Aspects of Group Field Theory

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Abstract

I review the basic ingredients of discretized gravity which motivate the introduction of Group Field Theory. Thus I describe the GFT formulation of some models and conclude with a few remarks on the emergence of noncommutative structures in such models.

keywords: Quantum Gravity, Noncommutative Geometry, Star Products

Introduction

This article is based on a talk given at the XX International Fall Workshop on Geometry and Physics in Madrid and it is aimed at illustrating the deep geometric roots of the Group Field Theory (GFT) approach to Quantum Gravity, together with the recent emergence of Noncommutative Geometry into the game.

The first part, partly based on [1], is dedicated to a review of the discrete formulation of gravity as a BF theory with constraints, including the Holst formulation [2] with the Barbero-Immirzi parameter [3, 4]. It is not meant to be exhaustive while it is directed to a public of non-specialists, with the accent put on geometric structures. I then introduce the GFT description of the discrete BF path integral and briefly review the implementation of constraints.

In the final part I discuss the emergence of noncommutative structures at various levels of the models considered.

1 Gravity as a BF theory with constraints

The starting point of this analysis is the well known reformulation of the Einstein-Hilbert action $S\,[\,g\,] = \int_M d^4x\ \sqrt{|g|}\ R$ in the first order formalism, as

$$S\,[\,e\,,\,\omega\,] = \int_M \mathrm{Tr}_{\,1}\,[\ (e\wedge e)\ \wedge F\ ] = \int_M \frac{1}{2}\,\epsilon_{ABCD}\,e^A\wedge e^B\wedge F^{CD} \qquad (1)$$ action

with $A,B,C,D = 1,..,4$. $F = D\,\omega$ is the curvature of the connection one-form of the principal $SO(3,1)$-bundle[1] while e are the tetrad one-forms implicitly defined by $g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{AB}$. The connection one-form ω and the tetrads e are to be regarded as independent variables. The equations of motion may be seen to be equivalent to Einstein's equations when the tetrads are non-degenerate. Let me recall that $e = e^A_\mu dx^\mu v_A$ is a vector valued one form of the associated vector bundle, so that, once a trivialization chosen $e\wedge e = e^A_\mu e^B_\nu dx^\mu\wedge dx^\nu \tau_{AB}$ with $\tau_{AB}\in so(3,1)$. Moreover, the connection and the curvature are locally of the form $\omega = \omega^{AB}_\mu dx^\mu \tau_{AB}$ $F = F^{AB}_{\mu\nu} dx^\mu\wedge dx^\nu \tau_{AB}$. Therefore $\mathrm{Tr}_{\,1}$ is to be intended as the bilinear nondegenerate form on the $so(3,1)$ algebra

$$< J_a, P_b >_1 = \eta_{ab}\ \ < J_a, J_b >_1 = < P_a, P_b >_1 = 0\ \ a,b = 1,..,3$$

[1] I will indicate Lie groups with capital letters, Lie algebras with lower-case letters, a generic Lie group with G and a generic Lie algebra with $\mathcal{L}$

with $J_a = J_A = \epsilon_A^{BC} \tau_{BC} \;\; A, B, C = 1, .., 3 \qquad P_a = P_A = \tau_{A,4}, \;\; A = 1, .., 3.$

The action (1) my be recast into the form of a BF action with constraints. The BF action

$$S\,[\,B\,,\,\omega\,] = \int_M \mathrm{Tr}_{\,1}\,(B \wedge F) \tag{2}$$

encodes a topological gauge theory described in terms of Lie algebra valued 2-forms, B, F, where F is the curvature. The equations of motion simply state that the connection ω is flat and $D_\omega B = 0$. The constraints $C(B)$ have to impose that the B field be simple, that is $B = e \wedge e$. The action becomes then

$$S_1\,[\,B\,,\,\omega\,,\,\phi\,] = \int_M B^{AB} \wedge F_{AB} + C(B). \tag{3}$$

In $so(3,1)$ we can define another trace $\mathrm{Tr}_{\,2} = < .,. >_2$,

$$< J_a, J_b >_2 = < P_a, P_b >_2 = \delta_{ab}, \;\; < J_a, P_b >_2 = 0$$

so that a new action is produced

$$S_2 = \int_M \mathrm{Tr}_{\,2}(e \wedge e \wedge F) = \int_M e^A \wedge e^B \wedge F_{AB}[\omega]. \tag{4}$$

When added to the action S_1 it doesn't change the equations of motion but it has quantum consequences. The full action $S_H = S_1 + \frac{1}{\gamma} S_2$ is the Palatini-Holst action [2]; the real parameter γ is the Barbero-Immirzi parameter [3, 4]. The relevance of this second term in the action is well known: it allows to introduce the Ashtekar connection which is at the basis of the canonical quantization programme for gravity.

