. Proof (i): $(X')^\dagger = (UXU^\dagger)^\dagger = (U^\dagger)^\dagger X^\dagger U^\dagger = UXU^\dagger = X'$; (ii): the trace is the sum of the eigenvalues and the eigenvalues are preserved in unitary transformations. Since X' is Hermitian and traceless, it can in turn be interpreted as a 3-component real vector $\mathbf{r}'$ (you are invited to prove this after reading on), and furthermore, if U has determinant 1 then X' has the same determinant as X so $\mathbf{r}'$ has the same length as $\mathbf{r}$. It follows that the transformation of $\mathbf{r}$ is either a rotation or a reflection. We shall prove that it is a rotation. To do this, it suffices to pick one of the spin rotation matrices; for convenience choose the z-rotation:

$$e^{i(\theta/2)\sigma_z} X e^{-i(\theta/2)\sigma_z} = \begin{pmatrix} z & e^{i\theta}(x-iy) \\ e^{-i\theta}(x+iy) & -z \end{pmatrix}.$$

The vector associated with this matrix is $(x\cos\theta + y\sin\theta,\ -x\sin\theta + y\cos\theta, z)$, which is $R_z\mathbf{r}$.

The relationship between the groups follows as before.

B. Spinors as eigenvectors

In this section we present an idea which is much used in quantum physics, but also has wider application because it is part of the basic mathematics of spinors. Every vector can be considered to be the eigenvector, with eigenvalue 1, of an orthogonal matrix, and a similar property applies to spinors. We will show that every spinor is the eigenvector, with eigenvalue 1, of a 2×2 traceless Hermitian matrix. But, we saw in the previous section that such matrices can be related to vectors, so we have another interesting connection. It will turn out that the direction associated with the matrix will agree with the flagpole direction of the spinor!

In fact, it is this result that motivated the assignment that we started with, eqn (1). The proofs connecting SU(2) matrices to SO(3) matrices do not themselves require any particular choice of the assignment of 3-vector direction to a complex 2-vector (spinor), only that it be assigned in a way that makes sense when rotations are applied. After all, we connected the groups in section II A without mentioning rank 1 spinors at all. The choice (1) either leads to, or, depending on your point of view, follows from, the considerations we are about to presemt.

Proof. First we show that for any 2-component complex vector $\boldsymbol{s}$ we can construct a matrix S such that $\boldsymbol{s}$ is an eigenvector of S with eigenvalue 1.

We would like S to be Hermitian. To achieve this, we make sure the eigenvectors are orthogonal and the eigenvectors real. The orthogonality we have in mind here is with respect to the standard definition of inner product in a complex vector space, namely

$$\boldsymbol{u}^\dagger \boldsymbol{v} = u_1^* v_1 + u_2^* v_2 = \langle u|\, v\rangle$$

where the last version on the right is in Dirac notation[12]. Beware, however, that we shall be introducing another type of inner product for spinors in section IV.

Let $\boldsymbol{s} = \begin{pmatrix} a \\ b \end{pmatrix}$. The spinor orthogonal[13] to $\boldsymbol{s}$ and with the same length is $\begin{pmatrix} -b^* \\ a^* \end{pmatrix}$ (or a phase factor times this). Let the eigenvalues be ± 1, then we have

$$SV = V\sigma_z$$

where

$$V = \frac{1}{s}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$

is the matrix of normalized eigenvectors, with $s = \sqrt{|a|^2 + |b|^2}$. V is unitary when the eigenvectors are normalized, as here. The solution is

$$S = V\sigma_z V^\dagger = \frac{1}{s^2}\begin{pmatrix} |a|^2 - |b|^2 & 2ab^* \\ 2ba^* & |b|^2 - |a|^2 \end{pmatrix}. \tag{34}$$

Comparing this with (32), we see that the direction associated with S is as given by (5). Therefore *the direction associated with the matrix S according to (32) is the same as the flagpole direction of the spinor $\boldsymbol{s}$ which is an eigenvector of S with eigenvalue 1. QED.*

Eq. (34) can be written

$$S = \mathbf{n}\cdot\boldsymbol{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$$

where n_x, n_y, n_z are given by equations (5) divided by s^2. We find that $\mathbf{n}$ is a unit vector (this comes from the choice that the eigenvalue is 1). The result can also be written $n_x = \boldsymbol{s}^\dagger\sigma_x\boldsymbol{s}/s^2$ and similarly. More succinctly, it is

$$\mathbf{n} = \frac{\boldsymbol{s}^\dagger\boldsymbol{\sigma}\boldsymbol{s}}{s^2} = \frac{\langle s|\, \boldsymbol{\sigma}\, |s\rangle}{s^2}, \tag{35}$$

Another useful way of stating the overall conclusion is

> For any unit vector $\mathbf{n}$, the Hermitian traceless matrix
>
> $$S = \mathbf{n}\cdot\boldsymbol{\sigma}$$
>
> has an eigenvector of eigenvalue 1 whose flagpole is along $\mathbf{n}$.

Since a rotation of the coordinate system would bring S onto one of the Pauli matrices, S is called a 'spin matrix' for spin along the direction $\mathbf{n}$.

Suppose now that we have another spinor related to the first one by a rotation: $\boldsymbol{s}' = U\boldsymbol{s}$. We ask the question, of which matrix is $\boldsymbol{s}'$ an eigenvector with eigenvalue 1? We propose and verify the solution $USU^\dagger$:

$$(USU^\dagger)\boldsymbol{s}' = USU^\dagger U\boldsymbol{s} = US\boldsymbol{s} = U\boldsymbol{s} = \boldsymbol{s}'.$$

Therefore the answer is

$$S' = USU^\dagger.$$

This is precisely the transformation that represents a rotation of the vector $\mathbf{n}$ (compare with (33)), so we have proved that the flagpole of $\boldsymbol{s}'$ is in the direction $R\mathbf{n}$, where R is the rotation in 3-space associated with U in the mapping between SU(2) and SO(3). Therefore U gives a rotation of the direction of the spinor.

We have here presented spinors as classical (in the sense of not quantum-mechanical) objects. If you suspect that the occasional mention of Dirac notation means that we are doing quantum mechanics, then please reject that impression. in this article a spinor is a classical object. It is a generalization of a classical vector.

III. LORENTZ TRANSFORMATION OF SPINORS

We are now ready to generalize from space to spacetime, and make contact with Special Relativity. It turns out that the spinor is already a naturally 4-vector-like quantity, to which Lorentz transformations can be applied.

We will adopt the font A, B, ... for 4-vectors, and use index notation where convenient. The inner product of 4-vectors is written either $\mathsf{A} \cdot \mathsf{B}$ or $\mathsf{A}^\mu \mathsf{B}_\mu$. The Minkowski metric is taken with signature $(-1, 1, 1, 1)$. Note, this is a widely used convention, but it is not the convention often adopted in particle physics where $(1, -1, -1, -1)$ is more common.

Let $\boldsymbol{s}$ be some arbitrary 1st rank spinor. Under a change of inertial reference frame it will transform as

$$\boldsymbol{s}' = \Lambda \boldsymbol{s} \tag{36}$$

where Λ is a 2×2 matrix to be discovered. To this end, form the outer product

$$\boldsymbol{s}\boldsymbol{s}^\dagger = \begin{pmatrix} a \\ b \end{pmatrix} (a^*,\, b^*) = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}. \tag{37}$$

This is (an example of) a 2nd rank spinor, and by definition it must transform as $\boldsymbol{s}\boldsymbol{s}^\dagger \to \Lambda \boldsymbol{s}\boldsymbol{s}^\dagger \Lambda^\dagger$. 2nd rank spinors (of the standard, contravariant type) are defined more generally as objects which transform in this way, i.e. $X \to \Lambda X \Lambda^\dagger$.

Notice that the matrix in (37) is Hermitian. Thus outer products of 1st rank spinors form a subset of the set of Hermitian 2×2 matrices. We shall show that the complete set of Hermitian 2×2 matrices can be used to represent 2nd rank spinors.

An arbitrary Hermitian 2×2 matrix can be written

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} = tI + x\sigma_x + y\sigma_y + z\sigma_z, \tag{38}$$

which can also be written

$$X = \sum_\mu \mathsf{X}^\mu \sigma^\mu$$

where we introduced $\sigma^0 \equiv I$. The summation here is written explicitly, because this is not a tensor expression, it is a way of creating one sort of object (a 2nd rank spinor) from another sort of object (a 4-vector).

Evaluating the determinant, we find

$$|X| = t^2 - (x^2 + y^2 + z^2),$$

which is the Lorentz invariant associated with the 4-vector X^μ. Consider the transformation

$$X \to \Lambda X \Lambda^\dagger. \tag{39}$$

To keep the determinant unchanged we must have

$$|\Lambda||\Lambda^\dagger| = 1 \qquad \Rightarrow \quad |\Lambda| = e^{i\lambda}$$

for some real number λ. Let us first restrict attention to $\lambda = 0$. Then we are considering complex matrices Λ with determinant 1, i.e. the group SL(2,C). Since the action of members of SL(2,C) preserves the Lorentz invariant quantity, we can associate a 4-vector $(t, \mathbf{r})$ with the matrix X, and we can associate a Lorentz transformation with any member of SL(2,C).

The more general case $\lambda \neq 0$ can be included by considering transformations of the form $e^{i\lambda/2}\Lambda$ where $|\Lambda| = 1$. It is seen that the additional phase factor has no effect on the 4-vector obtained from any given spinor, but it rotates the flag through the angle λ. This is an example of the fact that spinors are richer than 4-vectors. However, just as we did not include such global phase factors in our definition of 'rotation', we shall also not include it in our definition of 'Lorentz transformation'. In other words, the group of Lorentz transformation of spinors is the group of 2×2 complex matrices with determinant 1 (called SL(2,C)).

The extra parameter (allowing us to go from a 3-vector to a 4-vector) compared to eq. (32) is exhibited in the tI term. The resulting matrix is still Hermitian but it no longer needs to have zero trace, and indeed the trace is not zero when $t \neq 0$. Now that we don't require the trace of X to be fixed, we can allow non-unitary matrices to act on it. In particular, consider the matrix

$$\begin{aligned} e^{-(\rho/2)\sigma_z} &= \begin{pmatrix} e^{-\rho/2} & 0 \\ 0 & e^{\rho/2} \end{pmatrix} \\ &= \cosh(\rho/2)I - \sinh(\rho/2)\sigma_z. \end{aligned} \tag{40}$$

One finds that the effect on X is such that the associated 4-vector is transformed as

$$\begin{pmatrix} \cosh(\rho) & 0 & 0 & -\sinh(\rho) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\rho) & 0 & 0 & \cosh(\rho). \end{pmatrix}$$

This is a Lorentz boost along z, with rapidity ρ. You can check that $\exp(-(\rho/2)\sigma_x)$ and $\exp(-(\rho/2)\sigma_y)$ give Lorentz boosts along x and y respectively. (This must be the case, since the Pauli matrices can be related to one another by rotations). The general Lorentz boost for a spinor is, therefore, (for $\boldsymbol{\rho} = \rho\mathbf{n}$)

$$e^{-(\boldsymbol{\rho}/2)\cdot\boldsymbol{\sigma}} = \cosh(\rho/2)I - \sinh(\rho/2)\mathbf{n}\cdot\boldsymbol{\sigma}. \tag{41}$$

We thus find the whole of the structure of the restricted Lorentz group reproduced in the group SL(2,C). The relationship is a two-to-one mapping since a given Lorentz transformation (in the general sense, including rotations) can be represented by either $+M$ or $-M$, for $M \in$ SL(2,C). The abstract space associated with the group SL(2,C) has three complex dimensions and therefore six real ones (the matrices have four complex numbers and one complex constraint on the determinant). This matches the 6 dimensions of the manifold associated with the Lorentz group.

