Vacuum Fluctuations and the Small Scale Structure of Spacetime

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We show that vacuum fluctuations of the stress-energy tensor in two-dimensional dilaton gravity lead to a sharp focusing of light cones near the Planck scale, effectively breaking space up into a large number of causally disconnected regions. This phenomenon, called "asymptotic silence" when it occurs in cosmology, might help explain several puzzling features of quantum gravity, including evidence of spontaneous dimensional reduction at short distances. While our analysis focuses on a simplified two-dimensional model, we argue that the qualitative features should still be present in four dimensions.

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INTRODUCTION

A fundamental goal of quantum gravity is to understand the causal structure of spacetime—the behavior of light cones—at very small scales. One recent proposal [1, 2], based on the strong coupling approximation to the Wheeler-DeWitt equation [3, 4], is that quantum effects may lead to strong focusing near the Planck scale, essentially collapsing light cones and breaking space up into a large number of very small, causally disconnected regions. Such a phenomenon occurs in classical cosmology near a spacelike singularity, where it has been called "asymptotic silence" [5]; it can be viewed as a kind of anti-Newtonian limit, in which the effective speed of light drops to zero. Near the Planck scale, asymptotic silence could explain several puzzling features of quantum gravity, most notably the apparent dimensional reduction to two dimensions [1, 2, 6] that occurs in lattice models [7], in some renormalization group analyses [8], and elsewhere [9, 10]

In cosmology, the strong focusing of null geodesics comes from the presence of a singularity. For quantum gravity, a different source is required. Null geodesics are governed by the Raychaudhuri equation [11, 12]. For a congruence of null geodesics—a pencil of light—with affinely parametrized null normals ℓ^a, this equation tells us that

$$\ell^a\nabla_a\theta = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - 8\pi T_{ab}\ell^a\ell^b, \qquad (1)$$

where the expansion

$$\theta = \frac{1}{A}\ell^a\nabla_a A \qquad (2)$$

is the fractional rate of change of area of a cross section of the pencil, σ_{ab} is its shear, and ω_{ab} is its vorticity. The stress-energy tensor T_{ab} appears by virtue of the Einstein field equations, which relate it to the Ricci tensor. (We use natural units, $\hbar = c = G_N = 1$.) In particular, vacuum fluctuations with positive energy focus null geodesics, decreasing the expansion, while fluctuations with negative energy defocus them.

It has recently been shown that for many forms of matter, most vacuum fluctuations have negative energy [13]. These fluctuations have a strict lower bound, however, while the rarer positive fluctuations are unbounded. A key question is then which of these dominate the behavior of light cones near the Planck scale.

In this paper, we answer this question in the simplified context of two-dimensional dilaton gravity, a model obtained by dimensionally reducing general relativity to one space and one time dimension. We show that the positive energy fluctuations win, and cause a collapse of light cones in a time on the order of 15 Planck times. The reduction to two dimensions is, for the moment, required for exact calculations; it is only in this setting that the spectrum of vacuum fluctuations is fully understood. But we show that the qualitative features leading to strong focusing are also present in the full four-dimensional theory, strongly suggesting that our results should apply to full general relativity.

DILATON GRAVITY AND THE RAYCHAUDHURI EQUATION

Our starting point is two-dimensional dilaton gravity, a theory that can be described by the action [14, 15]

$$I = \int d^2x\sqrt{|g|}\left[\frac{1}{16\pi}\varphi R + V[\varphi] + \varphi L_m\right], \qquad (3)$$

where φ is a scalar field, the dilaton, with a potential $V[\varphi]$ whose details will be unimportant. This action can be obtained from standard general relativity by dimensional reduction, assuming either spherical or planar symmetry. A conformal redefinition of fields is needed to bring the action into this form; the dilaton coupling to the matter Lagrangian L_m is fixed provided the two-dimensional matter action is conformally invariant. From now on, we assume such an invariance.

