Generalized Uncertainty Principle: Approaches and Applications

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(Dated: October 2, 2014)

Abstract

We review some highlights from the String theory, the black hole physics and the doubly special relativity and some thought experiments which were suggested to probe the shortest distances and/or maximum momentum at the Planck scale. Furthermore, all models developed in order to implement the minimal length scale and/or the maximum momentum in different physical systems are analysed. We compare between them. They entered the literature as the Generalized Uncertainty Principle (GUP) assuming modified dispersion relation, and therefore are allowed for a wide range of Applications in estimating, for example, the inflationary parameters, Lorentz invariance violation, black hole thermodynamics, Saleker–Wigner inequalities, entropic nature of gravitational laws, Friedmann equations, minimal time measurement and thermodynamics of the high–energy collisions. One of the higher–order GUP approaches gives predictions for the minimal length uncertainty. A second one predicts a maximum momentum and a minimal length uncertainty, simultaneously. An extensive comparison between the different GUP approaches is summarized. We also discuss the GUP impacts on the equivalence principles including the universality of the gravitational redshift and the free fall and law of reciprocal action and on the kinetic energy of composite system. The existence of a minimal length and a maximum momentum accuracy is preferred by various physical observations. The concern about the compatibility with the equivalence principles, the universality of gravitational redshift and the free fall and law of reciprocal action should be addressed. We conclude that the value of the GUP parameters remain a puzzle to be verified.

Contents

I. SHORT HISTORY

The idea about the existence of a minimal length and/or time was speculated in the ancient time. In modern Physics, the *chronon*, the hypothetical fundamental or the indivisible interval of time with the value of the ratio between the diameter of the electron and the velocity of light proposed by Robert Levi [1] in 1927, would be the first minimum measurable time interval proposed, $\sim 10^{-24}$ s. Within this time interval, the Special Relativity (SR) and Quantum Mechanics (QM) are conjectured to unify in the framework of Quantum Field Theory (QFT). In light of this, the possible existence of a minimal length scale rose the awareness of the physicists. For example, the Planck time is given as $(G\hbar/c^5)^{1/2}$ with the dimensions of *"time"* which is to be formed from G, where c is speed of light, $\hbar$ is the Planck constant and G stands for the Newtonian gravitational constant.

Recdntly, the history of the minimal length scale scenarios have been reviewed [2, 3]. The main developments of minimal length were guided by:

- Singularities in fundamental theories, like Fermi theory of β–decay. This leads to cut–off and QM with a minimal length scale.

- Distasteful arbitrary procedure of cut–off. This leads to modification of the canonical commutation relations of position and momentum operators.

- Role of gravity to test physics at short distance and *"gedanken"* (thought) experiments. This leads to almost the approach that the minimal length scale is connected with some gravitational aspects.

- The trans-Planckian problem (black hole thermodynamic properties). This leads to modification in the dispersion relation and this an essential milestone.

- QM taking into account a minimal length scale and constructing QFT. This leads to modifications of the canonical commutation relations in order to accommodate a minimal length scale.

- The string theory, which leads to generalized uncertainty principle (GUP) based on string scattering in the *super-Planckian regime.*

Defining a fundamental length was necessary to overcome singularities in fundamental theories. First, regularization like <u>*cut-off*</u> or some dimensional quantities was implemented.

But since the cut-off would not be independent of the frame of reference, problems with the Lorentz invariance principle did appear. It dates back to 1930's [4, 5], where it was found that the effect of regularization with respect to cut-off should be the same as that of a fundamentally discrete space-time. At that time, neither a fundamental finite length nor a maximum frequency was known [6–9]. Thus, the fundamental length was thought as a realm of subatomic physics, 10^{-15} m.

Heisenberg refined the fundamental minimal length in 1938 [10]. He discovered that he Fermi theory of β–decay [11, 12] is non-normalizable. At high-energy and the four-fermion coupling the theory breaks down and should be replaced by the exchange of a gauge boson in electroweak interactions [4]. Heisenberg even connected the regularization problem with the breakdown of the perturbation expansion of the Fermi theory. He formulated in 1939 the idea that QM with a minimal length scale would be able to account for the discrete mass spectrum of the elementary particles [4]. Accordingly, the singularities in QFT became better understood [3]. Nevertheless, discrete approaches to space and time remained unappealing due to their problems with the Lorentz invariance principle [3].

Brinstein presented a novel idea in 1936 that the gravity might not be a fundamental force [13]. At that time, neither weak nor strong force had been discovered. It is apparent that gravity does not allow an arbitrarily high concentration of mass in a small region of space-time. This makes it fundamentally different than the electrodynamics. Nowadays, we know that the concentration of mass in a small region of space-time leads to Schwarzshild singularity [13]. The gravitational radius of a test–body $G\,V/c^2$ used in measuring the minimal distance should not be larger than its linear dimensions $V^{1/3}$ [14]. Thus, one obtains an upper bound for its mass density $\rho \lesssim c^2/G\,V^{2/3}$. In this region the measurement possibilities are even more restricted and differs from the quantum-mechanical commutation relations [15, 16]. Without a profound change of the classical notions, it seems hardly possible to extend the quantum theory of gravitation to this region. In 1960, the uncertainties in measuring the average values of Christoffel symbols due to the impossibility of concentrating a mass to a region smaller than its Schwarzschild radius were studied [17], i.e., the conclusion of Bronstein was approved [14].

In 1947, Snyder proposed that the momentum space cut-off should be achieved through a "distasteful arbitrary procedure" [18]. Therefore, instead of cut-off, he suggested modification of the canonical commutation relations of both position and momentum operators.

Accordingly, non-commutative space-time or modification of the commutation relations increase the Heisenberg uncertainty such that a kind of smallest possible resolution can be introduced. A minimal length scale does not need to be in conflict with the Lorentz invariance principle. Later on, the difficulties under inclusion of translations was criticized [19].

The idea of utilizing fundamental limits governing mass and size of any physical system in order to measure time dates back to nearly six decades. Salecker and Wigner suggested the use a quantum clock [20, 21] in measuring *distances* between events in space-time. The measuring rods are entirely avoided, as they are likely macroscopic objects [21]. The quantum clock is thought as constrains on the smallest accuracy and the maximum running time as functions of mass and position uncertainties. In light of this, Wigner second constrain is believed to be more severe than the Heisenberg uncertainty principle, which requires that only one single simultaneous measurement of both energy and time can be accurate. Wigner constrains assure that repeated measurements should not disturb the clock. On the other hand, the clock itself should be able to measure time over its total running period, accurately.

In 1964, another milestone has been put by Mead [22]. The peculiar role of gravity to test physics at short distances was proposed [23]. The role does not mean increasing in the Heisenberg uncertainty.

In mid 1970s, the trans-Planckian problem was introduced by Hawking [24]. Because of infinite blueshift of the photons approaching a black hole horizon, modes with energies exceeding the Planck scale have to be taken into account in calculating the emission rate. In order to solve this problem, Unruh suggested in 1995 [25] a modification in the *dispersion relation*. Therefore, the smallest possible wavelength is the one solving the *trans-Planckian problem*. Starting from a generalization of the Poincare algebra to Hopf algebra, Majid and Ruegg independently suggested a modification in the commutators of the space-time coordinates [26].

Kempf introduced a minimal length scale to the mathematical basis of QM [27, 29, 30]. This leads to different models employing modifications of the canonical commutation relations. Some of these approaches implement modification in the dispersion relation. Others implement a generalization of the Poincare algebra.

The impossibility to arbitrarily resolve the small structures with an object of finite extension was observed in the String theory [31–34]. The String scattering in the *super-Planckian*

regime would result in a generalized uncertainty principle (GUP). This apparently would prevent a localization to better than the String scale.

In supporting the phenomena that uncertainty principle would be affected by the QG, various examples can be mentioned. In the context of polymer quantization, the commutation relations are given in terms of the polymer mass scale [35]. Also, the standard commutation relations in QM are conjectured to be generalized (changed) at the length scales of the order of Planck length [29, 36–38]. Such modifications are supposed to play an essential role in the quantum gravitational corrections [39]. Accordingly, the standard uncertainty relation of QM is replaced by a gravitational uncertainty relation having a minimal observable length (of the order of Planck length) [40–43].

Nevertheless, we review argumentation against the GUP approaches insection IX. We first start with the *equivalence principle*, which is one of the five principles forming the basis of GR, where the motion of the gravitational test-particle in a gravitational field should be independent on the particle's mass and composition [291]. On the other hand, when taking into consideration the Strong (SEP) [66] and Weak Equivalence Principle (WEP) [66], the gravitational field should couple to everything [291].

II. INTRODUCTION

As introduced in section I, the existence of a minimal length is a great prediction from different approaches related to QG such as the black hole physics [44, 45] and the String theory [46, 47]. The mean idea is simply that the String is conjectured not to interact at distances smaller than its size, which is determined by its tension. For completeness, we highlight that the information about the String interactions would be included in the Polyakov action [48]. The existence of a minimal length leads to generalization of the Heisenberg uncertainty principle (GUP) [46]. At Planck (energy) scale, the corresponding Schwarzschild radius becomes comparable to the Compton wavelength. High energies (Planck energy) seem to result in further decrease in the Schwarzschild radius Δx in the presence of gravitational effects. In light of this, $\Delta x \approx \ell_{Pl}^2 \Delta p / \hbar$. This observation and the ones deduced from various *gedanken* experiments suggest that the GUP approach should be essential, especially at some concrete scales.