Now let

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{42}$$

For an arbitrary Lorentz transformation

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

we have

$$\Lambda^T \epsilon \Lambda = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \epsilon \tag{43}$$

It follows that for a pair of spinors $\boldsymbol{s}$, $\boldsymbol{w}$ the scalar quantity

$$\boldsymbol{s}^T \epsilon \boldsymbol{w} = s_1 w_2 - s_2 w_1$$

is Lorentz-invariant. Hence this is a useful inner product for spinors.

Equation (43) should remind you of the defining property of Lorentz transformations applied to tensors, "$\Lambda^T g \Lambda = g$" where g is the Minkowski metric tensor. The matrix ϵ satisfying (43) is called the **spinor Minkowski metric**.

A full exploration of the symmetries of spinors involves the recognition that the correct group to describe the symmetries of particles is not the Lorentz group but the Poincaré group. We shall not explore that here, but we remark that in such a study the concept of intrinsic spin emerges naturally, when one asks for a complete set of quantities that can be used to describe symmetries of a particle. One such quantity is the scalar invariant $\mathsf{P}\cdot\mathsf{P}$, which can be recognised as the (square of the) mass of a particle. A second quantity emerges, related to rotations, and its associated invariant is $\mathsf{W}\cdot\mathsf{W}$ where W is the Pauli-Lubanski spin vector.

$\boldsymbol{u}^\dagger \boldsymbol{u}$	zeroth component of a 4-vector
$\boldsymbol{u}^T \epsilon \boldsymbol{u}$	a scalar invariant (equal to zero)
$\overline{\boldsymbol{u}}^\dagger \boldsymbol{u}$	another way of writing $\boldsymbol{u}^T \epsilon \boldsymbol{u}$, with $\overline{\boldsymbol{u}} \equiv \epsilon \boldsymbol{u}^*$
$\boldsymbol{u}^T \boldsymbol{u}$	no particular significance

TABLE I: Some scalars associated with a spinor, and their significance.

A. Obtaining 4-vectors from spinors

By interpreting (37) using the general form (38) we find that the four-vector associated with the 2nd rank spinor obtained from the 1st rank spinor $\boldsymbol{s}$ is

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (|a|^2 + |b|^2)/2 \\ (ab^* + ba^*)/2 \\ i(ab^* - ba^*)/2 \\ (|a|^2 - |b|^2)/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \langle s|\, I\, |s\rangle \\ \langle s|\, \boldsymbol{\sigma}\, |s\rangle \end{pmatrix} \tag{44}$$

which can be written $(1/2)\langle s|\,\sigma^\mu\,|s\rangle$. Any constant multiple of this is also a legitimate 4-vector. In order that the spatial part agrees with our starting point (1) we must introduce[14] a factor 2, so that we have the result (perhaps the central result of this introduction)

obtaining a (null) 4-vector from a spinor

$$\mathsf{V}^\mu = \boldsymbol{v}^\dagger \sigma^\mu \boldsymbol{v} \tag{45}$$

This 4-vector is null, as we mentioned in the introductory section I. The easiest way to verify this is to calculate the determinant of the spinor matrix (37).

Since the zeroth spin matrix is the identity, we find that the zeroth component of the 4-vector can be written $\boldsymbol{v}^\dagger \boldsymbol{v}$. This and some other basic quantities are listed in table I.

The linearity of eq. (36) shows that the sum of two spinors is also a spinor (i.e. it transforms in the right way). The new spinor still corresponds to a null 4-vector, so it is in the light cone. Note, however, that the sum of two null 4-vectors is not in general null. So adding up two spinors as in $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ does not result in a 4-vector W that is the sum of the 4-vectors U and V associated with each of the spinors. If you want to get access to $\mathsf{U}+\mathsf{V}$, it is easy to do: first form the outer product, then sum: $\boldsymbol{u}\boldsymbol{u}^\dagger + \boldsymbol{v}\boldsymbol{v}^\dagger$. The resulting 2×2 matrix represents the (usually non-null) 4-vector $\mathsf{U}+\mathsf{V}$.

By using a pair of non-orthogonal null spinors, we can always represent a pair of orthogonal non-null 4-vectors by combining the spinors. Let the spinors be $\boldsymbol{u}$ and $\boldsymbol{v}$ and their associated 4-vectors be U and V. Let $\mathsf{P} = \mathsf{U}+\mathsf{V}$ and $\mathsf{W} = \mathsf{U}-\mathsf{V}$. Then $\mathsf{U}\cdot\mathsf{U} = 0$ and $\mathsf{V}\cdot\mathsf{V} = 0$ but $\mathsf{P}\cdot\mathsf{P} = 2\mathsf{U}\cdot\mathsf{V} \neq 0$ and $\mathsf{W}\cdot\mathsf{W} = -2\mathsf{U}\cdot\mathsf{V} \neq 0$. That is, as long as U and V are not orthogonal then P and W are not null. The latter are orthogonal to one another, however:

$$\mathsf{W}\cdot\mathsf{P} = (\mathsf{U}+\mathsf{V})\cdot(\mathsf{U}-\mathsf{V}) = \mathsf{U}\cdot\mathsf{U} - \mathsf{V}\cdot\mathsf{V} = 0.$$

Examples of pairs of 4-vectors that are mutually orthogonal are 4-velocity and 4-acceleration, and 4-momentum

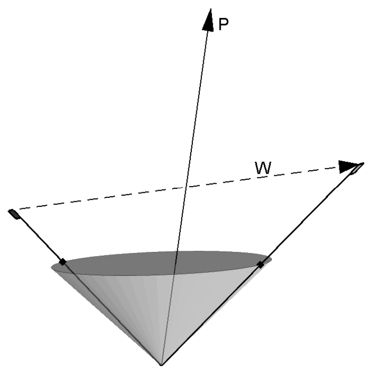

FIG. 4: Two spinors can represent a pair of orthogonal 4-vectors. The spacetime diagram shows two spinors. They have opposite spatial direction and are embedded in a null cone (light cone), including the flags which point around the cone. Their amplitudes are not necessarily equal. The sum of their flagpoles is a time-like 4-vector P; the difference is a space-like 4-vector W. P and W are orthogonal (on a spacetime diagram this orthogonality is shown by the fact that if P is along the time axis of some reference frame, then W is in along the corresponding space axis.)

and 4-spin (i.e. Pauli-Lubanski spin vector [1]). Therefore we can describe the motion and spin of a particle by using a pair of spinors, see figure 4. This connection will be explored further in section V.

To summarize:

rank 1 spinor	↔	null 4-vector
rank 2 spinor	↔	arbitrary 4-vector
pair of non-orthogonal rank 1 spinors	↔	pair of orthogonal 4-vectors

IV. CHIRALITY

We now come to the subject of *chirality*. This concerns a property of spinors very much like the property of *contravariant* and *covariant* applied to 4-vectors. In other words, chirality is essentially about *the way spinors transform under Lorentz-transformations*. Unfortunately, the name itself does not suggest that. It is a bad name. In order to understand this we shall discuss the transformation properties first, and then return to the terminology at the end.

First, let us notice that there is another way to construct a contravariant 4-vector from a spinor. Suppose that instead of (45) we try

$$\mathsf{V}^{\mu} = \begin{pmatrix} \langle\tilde{v}| - I\,|\tilde{v}\rangle \\ \langle\tilde{v}|\,\boldsymbol{\sigma}\,|\tilde{v}\rangle \end{pmatrix} = \langle\tilde{v}|\,\sigma_{\mu}\,|\tilde{v}\rangle\,, \tag{46}$$

for a spinor-like object $\tilde{\boldsymbol{v}}$. It looks at first as though we have constructed a covariant 4-vector and put the index 'upstairs' by mistake. However, what if we insist that this V really is contravariant? This amounts to saying that $\tilde{\boldsymbol{v}}$ is a new type of object, not like the spinors we talked about up till now. To explore this, observe that the same assignment can also be written

$$\mathsf{V}_{\mu} = \langle\tilde{v}|\,\sigma^{\mu}\,|\tilde{v}\rangle\,. \tag{47}$$

Index notation does not lend itself to the proof that (47) and (46) imply each other, but it can be seen readily enough by using a rectangular coordinate system and writing out all the terms, since there the Minkowski metric has the simple form $g_{ab} = \mathrm{diag}(-1,1,1,1)$ and its inverse is the same, $g^{ab} = \mathrm{diag}(-1,1,1,1)$. We deduce that the difference between $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ is that when combined with σ^{μ}, the former gives a contravariant and the latter gives a covariant 4-vector. Everything is consistent if we introduce the rule for a Lorentz transformation of $\tilde{\boldsymbol{v}}$ as

$$\text{if} \quad \boldsymbol{v}' = \Lambda\boldsymbol{v} \tag{48}$$

$$\text{then} \quad \tilde{\boldsymbol{v}}' = (\Lambda^{\dagger})^{-1}\tilde{\boldsymbol{v}}. \tag{49}$$

This is because, for a pure rotation $\Lambda^{\dagger} = \Lambda^{-1}$ so the two types of spinor transform the *same* way, but for a pure boost $\Lambda^{\dagger} = \Lambda$ (it is Hermitian) so we have precisely the inverse transformation. This combination of properties is exactly the relationship between covariant and contravariant 4-vectors.

The two types of spinor may be called *contraspinor* and *cospinor*. However, they are often called *right-handed* and *left-handed*. The idea is that we regard the Lorentz boost as a kind of 'rotation in spacetime', and for a *given* boost velocity, the contraspinor 'rotates' one way, while the cospinor 'rotates' the other. They are said to possess opposite *chirality*. However, given that we are also much concerned with real rotations in space, this terminology is regretable because it leads to confusion.

Equation (47) can be 'read' as stating that the presence of $\tilde{\boldsymbol{v}}$ acts to lower the index on σ^{μ} and give a covariant result.

The rule (46) was here introduced ad-hoc: what is to say there may not be further rules? This will be explored below; ultimately the quickest way to show this and other properties is to use Lie group theory on the generators, a method we have been avoiding in order not to assume familiarity with groups, but it is briefly sketched in section (VII).

A. Chirality, spin and parity violation

It is not too surprising to suggest that a spinor may offer a useful mathematical tool to handle angular momentum. This was the context in which spinors were first widely used. A natural way to proceed is simply to claim that there may exist fundamental particles whose intrinsic nature is not captured purely by scalar properties such as mass and charge, but which also have an angular-momentum-like property called spin, that is described by a spinor.

Having made the claim, we might propose that the 4-vector represented by the spinor flagpole is the Pauli-Lubanski spin vector [1]. The Pauli-Lubanski vector has components

$$\mathsf{W}^\mu = (\mathbf{s}\cdot\mathbf{p}, (E/c)\mathbf{s}) \tag{50}$$

for a particle with spin 3-vector $\mathbf{s}$, energy E and momentum $\mathbf{p}$. If this 4-vector can be extracted from a rank 1 spinor, then it must be a null 4-vector. This in turn implies the particle is massless, because

$$\mathsf{W}^\mu \mathsf{W}_\mu = 0 \implies E^2 = p^2c^2\cos^2\theta \tag{51}$$

where θ is the angle between $\mathbf{s}$ and $\mathbf{p}$ in some reference frame. But, for any particle, $E \geq pc$, so the only possibility is $E = pc$ and $\theta = 0$ or $\theta = \pi$. Therefore we look for a massless spin-half particle in our experiments. We already know one: it is the neutrino[15].

Thus we have a suitable model for intrinsic spin, that applies to massless spin-half particles. It is found in practice that it describes accurately the experimental observations of the nature of intrinsic angular momentum for such particles.