S_H may be regarded as a constrained BF action

$$S_H = \int_M B^{AB} \wedge F_{AB} + C(B) \tag{5}$$ sh

where the constraint has to implement

$$B^{AB} = \epsilon^{AB}_{CD} e^C \wedge e^D + \frac{1}{\gamma} e^A \wedge e^B. \tag{6}$$ constr

1.1 Gravity in three dimensions

In three space-time dimensions the Einstein-Hilbert action becomes in the first order formalism

$$S[\omega, e] = \int_M \mathrm{Tr}\;(e \wedge F) \tag{7}$$

with

- $\omega = \omega_\mu^A dx^\mu \tau_A$, the $SO(2,1)$ connection one-form $\qquad \tau_A \in so(2,1)$
- $F = F_{\mu\nu}^A dx^{\mu\nu}\tau_A$ the curvature of the connection one-form
- $e = e_\mu^A dx^\mu \tau_A$ triads, with the identification $so(2,1) \simeq V(M)$
- Tr $\leftrightarrow$ Killing form in $so(2,1)$

This is the BF action for the gauge group $SO(2,1)$, with $B = e$. Because the B field is a one-form there are no extra constraints to be imposed.

We can build BF models in any space-time dimensions, with gauge group the Lorentz group $SO(D-1,1)$. For BF in D dimensions F is always a Lie algebra valued 2-form (the curvature), while B is a $D-2$ Lie algebra valued form. In particular in $D = 2$ (with B a zero form)[2] and $D = 3$ (with B a one-form), the BF action reproduces the gravity action and it is a topological theory. In $D \geq 4$ BF + constraints reproduces gravity, and it is dynamical.

1.2 Discretization of the BF action

From now on we stick to the Riemannian case so that the Lorentz group is replaced by the rotation group and we often work with its covering group. We consider space-time triangulations with a D-dimensional simplicial complex $K_D = \{\sigma_D, ..., \sigma_0\}$. To discretize B which is a Lie algebra valued $D-2$ form, we integrate it on a $D-2$ simplex

$$B \longrightarrow E \in so(D), \quad E = \int_{\sigma_{D-2}} B.$$

To discretize the curvature 2-form we follow the prescription of Regge calculus where, in $D = 2$ the curvature is measured by the deficit angle when turning around a vertex (0-simplex)

$$\delta(v) = 2\pi - \sum_{\ell,\ell' \supset v} \theta_v(\ell, \ell').$$

Analogously, in $D = 3$ the curvature is measured by the deficit angle when turning around an edge (1-simplex). Therefore, in D generic the local curvature on the triangulated manifold is detected by the holonomy of the

[2]Notice however that in $D = 2$ the BF formulation requires the gauge group to be the Poncaré or the De Sitter group (for details see for example [5])

connection around a $D-2$ simplex. The closed path around the $D-2$ simplex is the boundary of a face, f_* in the dual discretization K_*. Therefore we have $h_{\ell_*} = P \exp \int_{\ell_* \subset \partial f_*} \omega$ and

$$H_{f_*} \equiv H_{\sigma_{D-2}} = \prod_{\ell_* \subset \partial f_*} h_{\ell_*} \tag{8}$$

where we have explicitly indicated the duality between dual faces and $D-2$ simplices. The two-dimensional sub-complex contained in K_* draws a graph $\mathcal{G}$. Each assigned $\mathcal{G} \subset |K_*|$ represents a specific discretization of space-time.