In QM, the physical observables are described by operators in Hilbert space. Given an observable A, we define an operator as a standard deviation of A [49]

$$\Delta A = A - \langle A \rangle, \tag{1}$$

where the expectation value is given by

$$\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2. \tag{2}$$

Using Schwartz inequality [50],

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2, \tag{3}$$

which is valid for any ket and bra state.

$$|\alpha\rangle = \Delta A |\alpha'\rangle, \tag{4}$$

$$|\beta\rangle = \Delta B |\beta'\rangle. \tag{5}$$

By using Dirac algebra, we get

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |\langle \Delta A \, \Delta B \rangle|^2, \tag{6}$$

which is known as Cauchy-Schwartz inequality. In other words,

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \Delta A \Delta B \rangle|. \tag{7}$$

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In Heisenberg algebra, the position $\hat{x}$ and momentum operator $\hat{p}$ satisfy the canonical commutation relation

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \tag{8}$$

As a consequence, for position and momentum uncertainties, Δx and Δp, respectively, of a given state, the Heisenberg uncertainty relation reads

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}. \tag{9}$$

To probe an arbitrarily small length scale, one has to utilize tools of sufficiently high energy (high momentum) and thus very short wavelength. This is the principle on which colliders/accelerators, such as the Relativistic Heavy-Ion Collider (RHIC) [51], Large Hadron Collider (LHC) [52], FermiLab [53], etc., are based. On the other hand, there are reasons to believe that at high energies, the gravity becomes important. In light of this, the former conclusion would be no longer true. In other words, the linear relation between energy and wavelength would be violated, as well.

The detectability of quantum space-time foam with gravitational wave interferometers has been discussed [54]. The limited measurability of the smallest quantum distances was criticized [54]. The authors gave an operative definition for the quantum distances and explained how to eliminate the contributions from the total quantum uncertainty [55]. Four decades later, Barrow applied Wigner inequalities [20, 21] in describing the quantum constrains on the black hole lifetime [56]. The black hole running time should be correspondent to the Hawking lifetime. The latter is to be calculated under the assumption that the black hole is a black body and therefore the utilization of Stefan-Boltzmann law is eligible. Also, it is found that the Schwarzschild radius is correspondent to the constrains on Wigner size. Furthermore, the information processing power of a black hole was estimated by emitted Hawking radiation [57].

A. Generalized (gravitational) uncertainty principle

The existence of a minimal length based on GUP introduces that the space in Hilbert space representation [29] can describe a non-commutative geometry. The non-commutative geometry can also arise as a momentum over curved spaces [58]. From various *gedanken* experiments, which have been designed to measure the area of the apparent horizon of a

black hole in QG [59], the uncertainty relation is found preformed [44]. The deformed or modified Heisenberg algebra introduces a relation between QG and Poincare algebra [59]. Under the effect of GUP in an n-dimension space, it is found that even the gravitational constant G [60] and the Newtonian law of gravity [66] are subject of modifications. The interpretation of QM through a quantization in 8-dimensional manifold implies the existence of an upper limit in the accelerated particles [61]. Nevertheless, GUP given in forms of quadratic [29, 44] and linear terms of momenta [62] assume that the momenta approach the maximum value at very high energy (Planck scale) [62].

Recently, a new GUP approach was proposed [63]. It seems to fit well with the String theory and the black hole physics (with quadratic term of momenta) and agrees well with the Doubly Special Relativity (DSR) (with linear term of momenta). This simultaneously predicts a minimal measurable length and a maximum measurable momentum and suggests that the space should be quantized and/or discritized. Nevertheless, the approach [63] shows some difficulties [64]. In light of this, a new GUP approach was proposed [64] to characterize a minimal length uncertainty and a maximal momentum, simultaneously. On the other hand, another approach is conjectured to absolve an extensive comparison with Kempf, Mangano and Mann (KMM) [29]. The latter has been performed in Hilbert space [65]. Here, a novel idea of minimal length modelled in terms of the quantized space-time was implemented. Thus, this new approach agrees with QFT and Heisenberg algebra, especially in context of non-commutative coherent states representation. The resulting GUP approach can be studied at UV finiteness of Feynman propagator [65].

B. Physics of generalized (gravitational) uncertainty principle

There are various observations showing that the GUP approaches offer a valuable possibility to study the influences of the minimal length on the properties of a wide range of physical systems, especially at the quantum scale [37, 44, 66]. The effects of linear GUP approach have been studied on compact stars [221], Newtonian law of gravity [67], inflationary parameters and thermodynamics of the early Universe [68], Lorentz invariance violation [69] and measurable maximum energy and minimum time interval [70]. Furthermore, the effects of QG on the quark-gluon plasma (QGP) are introduced [71]. It was found that the GUP can potentially explain the small observed violations of the weak equivalence principle in

neutron interferometry experiments [72], section IX, and also predicts a modified invariant phase space which is relevant to LT. It is suggested [73] that GUP can be measured directly in Quantum Optics Lab [74, 75]. Furthermore, deformed commutation relations would cause new difficulties in quantum as well as in classical mechanics. We give a list of some of these problems as follows.

- one-dimensional harmonic oscillator with minimal uncertainty in position [29] and minimal uncertainty in position and momentum [76] and the d-dimensional harmonic oscillator with minimal uncertainty in position [77, 78],

- problem of 3d Dirac oscillator [79] and the solution of $(1+1)$-d Dirac oscillator within Lorentz covariant algebra [80],

- 1d and 3d Coulomb problem within deformed Heisenberg algebra in perturbation theory [81–85],

- scattering problem in deformed space with minimal length [86],

- ultra-cold neutrons in gravitational field with minimal length [87–89],

- influence of minimal length on Lamb shift, Landau levels, and tunnelling current in scanning tunnelling microscope [75, 90]

- Casimir effect in a space with minimal length [91],

- effect of non-commutativity and the existence of a minimal length on the phase space of cosmological model [92],

- various physical consequences of non-commutative Snyder space-time geometry [93], and

- classical mechanics in a space with deformed Poisson brackets [94–96].

Furthermore, in sections VI and VII, we review the applications of the quadratic and linear GUP approaches, respectively, on physics of early Universe, inflation parameters, Lorentz invariance violation, black hole thermodynamics, compact stellar objects, Saleker-Wigner inequalities, entropic nature of gravitational force, time measurement and thermodynamics

of high-energy collisions.

The present review is organized as follows. In section II A, the generalized extended uncertainty principle (GEUP) is defined. The relationship between the minimal length and maximum momentum is also presented. As introduced in previous sections, there are various approaches to GUP presenting the existence of minimal length of non-zero value that leads to non-commutative geometry. The physics of GUP approaches is reviewed in section II B.

In section III, we summarize the behavior of some well-known expressions for GUP, such as thought experiments designed to study the effects of QG in modified Heisenberg uncertainty principle (HUP). These expressions contain quadratic term of momenta with a minimal uncertainty on position. In section III B, we shall investigate the modification of the uncertainty relation due to the high-energy fixed-angle scatterings at short length such as the String length. In section III C, the uncertainty relation through various *gedanken* experiments designed for the measurement of the area of the apparent horizon of black hole is reviewed. These thought experiments assume QG due to recording of many photons of the Hawking radiation emitted from the apparent horizon. Due to quantized space-time of the quantum field theory and the geometric approach to curvature of momentum space, an algebraic approach can be expressed in the co-products and the description of the Hopf-algebra [26] leading to modified commutation relation between the space and the momenta, section III D. In section III E, the modified de Broglie relation which leads to changing the commutation relation between space and momentum and the investigation of minimal length and/or non-zero minimal length. In section III F, a new commutation relation containing a linear term as an addition of the quadratic term of momenta and predicts of the maximum measurable of momenta, shall be investigated.

In section IV, the relations describing the minimal length uncertainty are outlined. Two proposals for the modification of the momentum operator are introduced. The proposal of a minimal length uncertainty with a further modification in the momentum shall be reviewed. The main features in Hilbert space representation of QM of the minimal length uncertainty will be studied. Furthermore, their difficulties are also listed out. Furthermore, we show how to overcome these difficulties, especially in Hilbert space representation.

In section V, the GUP approaches relating to the String theory and the black hole physics (lead to a minimum length) and the ones relating to DSR (suggest a similar modification

of commutators) shall be studied. The main features and difficulties in Hilbert space representation will be reviewed, as well, and we show how to overcome these difficulties.

Section VI is devoted to the applications of the quadratic GUP approach. We list out seven applications; physics of early Universe, inflation parameters, black hole thermodynamics, compact stellar objects, Saleker-Wigner inequalities, entropic nature of gravitational force and time measurement.

Section VII is devoted to the applications of the linear GUP approach on the same list of problems as given in section VI. Additional two problems are also discussed, namely the Lorentz invariance violation and thermodynamics of high-energy collisions.

In section VIII, other alternative approaches to GUP such as the one suggested by Nouicer, in which an exponential term of momentum and minimal length appears, shall be introduced. This approach agrees well with the GUP which is originated in the theories for QG. There is another approach coming up with higher orders of the minimal length uncertainty and maximal observable momentum. Finally, we compare between these approaches.

The effects of GUP on the principles of GR are studied in section IX. The results estimated in various thought experiments are compared with the potential effects of GUP. It is found that the GUP apparently changes the natural statement of the kinematic energy of the deformed system. Argumentation against the GUP approaches shall be reviewed in section IX. These can be divided into two groups; the equivalence principle and the kinetic energy of composite system. The first group includes the universality of the gravitational redshift and the free fall and the law of reciprocal action.