Now we shall, by 'waving a magic wand', discover a wonderful property of massless spin-half particles that emerges naturally when we use spinors, but does not emerge naturally in a purely 4-vector treatment of angular momentum (as, for example, in chapter 15 of [1]). By 'waving a magic wand' here we mean noticing something that is already built in to the mathematical properties of the objects we are dealing with, namely spinors. All we need to do is claim that the same spinor describes both the linear momentum and the intrinsic spin of a given neutrino. We claim that we don't need two spinors to do the job: just one is sufficient. There is a problem: since we can only allow one rule for extracting the 4-momentum and Pauli-Lubanski spin vector for a given type of particle, we shall have to claim that there is a restriction on the allowed combinations of 4-momentum and spin for particles of a given type. For massless particles the Pauli-Lubanski spin and the 4-momentum are aligned (either in the same direction or opposite directions, as we showed after eqn (51)), so there is already one restriction that emerges in either a 4-vector or a spinor analysis, but now we shall have to go further, and claim that *all massless spin-half particles of a given type have the same helicity* (equations (57) and (63)).

This is a remarkable claim, at first sight even a crazy claim. It says that, relative to their direction of motion, neutrinos are allowed to 'rotate' one way, but not the other! To be more precise, it is the claim that there exist in Nature processes whose mirror reflected versions never occur. Before any experimenter would invest the effort to test this (it is difficult to test because neutrinos interact very weakly with other things), he or she would want more convincing of the theoretical background, so let us investigate further.

Notation. We now have 3 vector-like quantities in play: 3-vectors, 4-vectors, and rank 1 spinors. We adopt three fonts:

entity	font	examples
3-vector	bold upright Roman	$\mathbf{s}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$
4-vector	sans-serif capital	S, U, V, W
spinor	bold italic	$\boldsymbol{s}$, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$

Processes whose mirror-reflected versions run differently (for example, not at all) are said to exhibit **parity violation**. We can prove that there are no such processes in classical electromagnetism, because Maxwell's equations and the Lorentz force equation are unchanged under the parity inversion operation. The 'parity-invariant' behaviour of the last two Maxwell equations, and the Lorentz force equation, involves the fact that $\mathbf{B}$ is an axial vector.

To investigate the possibilities for spinors, consider the Lorentz invariant

$$\mathsf{W}_\lambda \mathsf{S}^\lambda = \mathsf{W}_\lambda \boldsymbol{s}^\dagger \sigma^\lambda \boldsymbol{s}$$

where $\boldsymbol{s}$ is contravariant. Since, in the sum, each term W_λ is just a number, it can be moved past the $\boldsymbol{s}^\dagger$ and we have

$$\mathsf{W}_\lambda \mathsf{S}^\lambda = \boldsymbol{s}^\dagger \mathsf{W}_\lambda \sigma^\lambda \boldsymbol{s}. \tag{52}$$

The combination $\mathsf{W}_\lambda \sigma^\lambda = -\mathsf{W}^0 I + \mathbf{w}\cdot\sigma$ is a matrix. It can usefully be regarded as an operator acting on a spinor. We can prove that one effect of this kind of matrix, when multiplying a spinor, is to change the transformation properties. For, $\boldsymbol{s}$ transforms as

$$\boldsymbol{s} \to \Lambda \boldsymbol{s}$$

and therefore

$$\boldsymbol{s}^\dagger \to \boldsymbol{s}^\dagger \Lambda^\dagger.$$

Since $\mathsf{W}_\lambda \mathsf{S}^\lambda$ is invariant, we deduce from (52) that $\mathsf{W}_\lambda \sigma^\lambda \boldsymbol{s}$ must transform as

$$(\mathsf{W}_\lambda \sigma^\lambda \boldsymbol{s}) \to (\Lambda^\dagger)^{-1}(\mathsf{W}_\lambda \sigma^\lambda \boldsymbol{s}). \tag{53}$$

Therefore, for any W, if $\boldsymbol{s}$ is a contraspinor then $(\mathsf{W}_\lambda \sigma^\lambda \boldsymbol{s})$ is a cospinor, and *vice versa*.

If the 4-vector W is null, then it can itself be represented by a spinor $\boldsymbol{w}$. Let's see what happens when the matrix $\mathsf{W}_\lambda \sigma^\lambda$ multiplies the spinor representing W:

$$\mathsf{W}_\lambda \sigma^\lambda \boldsymbol{w} = \begin{pmatrix} -2|b|^2 & 2ab^* \\ 2a^*b & -2|a|^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{54}$$

where for convenience we worked in terms of the components ($\boldsymbol{w} = (a, b)^T$) in some reference frame. The result is

$$(-\mathsf{W}^0 + \mathbf{w}\cdot\boldsymbol{\sigma})\boldsymbol{w} = 0 \tag{55}$$

(N.B. in this equation $\mathbf{w}$ is a 3-vector whereas $\boldsymbol{w}$ is a spinor). This equation is important because it is (by construction) a Lorentz-covariant equation, and it tells us something useful about 1st rank spinors in general.

Suppose the 4-vector W is the 4-momentum of some massless particle. Then the equation reads

$$(E/c - \mathbf{p}\cdot\boldsymbol{\sigma})\boldsymbol{w} = 0. \tag{56}$$

This equation is called the first **Weyl equation** and in the context of particle physics, the rank 1 spinors are called *Weyl spinors*. The presence of $\boldsymbol{\sigma}$ in this equation invites us to guess that the equation might be interpreted also as a statement about intrinsic spin. This guess is very natural if we suppose that a single spinor can serve to encode both the 4-momentum and the 4-spin, for massless spin-half particles, and this is the interpretation proposed by Weyl. To adopt this interpretation, it is necessary to use a polar version of the vector $\boldsymbol{\sigma}$ when using (45) to relate $\boldsymbol{w}$ to linear momentum, and an axial version to extract the spin (which, being a form of angular momentum, must be axial). Therefore when the Weyl equation (56) is used in this context, $\mathbf{p}$ is polar but $\boldsymbol{\sigma}$ is axial. This means the equation transforms in a non-trivial way under parity inversion. In short, it is not parity-invariant. This was enough to make particle physicists very dubious of the claim that the equation could describe real physical behaviour—but it turns out that Nature does admit this type of behaviour, and neutrinos give an example of it.

For a massless particle, we have $E = pc$, so (56) gives

$$\frac{(\mathbf{p}\cdot\boldsymbol{\sigma})}{p}\boldsymbol{w} = \boldsymbol{w}. \tag{57}$$

This says that $\boldsymbol{w}$ is an eigenvector, with eigenvalue 1, of the spin operator pointing along $\mathbf{p}$. In other words, *the particle has positive helicity.*

Now let's explore another possibility: suppose the spinor representing the particle has the other chirality. Then the energy-momentum is obtained as

$$\mathsf{P}_\mu \;=\; \tilde{\boldsymbol{v}}^\dagger \sigma^\mu \tilde{\boldsymbol{v}}. \tag{58}$$

where the use of a different letter ($\boldsymbol{v}$) indicates that we are talking about a different particle, and the tilde acts as a reminder of the different transformation properties. The invariant is now

$$\mathsf{P}^\lambda \mathsf{S}_\lambda = \tilde{\boldsymbol{s}}^\dagger \mathsf{P}^\lambda \sigma^\lambda \tilde{\boldsymbol{s}} = \tilde{\boldsymbol{s}}^\dagger (\tilde{\boldsymbol{v}}^\dagger \sigma_\lambda \tilde{\boldsymbol{v}}) \sigma^\lambda \tilde{\boldsymbol{s}} \tag{59}$$

and the operator of interest is

$$\left(\tilde{\boldsymbol{v}}^\dagger \sigma_\lambda \tilde{\boldsymbol{v}}\right)\sigma^\lambda = E/c + \mathbf{p}\cdot\boldsymbol{\sigma}. \tag{60}$$

The version on the right hand side does not at first sight look like a Lorentz invariant, because of the absence of a minus sign, but as long as we use the operator with cospinors (left handed spinors) then Lorentz covariant equations will result. For example, the argument in (54) is essentially unchanged and we find

$$\left(\tilde{\boldsymbol{v}}^\dagger \sigma^\alpha \tilde{\boldsymbol{v}}\right)\sigma_\alpha \tilde{\boldsymbol{v}} \;=\; 0 \tag{61}$$

$$\text{i.e.} \qquad (E/c + \mathbf{p}\cdot\boldsymbol{\sigma})\tilde{\boldsymbol{v}} \;=\; 0. \tag{62}$$

This is called the 2nd Weyl equation. Since the particle is massless it implies

$$\frac{(\mathbf{p}\cdot\boldsymbol{\sigma})}{p}\tilde{\boldsymbol{v}} \;=\; -\tilde{\boldsymbol{v}}. \tag{63}$$

Therefore now the helicity is negative.

Overall, the spinor formalism suggests that there are two particle types, possibly related to one another in some way, but they are not interchangeable because they transform in different ways under Lorentz transformations. We are then forced to the 'parity-breaking' conclusion that one of these types of particle always has positive helicity, the other negative. This is born out in experiments. An experimental test involving the β-decay of cobalt nuclei was performed in 1957 by Wu *et al.*, giving clear evidence for parity non-conservation. In 1958 Goldhaber *et al.* took things further in a beautiful experiment, designed to allow the helicity of neutrinos to be determined. It was found that all neutrinos produced in a given type of process have the same helicity. This

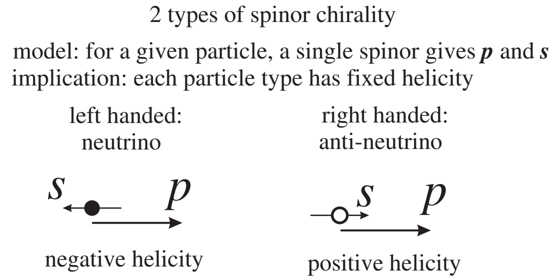

FIG. 5: The black and white circles represent two particle-like entities. Both are massless leptons with spin 1/2 and zero charge. Are they two examples of the same type of particle then, merely having the spin in opposite directions? The sequence of statements shown in the figure gives the logic. The black entity is found to have different chirality from the white entity. This is a subtle property, not easily illustrated by any diagram, since it refers to how the spinor transforms under a boost. However this property suffices to distinguish one entity from the other, and it is legitimate to give them different names ("neutrino" and "anti-neutrino") and draw them with different colours. The theoretical model asserts that the information about 4-spin and energy-momentum is contained in a single spinor for each entity. It then follows that the helicity is single-valued: always negative for the one we called "neutrino" and always positive for the one we called "anti-neutrino". Similar reasoning applied to electrons reaches a different conclusion. Each electron is not described by a single spinor, but by a pair of spinors, one of each chirality. Consequently the helicity of an electron can be of either sign, and is not Lorentz-invariant.

What is the difference between chirality and helicity?
Answer: helicity refers to the projection of the spin along the direction of motion, chirality refers to the way the spinor transforms under Lorentz transformations.
The word 'chirality' in general in science refers to *handedness*. A screw, a hand, and certain types of molecule may be said to possess *chirality*. This means they can be said to embody a rotation that is either left-handed or right-handed with respect to a direction also embodied by the object. When Weyl spinors are used to represent spin angular momentum and linear momentum, they also possess a handedness, which can with perfect sense be called an example of chirality. However, since the particle physicists already had a name for this (helicity), the word chirality came to be used to refer directly to the transformation property, such that spinors transforming one way are said to be 'right-handed' or of 'positive chirality', and those transforming the other way are said to be 'left-handed' or of 'negative chirality.' This terminology is poor because (i) it invites (and in practice results in) confusion between chirality and helicity, (ii) spinors can be used to describe other things beside spin, and (iii) the *transformation rule* has nothing in itself to do with angular momentum. The terminology is acceptable, however, if one understands it to refer to the Lorentz boost as a form of 'rotation' in spacetime.

is evidence that all neutrinos have one helicity, and anti-neutrinos have the opposite helicity. By convention those with positive helicity are called anti-neutrinos. With this convention, the process

$$\mathrm{n} \to \mathrm{p} + \mathrm{e} + \bar{\nu} \tag{64}$$

is allowed (with the bar indicating an antiparticle), but the process $\mathrm{n} \to \mathrm{p} + \mathrm{e} + \nu$ is not. Thus the properties of Weyl spinors are at the heart of the parity-non-conservation exhibited by the weak interaction.