The discretized BF action becomes

$$S(E_{\sigma_{D-2}}, H_{\sigma_{D-2}}) = \sum_{\sigma_{D-2} \in K} \mathrm{Tr}\, E_{\sigma_{D-2}} H_{\sigma_{D-2}}. \tag{9}$$

from which we derive the discretized partition function

$$A[K, K_*] = \int_{\mathcal{L}} \prod_{\sigma_{D-2} \in K} dE_{\sigma_{D-2}} \int_G \prod_{\ell_* \in K_*} dh_{\ell_*} \exp[i \,\mathrm{Tr}\,(E_{\sigma_{D-2}} \prod_{\ell_* \in \partial f_*} h_{\ell_*})] \tag{10}$$

where $\mathcal{L}$ is the Lie algebra of the appropriate rotation group G. The integral in the Lie algebra can be formally performed and we get

$$A[K_*] = \int \prod_{\ell_* \in K_*} dh_{\ell_*} \prod_{f_* \in K_*} \delta(\prod_{\ell_* \in \partial f_*} h_{\ell_*}) \tag{11}$$

bfampl

This result, valid in any dimension, is expressed solely in terms of the dual discretization. It can be interpreted as the amplitude of the graph $\mathcal{G}$ drawn in K_*. It is interesting to notice that the same result is obtained independently in the spin-foams approach [6], as the transition amplitude from a space geometry to another.

The natural question which arises is: what is, if any, the field theory which generates such Feynman graphs? Group field theory, introduced in the early 90's in [7] and later developed by [8, 9, 10] is a candidate to that.

2 Group Field Theory

Group Field Theories are a particular family of tensor models where the fields are tensors defined on the Lorentz group manifold. Tensor models are in turn the natural generalization of matrix models to higher dimensions, aimed at describing aleatory space-time geometries (for an up to date review on the subject and recent achievements see [11]). As in more general tensor models, GFT encode the space-time dimension in the order of the tensor-field while, specific to these models, the field arguments live on products of the Lorentz (rotation) Lie group

$$\phi : (g_1, ...g_D) \in [SO(D)]^D \to \phi(g_1, ...g_D).$$

Feynman amplitudes of a D dimensional GFT are dually associated with a discrete space-time via a specific triangulation and gluing rules given by the propagator and vertices of the theory. The functional integral formalism defines a weighted sum over triangulations with each weight (amplitude) related to a sum over geometries, therefore achieving a desired feature of any candidate quantum theory of gravity - a sum over both topologies and geometries.

The simplest GFT models generate amplitudes of the BF type, as in (11) . They are therefore topological models. A possible choice is to include the dynamics by implementing the constraints on the propagator, while the vertex of the theory would remain unchanged with respect to the topological theory. The other possibility is to constrain the vertex and leave the propagator unmodified. Following [12] I adopt here the first point of view .

The propagator C, is a Hermitian operator with Hermitian kernel $C(g_1, \ldots, g_D; g'_1, \ldots, g'_D)$:

$$[C\phi](g_1, \ldots, g_D) = \int dg'_1 \ldots dg'_D C(g_1, \ldots, g_D; g'_1, \ldots, g'_D)\phi(g'_1, \ldots, g'_D). \tag{12}$$

It is represented graphically as a stranded line with D strands and the precise form of C characterizes the different models. The vertex is the same for all models: its kernel is a product of delta functions matching strand arguments, so that each delta function joins two strands in two different lines. For instance, in three dimensions the SU(2) BF vertex is expressed as

$$\begin{aligned} S_{\text{int}}[\phi] &= \frac{\lambda}{4} \int \left(\prod_{i=1}^{12} dg_i \right) \phi(g_1, g_2, g_3)\phi(g_4, g_5, g_6)\phi(g_7, g_8, g_9) \\ &\quad \phi(g_{10}, g_{11}, g_{12})\ K(g_1, ..g_{12}), \end{aligned} \tag{13}$$

with

$$K(g_1, ..g_{12}) = \delta(g_3 g_4^{-1})\delta(g_2 g_8^{-1})\delta(g_6 g_7^{-1})\delta(g_9 g_{10}^{-1})\delta(g_5 g_{11}^{-1})\delta(g_1 g_{12}^{-1}) \tag{14}$$ 3vertex

It represents the gluing of four triangles to form a tetrahedron, the elementary space-time block in 3D. In D dimensions it is therefore replaced by a term proportional to ϕ^{D+1}. The propagator represents instead the gluing of two $D-$simplices along a common face.