III.   THE GENERALIZED (GRAVITATIONAL) UNCERTAINTY PRINCIPLE

The consistent unification of the classical description of GR with QM still an open problem. A first attempt assumes that the two theories can be used as a guiding principle to the search of a fundamental theory of QG. Secondly, several arguments starting from theoretical analysis in String theory to more sophisticated or even *gedanken* experiments to measure the minimal length, would conclude that there is a new contribution to the quantum uncertainty with a gravitational origin leading to a length scale as a Planck length in the determination of space-time coordinates.

On one hand, these approaches made essential predictions. We have listed out some of these in section II. Other applications shall be elaborated in sections VI and VII. DSR suggests a possibility to relate the transition from the quantum behavior at the microscopic level to the classical behavior at the macroscopic level with the modification of QM induced by a modification of the relativity principles. Thus, the laboratory tests should be able to judge about these theories. On the other hand, the predictions remain uncertain due to the limitations of the current technologies. Nevertheless, the minimal length has been observed in condensed matter and atomic physics experiments, such as Lamb shift [74, 90], Landau levels [74, 90], and the Scanning Tunnelling Microscope (STM) [74].

A.   Introduction

As discussed, it is widely accepted that HUP should break down at energies close to the Planck scale. Taking into account the gravitational effects, an emergence of a minimal measurable distance seems to be inevitable. More generally, the generalized (gravitational) uncertainty principle (GUP) reads [29]

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \alpha(\Delta x)^2 + \beta(\Delta p)^2 + \zeta \right), \tag{10}$$

where β and ζ are positive and independent on uncertainties Δx and Δp but may - in general - depend on the expectation values of the operators $\mathbf{x}$ and $\mathbf{p}$ in the way that $\zeta = \alpha \langle x \rangle^2 + \beta \langle p \rangle^2$.

In Fig. 1, the minimal momentum uncertainty Δp is given in dependence on the position uncertainty Δx. It is apparent that the position minimal uncertainty $\Delta x_0 \neq 0$ and $\Delta x_{min} \propto \Delta p_{max}$ [29]. Furthermore, the commutation relations read

$$[\mathbf{x}, \mathbf{p}] = i\hbar \left(1 + \alpha \mathbf{x}^2 + \beta \mathbf{p}^2 \right). \tag{11}$$
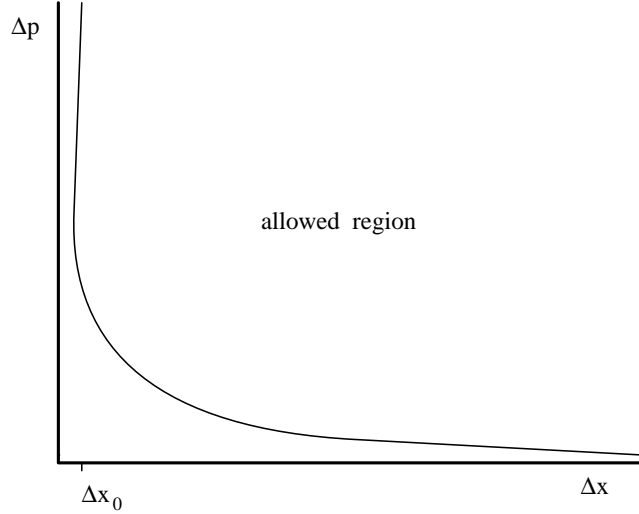
Fig. 1: The modified uncertainty relation apparently implies the existence of a minimal length of uncertainty Δx_0. The graph taken from Ref. [29].

In QM, both $\mathbf{x}$ and $\mathbf{p}$ could be represented as operators acting on position- and momentum-space wavefunctions, respectively,

$$\phi(x) = \langle x|\phi(x)\rangle, \tag{12}$$

$$\phi(p) = \langle p|\phi(p)\rangle, \tag{13}$$

where $|x\rangle$ and $|p\rangle$ are position and momentum eigenstates. The operators $\mathbf{x}$ and $\mathbf{p}$ are essentially self-adjoint. The eigenstates can be approximated to an arbitrary precision by sequences $|\phi_n\rangle$ of the physical states of increasing localization in position- or momentum-space

$$\lim_{n\to\infty} \Delta x_{|\phi_n\rangle} = 0, \tag{14}$$

$$\lim_{n\to\infty} \Delta p_{|\phi_n\rangle} = 0. \tag{15}$$

As pointed out in Refs. [27, 28], this situation changes drastically with the inclusion of minimal uncertainties $\Delta x_0 > 0$ and/or $\Delta p_0 > 0$. For example, a non-zero minimal uncertainty in position is given as

$$(\Delta x)^2_{|\phi_n\rangle} = \langle\phi|\left(\mathbf{x} - \langle\phi|x|\phi\rangle\right)^2|\phi\rangle \geq \Delta x_0, \qquad \forall \longrightarrow |\phi\rangle, \tag{16}$$

implying that no physical state with such a position eigenstate would exist [29]. The reason is that an eigenstate would - of course - have zero uncertainty in position. A minimal

uncertainty in position apparently means that the position operator is no longer essentially self-adjoint but merely symmetric. It is obvious that the preservation of symmetry assures that all expectation values are real. When self-adjointness abandoned, a golden path to introduce minimal uncertainties shall be planed [29].

Since there are then no longer position eigenstates $|x\rangle$ in representation of the Heisenberg algebra, the Heisenberg algebra will no longer find a Hilbert space representation on the position wavefunctions $\langle x|\phi(x)\rangle$ [29]. In light of this, the discussion should be restricted to the case $\Delta x_0 \neq 0$ and then $\alpha = 0$, where there is no minimal uncertainty in momentum.

In a similar manner, a minimal uncertainty in momentum abandons the momentum space wavefunctions [29]. This will allow us to work with the convenient representation of the commutation relations on momentum space wave functions. Then, the simplest GUP relation leads to

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \beta(\Delta p)^2 + \zeta\right), \tag{17}$$

where the constant ζ is positive and apparently related to the expectation value of momentum, $\zeta = \beta \langle p \rangle^2$.

1. Non-commutativity of space

HUP actually has a strong relationship to the canonical commutation or commutative phase space structures. When HUP should be broken down by GUP, an operational form of non-commutative (NC) phase space structures will be observed. The generic form [97] reads

$$[x_i, p_j] = i\hbar \left[\delta_{ij} \left(1 + \beta f_1 \left(\mathbf{p}^2\right)\right) + f_2 \left(\mathbf{p}^2\right) p_i p_j\right], \tag{18}$$

$$[x_i, x_j] = i\hbar f_{ij}(p) \neq 0. \tag{19}$$

The presence of a minimum length scale or a maximum momentum scale or both simply leads to the possibility of originating GUP with NC algebras. Both are likely consistent. Based on this, Kempf proposed the following algebraic relations [98]

$$[x_i, p_j] = i\hbar \left[\delta_{ij} \left(1 + \beta p^2\right) + \beta' p_i p_j\right], \tag{20}$$

$$[x_i, x_j] = i\hbar \left(\beta' - 2\beta\right) \left(x_i p_j - x_j p_i\right), \tag{21}$$

$$[p_i, p_j] = 0. \tag{22}$$

Other algebraic relations have been introduced in Ref. [29]

$$[x_i, p_j] = i\hbar \delta_{ij} \left(1 + \beta p^2\right), \tag{23}$$

$$[x_i, x_j] = -2 i\hbar \beta \left(x_i p_j - x_j p_i\right), \tag{24}$$

$$[p_i, p_j] = 0. \tag{25}$$

Recently, new algebraic relations have been presented in Ref. [99]

$$[x_i, p_j] = i\hbar \left[\delta_{ij} \left(1 + \beta p^2\right) + \beta' p_i p_j + O(\beta'^2, \beta^2) \right], \tag{26}$$

$$[x_i, x_j] = i\hbar \frac{\left(2\beta - \beta'\right) + \left(2\beta + \beta'\right)\beta p^2}{1 + \beta p^2} \left(x_i p_j - x_j p_i\right), \tag{27}$$

$$[p_i, p_j] = 0. \tag{28}$$

It is apparent that the GUP approaches which are consistent with NC algebras can open the possibility of space discreteness and/or quantization. In other words, the physical states of space should be non-commute. Although the physical states can not be measured, simultaneously, the space discretization seems to be accessible.

B. String theory

In String theory, a GUP approach was first proposed by Amati *et al.* [46]. The ultra high-energy scatterings of Strings were studied in order to check how the theory tackles in consistences of QG at the Planck scale [46]. Some interesting effects are compared to those which were found in *usual* field theories, especially the ones originating from the soft short-distance behavior of the String theory [46]. The hard processes are studied at a short distance as the high-energy fixed-angle scatterings. The latter are apparently not able to test distances shorter than the characteristic String length $\lambda_s = (\hbar\alpha)^{1/2}$, where α is the String tension.

