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Friedmann Equations from Entropic Force

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Abstract

In this note by use of the holographic principle together with the equipartition law of energy and the Unruh temperature, we derive the Friedmann equations of a Friedmann-Robertson-Walker universe.

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It has a long history that gravity is not regarded as a fundamental interaction in Nature. The earliest idea on this was proposed by Sakharov in 1967 [1]. In this so-called induced gravity, spacetime background emerges as a mean field approximation of underlying microscopic degrees of freedom, similar to hydrodynamics or continuum elasticity theory from molecular physics [2]. This idea has been further developed since the discovery of the thermodynamic properties of black hole in 1970s. Black hole thermodynamics tells us that a black hole has an entropy proportional to its horizon area and a temperature proportional to its surface gravity at the black hole horizon, and the entropy and temperature together with the mass of the black hole satisfy the first law of thermodynamics [3, 4, 5].

The geometric feature of thermodynamic quantities of black hole leads Jacobson to ask an interesting question whether it is possible to derive the Einstein's equations of gravitational field from a point of view of thermodynamics [6]. It turns out that it is indeed possible. Jacobson derived the Einstein's equations by employing the fundamental Clausius relation $\delta Q = TdS$ together with the equivalence principle. Here the key idea is to demand that this relation holds for all the local Rindler causal horizon through each spacetime point, with δQ and T interpreted as the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. In this way, the Einstein's equation is nothing, but an equation of state of spacetime.

Assuming the apparent horizon of a Friedmann-Robertson-Walker (FRW) universe has temperature T and entropy S satisfying $T = 1/2\pi\tilde{r}_A$ and $S = A/4G$, where $\tilde{r}_A$ is the radius of the apparent horizon and A is the area of the apparent horizon, one is able to derive Friedmann equations of the FRW universe with any spatial curvature by applying the Clausius relation to apparent horizon [7]. This works not only in Einstein gravitational theory, but also in Gauss-Bonnet and Lovelock gravity theories. Here a key ingredient is to replace the entropy area formula in Einstein theory by using entropy expressions of black hole horizon in those higher order curvature theories. Recently the Hawking temperature associated with the apparent horizon of FRW universe has been showed [8]. There exist a lot of papers investigating the relation between the first law of thermodynamics and the Friedmann equations of FRW universe in various gravity theories. For more references see, for example [9, 10] and references therein.

Another hint appears on the relation between thermodynamics and gravitational dynamics by investigating the relation between the first law of thermodynamics and gravitational field equation in the setup of black hole spacetime. Padmanabhan [11] first noticed that the gravitational field equation in a static, spherically symmetric spacetime can be rewritten as a form of the ordinary first law of thermodynamics at a black hole horizon. This indicates that Einstein's equation is nothing but a thermodynamic identity. This observation was then extended to the cases of stationary axisymmetric horizons and evolving spherically symmetric horizons in Einstein gravity [12], static spherically symmetric horizons [13] and dynamical apparent horizons [14] in Lovelock gravity, and three dimensional BTZ black hole horizons [15]. Very recently it has been showed it also holds in Horava-Lifshitz gravity [16]. For a recent review on this topic and some relevant issues, see [17].

In a very recent paper by Verlinde [18], the viewpoint of gravity being not a fundamental interaction has been further advocated. Gravity is explained as an entropic force caused by changes in the information associated with the positions of material bodies. Among various interesting observations made by Verlinde, here we mention two of them. One is that with the assumption of the entropic force together with the Unruh temperature [19], Verlinde is able to derive the second law of Newton. The other is that the assumption of the entropic force together with the holographic principle and the equipartition law of energy leads to the Newton's law of gravitation. Similar observations are also made by Padmanabhan [20]. He observed that the equipartition law of energy for the horizon degrees of freedom combing with the thermodynamic relation $S = E/2T$, also leads to the Newton's law of gravity, here S and T are thermodynamic entropy and temperature associated with the horizon and E is the active gravitational mass producing the gravitational acceleration in the spacetime [21].

In this short note we are going to derive the Friedmann equations governing the dynamical evolution of the FRW universe from the viewpoint of entropic force together with the equipartition law of energy and the Unruh temperature.