In two dimensions, the Raychaudhuri equation (1) does not quite make sense, since there are no transverse directions in which to define the area or the expansion. Its generalization to dilaton gravity is simple, however. For any model of dilaton gravity obtained by dimensional reduction, the dilaton has a direct physical interpretation as the transverse area in the "missing" dimensions. We can therefore define a generalized expansion

$$\bar{\theta} = \frac{1}{\varphi}\ell^a\nabla_a\varphi = \frac{d}{d\lambda}\ln\varphi, \tag{4}$$

where λ is the affine parameter. It is then an easy consequence of the dilaton gravity field equations that

$$\frac{d\bar{\theta}}{d\lambda} = -\bar{\theta}^2 - 8\pi T_{ab}\ell^a\ell^b = -\bar{\theta}^2 - 16\pi T_L, \tag{5}$$

where T_L is the left-moving component of the stress-energy tensor.

We next need the vacuum fluctuations of the stress-energy tensor. For conformally invariant matter in two dimensions, Fewster, Ford, and Roman have found these exactly [13]. A conformally invariant field is characterized by a central charge c; a massless scalar, for instance, has $c = 1$. To obtain a finite value for the stress-energy tensor, one must smear it over a small interval; Fewster et al. use a Gaussian smearing function with width τ. The probability distribution for the quantity T_L in the Minkowski vacuum is then given by a shifted Gamma distribution

$$Pr(T_L = \omega) = \vartheta(\omega+\omega_0)\frac{(\pi\tau^2)^\alpha(\omega+\omega_0)^{\alpha-1}}{\Gamma(\alpha)}e^{-\pi\tau^2(\omega+\omega_0)} \tag{6}$$

where

$$\omega_0 = \frac{c}{24\pi\tau^2}, \quad \alpha = \frac{c}{24}, \tag{7}$$

and ϑ is the Heaviside step function. As noted earlier, this distribution is peaked at negative values of the energy; for $c = 1$, the probability of a positive fluctuation is only .16. There is, however, a long positive tail, as there must be in order that the average $\langle T_L\rangle$ be zero.

Before proceeding with the calculation, it is useful to look at the behavior of the Raychaudhuri equation (5) with a constant source $T_L = \omega$. It is easy to see that

$$\bar{\theta}(\lambda) = \begin{cases} -\sqrt{16\pi\omega}\tan\sqrt{16\pi\omega}(\lambda-\lambda_0) \\ \qquad \text{if } \omega > 0 \\ \sqrt{|16\pi\omega|}\tanh\sqrt{|16\pi\omega|}(\lambda-\lambda_0) \\ \qquad \text{if } \omega < 0 \text{ and } |\bar{\theta}(0)| < \sqrt{|16\pi\omega|} \\ \pm\sqrt{|16\pi\omega|} \\ \qquad \text{if } \omega < 0 \text{ and } \bar{\theta}(0) = \pm\sqrt{|16\pi\omega|} \\ \sqrt{|16\pi\omega|}\coth\sqrt{|16\pi\omega|}(\lambda-\lambda_0) \\ \qquad \text{if } \omega \le 0 \text{ and } |\bar{\theta}(0)| > \sqrt{|16\pi\omega|} \end{cases} \tag{8}$$

where the integration constant λ_0 is determined from the initial value of $\bar{\theta}$.

As noted earlier, positive energy fluctuations can quickly drive the expansion to $-\infty$. If $\bar{\theta}$ is initially negative, even small negative energy fluctuations—those on the coth branch of (8)—cannot overcome the nonlinearities that also drive the expansion to $-\infty$. Larger negative energy fluctuations tend to defocus null geodesics, increasing the expansion, but even these have a limited effect. The lower bound $-\omega_0$ for energy in the distribution (6) determines a maximum asymptotic value

$$\bar{\theta}_+ = \sqrt{16\pi\omega_0} = \sqrt{2c/3\tau^2}, \tag{9}$$

and if the expansion starts below this value, it will asymptote to at most $\bar{\theta}_+$.