Another scale is dynamically generated. The D-dimensional gravitational Schwarzschild radius $R(E) \sim (G_N E)^{1/(D-3)}$ approaches towards the String length λ_s [46]. This depends on whether $R(E)$ smaller or greater than λ_s. If $R(E) > \lambda_s$, then new contributions at distances of the order of $R(E)$ appear. This indicates a classical gravitational instability that can be attributed to the black hole formation. If the opposite should be the case ($R(E) < \lambda_s$), then their contributions are irrelevant. There are no black holes with a radius smaller than the

String length. In this case, the analysis of short distances can go on. It has been shown that the larger momentum transfers do not always correspond to shorter distances. Precisely, the analysis of the angle distance relationship suggests the existence of a scattering angle θ_M. When the scattering should take place at $\theta < \theta_M$, then the relation between the interaction distance and the momentum transfer is the classical one, i.e. follows the Heisenberg relation with $q \sim \hbar/b$, where b is the impact parameter. But when $\theta \gg \theta_M$, then the classical picture is no longer valid. An important new regime where $\langle q \rangle \sim b$ would be constructed. This suggests a modification of the uncertainty relation at the Planck scale [46]

$$\Delta x \sim \frac{\hbar}{\Delta p} + Y \, \alpha \, \Delta p, \tag{29}$$

where Y is a suitable constant. Consequently, the existence of a minimal observable length of the order of String size λ_s is likely.

C.   Black hole physics

Several works have been devoted to perform the uncertainty relations and their measurability bounds in QG [44]. Gedanken (thought) experiments have been proposed to measure the area of apparent horizon of a black hole [44]. Accordingly, a generalization of the uncertainty principle has been concluded. The GUP approach agrees well with the one which is deduced from the String theories [44, 46, 100]. Also, in String theories, the tool of gedanken String collisions at Planck energy was very useful [45, 46]. In addition to these, the renormalization group analysis has been applied to the String [47]. A main physical ingredient was the Hawking radiation [24]. The black hole approach to GUP, which is a rather model independent approach, agrees, especially in its functional form, with the one obtained in framework of the String theory.

The gedanken experiment proceeds by observing the photons scattered by the studied black hole. The main physical hypothesis of the experiment is that the black hole emits Hawking radiation. Detecting the Hawking radiation, it turns to be possible to span an "image" of the black hole [24]. Besides, measuring the direction of the propagating photons that are emitted at different angles and tracing them back, we can - in principle - locate the position of the center of the hole [24]. In this way, we make a measurement of the radius R_h of the horizon of the hole. Apparently, this measurement has two sources of uncertainty

[44].

- The first one is based on the fact that a photon with wavelength λ cannot carry information about a more detailed scale than λ itself [44]. As in the classical Heisenberg analysis, the resolving power of the microscope gives the minimum error

$$\Delta x^{(1)} \sim \frac{\lambda}{\sin \theta}, \tag{30}$$

where θ is the scattering angle. Then, the final momenta should have the uncertainty $\Delta p \sim h \sin(\theta)/\lambda$. During the emission process, the mass of the black hole varies from M to $M - \Delta M$ [44], where $\Delta M = h/(c\,\lambda)$. The radius of the horizon changes, accordingly. The corresponding uncertainty is intrinsic to the measurement.

For example, the metric field of the Reissner black hole [101] is given as

$$ds^2 = -\left(1 - \frac{2G}{r} + \frac{GQ}{r^2}\right) dt^2 + \left(1 - \frac{2G}{r} + \frac{GQ}{r^2}\right) dr^2 + r^2 d\Omega^2. \tag{31}$$

Also, the apparent horizon is defined as the outer boundary of a region of closed trapped surfaces. In spherical topology and Boyer-Lindquist coordinates [102], the apparent horizon is located at $r = R_h$

$$R_h = G\,M \left[1 + \left(1 - \frac{Q^2}{G\,M^2}\right)^{1/2}\right]. \tag{32}$$

The Boyer-Lindquist coordinates are a generalization of the coordinates used for the metric of a Schwarzschild black hole. This can be used to express the metric of a Kerr black hole [103]. Accordingly, the line element for a black hole with mass M, angular momentum J, and charge Q reads

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - K\,\sin^2(\theta)\,d\phi\right)^2 + \frac{\sin^2(\theta)}{\Sigma}\left(\left(R^2 + K^2\right)d\phi - K\,dt\right)^2 + \frac{\Sigma}{\Delta}\,dR^2 + \Sigma\,d\theta^2, \tag{33}$$

where

$$\Delta = R^2 - 2MR + K^2 + Q^2, \tag{34}$$

$$\Sigma = R^2 + K^2 \cos^2(\theta), \tag{35}$$

$$K = \frac{J}{M}. \tag{36}$$

In Boyer-Lindquist coordinates, the Hamiltonian of a test particle is separable in Kerr space-time. From Hamilton-Jacobi theory, a fourth constant of the motion can be derived. This is known as Carter's constant [104]

- The second source of uncertainty is the case, when $1 - 2G/r + GQ/r^2$ vanishes. In one dimension, for $M \gg \Delta M$ and $Q^2 = G M^2$

$$\Delta x^{(2)} = GM \pm \sqrt{G^2(M + \Delta M)^2 - GQ^2}, \tag{37}$$

$$\Delta x^{(2)} > G\sqrt{2M\Delta M} \geq \frac{2G}{c^2}\Delta M = \frac{2G}{c^3}\frac{h}{\lambda}. \tag{38}$$

By means of inequality $\lambda/\sin\theta \geq \lambda$, the uncertainty in $\Delta x^{(2)}$ and the quantity itself can be combined, linearly

$$\Delta x \gtrsim \lambda + \kappa\frac{l_p^2}{\lambda} \tag{39}$$

$$\Delta x \gtrsim \frac{\hbar}{\Delta p} + c\, G\Delta p, \tag{40}$$

$$\Delta x \gtrsim \frac{\hbar}{\Delta p} + \beta\Delta p, \tag{41}$$

where κ is a constant. The other numerical constant β cannot be predicted by the model-independent arguments presented so far. It is natural to investigate whether the relation given in Eq. (39) reproduces what was obtained considering only a very specific measurement. This principle would assure that the results should have a more general validity in QG.

There is another approach obtained in a *gedanken* experiment of a micro black hole in four-dimensions [66]. This approach is given as a function of time and energies. When position with a precision of order Δx is measured, the quantum fluctuations of the metric field around the measured position with energy amplitude can be expected as

$$\Delta E \sim \frac{c\,\hbar}{2\,\Delta x}. \tag{42}$$

The Schwarzschild radius associated with the energy fluctuation ΔE,

$$R_s = \frac{2\,G_N\,\Delta E}{c^4}. \tag{43}$$

The energy fluctuation ΔE would grow up and the corresponding the radius R_s would become larger and larger, until it reaches the same size as Δx. As it is well known, the critical length is the Planck length, $R_s = \Delta x \equiv l_p$, where $l_p^2 = G_N\hbar/c^3$ and the associated energy is the Planck energy $\epsilon_p = \hbar\, c/(2\, l_p) = \sqrt{\hbar\, c^5/G_N}/2$.

When the discussion is limited to the Planck energy, this would enlarge further the Schwarzschild radius R_s. The situation can be summarized by the inequalities

$$\Delta x \gtrsim \frac{c\hbar}{2\Delta E} \implies \Delta E \ll \epsilon_p, \tag{44}$$

$$\Delta x \gtrsim \frac{2G_N \Delta E}{c^4} \implies \Delta E \sim \epsilon_p. \tag{45}$$

If these two inequalities are combined linearly, then

$$\Delta x \gtrsim \frac{c\hbar}{2\Delta E} + \frac{2G_N \Delta E}{c^4}. \tag{46}$$

This is a generalization of the uncertainty principle to the cases in which gravity gets very important, i.e., to energies of the order of ϵ_p. We have discussed this in connection with the various colliders and the indirect relation between energy and wavelength. We noticed that this relation might be violated at very high energy, due to the dominant role of gravity at this energy scale. It is obvious that the minimum value of Δx is reached for $\Delta E_{max} \sim \epsilon_P$,

$$\Delta x_{min} = 2\, l_p \tag{47}$$

D.   Snyder form

Hopf algebra [26] introduces a relationship between a dual structure and the associated product rules fulfilling certain compatibility conditions [26]. A additional structure was found in the geometric approach. The curvature of momentum space is expressed in the algebraic approach in co-products and antipodes of Hopf algebra [26]. As in the geometric approach, there is an ambiguity in the choice of coordinates in the phase space.

Snyder presented a theory for quantized space-time [105, 106]. Different possibilities are investigated in resolving the infinities problem in early days of QFT. Snyder considered a de-Sitter space with real coordinates $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4)$. In addition to the various choices of position space coordinates, one can also use different coordinates in the momentum space, by choosing different parametrizations of the hypersurface than the ones of Snyder [26]. One such parametrizations is using coordinates π_ν, which are related to Snyder basis [26]

$$\eta_0 = -m_p \sinh\left(\frac{\pi_0}{m_p}\right) - \frac{\vec{\pi}^2}{2\, m_p} \exp\left(\frac{\pi_0}{m_p}\right), \tag{48}$$

$$\eta_i = -\pi_i \exp\left(\frac{\pi_0}{m_p}\right), \tag{49}$$

$$\eta_4 = -m_p \cosh\left(\frac{\pi_0}{m_p}\right) - \frac{\vec{\pi}^2}{2\, m_p} \exp\left(\frac{\pi_0}{m_p}\right), \tag{50}$$

where η_4 is not constant on the hypersurface and π_ν is the bicrossproduct basis of the Hopf algebra [26].