B. Reflection and Lorentz transformation

Recall the relationship between a spinor $\boldsymbol{s}$ and the 3-vector of its flagpole:

$$\mathbf{r} = \boldsymbol{s}^\dagger \boldsymbol{\sigma} \boldsymbol{s}.$$

Taking the complex conjugate yields

$$\mathbf{r}^* = \mathbf{r} = (\boldsymbol{s}^*)^\dagger \boldsymbol{\sigma}^* \boldsymbol{s}^*.$$

Now, since σ_x and σ_z are real while σ_y is imaginary, $\boldsymbol{\sigma}^* = (\sigma_x,\ -\sigma_y,\ \sigma_z)$. Therefore

$$(\boldsymbol{s}^*)^\dagger \boldsymbol{\sigma} \boldsymbol{s}^* = (x,\ -y,\ z).$$

In other words, taking the complex conjugate of a spinor corresponds to a reflection in the xz plane. An inversion through the origin (parity inversion) is obtained by such a reflection followed by a rotation about the y axis through $180°$, i.e. the transformation

$$\boldsymbol{s} \to e^{i(\pi/2)\sigma_y} \boldsymbol{s}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{s}^* = \epsilon \boldsymbol{s}^* \tag{65}$$

where the last version uses the spinor Minkowski metric ϵ introduced in eq. (42).

We now have four possibilities: for a given $\boldsymbol{s}$ we can construct three others by use of complex conjugation and multiplication by the metric. These are $\boldsymbol{s}^*$, $\epsilon\boldsymbol{s}$ and $\epsilon\boldsymbol{s}^*$. They transform under a general Lorentz transformation as:

$$\begin{aligned} \boldsymbol{s} &\to \Lambda \boldsymbol{s} \\ \boldsymbol{s}^* &\to \Lambda^* \boldsymbol{s}^* \\ (\epsilon \boldsymbol{s}) &\to (\Lambda^T)^{-1} (\epsilon \boldsymbol{s}) \\ (\epsilon \boldsymbol{s}^*) &\to (\Lambda^\dagger)^{-1} (\epsilon \boldsymbol{s}^*) \end{aligned} \tag{66}$$

The second result follows immediately from the first by complex conjugation. The third result uses $\epsilon\Lambda = (\Lambda^T)^{-1}\epsilon$ from (43); the fourth then follows by complex conjugation. The last result shows that parity inversion changes the chirality. That is, under parity inversion, a right handed spinor changes into a left handed one, and *vice versa*.

A pure boost such as $\exp(-(\rho/2)\sigma_z)$ is Hermitian. From (40) we have

$$e^{(-\rho/2)\sigma_z} = \cosh(\rho/2) I - \sinh(\rho/2)\sigma_z$$

and therefore

$$\left(e^{(-\rho/2)\sigma_z}\right)^{-1} = e^{(\rho/2)\sigma_z}. \tag{67}$$

This confirms, as expected, that the inverse Lorentz boost is obtained by reversing the sign of the velocity. Thus we deduce that $\epsilon\boldsymbol{s}^*$ transforms the same way as $\boldsymbol{s}$ under rotations, but it transforms the inverse way (i.e. with opposite velocity sign) under boosts. This confirms that it is the covariant partner to $\boldsymbol{s}$. Sometimes $\epsilon\boldsymbol{s}^*$ is called the **dual** of $\boldsymbol{s}$ and is written $\overline{\boldsymbol{s}} \equiv \epsilon\boldsymbol{s}^*$.

The results are summarized in table II. The Lorentz transformation for the ϵs^* case can also be written

$$(\Lambda^\dagger)^{-1} = \epsilon \Lambda^* \epsilon^{-1} \tag{68}$$

(this is quickly proved for arbitrary $\Lambda \in$ SL(2,C) using (42).)

In short, we have discovered that there are four types of spinor, distinguished by how they behave under a change of inertial reference frame. These are best described as two types, plus their mirror inversions:

spinor	any transformation	pure rotation	pure boost	chirality
$\boldsymbol{u}$	$\Lambda \boldsymbol{u}$	$U\boldsymbol{u}$	$L\boldsymbol{u}$	$+$
$\boldsymbol{v} = (\epsilon \boldsymbol{u}^*)$	$(\Lambda^\dagger)^{-1}\boldsymbol{v}$	$U\boldsymbol{v}$	$L^{-1}\boldsymbol{v}$	$-$
$\boldsymbol{s} = \boldsymbol{u}^*$	$\Lambda^*\, \boldsymbol{s}$	$U^*\, \boldsymbol{s}$	$L^T\, \boldsymbol{s}$	$+$
$\boldsymbol{t} = (\epsilon \boldsymbol{u})$	$(\Lambda^{-1})^T \boldsymbol{t}$	$U^* \boldsymbol{t}$	$(L^{-1})^T \boldsymbol{t}$	$-$

TABLE II: Four types of spinor and their transformation. In principle, any expression can be written using just one of these types of spinor, by including explicit use of ϵ and complex conjugation (see exercise 4). In practice, a notation such as $\boldsymbol{u}$ and $\tilde{\boldsymbol{v}}$ or ϕ_R, χ_L is more convenient to write the first two types, then complex conjugation suffices to express the other two types where needed.

1. Type I, called 'right handed spinor' or 'positive chirality spinor'

$$\phi_R \to \Lambda \phi_R = \exp\left(i \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\theta}}{2} - \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\rho}}{2} \right) \phi_R$$

2. Type II, called 'left handed spinor' or 'negative chirality spinor'

$$\phi_L \to (\Lambda^\dagger)^{-1} \phi_L = \exp\left(i \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\theta}}{2} + \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\rho}}{2} \right) \phi_L$$

C. Index notation*

Suppose we take a 2nd rank spinor X obtained from a four-vector following the prescription in eq. (38), and a first rank spinor $\boldsymbol{u}$ transforming as $\boldsymbol{u} \to \Lambda \boldsymbol{u}$. We might be tempted to evaluate the product $X\boldsymbol{u}$ (i.e. a 2×2 matrix multiplying a column vector), but we must immediately check whether or not the result is a spinor. It is not. Proof: $X \to \Lambda X \Lambda^\dagger$ and $\boldsymbol{u} \to \Lambda \boldsymbol{u}$ so $X\boldsymbol{u} \to \Lambda X \Lambda^\dagger \Lambda \boldsymbol{u}$ which is not equal to $\Lambda X \boldsymbol{u}$ nor does it correspond to any of the other transformations listed in table II.

A similar issue arises with 4-vectors and tensors, and it is handled by involving the metric tensor g. The index notation signals the presence of g by a lowered index; the matrix notation signals the presence of g by the dot notation for an inner product. When using index notation, a contraction is only a legal tensor operation if it involves a pair consisting of one upper and one lower index. For spinor manipulations, a similar notation is available, but we have a further complication: there are four kinds of basic spinor, not just two. This leads to 16 kinds of 2nd rank spinors, 64 kinds of 3rd rank spinors, and so on. Fortunately, just as with tensors, the higher rank spinors all transform in the same way as outer products of lower rank spinors, so the whole system can be 'tamed' by the use of index notation.

We show in table III all the possible types of 2nd rank spinor, in order to convey the essential idea. We introduce $\bullet$ and $\star$ symbols attached to a letter M to serve as a 'code' to show what type of spinor is represented by the matrix M. For example, consider the entry in the first row, second column: $M^{\bullet}{}_{\bullet} = \boldsymbol{u}(\epsilon \boldsymbol{v})^T = \boldsymbol{u}\boldsymbol{v}^T \epsilon^T$. When $\boldsymbol{u} \to \Lambda \boldsymbol{u}$ and $\boldsymbol{v} \to \Lambda \boldsymbol{v}$ we have

$$M^{\bullet}{}_{\bullet} \to \Lambda \boldsymbol{u}\boldsymbol{v}^T \Lambda^T \epsilon^T = \Lambda \boldsymbol{u}\boldsymbol{v}^T \epsilon^T \Lambda^{-1} = \Lambda M^{\bullet}{}_{\bullet} \Lambda^{-1}$$

where the second step used the complex conjugate of eq. (43), namely

$$\Lambda^T \epsilon^T \Lambda = \epsilon^T.$$

By similar arguments you can prove any other entry in the table. In practice one does not need all the different types of spinor, and we shall be mostly concerned with the types $M^{\bullet\star}$ and $M^{\bullet}{}_{\bullet}$. When we go over to index notation, the $\bullet$ will be replaced by a generic greek letter such as μ, and the $\star$ by a barred letter such as $\bar{\nu}$.

The important result of this analysis is that to ensure we only carry out legal spinor manipulations, it is sufficient to follow the rule that only indices of the same type can be summed over, and they must be one up, one down. That this is true in general, for spinors of all ranks, follows immediately from (43) and its complex conjugate ($\Lambda^\dagger \epsilon \Lambda^* = \epsilon$) as long as we arrange (as we have done) that the index lowering operation is achieved by premultiplying by ϵ or postmultiplying by ϵ^T, and the index raising operation is achieved by premultiplying by $\epsilon^{-1} = \epsilon^T$ or postmultiplying by ϵ. This applies to either type of index:

$$M_{\bullet\bullet} = \epsilon M^{\bullet}{}_{\bullet} = M_{\bullet}{}^{\bullet} \epsilon^T, \qquad M_{\star\bullet} = \epsilon M^{\star}{}_{\bullet},$$

etc. and

$$M^{\bullet\bullet} = \epsilon^T M_{\bullet}{}^{\bullet} = M^{\bullet}{}_{\bullet} \epsilon, \qquad M^{\star\bullet} = \epsilon^T M_{\star}{}^{\bullet},$$

etc. The proof that contraction can be applied to spinors of higher rank is simple: we *define* spinors of arbitrary rank to be entities that transform in the same way as outer products of rank 1 spinors.

We have seen how to raise and lower indices. One may also want to ask, can we convert a spinor with one type of index to a spinor with another type? For example, can we convert between $M^{\bullet\bullet}$ and $M^{\bullet\star}$? The answer is that it is not possible to do this. There is no simple relationship between these two types of spinor. However, it is possible to change the type of all the indices at once: the matrix $M^{\star\star}$, for example, is the complex conjugate of the matrix $M^{\bullet\bullet}$, and $M^{\bullet\star}$ is the complex conjugate of $M^{\star\bullet}$.