Moreover, the same bound on negative energy fluctuations means that the positive contribution to the right-hand side of (5) cannot be larger than $16\pi\omega_0 = {\bar{\theta}_+}^2$. The expansion thus has a critical negative value. Any fluctuation that brings $\bar{\theta}$ to a value lower than

$$\bar{\theta}_{crit} = -\sqrt{\frac{2c}{3\tau^2}} = -\bar{\theta}_+ \tag{10}$$

is irreversible: once the expansion becomes this negative, it will necessarily continue to decrease. This gives a qualitative answer to our central question—in the long run, the positive energy fluctuations will always win, and the expansion will diverge to $-\infty$. Note the crucial role of the nonlinear term in (5); a similar problem was considered in [16], but only in the approximation that $\bar{\theta}$ was small enough that this term could be neglected.

To estimate the time to this "collapse" of the light cones, let us assume that the initial negative energy fluctuations push $\bar{\theta}$ to near its peak value of $\bar{\theta}_+$. We can then ask for the probability of a positive energy fluctuation large enough to drive $\bar{\theta}$ to $\bar{\theta}_{crit}$ within a characteristic time τ. This can be determined from (6); for a massless scalar field ($c = 1$), it is $\rho \approx .065$. If we now treat these fluctuations as a Poisson process—i.e., independent events occurring randomly with probability $\rho\Delta t/\tau$ in any interval Δt—and ignore all smaller fluctuations, the time to "collapse" will be given by an exponential distribution $(\rho/\tau)e^{-\rho t/\tau}$, with a mean of approximately 15.4τ.

This is, of course, an oversimplified picture of the combined effect of many vacuum fluctuations. To obtain a more precise result, we next turn to a numerical analysis of the Raychaudhuri equation.

QUANTITATIVE RESULTS

Our analysis so far has been semiclassical: we consider quantum fluctuations of matter, but ignore purely quantum gravitational effects. We can, in principle, incorporate weak quantum fluctuations of the metric into the stress-energy tensor in (5), but there is no reason to trust our methods at scales at or below the Planck scale. We therefore choose the width τ of the Gaussian smearing function to be the Planck length. The overall system (5)–(6) is invariant under a simultaneous rescaling of τ and the affine parameter λ, so this choice is not critical; given any cutoff τ, our results can be interpreted as giving the focusing time in units of τ. For our matter field, we choose a massless scalar, that is, a conformal field with central charge $c = 1$.

We proceed as follows:

1. We choose an initial condition $\bar{\theta}_0 = 0$, and select a random value of the vacuum fluctuation ω, with a probability given by the distribution (6).

2. We evolve forward one Planck time using (8), to determine a new value $\bar{\theta}_1$.

3. Using $\bar{\theta}_1$ as an initial condition and choosing a new value of ω, we evolve another Planck time to determine $\bar{\theta}_2$. We repeat the process, keeping track of the number of iterations, until we land on a negative branch of (8) with the expansion diverging to $-\infty$ during the step.

Using Mathematica [17], we have performed ten million runs of this simulation. Figure 1 shows the probability density of light cone "collapse" as a function of time in Planck units. The mean time to collapse is $14.73\,t_P$, where t_P is the Planck time; the standard deviation is $14.53\,t_P$. These are perhaps large enough to justify our neglect of strong quantum gravitational effects, but small enough to probe the causal structure of spacetime in a physically very interesting region.

We see from the figure that the probability for "collapse" at the first step is approximately .051. This provides a useful check, since this probability can be obtained directly from the distribution (6): it is just the probability that the argument of the tangent in the collapsing branch of (8) is at least $\pi/2$ when $\lambda = 1$, $\lambda_0 = 0$. The exact result matches our simulation.

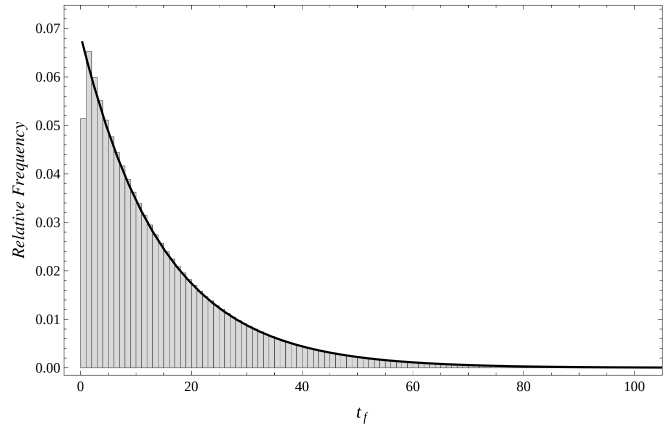

FIG. 1: Probability of the expansion diverging to $-\infty$ as a function of Planck time steps. The solid line is the exponential distribution (11).