We make a natural choice for the algebraic approach and define the position X and time T operators [105, 106]

$$X_i = i\,a\left(\eta_4\frac{\partial}{\partial\,\eta_i} - \eta_i\frac{\partial}{\partial\,\eta_4}\right), \tag{51}$$

$$T = \frac{i\,a}{c}\left(\eta_4\frac{\partial}{\partial\,\eta_i} + \eta_i\frac{\partial}{\partial\,\eta_4}\right), \tag{52}$$

where $i = 1, 2, 3$ and a is a natural unit of length. These operators should act on functions of variables $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4)$. In additional, the energy and momentum operators [105, 106]

$$P_T = \frac{\hbar\,\eta_0}{a\,\eta_4}, \tag{53}$$

$$P_i = \frac{\hbar\,\eta_i}{a\,\eta_4}. \tag{54}$$

Thus, the commutators between positions and momenta are given by

$$[X_i,\,P_j] = i\,\hbar\left(1 + \left(\frac{a}{\hbar}\right)^2 P^2\right), \tag{55}$$

where $P^2 = \sum_j^3 P_j\,P_j$. These algebraic relations (described by Snyder) are close to the generalized uncertainty commutation relation [105, 106], represent an essential milestone on the path to construct theoretical framework for GUP.

E. Modified de Broglie relation

The modified de Broglie relation has been investigated by Hossenfelder *et al.* [107]. It is assumed that the wave number $\kappa(p)$ is an odd function and nearly linear for small values of p and approaching asymptotically some upper limit which is proportional to a minimal length $M_p \sim L_p^{-1}$ [107]. Such a function will have an expansion in p as follows.

$$\kappa = p - \sigma\frac{p_3}{m_p^2}, \tag{56}$$

Taking into consideration the commutation relation between x and $\kappa(p)$, the generalized uncertainty principle can be deduced [107]

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}\left\langle\frac{\partial p}{\partial\kappa}\right\rangle, \tag{57}$$

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}\left(1 + \sigma\frac{<p^2>}{M_p}\right). \tag{58}$$

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This gives the commutation relation

$$[X_i, P_j] = i\hbar \left(1 + \sigma \frac{p^2}{M_p}\right). \tag{59}$$

Obviously, these algebraic relations (de Broglie relations) are close to the generalized uncertainty commutation relation presented in Refs. [29, 44, 58, 66].

F. Doubly Special Relativity

The doubly relativistic theory are a group of transformations with two invariants [62]. In additional to the constant speed of light, it also assumed that an invariant energy scale should exist. Nevertheless, this group of transformation remains Lorentzian. A non-linear realization of the Lorentz transformations in energy-momentum (E, p) space parametrized by an invariant length l was defined in Ref. [108]

$$\epsilon = E f \left(l E, l^2 p^2\right), \tag{60}$$

$$\pi_i = P_i g \left(l E, l^2 p^2\right), \tag{61}$$

where (ϵ, π) are auxiliary linearly transforming variables which define the non-linear Lorentz transformation of the physical energy-momentum (E, p). Then, we get two functions with two variables f and g. These functions are able to parametrize more general non-linear realization of the Lorentz transformations with rotations realized as linearly depending on the dimensional scale [62].

The condition to recover the special relativistic theory at low energy reduces to the condition $f(0; 0) = g(0; 0) = 1$. Each choice of the two functions f and g will lead to a generalization of the relativity principle with an invariant length scale. The Lorentz transformations connecting the energy-momentum of a particle in different inertial frames differ from the standard special relativistic linear transformations which are recovered when $l E \ll 1$ and $l^2 p^2 \ll 1$ a non-linear realization of the Lorentz transformations corresponds to the choice of the two functions [36, 37, 109]

$$f = \frac{1}{2} \left[\left(1 + l^2 p^2\right) \frac{e^{l E}}{l E} - \frac{e^{-l E}}{l E}\right], \tag{62}$$

$$g = e^{l E}. \tag{63}$$

For a particle of mass m, the relation between the energy and momentum is given by [36, 37, 109]

$$\left(1 - l^2\, p^2\right) e^{l\, E} + e^{-l\, E} \; = \; e^{l\, m} + e^{-l\, m}. \tag{64}$$

Accordingly,

$$e^{l\, E} = \frac{\cosh(l\, m) + \sqrt{\cosh^2(l\, m) - (1 - l^2 p^2)}}{(1 - l^2 p^2)}. \tag{65}$$

Furthermore, an upper bound on the momentum can be defined as

$$p_{max}^2 < \frac{1}{l^2}, \tag{66}$$

suggesting the existence of a minimal measurable length which would restrict the momentum of the test particle to take any arbitrary value. This leads to an upper bound, P_{max}, on this momentum. This means that there is a maximal momentum of the particle due to the fundamental structure of space-time at the Planck scale [62].

The commutation relation between the canonical variables x and p was suggested [62]

$$[X_i\,, P_j] \; = \; i\,\hbar \; \left[e^{-l\, E} \delta_{i\, j} + \frac{l^2}{\cosh(l\, m)} p_i p_j \right]. \tag{67}$$

It is obvious that when the mass m gets much larger than the inverse of the length scale l, a classical phase space is approached. This result simply suggests the possibility to relate the transition from the quantum behavior at the microscopic level to the classical behavior at the macroscopic level with the modification of QM induced by a modification of the relativity principle [62]. If we consider the massless particle, then

$$e^{l\, E} = \frac{1}{1 - l\, |\mathbf{p}|}. \tag{68}$$

It is found that the commutation relation between the canonical variables x and p [62] should be modified in doubly special relativity

$$[X_i,\, P_j] \; = \; i\,\hbar \; \left[(1 - l\, |\mathbf{p}|)\, \delta_{ij} + l^2\, p_i\, p_j \right]. \tag{69}$$

It is apparent that when the momentum approaches its maximum value, one has a non-trivial limit for the canonical commutation relation [62].

IV. MINIMAL LENGTH UNCERTAINTY RELATION

In framework of HUP, there is no restriction on the measurement precision of the particle's position, Δx. This minimal position uncertainty can be made arbitrarily small even vanishes [37]. The theoretical motivations to avoid such a limit are reviewed in section I. It is obvious that going down to such a limit is not essentially the case of the framework of GUP, because of the existence of a minimal length uncertainty, section I, which obviously modifies the Hamiltonian of the physical system leading to modifications, especially at the Planck scale of the energy spectrum of quantum systems, which in turn predicts small corrections in the measurable quantities. As discussed in section III, this has been observed in condensed matter and atomic physics experiments, such as Lamb shift [74, 90], Landau levels [74, 90], and the Scanning Tunnelling Microscope (STM) [74]. Thus, a hope arises that the quantum gravity effects may be observable in the laboratory.

We show two GUP approaches suggesting the existence of minimal length uncertainty. We shall summarize the features to each of them. In section IV A, we show the proposal of the minimal length uncertainty with momentum modification [74, 90]. In section IV B, we study the main features in Hilbert space representation of QM of the minimal length uncertainty [29].

A. Momentum modification

Via Jacobi identity, the GUP approaches modify Heisenberg algebra

$$[x_i, \, p_j] = i \, \hbar \, \left(\delta_{ij}(1 + \beta \, p^2) + 2 \, \beta \, p_i \, p_j \right), \tag{70}$$

These ensure [74, 90] that

$$[x_i, \, x_j] = \, 0 \, = [p_i, \, p_j] \,. \tag{71}$$

Thus, the position and momentum operators can be, respectively, defined

$$X_i \, = \, x_{0i}, \tag{72}$$

$$P_j \, = \, p_{0j} \, (1 + \beta \, p_0^2). \tag{73}$$

We note that $p_0^2 = \sum_j^3 p_{0j} p_{0j}$ satisfies the canonical commutation relations $[x_{0i}, \, p_{0j}] = i \, \hbar \, \delta_{ij}$ and p_{0j} is defined as the momentum at low energy scale which is represented by $p_{0j} = -i \, \hbar \partial / \partial \, x_{0j}$ while P_j is considered as the momentum at the higher energy scales.

As discussed, the introduction of a minimal length leads to modification in the canonical commutation relations, while the position space at Planck scale must differ from the position in the canonical system, because the absence of zero-state in the position eigenstates. Thus, it is useful to modify the position space rather to allow for modification in momentum space. The latter leads to non-commutation of space $[x_i, x_j] \neq 0$.

From the assumptions given in Eqs. (72) and (73), it is impossible to utilize Hilbert representation for position space, since no zero physical state exists. With the definition of the modified momentum at the highest energy scales, Eq. (73), the non-commutative values of momentum states $[p_i, p_j] \neq 0$. Thus, the representation in Hilbert space is not available.

B. Hilbert space representation

We discuss a generalized quantum theoretical framework, which should be able to implement the appearance of a non-zero minimal uncertainty in position. The discussion can be confined to exploring the applications of such a minimal uncertainty in the context of non-relativistic QM. Various new features of Hilbert space representation of QM, especially at the Planck scale, are introduced [29].

$$\Delta x \, \Delta p \ \geq \ \frac{\hbar}{2} + \beta_0 \, l_p^2 \, \frac{(\Delta p)^2}{\hbar^2}. \tag{74}$$

The additional term $\beta_0 \, l_p^2 \, (\Delta p)^2/\hbar^2$ has its origin in the very nature of spacetime at the Planck energy scale ϵ_p of 10^{39} GeV [29, 37]. The simplest GUP relation which implies the appearance of a non-zero minimal uncertainty Δx_0 is encoded in the following approach:

$$\Delta x \ \Delta p \geq \tfrac{\hbar}{2} \left(1 + \beta \, (\Delta p)^2\right), \tag{75}$$

where β is the GUP parameter $\beta = \beta_0/(M_p \, c^2) = \beta_0 \, l_p^2/\hbar^2$.