Although the Lorentz transformation Λ is not itself a spinor (it cannot be written in any given reference frame, it is a bridge between reference frames), it is convenient to write it in index notation as $\Lambda^{\mu}{}_{\nu}$. Then the transformation of the standard right-handed spinor can be

$M^{\bullet\bullet}$ $= \boldsymbol{u}\boldsymbol{v}^T$	$v^\bullet$ $\boldsymbol{v}$	$v_\bullet$ $(\epsilon\boldsymbol{v})$	$v^\star$ v^*	$v_\star$ $(\epsilon\boldsymbol{v}^*)$
$u^\bullet = \boldsymbol{u}$	$\Lambda M^{\bullet\bullet}\Lambda^T$	$\Lambda M^{\bullet}{}_{\bullet}\Lambda^{-1}$	$\Lambda M^{\bullet\star}\Lambda^\dagger$	$\Lambda M^{\bullet}{}_{\star}(\Lambda^*)^{-1}$
$u_\bullet = \epsilon\boldsymbol{u}$	$(\Lambda^T)^{-1}M_{\bullet}{}^{\bullet}\Lambda^T$	$(\Lambda^T)^{-1}M_{\bullet\bullet}\Lambda^{-1}$	$(\Lambda^T)^{-1}M_{\bullet}{}^{\star}\Lambda^\dagger$	$(\Lambda^T)^{-1}M_{\bullet\star}(\Lambda^*)^{-1}$
$u^\star = \boldsymbol{u}^*$	$\Lambda^* M^{\star\bullet}\Lambda^T$	$\Lambda^* M^{\star}{}_{\bullet}\Lambda^{-1}$	$\Lambda^* M^{\star\star}\Lambda^\dagger$	$\Lambda^* M^{\star}{}_{\star}(\Lambda^*)^{-1}$
$u_\star = \epsilon\boldsymbol{u}^*$	$(\Lambda^\dagger)^{-1}M_{\star}{}^{\bullet}\Lambda^T$	$(\Lambda^\dagger)^{-1}M_{\star\bullet}\Lambda^{-1}$	$(\Lambda^\dagger)^{-1}M_{\star}{}^{\star}\Lambda^\dagger$	$(\Lambda^\dagger)^{-1}M_{\star\star}(\Lambda^*)^{-1}$

TABLE III: Transformation rules for 2nd rank spinors. The first row and column show the four types of rank-1 spinor. In the table, the M symbols are 2nd rank spinors formed from the outer product of the rank-1 spinor of each row and column. For example, $M^{\bullet}{}_{\bullet} = \boldsymbol{u}(\epsilon\boldsymbol{v})^T$. The dots and stars attached to M symbols serve as generic indices of one of two types. The entries in the table show how each M transforms under a change of reference frame (see text). The table shows, for example, that a matrix product such as $X^{\bullet}{}_{\bullet}Y^{\bullet\star}$ is legal because the transformation carries it to $\Lambda X^{\bullet}{}_{\bullet}\Lambda^{-1}\Lambda Y^{\bullet\star}\Lambda^\dagger = \Lambda X^{\bullet}{}_{\bullet}Y^{\bullet\star}\Lambda^\dagger$ and furthermore the object that results is one which transforms as $W^{\bullet\star}$. Thus legal operations and the class of the result are easily identified by paying attention to the placement and type of index. A spinor of type $M^{\bullet\star}$ would be written in index notation as $M^{\mu\bar{\nu}}$.

written $u'^\mu = \Lambda^\mu{}_\alpha u^\alpha$. The standard left-handed spinor would be written $v_{\bar{\mu}}$ so its transformation rule should be $v'_{\bar{\mu}} = \Lambda_{\bar{\mu}}{}^{\bar{\alpha}} v_{\bar{\alpha}}$. The relationship between these two Lorentz transformations is

$$\Lambda_{\bar{\mu}}{}^{\bar{\nu}} = (\epsilon_{\mu\alpha}\Lambda^\alpha{}_\beta\epsilon^{\beta\nu})^*. \tag{69}$$

This is eq (68), so everything is consistent (the overall complex conjugation on the right hand side causes the indices to change from unbarred to barred).

D. Invariants

The most basic spinor invariant is

$$u^\mu u_\mu = 0. \qquad [= \boldsymbol{u}^T\epsilon\boldsymbol{u}$$

That is, the 'length' of a spinor, as indicated by this type of scalar product, is zero. This is consistent with the fact that the flagpole of a spinor is a null 4-vector. To prove the result you can use $u^\mu u_\mu = \boldsymbol{u}^T\epsilon\boldsymbol{u} = u^1u^2 - u^2u^1 = 0$ or use the general property that the scalar product of spinors is anticommutative:

$$\begin{aligned} u^\mu v_\mu = u^\mu\epsilon_{\mu\alpha}v^\alpha &= \epsilon_{\mu\alpha}u^\mu v^\alpha \\ &= -\epsilon_{\alpha\mu}u^\mu v^\alpha = -u_\alpha v^\alpha. \end{aligned}$$

Note that this shows we have a 'see-saw rule' as long as a minus sign is introduced whenever a see-saw is performed. The minus sign comes from the fact that the metric spinor $\epsilon_{\mu\nu}$ is antisymmetric (it is a Levi-Civita symbol). Setting $v^\mu = u^\mu$ we find

$$u^\mu u_\mu = -u^\mu u_\mu$$

and therefore $u^\mu u_\mu = 0$ as before.

We should expect the scalar invariant $u^\mu v_\mu$ to be something to do with the scalar product of the associated 4-vectors, and you can confirm using (45) that

$$|u^\mu v_\mu|^2 = -\frac{1}{2}\mathsf{U}\cdot\mathsf{V}. \qquad [= |\boldsymbol{u}^T\epsilon\boldsymbol{v}|^2 \tag{70}$$

A Hermitian matrix formed from a 4-vector as in (38) is of type $X^{\mu\bar{\nu}}$. Therefore the trace is not a Lorentz scalar (that is, we can't set the two indices equal and sum). To obtain a Lorentz scalar we can use $X^{\alpha\bar{\beta}}X_{\alpha\bar{\beta}}$. More generally, for a pair of such spinors, you are invited to verify that

$$X^{\alpha\bar{\beta}}Y_{\alpha\bar{\beta}} = -2\mathsf{X}\cdot\mathsf{Y}. \tag{71}$$

This result makes it easy to convert some familiar tensor results into spinor notation. For example, the continuity equation is

$$\partial^{\alpha\bar{\beta}}J_{\alpha\bar{\beta}} = 0 \tag{72}$$

where

$$\partial^{\mu\bar{\nu}} = \sum_\lambda \partial^\lambda\sigma^\lambda = \begin{pmatrix} -\frac{\partial}{c\partial t} + \frac{\partial}{\partial z}, & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, & -\frac{\partial}{c\partial t} - \frac{\partial}{\partial z} \end{pmatrix} \tag{73}$$

and

$$J^{\mu\bar{\nu}} = \begin{pmatrix} \rho c + j_z & j_x - ij_y \\ j_x + ij_y & \rho c - j_z \end{pmatrix}. \tag{74}$$

Using (71) again, the D'Alembertian can be written

$$\Box^2 = -\frac{1}{2}\partial^{\alpha\bar{\beta}}\partial_{\alpha\bar{\beta}}.$$

You wouldn't ever want to write it like that, of course, since it is a scalar so you may as well just write $\Box^2$ and convert it to $-(\partial/c\partial t)^2 + (\partial/\partial x)^2 + (\partial/\partial y)^2 + (\partial/\partial z)^2$ when needed.

V. APPLICATIONS

We illustrate the application of spinors to something other than spin by writing down Maxwell's equations in spinor notation. This is merely to demonstrate that it can be done. We won't pursue whether or not much can be learned from this, it is just to demonstrate that spinors are a flexible tool.

To this end, introduce the quantity $\mathbf{F} = \mathbf{E} - ic\mathbf{B}$ where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. Form the mixed 2nd rank spinor

$$F_{\bar{\mu}}{}^{\bar{\nu}} = \begin{pmatrix} F_z & F_x - iF_y \\ F_x + iF_y & -F_z \end{pmatrix}, \tag{75}$$

then Maxwell's equations can be written

$$\boxed{\partial^{\mu\bar{\alpha}} F_{\bar{\alpha}}{}^{\bar{\nu}} = c\mu_0 J^{\mu\bar{\nu}}.} \tag{76}$$

For example, the $\mu = 1, \nu = 1$ term on the left hand side is

$$-\frac{\partial F_z}{c\partial t} + \frac{\partial F_z}{\partial z} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + i\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = \nabla \cdot \mathbf{F} - \left(\frac{\partial \mathbf{F}}{c\partial t} - i\nabla \wedge \mathbf{F}\right)_z.$$

The real and imaginary parts of this are

$$\nabla \cdot \mathbf{E} + c(\nabla \wedge \mathbf{B})_z - \frac{\partial E_z}{\partial t} \quad \text{and} \quad -c\nabla \cdot \mathbf{B} + (\nabla \wedge \mathbf{E})_z + \frac{\partial B_z}{\partial t}.$$

Eq (76) says the first of these is equal to $c\mu_0(\rho c + j_z)$, and the second is equal to zero (since the $1,1$ term of the right hand side is real). By evaluating the $2,2$ term you can find similarly that

$$\begin{aligned} \nabla \cdot \mathbf{E} - c(\nabla \wedge \mathbf{B})_z + \frac{\partial E_z}{\partial t} &= c\mu_0(\rho c - j_z) \\ \text{and} \quad -c\nabla \cdot \mathbf{B} - (\nabla \wedge \mathbf{E})_z - \frac{\partial B_z}{\partial t} &= 0. \end{aligned} \tag{77}$$

By taking sums of these simultaneous equations we find $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. By taking differences we find the z-component of the other two Maxwell equations. You can check that the $1,2$ and $2,1$ terms of the spinor equation yield the x and y components of the remaining Maxwell equations.

We now have a spinor-based method to obtain the transformation law for electric and magnetic fields: just transform $F_{\bar{\mu}}{}^{\bar{\nu}}$. The result is exactly the same as one may obtain by using tensor analysis to transform the Faraday tensor. It follows that any antisymmetric second rank tensor can similarly be 'packaged into' a 2nd rank spinor whose indices are both of the same type.

A spinor for the 4-vector potential can also be introduced, and it is easy to write the Lorenz gauge condition and wave equations, etc. The Lorentz force equation is slightly more awkward—see exercises. Some solutions of Maxwell's equations can be found relatively easily using spinors. An example is the radiation field due to an accelerating charge: something that requires a long calculation using tensor methods.

VI. DIRAC SPINOR AND PARTICLE PHYSICS

We already mentioned in section (III A) that a pair of spinors can be used to represent a pair of mutually orthogonal 4-vectors. A good way to do this is to use a pair of spinors of opposite chirality, because then it is possible to construct equations possessing invariance under parity inversion. Such a pair is called a *bispinor* or *Dirac spinor*. It can conveniently be written as a 4-component complex vector, in the form

$$\Psi = \begin{pmatrix} \phi_R \\ \chi_L \end{pmatrix} \tag{78}$$

where it is understood that each entry is a 2-component spinor, ϕ_R being right-handed and χ_L left-handed. (Following standard practice in particle physics, we won't adopt index notation for the spinors here, so the subscript L and R is introduced to keep track of the chirality). Under change of reference frame Ψ transforms as

$$\Psi \to \begin{pmatrix} \Lambda(v) & 0 \\ 0 & \Lambda(-v) \end{pmatrix} \Psi \tag{79}$$

where each entry is understood to represent a 2×2 matrix, and we wrote $\Lambda(v)$ for $\exp(i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}/2 - \boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2)$ and $\Lambda(-v)$ for $(\Lambda(v)^\dagger)^{-1} = \exp(i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}/2 + \boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2)$. It is easy to see that the combination

$$\begin{pmatrix} \phi_R^\dagger, \chi_L^\dagger \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \phi_R \\ \chi_L \end{pmatrix} = \phi_R^\dagger \chi_L + \chi_L^\dagger \phi_R \tag{80}$$

is Lorentz-invariant.

We will show how Ψ can be used to represent the 4-momentum and 4-spin (Pauli-Lubanski 4-vector) of a particle. First extract the 4-vectors given by the flagpoles of ϕ_R and χ_L:

$$\mathsf{A}^\mu = \langle \phi_R | \sigma^\mu | \phi_R \rangle, \qquad \mathsf{B}_\mu = \langle \chi_L | \sigma^\mu | \chi_L \rangle. \tag{81}$$

Note that B_μ has a lower index. This is because, in view of the fact that χ_L is left-handed (i.e. negative chirality), under a Lorentz transformation its flagpole behaves as a covariant 4-vector. We would like to form the difference of these 4-vectors, so we need to convert the second to contravariant form. This is done via the metric tensor $g_{\mu\nu}$:

$$\mathsf{U}^\mu = (\mathsf{A}^\mu - \mathsf{B}^\mu) = \langle \phi_R | \sigma^\mu | \phi_R \rangle - g^{\mu\alpha} \langle \chi_L | \sigma^\alpha | \chi_L \rangle.$$

(The notation is consistent if you keep in mind that the $_L$'s have lowered the index on σ in the second term). In terms of the Dirac spinor Ψ, this result can be written as

$$\mathsf{U}^0 = \Psi^\dagger \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \Psi = \phi_R^\dagger \phi_R + \chi_L^\dagger \chi_L \tag{82}$$

for the time component, and

$$\mathsf{U}^i = \Psi^\dagger \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \Psi \tag{83}$$

for the spatial components.