2.1 GFT for BF theories

The propagator for BF theories is just the projection on gauge invariant fields,

$$\mathbb{P}(\phi) = \int_{SO(D)} dh \phi(g_1 h, \ldots, g_D h), \tag{15}$$

It verifies $\mathbb{P}^2 = \mathbb{P}$ so that the only eigenvalues are 0 and 1. This is another manifestation of the fact that BF models have no dynamical content. The operator $\mathbb{P}$ is Hermitian with kernel

$$\mathbb{P}(g_1, ..., g_D; g'_1, ...g'_D) = \int dh \prod_{i=1}^{D} \delta(g_i h (g'_i)^{-1}). \tag{16}$$

prop

The full GFT action for these models may be synthetically represented as

$$S[\phi] = \int dg \phi^2 + \lambda \int dg \phi^{D+1} \tag{17}$$

GFTaction

where the fields are functions of D copies of the group and the integration is performed on as many copies of the group as needed, so that dg stands for the appropriate power of the Haar measure. To compute the amplitude of a given graph we assign to each propagator the definition in Eq. (16) and to each vertex the appropriate generalization of Eq. (14). We choose for simplicity graphs with no external legs. After integration over all group variables associated to the strands of propagators we obtain

$$A_{\mathcal{G}} = \int \prod_{\ell \in L_{\mathcal{G}}} dh_\ell \prod_{f \in \mathcal{F}_{\mathcal{G}}} \delta\left(\vec{\prod}_{\ell \in f} h_\ell^{\eta_{\ell f}} \right), \tag{18}$$

gftampl

where we have omitted the star labeling the dual discretization. The incidence matrix $\eta_{\ell f}$ has value $+1$ if the face f goes through the edge ℓ in the same direction, -1 if the face f goes through the edge ℓ in the opposite direction, 0 otherwise. Let us that notice that the total amplitude Eq. (18) is factorized as a product of face amplitudes and it reproduces the result that we obtained for BF amplitudes in Eq. (11). We therefore have a positive answer to the question we posed at the end of section 1, at least for simple models: we have a field theory which generates the transition amplitudes for space-time geometries in the absence of constraints. It can be easily shown that, when using the Peter-Weyl decomposition for group variables we obtain an expression for the amplitude (18) in terms of 6j-symbols which is exactly the Ponzano-Regge model [13].

In four dimensions we have to implement the constraints described in section 1, at the level of the discrete theory. There are various proposals, the first being the Barrett-Crane model [14] while the inclusion of the Barbero-Immirzi parameter led to the EPRL/FK model [15, 16]. Here the Barrett-Crane model is recovered in the $\gamma \to \infty$ limit. Interestingly, recently new models have appeared, with and without the Barbero-Immirzi parameter [17], [18] which are based on the noncommutative algebra of flux variables. Here we just sketch the EPRL/FK model.

2.2 Models of 4D gravity

The EPRL/FK model [15, 16] implements in two steps the constraints in Eq. (6) with a non trivial value of the Barbero-Immirzi parameter γ. In order to describe the model we introduce $SU(2)$ coherent states [19]

$$|j, g\rangle \equiv g|j, j\rangle = \sum_m |j, m\rangle [R^{(j)}]_j^m(g).$$

with $|j, m>$ the eigenstates of the Lie algebra generators and $[R^{(j)}]_j^m(g)$ the spin-j representation of the group element g. We have for the partition of unity

$$1_j = \mathrm{d}_j \int_{\mathrm{SU(2)}} dg\, |j, g\rangle\langle j, g| = \mathrm{d}_j \int_{G/H=S_2} dn\, |j, n\rangle\langle j, n|$$

with $|j, n\rangle = g_n|j, j\rangle$. In four dimensions we use the $SU(2) \times SU(2)$ coherent states $\quad |j_+, n_+\rangle \otimes |j_-, n_-\rangle$

$$1_{j_+} \otimes 1_{j_-} = \mathrm{d}_{j_+}\mathrm{d}_{j_-} \int dn_+ dn_- |j_+, n_+\rangle \otimes |j_-, n_-\rangle\langle j_+, n_+| \otimes \langle j_-, n_-|$$