As a non-trivial assumption, we propose that the minimal observable length has also a minimal but non-zero uncertainty. Therefore, the Hilbert space representation on position space wave functions of the ordinary QM [29] is no longer possible, because no physical system with a vanishing position eigenstate $|x\rangle$ is allowed [29]. In light of this, we must construct a new Hilbert space representation which should be compatible with the commutation relation in GUP, Eq. (75). This means that we can explore the physical applications

of the minimal length by working with the convenient representation of the commutation relations on momentum space wave functions [29].

The Heisenberg algebra of GUP is given as [27–30, 37, 38, 44, 110]

$$[x, \, p] \; = \; i \, \hbar \left(1 + \beta \, p^2 \right). \tag{76}$$

In fact, the Heisenberg algebra can be represented in momentum space wave functions $\phi(p) = \langle p | \phi(p) \rangle$ and $\partial_p = i\hbar \frac{\partial}{\partial x}$

$$\mathbf{P} \cdot \phi(p) \; = \; p \, \phi(p), \tag{77}$$

$$\mathbf{X} \cdot \phi(p) \; = \; i \, \hbar \left(1 + \beta \, p^2 \right) \partial_p \phi(p), \tag{78}$$

where $\mathbf{X}$ and $\mathbf{P}$ are symmetric operators on the dense domain S_∞ with respect to the following scalar product

$$\langle \phi | \psi \rangle \; = \; \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} \phi^*(p) \psi(p). \tag{79}$$

The identity operator would be represented as

$$\int_{-\infty}^{\infty} \frac{dp}{1 + \beta \, p^2} |p\rangle \langle p| = 1 \tag{80}$$

and the scalar product of the momentum eigenstates changes to

$$\langle p | p' \rangle = \left(1 + \beta \, p^2 \right) \delta \left(p - p' \right). \tag{81}$$

While the momentum operator essentially still self-adjoint, the functional analysis of the position operator, as expected from the appearance of the minimal uncertainty in positions, changes.

To obtain a minimum measurable uncertainty in position, the relation $(\Delta \, p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ can be utilized [29]

$$\Delta x \, \Delta p \; \geq \; \frac{\hbar}{2} \left(1 + \beta \left(\Delta p \right)^2 + \beta \left\langle p \right\rangle^2 \right). \tag{82}$$

This relation can be rewritten as a second order equation for Δp. Then, the solutions for Δp are as follows [29].

$$\Delta p \; = \; \left(\frac{\Delta x}{\hbar \, \beta} \right) \pm \sqrt{ \left(\frac{\Delta x}{\hbar \, \beta} \right)^2 - \frac{1}{\beta} - \langle p \rangle^2 }. \tag{83}$$

A minimum value Δx can be deduced

$$\Delta x_{min}(\langle p \rangle) \; = \; \hbar \, \sqrt{\beta} \sqrt{1 + \beta \, \langle p \rangle^2}. \tag{84}$$

Therefore, the absolutely smallest uncertainty in position, where $\langle p \rangle = 0$, reads

$$\Delta x_0 = \hbar \, \sqrt{\beta}. \tag{85}$$

There is a non-vanishing minimal uncertainty in momentum as accepted from Fig. 1.

For the construction of Hilbert space representations of this general case, one cannot work on the position space. One has to resort a generalized Bargmann Fock representation [111, 112]. Here, we specify to the situation with non-zero minimal position uncertainties. For n dimensions, the generalised Heisenberg algebra, Eq. (75), reads [27–30, 37, 38, 44, 110]

$$[x_i, \, p_j] = i \, \hbar \, \left(1 + \beta \, \vec{p}^2\right). \tag{86}$$

We require that

$$[p_i, \, p_j] = 0, \tag{87}$$

which allows us to generalize the momentum space representation, straightforwardly [29].

$$\mathbf{P}_i \cdot \phi(p) \; = \; p_i \, \phi(p), \tag{88}$$

$$\mathbf{X}_i \cdot \phi(p) \; = \; i \, \hbar \, \left(1 + \beta \vec{\mathbf{p}}^2\right) \partial_{p_i} \phi(p), \tag{89}$$

and $\partial_{p_i} = i \, \hbar \, \partial / \partial p_i$. Then, it is obvious to show that

$$[\mathbf{X}_i, \, \mathbf{X}_j] \; = \; 2 \, i \, \hbar \, \beta \, \left(\mathbf{P}_i \, \mathbf{X}_j - \mathbf{P}_j \, \mathbf{X}_i\right), \tag{90}$$

leads to a non-commutative geometric generalization of position space.

Furthermore, the commutation relations, Eqs. (86), (87) and (90) do not break the rotational symmetry [29]. In fact, the generators of rotations can be expressed in terms of position and momentum operators as [29]

$$\mathbf{L}_{ij} = \frac{1}{\left(1 + \beta \, \vec{\mathbf{p}}^2\right)} \left(\mathbf{X}_i \, \mathbf{P}_j - \mathbf{X}_j \, \mathbf{P}_i\right), \tag{91}$$

where the representation of the rotation generators in momentum wavefunctions reads

$$\mathbf{L}_{ij} \, \psi(p) = - \, i \, \hbar \, \left(p_i \, \partial_{p_j} - p_j \, \partial_{p_i}\right) \psi(p) \tag{92}$$

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and are essentially the same as one encounters in ordinary QM. However, the main change now appears in the relation

$$[x_i, \, x_j] = - \, 2 \, i \, \hbar \, \beta \, \left(1 + \beta \, \vec{\mathbf{p}}^2 \right) \, L_{ij}. \tag{93}$$

Once again, this relation reflects the noncommutative nature of the spacetime manifold at the Planck scale.

1. *Eigenstates of position operator in momentum space*

The position operator generating the momentum-space eigenstates [29]

$$\mathbf{X} \, \phi_\lambda(p) \; = \; \lambda \, \phi_\lambda(p), \tag{94}$$

$$i \, \hbar \, \left(1 + \beta \, p^2 \right) \, \partial_p \, \phi_\lambda(p) \; = \; \lambda \, \phi_\lambda(p). \tag{95}$$

This differential equation can be solved to obtain formal position eigenvectors [29]

$$\phi_\lambda(p) \; = \; C \, e^{-i \frac{\lambda}{\hbar \sqrt{\beta}} \, \tan^{-1} \sqrt{\beta} \, p}. \tag{96}$$

By applying the normalization condition, we can find out the formal position eigenvectors in momentum-space

$$\phi_\lambda(p) \; = \; \sqrt{\frac{\sqrt{\beta}}{\pi}} \, e^{-i \frac{\lambda}{\hbar \sqrt{\beta}} \, \tan^{-1} \sqrt{\beta} \, p}. \tag{97}$$

This is the generalized momentum-space eigenstate of the position operator in the presence of both a minimal length and a maximal momentum. To this end, we calculate the scalar product of the momentum space eigenstate of the position operator $|\phi_\lambda(p)\rangle$ [29]

$$\langle \phi_{\lambda'} | \phi_\lambda \rangle \; = \; \frac{\sqrt{\beta}}{\pi} \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} e^{-i \frac{\lambda - \lambda'}{\hbar \sqrt{\beta}} \, \tan^{-1} \sqrt{\beta} p}. \tag{98}$$

Thus,

$$\langle \phi_{\lambda'} | \phi_\lambda \rangle \; = \; \frac{2 \, \hbar \, \sqrt{\beta}}{\pi \left(\lambda - \lambda' \right)} \sin \left(\frac{(\lambda - \lambda')}{2 \, \hbar \, \sqrt{\beta}} \pi \right). \tag{99}$$

Fig. 2 compares the behavior of $\langle \phi_{\lambda'} | \phi_\lambda \rangle$ with $\lambda - \lambda'$ normalized to $\hbar \sqrt{\beta}$. The curve represents the special case Δp_0 and the sets of eigenvectors parametrised by $\lambda \in [-1, 1[$ [29]. It apparent that the formal position eigenstates are generally no longer orthogonal. This

is however should not the case, since the formal position eigenvectors $|\phi_\lambda\rangle$ are not physical states. They are not in the domain of p, which physically means that they have infinite uncertainty in momentum and - in particular - infinite energy

$$\left\langle \phi_\lambda \left| \frac{\mathbf{p}^2}{2\,m} \right| \phi_\lambda \right\rangle = \texttt{divergent}, \tag{100}$$

The main difficulty in this approach is the divergence of the energy spectrum of the position operator eigenfunctions. In other words, the energy of the short wavelength modes seems to diverge [29].
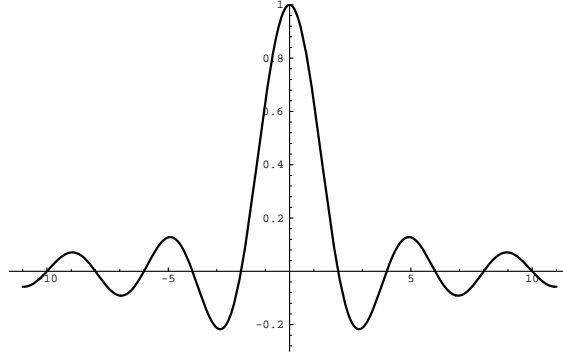
Fig. 2: $\langle\phi_{\lambda'}|\phi_\lambda\rangle$ is plotted versus $\lambda - \lambda'$ normalized to $\hbar\sqrt{\beta}$. The graph taken from Ref. [29].