Now introduce the 4×4 matrices, called **Dirac matrices**:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \;\; \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{84}$$

Here we are writing these matrices in the 'chiral' basis implied by the form (78); see exercise 13 for another representation. Using these, we can write (82) and (83) both together, as

$$\mathsf{U}^\mu = \Psi^\dagger \gamma^0 \gamma^\mu \Psi. \tag{85}$$

We now have a 4-vector extracted from our Dirac spinor. It can be of any type—it need not be null. If in some particular reference frame it happens that $\phi_R = \pm\chi_L$ (this equation is not Lorentz covariant so cannot be true in all reference frames, but it can be true in one), then from eqn (81) we learn that in this reference frame the components of A^μ are equal to the components of B_μ so the spatial part of U is zero, while the time part is not. Such a 4-vector is proportional to a particle's 4-velocity in its rest frame. In other words, the Dirac spinor can be used to describe the 4-velocity of a massive particle, and in this application it must have either $\phi_R = \chi_L$ or $\phi_R = -\chi_L$ in the rest frame. In other frames the 4-velocity can be extracted using (85).

Next consider the sum of the two flagpole 4-vectors. Let

$$\mathsf{W} = mcS\,(\mathsf{A} + \mathsf{B}) \tag{86}$$

where S is the size of the intrinsic angular momentum of the particle, and introduce

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{87}$$

By using

$$\Sigma^\mu \;=\; \gamma^0\gamma^\mu\gamma^5 = \left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \right), \tag{88}$$

we can write

$$\mathsf{W}^\mu = mcS\Psi^\dagger \Sigma^\mu \Psi. \tag{89}$$

This 4-vector is orthogonal to U. It can therefore be the 4-spin, if we choose ϕ_R and χ_L appropriately. What is needed is that the spinor ϕ_R be aligned with the direction of the spin angular momentum in the rest frame. We already imposed the condition that either $\phi_R = \chi_L$ or $\phi_R = -\chi_L$ in the rest frame, so it follows that either both spinors are aligned with the spin angular momentum in the rest frame, or one is aligned and the other opposed.

We now have a complete representation of the 4-velocity and 4-spin of a particle, using a single Dirac spinor. A spinor equal to $(1, 0, 1, 0)/\sqrt{2}$ in the rest frame, for example, represents a particle with spin directed along the z direction. A spinor $(0, 1, 0, 1)/\sqrt{2}$ represents a particle with spin in the $-z$ direction. More generally, $(\phi, \phi)/\sqrt{2}$ is a particle at rest with spin vector $\phi^\dagger \boldsymbol{\sigma} \phi$. Table IV gives some examples.

The states having $\phi_R = \chi_L$ in the rest frame cover half the available state space; the other half is covered by $\phi_R = -\chi_L$ in the rest frame. The spinor formalism is here again implying that there may exist in Nature two types of particle, similar in some respects (such as having the same mass), but not the same. In quantum field theory, it emerges that if the first set of states are particle states, then the other set can be interpreted as antiparticle states. This interpretation only emerges fully once we look at physical *processes*, not just states, and for that we need equations describing interactions and the evolution of the states as a function of time—the equations of quantum electrodynamics, for example. However, we can already note that the spinor formalism is offering a natural language to describe a universe which can contain both matter and anti-matter. The structure of the mathematics matches the structure of the physics in a remarkable, elegant way. This inspires wonder and a profound sense that there is more to the universe than a merely adequate collection of ad-hoc rules.

As long as the spin and velocity are not exactly orthogonal, in the high-velocity limit it is found that one of the two spinor components dominates. For example, if we start from $\phi_R = \chi_L = (1, 0)$ in the rest frame, then transform to a reference frame moving in the positive z direction, then ϕ_R will shrink and χ_L will grow until in the limit $v \to c$, $\phi_R \to 0$. This implies that a massless particle can be described by a single (two-component) spinor, and we recover the description we saw in section IV in connection with the Weyl equations. In particular, we find that a Weyl spinor has helicity of the same sign as its chirality. (The example we just considered had negative helicity because the spin is along z but the particle's velocity is in the negative z direction in the new frame.)

A parity inversion ought to change the direction in space of U (since its spatial part is a polar vector) but leave the direction in space of W unaffected (since its spatial part is an axial vector). You can verify that this is satisfied if the parity inversion is represented by the matrix

$$P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{90}$$

The effect of P acting on Ψ is to swap the two parts, $\phi_R \leftrightarrow \chi_L$. You can now verify that the Lorentz invariant given in (80) is also invariant under parity so it is a true scalar. The quantity $\Psi^\dagger\gamma^0\gamma^5\Psi = \phi_R^\dagger \chi_L - \chi_L^\dagger \phi_R$

Ψ	U	$2\mathsf{W}/(mcS)$	
$(1,0,1,0)/\sqrt{2}$	$(1,0,0,0)$	$(0,0,0,1)$	at rest, spin up
$(0,1,0,1)/\sqrt{2}$	$(1,0,0,0)$	$(0,0,0,-1)$	at rest, spin down
$(1,1,1,1)/2$	$(1,0,0,0)$	$(0,1,0,0)$	at rest, spin along $+x$
$(1,-1,1,-1)/2$	$(1,0,0,0)$	$(0,-1,0,0)$	at rest, spin along $-x$
$(1,0,0,0)$	$(1,0,0,1)$	$(1,0,0,1)$	$v_z = c$, +ve helicity
$(0,1,0,0)$	$(1,0,0,-1)$	$(1,0,0,-1)$	$v_z = -c$, +ve helicity
$(0,0,1,0)$	$(1,0,0,-1)$	$(-1,0,0,1)$	$v_z = -c$, −ve helicity
$(0,0,0,1)$	$(1,0,0,1)$	$(-1,0,0,-1)$	$v_z = c$, −ve helicity

TABLE IV: Some example Dirac spinors and their associated 4-vectors.

$\Psi^\dagger\gamma^0\Psi$	scalar
$\Psi^\dagger\gamma^0\gamma^5\Psi$	pseudoscalar
$\Psi^\dagger\gamma^0\gamma^\mu\Psi$	4-vector U, difference of flagpoles
$\Psi^\dagger\gamma^0\gamma^\mu\gamma^5\Psi$	axial 4-vector W, sum of flagpoles
$\Psi^\dagger\gamma^0(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\Psi$	antisymmetric tensor

TABLE V: Various tensor quantities associated with a Dirac spinor. The notation $\overline{\Psi} = \Psi^\dagger\gamma^0$ (called *Dirac adjoint*) can also be introduced, which allows the expressions to be written $\overline{\Psi}\Psi$, $\overline{\Psi}\gamma^5\Psi$, and so on.

is invariant under Lorentz transformations but changes sign under parity, so is a pseudoscalar. The results are summarised in table V.

A. Moving particles and classical Dirac equation

So far we have established that a Dirac spinor representing a massive particle possessing intrinsic angular momentum ought to have $\phi_R = \chi_L$ in the rest frame, with both spinors aligned with the intrinsic angular momentum. We have further established that under Lorentz transformations the Dirac spinor will continue to yield the correct 4-velocity and 4-spin using eqs (85) and (89).

Now we shall investigate the general form of a Dirac spinor describing a moving particle. All we need to do is apply a Lorentz boost. We assume the Dirac spinor $\Psi = (\phi_R(v), \chi_L(v))$ takes the form $\phi_R(0) = \chi_L(0)$ in the rest frame, and that it transforms as (79). Using (41) the Lorentz boost for a Dirac spinor can be written

$$\Lambda = \cosh\Big(\frac{\rho}{2}\Big)\begin{pmatrix} I - \mathbf{n}\cdot\boldsymbol{\sigma}\tanh(\rho/2) & 0 \\ 0 & I + \mathbf{n}\cdot\boldsymbol{\sigma}\tanh(\rho/2)\end{pmatrix}.$$

Now (in units where $c=1$) $\cosh\rho = \gamma = E/m$ where E is the energy of the particle, and $\cosh\rho = 2\cosh^2(\rho/2) - 1 = 2\sinh^2(\rho/2)+1$ so

$$\begin{aligned} \cosh(\rho/2) &= \left(\frac{E+m}{2m}\right)^{1/2}, \\ \sinh(\rho/2) &= \left(\frac{E-m}{2m}\right)^{1/2}, \\ \tanh(\rho/2) &= \left(\frac{E-m}{E+m}\right)^{1/2} = \frac{p}{E+m}. \end{aligned} \tag{91}$$

Therefore we can express the Lorentz boost in terms of energy and momentum (of a particle boosted from its rest frame):

$$\Lambda = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} I + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} & 0 \\ 0 & I - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\end{pmatrix} \tag{92}$$

where the sign is set such that $\mathbf{p}$ is the momentum of the particle in the new frame.

For example, consider the spinor $\Psi_0 = (1,0,1,0)/\sqrt{2}$, i.e. spin up along z in the rest frame. Then in any other frame,

$$\Psi = \frac{1}{\sqrt{4m(E+m)}}\begin{pmatrix} E+m+p_z \\ p_x + ip_y \\ E+m-p_z \\ -p_x - ip_y \end{pmatrix} \tag{93}$$

(with $c=1$). Suppose the boost is along the x direction. Then, as p_x grows larger, $p_x \to E$, so the positive chirality part has a spinor more and more aligned with $+x$, and the negative chirality part has a spinor more and more aligned with $-x$. For a boost along z, one of the chirality components vanishes in the limit $|v| \to c$. This is the behaviour we previously discussed in relation to table IV.

Next we shall present the result of a Lorentz boost another way. We will construct a matrix equation satisfied by Ψ that has the same form as the Dirac equation of particle physics. Historically, Dirac obtained his equation via a quantum mechanical argument. However, the classical version can prepare us for the quantum version, and help in the interpretation of the solutions.

We already noted that the Lorentz boost takes the form

$$\begin{aligned} \phi_R(\boldsymbol{v}) &= (I\cosh(\rho/2) - \boldsymbol{\sigma}\cdot\mathbf{n}\sinh(\rho/2))\,\phi_R(0), \\ \chi_L(\boldsymbol{v}) &= (I\cosh(\rho/2) + \boldsymbol{\sigma}\cdot\mathbf{n}\sinh(\rho/2))\,\chi_L(0). \end{aligned}$$

Using eq. (91), and multiplying top and bottom by $(E+m)^{1/2}$, we find

$$\begin{aligned} \phi_R(\mathbf{p}) &= \frac{E+m+\boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(E+m)]^{1/2}}\phi_R(0), \\ \chi_L(\mathbf{p}) &= \frac{E+m-\boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(E+m)]^{1/2}}\chi_L(0). \end{aligned}$$

where $\mathbf{p} = -\gamma m\mathbf{v}$ is the momentum of the particle in the new frame. (The same result also follows immediately from eqn (92).) Introducing the assumption $\phi_R(0) = \chi_L(0)$ we obtain from these two equations, after some algebra[16],

$$\begin{aligned}(E - \boldsymbol{\sigma}\cdot\mathbf{p})\phi_R(\mathbf{p}) &= m\chi_L(\mathbf{p}),\\ (E + \boldsymbol{\sigma}\cdot\mathbf{p})\chi_L(\mathbf{p}) &= m\phi_R(\mathbf{p}).\end{aligned}$$

The left hand sides of these equations are the same as in the Weyl equations; the right hand sides have the requisite chirality. As a set, this coupled pair of equations is parity-invariant, since under a parity inversion the sign of $\boldsymbol{\sigma}\cdot\mathbf{p}$ changes and χ and ϕ swap over. In matrix form the equations can be written

$$\begin{pmatrix} E - \boldsymbol{\sigma}\cdot\mathbf{p} & -m \\ -m & E + \boldsymbol{\sigma}\cdot\mathbf{p}\end{pmatrix}\begin{pmatrix}\phi_R(\mathbf{p})\\ \chi_L(\mathbf{p})\end{pmatrix} = 0. \tag{94}$$

This equation is very closely related to the Dirac equation. One may even go so far as to say that (94) "is" the Dirac equation in free space, if one re-interprets the terms, letting E and p be the frequency and wave-vector of a plane wave, up to factors of $\hbar$. In the present context the equation represents a constraint that must be satisfied by any Dirac spinor that represents 4-momentum and intrinsic spin of a given particle.