Consider the FRW unverse with metric

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2), \tag{1}$$

where $a(t)$ is the scale factor of the universe. Following [18], consider a compact spatial region $\mathcal{V}$ with a compact boundary $\partial\mathcal{V}$, which is a sphere with physical radius $\tilde{r} = ar$. The compact boundary $\partial\mathcal{V}$ acts as the holographic screen. The number of bits on the screen is assumed as

$$N = \frac{Ac^3}{G\hbar}, \tag{2}$$

where A is the area of the screen (note that here there is a difference 1/4 in the coefficient of the formula, compared to the Bekenstein-Hawking area entropy formula of black hole). Assuming the temperature on the screen is T, and then according to the equipartition law of energy, the total energy on the screen is

$$E = \frac{1}{2} N k_B T. \tag{3}$$

Further just as in [18], we need the relation

$$E = Mc^2, \tag{4}$$

where M represents the mass that would emerge in the compact spatial region $\mathcal{V}$ enclosed by the boundary screen $\partial\mathcal{V}$.

Suppose the matter source in the FRW universe is a perfect fluid with stress-energy tensor

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}. \tag{5}$$

Due to the pressure, the total mass $M = \rho V$ in the region enclosed by the boundary $\partial\mathcal{V}$ is no longer conserved, the change in the total mass is equal to the work made by the pressure $dM = -p dV$, which leads to the well-known continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{6}$$

where $H = \dot{a}/a$ is the Hubble parameter.

The total mass in the spatial region $\mathcal{V}$ can be expressed as

$$M = \int_{\mathcal{V}} dV (T_{\mu\nu} u^\mu u^\nu), \tag{7}$$

where $T_{\mu\nu} u^\mu u^\nu$ is the energy density measured by a comoving observer. On the other hand, the acceleration for a radial comoving observer at r, namely at the place of the screen, is

$$a_r = -d^2\tilde{r}/dt^2 = -\ddot{a}\ r, \tag{8}$$

where the negative arises because we consider the acceleration is caused by the matter in the spatial region enclosed by the boundary $\partial\mathcal{V}$. Note that the proper acceleration vanishes for a comoving observer. However, the acceleration (8) is crucial in the following discussions [In fact, a_r is just the the acceleration of geodesic deviation vector [24]]. According to the Unruh formula, we assume that the acceleration corresponds to a temperature

$$T = \frac{1}{2\pi k_B c} \hbar a_r. \tag{9}$$

Next it follows from eqs. (2), (3), (4), (7) and (9) to derive the following equation

$$\ddot{a} = -\frac{4\pi G}{3} \rho a. \tag{10}$$

This is nothing, but the dynamical equation for Newtonian cosmology (Page 10 in [22]). Note that the reference [22] derives (10) from the Newtonian gravity law, while we obtain (10) by using the holographic principle and the equipartition law of energy in statistical physics. To produce the Friedmann equations of FRW universe in general relativity, let us notice that producing the acceleration is the so-called active gravitational mass $\mathcal{M}$ [21], rather than the total mass M in the spatial region $\mathcal{V}$. The active gravitational mass is also called Tolman-Komar mass, defined as

$$\mathcal{M} = 2\int_{\mathcal{V}} dV \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) u^\mu u^\nu. \tag{11}$$

Replacing M by $\mathcal{M}$, we have in this case

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right). \tag{12}$$

This is just the acceleration equation for the dynamical evolution of the FRW universe. Multiplying $\dot{a}a$ on both sides of eq. (12), and using the continuity equation (6), we integrate the resulting equation and obtain

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho. \tag{13}$$

Note that k appears in (13) as an integration constant, but it is clear that the constant k has the interpretation as the spatial curvature of the region $\mathcal{V}$ in the Einstein theory of gravity. $k = 1$, 0 and -1 correspond to a close, flat and open FRW universe, respectively.

The above discussion can be extended to any spacetime dimension $d \geq 4$. In that case, the number of bits on the screen is changed to [18]

$$N = \frac{1}{2}\frac{d-2}{d-3}\frac{Ac^3}{G\hbar}, \tag{14}$$

the continuity equation becomes $\dot{\rho} + (d-1)H(\rho + p) = 0$, and the active mass $\mathcal{M}$ is defined as

$$\mathcal{M} = \frac{d-2}{d-3}\int_{\mathcal{V}} dV \left(T_{\mu\nu} - \frac{1}{d-2} T g_{\mu\nu} \right) u^\mu u^\nu. \tag{15}$$

The acceleration equation (12) is changed to

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{(d-1)(d-2)}\left((d-3)\rho + (d-1)p\right). \tag{16}$$

Integrating (16) we then have

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{(d-1)(d-2)}\rho. \tag{17}$$

This is just the Friedmann equation of the FRW universe in d dimensions.

Note added: When we are in the final stage of writing the manuscript, two papers appear [23, 24] in the preprint archive, which discuss some relevant issues and have some overlap with our discussions in this paper.

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