As shown in Figure 1, after the first few steps the distribution can be fit very accurately to an exponential distribution:

$$Pr(t_f = n) = \rho e^{-\rho n} \qquad \text{with } \rho \approx .0686. \tag{11}$$

This is surprisingly close to our approximation at the end of the preceding section. Moreover, while the details of individual runs vary widely, almost all show $\bar{\theta}$ rising quickly to near its maximum value of $\bar{\theta}_+$, confirming the starting assumption of our approximation.

Two-dimensional dilaton gravity thus appears to exhibit short distance asymptotic silence, with vacuum fluctuations causing a rapid convergence of null cones.

GENERALIZATIONS AND IMPLICATIONS

Our computations have been restricted to two-dimensional dilaton gravity, primarily because this is the only setting in which the probability distribution (6) is known. We have also restricted ourselves to conformally invariant matter, for the same reason. But the qualitative features of our results are largely independent of such details, and we expect them to carry over to the full four-dimensional theory. In particular,

1. The vacuum fluctuations of the stress-energy tensor in four dimensions are also expected to have a strict lower bound and an infinitely long positive tail [13]. If anything, the positive tail seems to fall off more slowly than in two dimensions. Moreover, while our detailed results used a Gaussian test function to smear the stress-energy tensor, these features hold for any sufficiently compact test function [18].

2. If the stress-energy tensor is bounded below, the expansion in four dimensions has bounds exactly

analogous to those of (8), even taking the same functional form [19].

3. Adding real matter would change the quantitative details of our results. But as long as that matter satisfies the null energy condition, it can only lead to further focusing.

We thus expect something akin to short distance asymptotic silence in four-dimensional general relativity. We should, however, add three caveats. First, we have treated our sequence of vacuum fluctuations as if they were statistically independent. This is not quite right: the results of [13] imply that the (Gaussian smeared) stress-energy tensor T_L at coordinate u is weakly anti-correlated with the same object at $u+\tau$. The correlation drops off sharply with distance, and should not affect our qualitative results. It may be possible, though, to take this effect into account to produce a more accurate quantitative picture. Work on this question is in progress.

Second, the vacuum fluctuations found in [13] are fluctuations of the Minkowski vacuum (or, by conformal invariance, of any conformally equivalent vacuum). While any curved spacetime is approximately flat at short enough distances, this is not enough to determine a unique vacuum, and we do not know how sensitive our results are to this choice.

Third, we have by necessity neglected purely quantum gravitational effects, which could also compete with the vacuum fluctuations of matter. This is not independent of the problem of choosing a vacuum; for instance, it is known that if one chooses the Unruh vacuum near a black hole horizon, the renormalized value of the shear term $\sigma_{ab}\sigma^{ab}$ in (1) is negative [20].

If a proper handling of these caveats does not drastically change our conclusions, though, we have learned something very interesting about the small scale structure of spacetime. The strong focusing of null cones means that "nearby" neighborhoods of space are no longer in causal contact. This sort of breakup of the causal structure has been studied in cosmology [5], where it leads to BKL behavior [21]: each small neighborhood spends most of its time as an anisotropically expanding Kasner space with essentially random expansion axes and speeds, but periodically undergoes a chaotic "bounce" to a new Kasner space with different axes and speeds. It was argued in [1, 2] that such a local Kasner behavior could explain the apparent spontaneous dimensional reduction of spacetime near the Planck scale, while preserving Lorentz invariance at large scales. Whether or not this proves to be the case, the short distance collapse of light cones suggests both a new picture of spacetime and a new set of approximations for short distances.

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