If we premultiply (94) by γ^0 then we have the form

$$\begin{pmatrix} -m & E + \boldsymbol{\sigma}\cdot\mathbf{p} \\ E - \boldsymbol{\sigma}\cdot\mathbf{p} & -m\end{pmatrix}\begin{pmatrix}\phi_R(\mathbf{p})\\ \chi_L(\mathbf{p})\end{pmatrix} = 0. \tag{95}$$

This can conveniently be written

$$(-\gamma^\lambda P_\lambda - m)\Psi = 0 \tag{96}$$

(see also exercise 12.)

Our discussion has been entirely classical (in the sense of not quantum-mechanical). In quantum field theory the spinor plays a central role. One has a spinor field, the excitations of which are what we call spin 1/2 particles. The results of this section reemerge in the quantum context, unchanged for energy and momentum eigenstates, and in the form of mean values or 'expectation values' for other states.

B. The standard representation

In order to present Dirac spinors, we found it helpful to write them in component form as in eqn (78). This amounts to choosing a basis. The choice we made is called the *chiral basis*. It is the most natural choice in which to discuss chirality, and it gives the convenient fact that in this basis the Lorentz transformation matrix ((79) and (92)) is block-diagonal. In the application to particle physics, especially in the case of slow-moving particles, another basis is convenient. This is called the 'standard representation' or 'Dirac representation', whose basis vectors are related to the chiral basis by the transformation

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I\end{pmatrix}.$$

For example, in the standard representation, Ψ as given in eqn (78) would be written

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_R + \chi_L \\ \phi_R - \chi_L\end{pmatrix}.$$

For a particle with spin described by a spinor ψ in the rest frame, we have $\phi_R = \chi_L = \psi/\sqrt{2}$ in the rest frame, and therefore in other frames,

$$\begin{aligned}\phi_R &= (E + m + \boldsymbol{\sigma}\cdot\mathbf{p})\psi(4m(E+m))^{-1/2},\\ \chi_L &= (E + m - \boldsymbol{\sigma}\cdot\mathbf{p})\psi(4m(E+m))^{-1/2},\end{aligned}$$

using eqn (92). Thus, when expressed in the standard representation, the resulting Dirac spinor is

$$\Psi = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}(E+m)\psi \\ \mathbf{p}\cdot\boldsymbol{\sigma}\psi\end{pmatrix}. \tag{97}$$

For low velocities, $v \ll c$, we have $E = m + O(v^2)$ and hence, to first order in v,

$$\Psi \simeq \begin{pmatrix}\psi \\ \mathbf{v}\cdot\boldsymbol{\sigma}\psi/2\end{pmatrix}. \tag{98}$$

C. Electromagnetic interactions and $g = 2$

Introduce the matrices $\beta \equiv \gamma^0$ and $\alpha^i \equiv \gamma^0\gamma^i$. A useful way to write the Dirac equation (95) is (exercise 12)

$$H\Psi = E\Psi \tag{99}$$

where $H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m$. Eqn (99) suggests that we should regard H as a Hamiltonian. The standard way to treat the motion of a charged particle in an electromagnetic field in special relativity is to add a potential energy term $q\phi$ to the Hamiltonian, and replace $\mathbf{p}$ in the Hamiltonian by $\tilde{\mathbf{p}} - q\mathbf{A}$ where $\mathbf{A}$ is the vector potential and $\tilde{\mathbf{p}}$ is the canonical momentum [1]. It is standard practice, in quantum mechanics and particle physics, to use the symbol $\mathbf{p}$ for canonical momentum, so we shall adopt that notation, and use the symbol $\mathbf{p}_{\rm k}$ for the kinetic momentum, such that

$$\mathbf{p}_{\rm k} = \mathbf{p} - q\mathbf{A} = \gamma m\mathbf{v}.$$

We thus obtain the Hamiltonian

$$H = \boldsymbol{\alpha}\cdot(\mathbf{p} - q\mathbf{A}) + \beta m + q\phi. \tag{100}$$

In the standard representation, eqn (99) now reads

$$\begin{pmatrix} E - V - m & -\boldsymbol{\sigma}\cdot(\mathbf{p} - q\mathbf{A}) \\ -\boldsymbol{\sigma}\cdot(\mathbf{p} - q\mathbf{A}) & E - V + m\end{pmatrix}\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = 0 \tag{101}$$

where we introduced $\psi_\pm \equiv (\phi_R \pm \chi_L)/\sqrt{2}$ and the potential energy $V = q\phi$.

Let us treat motion in a pure magnetic field, so $V = 0$. The second row of (101) tells us that

$$\psi_- = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_\mathrm{k}}{E+m}\psi_+ \simeq \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_\mathrm{k}}{2m}\psi_+$$

where the second version is valid at low speeds (c.f. eqn (98)). Since $p \simeq mv$ at $v \ll 1$, we then have that $|\phi_-| \simeq (v/2)|\psi_+|$. For this reason ψ_+ and ψ_- are called the 'large' and 'small' components in this context. The first row of (101) gives

$$(E-m)\psi_+ = \boldsymbol{\sigma}\cdot\mathbf{p}_\mathrm{k}\psi_- \simeq \frac{(\boldsymbol{\sigma}\cdot\mathbf{p}_\mathrm{k})(\boldsymbol{\sigma}\cdot\mathbf{p}_\mathrm{k})}{2m}\psi_+. \quad (102)$$

Now using the identity

$$(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\boldsymbol{\sigma}\cdot\mathbf{a}\wedge\mathbf{b} \quad (103)$$

we have

$$(\boldsymbol{\sigma}\cdot\mathbf{p}_\mathrm{k})(\boldsymbol{\sigma}\cdot\mathbf{p}_\mathrm{k}) = p_\mathrm{k}^2 + i\boldsymbol{\sigma}\cdot(\mathbf{p}-q\mathbf{A})\wedge(\mathbf{p}-q\mathbf{A}). \quad (104)$$

In classical physics, the second term here would be zero, but in quantum physics it is not. Although our whole presentation has been classical up till now, we will now make a small foray into quantum mechanics. We regard $\mathbf{p}$ has an operator, which in the position representation is expressed $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$, and now the spinor ψ_+ has to be thought of as a function of position—it is a wavefunction with two components. We have

$$\begin{aligned}[(\mathbf{p}-q\mathbf{A}) \;\wedge\; (\mathbf{p}-q\mathbf{A})]\psi_+ & \\ &= -i\hbar q\left(\boldsymbol{\nabla}\wedge(\mathbf{A}\psi_+) + \mathbf{A}\wedge(\boldsymbol{\nabla}\psi_+)\right) \\ &= -i\hbar q(\boldsymbol{\nabla}\wedge\mathbf{A})\psi_+ \quad (105)\end{aligned}$$

so eqn (102) reads

$$(E-m)\psi_+ = \left(\frac{(\mathbf{p}-q\mathbf{A})^2}{2m} + \frac{\hbar q}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}\right)\psi_+. \quad (106)$$

This is the time-independent Schrödinger equation for a particle interacting with a magnetic field, such that the operator $(-\hbar q/2m)\boldsymbol{\sigma}$ represents the magnetic dipole moment of the particle. One finds that the angular momentum of the particle is represented by the operator $\boldsymbol{\sigma}\hbar/2$ (exercise 14), so the gyromagnetic ratio is $g = 2$. Thus the value of the g-factor of a spin half particle described by the Dirac equation is not an independent variable but is constrained to take the value 2.

VII. SPIN MATRIX ALGEBRA (LIE ALGEBRA)*

We introduced the Pauli spin matrices abruptly at the start of section II, by giving a set of matrices and their commutation relations. By now the reader has some idea of their usefulness.

In group theory, these matrices are called the *generators* of the group SU(2), because any group member can be expressed in terms of them in the form $\exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2)$. More precisely, the Pauli matrices are the generators of one *representation* of the group SU(2), namely the representation in terms of 2×2 complex matrices. Other representations are possible, such that there is an isomorphism between one representation and another. Each representation will have generators in a form suitable for that representation. They could be matrices of larger size, for example, or even differential operators. In every representation, however, the generators will have the same behaviour when combined with one another, and this behaviour reveals the nature of the group. This means that the innocent-looking commutation relations (9) contain much more information than one might have supposed: they are a 'key' that, through the use of $\exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2)$, unlocks the complete mathematical behaviour of the group. In Lie group theory these equations describing the generators are called the 'Clifford algebra' or 'Lie algebra' of the group.

If a Lie group does not have a matrix representation it can be hard or impossible to give a meaningful definition to $\exp(M)$ where M is a member of the group. In this case one uses the form $I + \epsilon M$ to write a group member infinitesimally close to the identity for $\epsilon \to 0$. The generators are a subgroup such that any member close to I can be written $I + \epsilon G$ where G is in the generator group. This is the more general definition of what is meant by the generators.

The generators of rotations in three dimensions (30) have the commutation relations

$$[J_x,\, J_y] = iJ_z \quad \text{and cyclic permutations.} \quad (107)$$

By comparing with (9) one can immediately deduce the relationship between SU(2) and SO(3), including the angle doubling!

The restricted Lorentz group has generators K_i (for boosts) and J_i (for rotations). A matrix representation (suitable for a rectangular coordinate system) is

$$J_x = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix}, \quad K_x = \begin{pmatrix} 0&i&0&0\\ i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad (108)$$

$$J_y = \begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0&0&i&0\\ 0&0&0&0\\ i&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad (109)$$

$$J_z = \begin{pmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&i&0&0\\ 0&0&0&0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0&0&0&i\\ 0&0&0&0\\ 0&0&0&0\\ i&0&0&0 \end{pmatrix}. \quad (110)$$

The commutation relations are (with cyclic permutations)

$$
\begin{aligned}
[J_x,\, J_y] &= iJ_z \\
[K_x,\, K_y] &= -iJ_z \\
[J_x,\, K_x] &= 0 \\
[J_x,\, K_y] &= iK_z \\
[J_x,\, K_z] &= iK_y.
\end{aligned}
$$

The second result shows that the Lorentz boosts on their own do not form a closed group, and that two boosts can produce a rotation: this is the Thomas precession. If we now form the combinations

$$\mathbf{A} = (\mathbf{J}{+}i\mathbf{K})/2, \qquad \mathbf{B} = (\mathbf{J}{-}i\mathbf{K})/2,$$

then the commutation relations become

$$
\begin{aligned}
[A_x,\, A_y] &= iA_z \\
[B_x,\, B_y] &= iB_z \\
[A_i,\, B_j] &= 0, \quad (i, j = x, y, z).
\end{aligned}
$$

This shows that the Lorentz group can be divided into two groups, both SU(2), and the two groups commute. This is another way to deduce the existence of two types of Weyl spinor and thus to define chirality.