In this language the constraints are implemented as [16]
$j_+/j_- = (1+\gamma)/(1-\gamma)$ and $\quad n_+ = n_- = n$

$$\gamma > 1 \qquad\qquad j_\pm = \frac{\gamma \pm 1}{2} j,$$

$$\gamma < 1 \qquad\qquad j_\pm = \frac{1 \pm \gamma}{2} j.$$

We consider now the propagator of the 4D BF theory, with gauge group $SO(4)$, which is a natural generalization of the 3D propagator Eq. (16), when represented in the coherent states basis

$$\begin{aligned} \mathbb{P}(g; g') \;\; &= \;\; \int_{SU(2)\times SU(2)} dudv \prod_{f=1}^{4} \sum_{j_{f+}, j_{f-}} \mathrm{d}_{j_{f+}}\mathrm{d}_{j_{f-}} \\ &\quad \mathrm{Tr}_{\,V_{j_{f+}}\otimes V_{j_{f-}}} \left(u g_f (g'_f)^{-1} v^{-1} 1_{j_{f+}} \otimes 1_{j_{f-}} \right). \end{aligned} \tag{19}$$

In each strand the identity $1_{j_+} \otimes 1_{j_-}$ is replaced by a projector T_j^γ

$$T_j^\gamma = d_{j_+ + j_-} \left[\delta_{j_-/j_+ = (1-\gamma)/(1+\gamma)}\right] \int dn |j_+, n\rangle \otimes |j_-, n\rangle\langle j_+, n| \otimes \langle j_-, n|. \quad (20)$$

which verifies $(T_j^\gamma)^2 = T_j^\gamma$. Grouping the four strands of a line defines a $\mathbb{T}^\gamma$ operator that acts separately and independently on each strand of the propagator:

$$\mathbb{T}^\gamma = \oplus_{j_f} \otimes_{f=1}^4 T_{j_f}^\gamma \quad (21)$$

so that the EPRL/FK propagator is

$$C = \mathbb{P}\mathbb{T}^\gamma\mathbb{P}. \quad (22)$$

The operator C is symmetric. This implies that Feynman amplitudes are independent of the orientations of faces and propagators. Since $\mathbb{T}^\gamma$ and $\mathbb{P}$ do not commute, the propagator C can have non-trivial *spectrum* (with eigenvalues between 0 and 1). Moreover, since $\mathbb{T}^\gamma$ is a projector, the propagator C of the EPRL/FK theory is bounded in norm by the propagator of the BF theory, as well as Feynman amplitudes.

To obtain the amplitude of a given graph $\mathcal{G}$ we combine the propagator and the vertex expressions as in usual quantum field theory and integrate over all g, g' group variables (see [12] for details). The total amplitude may be seen to be factorized as (the integral of) a product of face amplitudes

$$A_{\mathcal{G}} = \int \prod_{\ell \in L_{\mathcal{G}}} du_\ell dv_\ell \prod_{f \in \mathcal{F}_{\mathcal{G}}} \mathcal{A}_f \quad (23)$$

with $\ell \in L_{\mathcal{G}}$ the edges of our graph, and $\mathcal{A}_f$ given by

$$\mathcal{A}_f = \sum_{j_f \le \Lambda} d_{j_{f+}} d_{j_{f-}} \operatorname{Tr}_{j_{f+} \otimes j_{f-}} \prod_{a=1}^{p} \left(h_{\ell_a, v_a}^{\eta_{\ell_a f}} h_{\ell_a, v_{a+1}}^{\eta_{\ell_a f}} T_{j_f}^\gamma \right) \quad (24)$$

It can be seen that we recover the $SU(2)$ BF model in the limit $\gamma \to 1$. At this point we have all the ingredients of a quantum field theory. Specific graphs have been computed (see for example [12]) and their degree of divergence analyzed. There is however no understanding on the perturbative expansion of the partition function and a full renormalization group analysis is still lacking. A modification of the model, which introduces colors for the fields has recently been introduced. It allows for a better control of the kind of topologies which are dually associated to the graphs (see [20] and references therein).

3 Noncommutative structures

In this section we will only consider the three dimensional case, although some of the results we will describe have been extended to the full 4d case [17, 18].

As we have seen, in three dimensions gravity is described by a BF theory with $SU(2)$ group and the group field theory associated to its discretization is represented by the Boulatov model, with action in Eq. (17) (with D=3).