2. *Maximal localization states*

The states $|\phi_\zeta^{ml}\rangle$ of a maximum localization around ζ position can be calculated as [29]

$$\left\langle \phi_\zeta^{ml} \left| \hat{X} \right| \phi_\zeta^{ml} \right\rangle = \zeta, \tag{101}$$

and $\Delta x_{min} = \Delta x_0$ depends on $\langle p\rangle$. These states also satisfy [29]

$$\left\| \left((x - \langle x\rangle) + (p - \langle p\rangle)\frac{[x,p]}{2(\Delta p)^2} \right) |\phi\rangle \right\| \geq 0, \tag{102}$$

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which immediately implies that

$$\Delta x \, \Delta p \; \geq \; \left(\frac{|\langle [x, p] \rangle|}{2} \right). \tag{103}$$

For first order of the GUP parameter, we can use the approximate relation [29, 37, 74, 90]

$$|\langle [x, \, p] \rangle| \; \approx \; i \, \hbar \, \left(1 + \beta (\Delta p)^2 + \beta \, \langle p \rangle^2 \right). \tag{104}$$

Only if it obeys Eq. (102), it is on the boundary of the physically allowed region. From Eqs. (77) and (78) and in momentum space, this takes the form of a differential equation [29]

$$\left\{ \left[i\hbar(1 + \beta p^2) \, \partial_p - \langle x \rangle \right] + i \, \hbar \, \frac{(1 + \beta(\Delta p)^2 + \beta \langle p \rangle^2)}{2(\Delta p)^2} (p - \langle p \rangle) \right\} \phi(p) \; \approx \; 0, \tag{105}$$

which can be solved as

$$\phi(p) \; \approx \; C(1 + \beta p^2)^{\frac{-\left(1 + \beta(\Delta p)^2 + \beta \langle p \rangle^2\right)}{4 \, \beta \, (\Delta p)^2}}$$
$$\exp \left[\left(\frac{\langle x \rangle}{i \, \hbar \, \sqrt{\beta}} - \frac{(1 + \beta \, (\Delta p)^2 + \beta \, \langle p \rangle^2) \, \langle p \rangle}{2 \, \sqrt{\beta}(\Delta p)^2} \right) \, \tan^{-1}(\sqrt{\beta} \, p) \right]. \tag{106}$$

For $\langle p \rangle = 0$ and the critical momentum uncertainty $(\Delta p)^2 = 1/\beta$, the absolutely maximal localization states can only be obtained [29]

$$\phi_\zeta^{ml}(p) \; \approx \; C \, (1 + \beta \, p^2)^{-\frac{1}{2}} \, \exp \left(-i \, \frac{\langle x \rangle \tan^{-1}(\sqrt{\beta} p)}{\hbar \, \sqrt{\beta}} \right). \tag{107}$$

By applying the normalized condition, the factor C can be determined [29],

$$1 \; = \; C C^* \int_{-\infty}^{\infty} \frac{dp}{(1 + \beta p^2)^2} = C^2 \frac{\pi}{2\sqrt{\beta}}. \tag{108}$$

Thus, the momentum space wavefunctions $|\phi_\zeta^{ml}\rangle$ of a maximum localization around ζ position reads

$$\phi_\zeta^{ml}(p) \; = \; \sqrt{\frac{2\sqrt{\beta}}{\pi}} \, (1 + \beta p^2)^{-\frac{1}{2}} \, \exp \left(-i \frac{\zeta \tan^{-1}(\sqrt{\beta} p)}{\hbar \sqrt{\beta}} \right). \tag{109}$$

These states generalize the plane waves in momentum-space which would describe a maximal localization in the ordinary QM, which are now the proper physical states of finite energy [29]

$$\left\langle \phi_\zeta^{ml} \left| \frac{\hat{\mathbf{P}}^2}{2m} \right| \phi_\zeta^{ml} \right\rangle \; = \; \frac{2\sqrt{\beta}}{\pi} \int_{-\infty}^{\infty} \frac{dp}{(1 + \beta p^2)^2} \frac{p^2}{2m} = \frac{1}{2m\beta}. \tag{110}$$
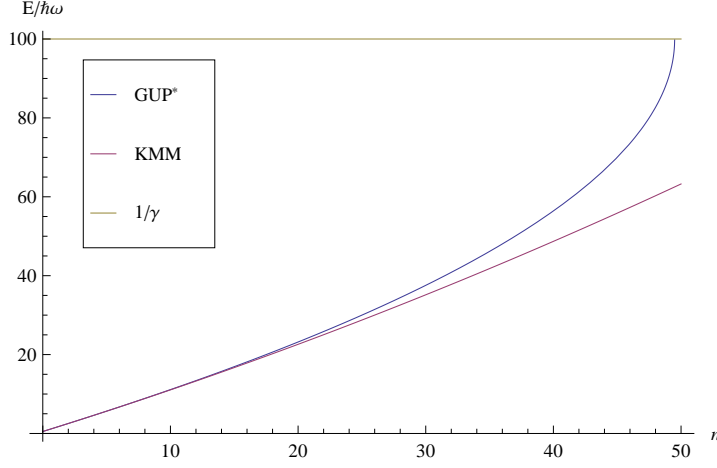
Fig. 3: The wavelength of the quasiposition wave is given as a function of the momentum eigenstate, Eq. 113, in ordinary QM and GUP approach at $\beta = 0.2$. The graph taken from Ref. [64].

3. Transformation to quasi-position wavefunctions

Through projecting arbitrary states on maximally localized states, we obtain the probability amplitude for the particle being maximally localized around a position. For quasiposition wavefunction $\phi(\zeta)$, it is defined [29]

$$\phi(\zeta) = \langle \phi_\zeta^{ml} | \phi \rangle, \tag{111}$$

where in the limit $\beta \to 0$, the ordinary position wave function $\phi(\zeta) = \langle \zeta | \phi \rangle$. Now, the transformation of the wavefunction in momentum representation into its counterpart quasi-position wavefunction reads [29]

$$\phi(\zeta) = \sqrt{\frac{2\sqrt{\beta}}{\pi}} \int_{-\infty}^{\infty} \frac{dp}{(1 + \beta p^2)^{\frac{3}{2}}} \exp\left[i \frac{\zeta \tan^{-1}(\sqrt{\beta} p)}{\hbar \sqrt{\beta}}\right] \phi(p). \tag{112}$$

The quasiposition wavefunction of a momentum eigenstate $\phi_{\widetilde{p}}(P) = \delta(p - \widetilde{p})$ with energy $E = \widetilde{p}^2/2m$ is characterized as a plane wave. In terms of modified dispersion relation, the wavelength is given as [29]

$$\lambda(E) = \frac{2\pi\hbar\sqrt{\beta}}{\tan^{-1}\sqrt{2m\beta E}}. \tag{113}$$

No wavelength components smaller than the wavelength are possible in absence of GUP

$$\lambda_0 = 4\hbar\sqrt{\beta}. \tag{114}$$

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No arbitrarily fine ripples are possible, since the energy of short wavelength modes diverges as the wavelength approaches the finite value λ_0 with the energy

$$E(\lambda) = \frac{1}{2m\beta} \left(\tan \frac{2\pi\hbar\sqrt{\beta}}{\lambda} \right)^2 . \tag{115}$$

Fig. 3 depicts $\lambda(E)$ versus $m\,E$ in ordinary QM and GUP approach at $\beta = 0.2$. Since Eq. (113) is bounded from below, there exists a nonzero minimal wavelength [29]. Benause the transformation, Eq. (112), like the generalized Fourier transformation, is invertible, the transformation of a quasiposition wavefunction into a momentum-space wavefunction is given as [29]

$$\phi(p) = \frac{1}{\sqrt{8\,\pi\,\sqrt{\beta}}\hbar} \int_{-\infty}^{\infty} \frac{d\zeta}{(1+\beta\,p^2)^{\frac{-1}{2}}} \exp\left[-i\,\frac{\zeta\,\tan^{-1}(\sqrt{\beta}\,p)}{\hbar\,\sqrt{\beta}} \right] \phi(\zeta). \tag{116}$$

V.   MINIMAL LENGTH UNCERTAINTY: MAXIMAL MOMENTUM

A.   Momentum modification

Based on DSR, GUP suggests modifications of commutators [62]

$$[x_i, p_j] = i\hbar \left(\delta_{ij}(1 + \beta p^2) + 2\beta p_i p_j\right), \tag{117}$$

$$[x_i, p_j] = i\hbar \left[(1 - l|\mathbf{p}|)\delta_{ij} + l^2 p_i p_j\right]. \tag{118}$$

Both relations can be combined

$$[x_i, p_j] = i\hbar \left[\delta_{ij} + \alpha_1 p\,\delta_{ij} + \alpha_2 \frac{p_i\, p_j}{p} + \beta_1 p^2\, \delta_{ij} + \beta_2\, p_i\, p_j\right]. \tag{119}$$

It follows from the Jacobi identity that

$$- [[x_i, x_j], p_k] = 0 = [[x_j, p_k], x_i] + [[p_k, x_i], x_j], \tag{120}$$

$$\left[\left(\frac{\alpha_1 - \alpha_2}{p}\right) + \left(\alpha_1^2 + 2\beta_1 - \beta_2\right)\right]\Delta_{jki} = 0, \tag{121}$$

where $\Delta_{jki} = p_i \delta_{jk} - p_j \delta_{ik}$. We should assume that $\alpha_1 = \alpha_2 = -\alpha$, where the negative sign appearing in Eq. (120) or Eq. (117) with $\alpha > 0$ and $\alpha_1^2 + 2\beta_1 - \beta_2 = 0$ and assume that $\beta_1 = \alpha^2$ and $\beta_2 = 3\alpha^2$ with $\alpha^2 = \beta$. We get commutators consistent with the String theory, black holes physics and DSR

$$[x_i, p_j] = i\hbar \left[\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2\left(p^2\delta_{ij} + 3p_i p_j\right)\right], \tag{122}$$

and via the Jacobi identity

$$[x_i, x_j] = 0 = [p_i, p_j], \tag{123}$$

where $\alpha = \alpha_0\, \ell_{pl}/\hbar = \alpha_0/(M_{pl}\, c)$ and the Planck length $\ell_{pl} \approx 10^{-35}$ m and energy $\epsilon_{pl} = M_{pl}c^2 \approx 10^{19}$ GeV.