The Clifford algebra satisfied by the Dirac matrices γ^0, γ^1, γ^2, γ^3 is

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} I \tag{111}$$

where $\eta^{\mu\nu}$ is the Minkowski metric and I is the unit matrix. That is, γ^0 squares to I and γ^i $(i = 1, 2, 3)$ each square to $-I$, and they all anticommute among themselves. These anti-commutation relations are normally taken to be the *defining property* of the Dirac matrices. A set of quantities γ^μ satisfying such anti-commutation relations can be represented using 4×4 matrices in more than one way—c.f. exercise 13. If the metric of signature $(1, -1, -1, -1)$ is used, then the minus sign on the right hand side of (111) becomes a plus sign.

A. Dirac spinors from group theory*

We conclude with a demonstration of how to establish the main properties of Dirac spinors by using group theory.

First let's take a fresh perspective on the six generators of the Lorentz group in 4-dimensional spacetime, i.e. the J_i and K_i matrices defined in eqs. (108)–(110). There are six entities, three of which are used to form a polar vector, and three an axial vector. You guessed it: we can gather them together into antisymmetric tensor $\mathcal{M}^{\mu\nu}$. This is a tensor of matrices, or, to be more general, of objects that behave like matrices having given commutation relations. You can check that the J_i and K_i matrices can all be written

$$(\mathcal{M}^{ab})^c{}_d = \eta^{bc}\delta^a_d - \eta^{ac}\delta^b_d$$

where by picking the 6 combinations $(a, b) = (0, 1),\ (0, 2),\ (0, 3),\ ,(1, 2),\ (1, 3),\ (2, 3)$ the expression gives the 6 matrices. For example, $\mathcal{M}^{01}$ is $-iK_x$, $\mathcal{M}^{12}$ is $-iJ_z$, etc. We can now write any Lorenz transformation as

$$\Lambda = \exp\left(\tfrac{1}{2}\theta_{\mu\nu}\mathcal{M}^{\mu\nu}\right)$$

where, don't forget, each $\mathcal{M}^{ab}$ is a 4×4 matrix. θ_{ab} provides 6 numbers telling which transformation we want.

These generators of the Lorentz group obey the following Clifford algebra, which is called the *Lorentz Lie algebra*:

$$[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = \eta^{\mu\rho}\mathcal{M}^{\nu\sigma} - \eta^{\nu\rho}\mathcal{M}^{\mu\sigma} + \eta^{\nu\sigma}\mathcal{M}^{\mu\rho} - \eta^{\mu\sigma}\mathcal{M}^{\nu\rho}. \tag{112}$$

Now we can connect the Clifford algebra to the Lorentz group. Let

$$\mathbb{S}^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\ \gamma^\nu] = \frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu,\ \gamma^\mu),$$

then *the matrices* $S^{\mu\nu}$ *form a representation of the Lorentz algebra*:

$$[\mathbb{S}^{\mu\nu}, \mathbb{S}^{\rho\sigma}] = \eta^{\nu\rho}\mathbb{S}^{\mu\sigma} - \eta^{\mu\rho}\mathbb{S}^{\nu\sigma} + \eta^{\mu\sigma}\mathbb{S}^{\nu\rho} - \eta^{\nu\sigma}\mathbb{S}^{\mu\rho}.$$

Proof: exercise for the reader! You may find it helpful first to obtain

$$
\begin{aligned}
\mathbb{S}^{ab} &= \tfrac{1}{2}(\gamma^a\gamma^b + \eta^{ab}), \\
\left[\mathbb{S}^{ab}, \gamma^c\right] &= \gamma^b\eta^{ca} - \gamma^a\eta^{bc}.
\end{aligned}
$$

It follows that if we introduce an object ψ that can be acted upon by the matrices $\mathbb{S}$, then we shall be able to Lorentz-transform it using

$$\psi \to \exp\left(\tfrac{1}{2}\theta_{\mu\nu}\mathbb{S}^{\mu\nu}\right)\psi.$$

The object ψ can be a set of four complex numbers. It is a Dirac spinor.

Let

$$\tilde{\Lambda} \equiv \exp\left(\tfrac{1}{2}\theta_{\mu\nu}\mathbb{S}^{\mu\nu}\right)$$

(the tilde is to distinguish this from the Lorentz transformation of 4-vectors). You can prove that

$$\tilde{\Lambda}^\dagger = \gamma^0\tilde{\Lambda}^{-1}\gamma^0.$$

Then with some further effort one can obtain central properties such as the Lorentz invariance of $\psi^\dagger\gamma^0\psi$, and that $\psi^\dagger\gamma^0\gamma^\mu\psi$ is a 4-vector, etc.

VIII. EXERCISES

1. Show that the Pauli matrices all square to 1, i.e. $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$. Hence, using $\exp(M) \equiv \sum_n M^n/n!$, prove eq. (14).

2. (a) Prove that any SU(2) matrix can be written $aI + ib\sigma_x + ic\sigma_y + id\sigma_z$ where a, b, c, d are real and $a^2 + b^2 + c^2 + d^2 = 1$. [e.g. start from an arbitrary 2×2 matrix M and show that if $M^{-1} = M^\dagger$ and $|M| = 1$ then $M_{11} = M_{22}^*$ and $M_{12} = -M_{21}^*$].
(b) Show that any SU(2) matrix can be written $\exp(i\boldsymbol{\theta} \cdot \boldsymbol{\sigma}/2)$. [e.g. prove that this form always gives an SU(2) matrix and spans the space].

3. Show that $g_{\mu\alpha}(\boldsymbol{u}^\dagger \sigma^\alpha \boldsymbol{u}) = (\epsilon \boldsymbol{u}^*)^\dagger \sigma^\mu (\epsilon \boldsymbol{u}^*)$ and interpret this result (c.f. eqs (45) and (47), and the caption to table II). [Method: either manipulate the matrices, or just write $\boldsymbol{u} = (a, b)$ and evaluate all the terms].

4. Find the flagpole 4-vectors for the following spinors, and confirm that they are null: $(1, 1)$, $(-2, 1)$, $(2, 1 + i)$.
Ans (2,2,0,0); $(5, -4, 0, 3)$; $(6, 4, 4, 2)$.

5. Prove the statement after eqn (40).

6. Starting from eqs. (85) and (89), show that the effect of swapping ϕ_R and χ_L is to change the sign of the spatial part of U and the time part of W.

7. Starting from eqs. (85) and (89), confirm that U and W are orthogonal.

8. Bearing in mind the commutation relations for Pauli matrices, show that, for any 3-vector $\mathbf{w}$, $(\boldsymbol{\sigma} \cdot \mathbf{w})^2 = w^2$, and hence complete the steps leading to the Dirac equation (94).

9. An electron moving along the x axis with speed $0.8c$ has its spin in the $(1, 0, 1)$ direction in the lab frame. Adopting units where $c = 1$ and the size of the spin is $1/2$, construct a Dirac spinor appropriate to describe this electron in the lab frame. [Method: first obtain the 4-velocity U and 4-spin W, hence obtain the two flagpoles and hence the two spinors]. Transform this spinor to the rest frame, and find the direction of the spin in the rest frame.

10. Find the two null 4-vectors (flagpoles) associated with the Dirac spinor $\Psi = (1.17005,\ 0.204124,\ 0.462943,\ -0.204124)$. Assuming this spinor represents the motion of a particle, find the 4-velocity and the direction of the spin in the rest frame.

11. Investigate $qF^\mu_{\ \alpha} U^{\alpha\bar{\nu}}$ where F is the electromagnetic field spinor and $U^{\alpha\bar{\nu}}$ is the 4-velocity spinor. The result is not simply the Lorentz force, but can be understood in terms of the Lorentz force and a force obtained from the dual of the Faraday tensor:
$$icf^{\mu\bar{\nu}} + c\tilde{f}^{\mu\bar{\nu}} = -qF^\mu_{\ \alpha} U^{\alpha\bar{\nu}}$$
[To obtain $F^\mu_{\ \nu}$ from $F_{\bar{\mu}}^{\ \bar{\nu}}$, first take the complex conjugate then raise and lower indices. One finds that the matrix is unchanged except the sign of $\mathbf{E}$ is reversed.]

12. Let $\beta \equiv \gamma^0$ and $\alpha^i \equiv \gamma^0 \gamma^i$. Show that, using these matrices, the Dirac equation (95) may be written $H\Psi = E\Psi$ where $H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$.

13. Show that, in the standard representation, the Dirac matrices take the form
$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

14. In quantum mechanics, the Dirac equation forces us to conclude that the operator representing spin angular momentum is $(\hbar/2)\boldsymbol{\sigma}$, not some other multiple. Prove this, as follows. Treat the Dirac equation in free space, so $H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$. Let $\mathbf{J} \equiv \mathbf{L} + \mathbf{S}$ be the total angular momentum operator, with $\mathbf{L} \equiv \mathbf{r} \wedge \mathbf{p}$ and $\mathbf{S}$ to be discovered. Show that $[\mathbf{L}, H] = i\hbar \boldsymbol{\alpha} \wedge \mathbf{p}$ (e.g. treat just the z component and the others follow). Clearly, this is non-zero so we do not have overall rotational invariance of the energy unless $\mathbf{S}$ also contributes to the angular momentum, such that $[\mathbf{S}, H] = -i\hbar \boldsymbol{\alpha} \wedge \mathbf{p}$. Verify that the solution is
$$\mathbf{S} = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

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[1] A. M. Steane, *Relativity made relatively easy* (Oxford U.P., Oxford, 2012).
[2] W. T. Payne, Am. J. Phys. **20**, 253 (1952).
[3] W. L. Bade and H. Jehle, Rev. Mod. Phys. **25**, 714 (1953).
[4] R. Feynman, *Theory Of Fundamental Processes* (Westview Press, 1998).
[5] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman & Company, 1973).
[6] L. H. Ryder, *Quantum field theory* (Cambridge University Press, 1996).
[7] This article has been prepared as a chapter of a textbook. A preliminary, somewhat cut-down, version has been available on the author's web-site for a few years. It there began to generate requests for further material, so it was decided to make it more widely available. Sources include [1–6] and some web-based material which I did not keep a note of. Starred sections can be omitted at first reading.
[8] A *unitary* matrix is one whose Hermitian conjugate is its inverse, i.e. $UU^\dagger = I$.
[9] An *orthogonal* matrix is one whose transpose is its inverse, i.e. $RR^T = I$.
[10] We will occasionally exhibit Dirac notation alongside the

vector and matrix notation, for the benefit of readers familiar with it. If you are not such a reader than you can safely ignore this. It is easily recognisable by the presence of $\langle\,|\,\rangle$ angle bracket symbols.

[11] This is why (16) and (17) have a rotation angle of opposite sign to (2) and (6).

[12] We will occasionally exhibit Dirac notation alongside the vector and matrix notation, for the benefit of readers familiar with it. If you are not such a reader than you can safely ignore this. It is easily recognisable by the presence of $\langle\,|\,\rangle$ angle bracket symbols.

[13] The two eigenvectors, considered as vectors in a complex vector space, are orthogonal to one another (because $S_{\mathbf{n}}$ is Hermitian), but their associated flagpole directions are opposite. This is an example of the angle doubling we already noted in the relationship between SU(2) and SO(3).

[14] When moving between the 4-vector and the complex number representation, the overall scale factor is a matter of convention. The convention adopted here slightly simplifies various results. Another possible convention is to retain the factor 1/2 as in (44).

[15] There now exists strong evidence that neutrinos possess a small non-zero mass. We proceed with the massless model as a valid theoretical device, which can also serve as a first approximation to the behaviour of neutrinos.

[16] Let $\eta = \boldsymbol{\sigma}\cdot\mathbf{p}$. Premultiply the first equation by $E+m-\eta$ and the second by $E+m+\eta$ to obtain $(E+m-\eta)\phi_R = (E+m+\eta)\chi_L$; then premultiply the first equation by $E-m-\eta$ and the second by $-E+m-\eta$ to obtain $(E-m-\eta)\phi_R = (-E+m-\eta)\chi_L$ (after making use of $\eta^2 = p^2$). The sum and difference of these equations gives the result.