We can define on the group manifold coordinate functions

$$p^i = -i \,\mathrm{Tr}\, g\sigma^i, \quad i = 1, .., 3 \tag{25}$$

where σ are the Pauli matrices, and we parametrize $g \in SU(2)$ as $g = p^0 I + i\sigma_i p^i$, with $(p^0)^2 + \sum_i (p^i)^2 = 1$. We indicate with x_i the conjugate variables which live on the fibers of the cotangent bundle $T^*SU(2)$. The confusing notation for the base and fiber coordinates is linked to the physical interpretation from the gravity point of view. The canonical Poisson brackets on the cotangent bundle are

$$\{p^i, p^j\} = 0 \tag{26}$$

$$\{x_i, x_j\} = \epsilon_{ij}^k x_k \tag{27}$$

$$\{p^i, x_j\} = 2(-\delta_{ij}\sqrt{1 - |\vec{p}|^2} + \epsilon_{ijk} p^k) \tag{28}$$

They describe the dynamics of many interesting physical systems, as for example the dynamics of the rigid rotor with x_i associated to the angular momentum components and p^i to the orientation of the rotor, or, when generalized to field theory, the Poisson algebra of currents for the principal chiral model.

As a group $T^*SU(2)$ is the semidirect product of $SU(2)$ and the abelian group $\mathbb{R}^3$, with Lie algebra the semidirect sum represented by

$$[J_i, J_j] = \epsilon_{ij}^k J_k \tag{29}$$

$$[P_i, P_j] = 0 \tag{30}$$

$$[J_i, P_j] = \epsilon_{ij}^k P_k. \tag{31}$$

The non-trivial Poisson bracket on the fibres of the bundle, (27), is usually understood in terms of coadjoint action of the group $SU(2)$ on its dual algebra $\mathcal{L}^* = (\mathbb{R}^3)^* \simeq \mathbb{R}^3$ and it reflects the non-triviality of the Lie bracket (29)[3].

[3] The Lie algebra generators J_i are identified with the linear functions on the dual algebra

The question arises whether the non-trivial Poisson bracket on $\mathcal{F}(\mathcal{L}^*)$ may be quantized yielding a noncommutative star-product in the spirit of deformation quantization. This is relevant to our problem because, in the BF picture, the group variables are associated to the holonomies while the x_i variables are associated to the triad components [21].

I am aware of essentially two different answers and it is not clear at the moment what is the relation among them.

The first approach consists in regarding the algebra $\mathcal{F}(\mathcal{L}^*)$ as a subalgebra of the algebra of quadratic functions on $\mathbb{R}^4$. This is known as the classical Jordan-Schwinger map or symplectic realization. For details we refer to the existing literature [22, 23]. Once such an immersion is realized, one can use the Moyal product on $\mathcal{F}(R^4)$ or variations of it (see for example [24] where the Voros product has been used and [25] for a recent application) to induce a star product on $\mathcal{F}(\mathcal{L}^*)$. It can be shown that the subalgebra is closed under the product. The symplectic realization of the coordinate functions x_i is

$$x_i = \bar{z}^a \sigma_i^{ab} z^b, \quad x_0 = \bar{z}^a \delta^{ab} z^b \tag{32}$$

with $x_0 = |\vec{x}|$ in the kernel of the projection, $a, b \in 1, 2$ and we have made the identification $\mathbb{R}^4 \simeq \mathbb{C}^2$ with canonical symplectic structure

$$\{\bar{z}^a, z^b\} = i \; . \tag{33}$$

Let us notice that similar realizations of a 3d Lie algebra as Poisson subalgebra of quadratic functions on $\mathbb{R}^4$ have been derived for all the 3d Lie algebras [22]. The Moyal star product

$$\phi \star_M \psi(\bar{z}, z) = \phi \exp\left(\frac{\theta}{2} \overleftarrow{\partial}_{z_a} \overrightarrow{\partial}_{\bar{z}_a} - \overleftarrow{\partial}_{\bar{z}_a} \overrightarrow{\partial}_{z_a}\right) \psi \tag{34}$$

induces in $\mathcal{F}(\mathcal{L}^*)$ the product

$$(x_i \star_M \phi)(x) = \left\{ x_i - i\frac{\theta}{2}\epsilon_{ijk} x_j \partial_k - \frac{\theta^2}{8}[(1 + x \cdot \partial)\partial_i - \frac{1}{2} x_i \partial \cdot \partial] \right\} \phi(x) \tag{35}$$

which implies for coordinate functions

$$x_i \star_M x_j = x_i \cdot x_j + i\frac{\theta}{2}\epsilon_{ijk} x_k - \frac{\theta^2}{8}\delta_{ij} \tag{36}$$

Once again, similar expressions exist not only for $SU(2)$ but for all 3d cases [23]. If we replace the Moyal product with the Voros product

$$\phi \star_V \psi(x) = \phi \exp\left(\theta \overleftarrow{\partial}_{z_a} \overrightarrow{\partial}_{\bar{z}_a}\right) \psi \tag{37}$$

we get instead

$$x_i \star_V x_j = x_i \cdot x_j + \theta(|\vec{x}| + i\epsilon_{ijk}x_k) \tag{38}$$

starvoros

with $x_0 = |\vec{x}|$. Let us point out that indeed a whole family of star products can be derived, corresponding to different ordering choices in the quantization procedure on the plane. These products, Moyal and Voros products being just two representatives, are characterized by being translation invariant, therefore reproducing the same star commutator [26].

The second approach consists in defining the star product for $\mathcal{F}(\mathcal{L}^*)$ in terms of a group Fourier transform

$$\tilde{\phi}(x) = \int dg \phi(g) e^{\,\mathrm{Tr}\,(g\vec{\sigma})\cdot\vec{x}} \tag{39}$$

with

$$e^{\,\mathrm{Tr}\,(g_1\vec{\sigma})\cdot\vec{x}} \star_F e^{\,\mathrm{Tr}\,(g_2\vec{\sigma})\cdot\vec{x}} := e^{i\,\mathrm{Tr}\,(g_1 g_2\vec{\sigma})\cdot\vec{x}} \tag{40}$$

It was first introduced in [27], then further investigated in [28, 29]. It has been adapted to GFT in [21] and recently extended to the four dimensional case in [17, 18]. We refer to the literature for a proper definition of the product, limiting ourselves to observe that the induced star product among coordinates does not coincide with the Moyal-induced one. We have instead

$$x_i \star_F x_j = x_i \cdot x_j + i\kappa\epsilon_{ijk}x_k \tag{41}$$

Fstar

with κ a suitable constant, needed to fix the dimensionality. The interesting feature of this product is that it naturally arises in the GFT action for the Boulatov model, when we pass to the Fourier transform [21].

It would be interesting to understand what is the relation between all these products, given that they realize the same commutation relations

$$x_i \star x_j - x_j \star x_i = i\epsilon_{ijk}x_k \tag{42}$$

up to multiplicative constants. In particular we would like to understand whether the Fourier-related star product in Eq. (41) may be induced from one of the translation invariant star products on the algebra $\mathcal{F}(\mathbb{R}^4)$, via symplectic realization.

To conclude this section on noncommutative structures in GFT let me speculate on the issue of recovering the cosmological term in the GFT action.

It is known that, at the level of spin-foam amplitudes, the cosmological constant is taken into account on replacing the group $SU(2)$ with its quantum analogue $SU_q(2)$. This is the Turev-Viro model [30]. On the other hand, at the classical level, it is known since the famous paper of Witten

[31] that the cosmological constant is easily introduced in the 3D action of gravity if one regards gravity with zero cosmological constant as a Chern-Simons theory for the Poincaré gauge group $ISO(2,1)$. Then one deforms the algebra of $ISO(2,1)$ into a fully nonabelian one ($SO(3,1)$ or $SO(2,2)$, depending on the sign of the cosmological constant).

If we look back at the starting Poisson algebra of coordinate functions Eqs. (26)-(28) and its Lie algebra counterparts Eqs (29)-(31), we realize that this amounts to modify Eq. (30) in the Lie algebra and dually Eq. (26) in the Poisson algebra. This makes $SU(2)$ into a Lie-Poisson group. Its full quantization should give the desired quantum group and allow to recover the cosmological term at the GFT level. We shall come back to this issue in a separate publication.

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