In one dimension, this GUP approach was formulated as [63, 113]

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}\left(1 - 2\alpha\, \langle p \rangle + 4\alpha^2 \langle p^2 \rangle\right). \tag{124}$$

Apparently, $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$, then we get

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}\left[1 + \left(\frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2\right)(\Delta p)^2 + 4\alpha^2\, \langle p \rangle^2 - 2\alpha\sqrt{\langle p^2 \rangle}\right]. \tag{125}$$

Commutators and inequalities similar to the ones given in Eqs. (122) and (124) were proposed and derived [63, 113], which - in turn - imply a minimum measurable length and a maximum measurable momentum, simultaneously

$$\Delta x \ \geq \ (\Delta x)_{min} \approx \alpha \, \hbar \approx \alpha_0 \, \ell_{pl}, \tag{126}$$

$$\Delta p \ \leq \ (\Delta p)_{max} \approx \frac{1}{\alpha} \approx \frac{M_{pl} \, c}{\alpha_0}, \tag{127}$$

and define

$$X_i \ = \ x_{0i}, \tag{128}$$

$$P_j \ = \ p_{0j}(1 - \alpha \, p_0 + 2 \, \alpha^2 \, p_0^2). \tag{129}$$

We note that $p_0^2 = \sum_j^3 p_{0j} \, p_{0j}$ satisfies the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$ and p_{0j} is defined as the momentum at low energy scale which is represented by $p_{0j} = -i\hbar \, \partial/\partial x_{0j}$, while p_j is considered as the momentum at high energy. It is normally assumed that the dimensionless parameter α_0 is of the order of unity. In this case the α-dependent terms are important only when the energies (momenta) are comparable to the Planck energy (momentum), and the lengths are comparable to the Planck length.

1. Main difficulties with this approach

The GUP approach introduced in Ref. [63] contains linear and quadratic terms of momenta with a minimum measurable length and a maximum measurable momentum. Furthermore, it was suggested that when the energy gets comparable to the Planck energy, there should be a modification in Eq. (129) and this should ensure the commutators of space, Eq. (123), as the canonical system can predict the measurable length and a maximum measurable momentum simultaneous. In fact, this assures wide applications in different physical systems. Some of the main difficulties which appear because of the GUP approach are listed out in Ref. [64]:

- it is a perturbative approach (therefore, it is only valid for small values of the GUP parameter α),

- it can not approach the non-commutative geometry, see Eq. (123),

- it suggests a minimal length uncertainty which can be interpreted as the minimal length. The maximal momentum uncertainty differs from the idea of the maximal momentum which is required in DSR theories, where the maximal momentum given in uncertainty not on the value of the observed momentum, see Eq. (127),

- it suggests momentum modification given in Eq. (129), but does not achieve the commutator relation of the momentum space $[p_i, p_j] \neq 0$,

- its minimal length uncertainty with maximal momentum results in uncertainty instead of maximum observed momentum, see Eqs. (126) and (127), and

- the introduction of minimal length (non-varnishing value) allows the study for the Hilbert space representation corresponding to the momentum wavefunction $\psi(p)$.

B. Hilbert space representation

The first term in Eq. (124) is related to the particles momentum and has its origin in the existence of a maximal momentum. In this term, various differences between the Hilbert space representation and the work of KMM [29] can be originated.

Assuming that the minimal observable length has a non-vanishing uncertainty, we should construct a new Hilbert space representation which is compatible with the commutation relation accompanied with the GUP approach

$$[x_i, p_j] = i\hbar\delta_{ij}\left(1 - \alpha p + 2\alpha^2 \vec{p}^2\right). \tag{130}$$

Fortunately, when neglecting the presence of a minimal momentum uncertainty, there would still exist a continuous momentum space representation, which means that we can explore physical applications of the minimal length by implementing convenient representation of the commutation relations on momentum-space wavefunctions [114], Then, the momentum $\mathbf{P}$ and position $\mathbf{X}$ operators read

$$\mathbf{X}_i\,\phi(p) = x_{0i}(1 - \alpha p_0 + 2\alpha^2\,\vec{p_0}^2)\,\phi(p), \tag{131}$$

$$\mathbf{P}_j\,\phi(p) = p_{0j}\,\phi(p), \tag{132}$$

where $p_0^2 = \sum_j^3 p_{0j}\,p_{0j}$ satisfying the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$ and p_{0j} is defined as the momentum at low energy scale which is represented by $x_{0i} =$

$i\,\hbar\,\partial_{p_i}$. These commutation relations imply a nonzero minimal uncertainty in each position coordinate. As in ordinary QM, $[p_i,\,p_j]=0$. Then, it is straightforward to show that

$$[x_i,\,x_j]\;=\;i\,\hbar\,\alpha\,\left(4\,\alpha-\frac{1}{P}\right)\,\left(\mathbf{P}_i\mathbf{X}_j-\mathbf{P}_j\,\mathbf{X}_i\right). \tag{133}$$

In light of this, one would worry about the divergence in KMM formalism [29], at vanishing momentum. "Singularity" is likely, because the derivative diverges at $p=0$. The commutation relations do not break the rotational symmetry. In fact, the generators of rotations can still be expressed in terms of position and momentum operators

$$L_{ij}\;=\;\frac{\mathbf{X}_i\,\mathbf{P}_j-\mathbf{X}_j\,\mathbf{P}_i}{1-\alpha\,p_0+2\,\alpha^2\,\vec{p_0}^{\,2}}. \tag{134}$$

The action on a momentum-space wave function reads

$$L_{ij}\phi(p)\;=\;-\,i\,\hbar\,\left(p_i\,\partial_{p_j}-p_j\,\partial_{p_i}\right)\,\phi(p). \tag{135}$$

This is essentially the same as in ordinary QM. However, the main difference appears in the relation

$$[x_i,\,x_j]\;=\;i\,\hbar\,\alpha\,\left(4\,\alpha-\frac{1}{P}\right)\,\left(1-\alpha\,p_0+2\alpha^2\,\vec{p_0}^{\,2}\right)\,L_{ij}. \tag{136}$$

The $1/P$-term was absent in the original KMM formalism [29]. It gives a trace of the existence of maximal momentum. Equation (136) expresses the noncommutative nature of the spacetime manifold in the Planck scale. The presence of an upper bound of the momentum agrees with DSR theories.

The scalar product in this representation should be modified due to the presence of the additional factor $(1-\alpha\,p_0+2\,\alpha^2\,\vec{p_0}^{\,2})$ and the existence of maximal momentum. The integrals calculated from the limits of the Planck momentum from $-p_{pl}$ to $+p_{pl}$ differs than the one in the KMM formalism [29, 114]. This implies the existence of a maximal momentum (Planck momentum), $p_{pl}\equiv M_{pl}\,c$.

$$\langle\phi|\psi\rangle\;=\;\int_{-p_{pl}}^{+p_{pl}}\frac{\phi^*(p)\,\psi(p)}{(1-\alpha\,p_0+2\,\alpha^2 p_0^2)}\,dp. \tag{137}$$

The identity operator would be represented as [114]

$$\int_{-p_{pl}}^{+p_{pl}}\frac{|p\rangle\langle p|}{(1-\alpha\,p_0+2\,\alpha^2\,p_0^2)}\,dp=1, \tag{138}$$

and the scalar product of the momentum eigenstates changes to

$$\langle p|p'\rangle=\left(1-\alpha\,p_0+2\,\alpha^2\,p_0^2\right)\,\delta\left(p-p'\right). \tag{139}$$

The position operator acting on momentum-space eigenstates gives [29, 114].

$$\mathbf{X} \cdot \phi_\xi(p) = \xi \, \phi_\xi(p), \tag{140}$$

where $\phi_\xi(p) = \langle \xi | p \rangle$ is a formal position eigenstate and $| \xi \rangle$ is an arbitrary state

$$i\,\hbar \left(1 - \alpha p_0 + 2\alpha^2 p_0^2\right) \partial_p \, \phi_\xi(p) = \xi \, \phi_\xi(p). \tag{141}$$

By solving this differential equation, we obtain the formal position eigenvectors [114]

$$\phi_\xi(p) = C \, \exp \left[-i \frac{2\,\xi}{\alpha\,\hbar\,\sqrt{7}} \left(\tan^{-1} \frac{1}{\sqrt{7}} + \tan^{-1} \frac{4\,\alpha\,p - 1}{\sqrt{7}} \right) \right]. \tag{142}$$

Also by applying normalized condition to extract the factor C, the formal position eigenvectors in momentum-space [114]

$$\phi_\xi(p) = \sqrt{\frac{\alpha\,\sqrt{7}}{2}} \left[\tan^{-1} \left(\frac{4\,\alpha\,p_{pl} - 1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\,\alpha\,p_{pl} + 1}{\sqrt{7}} \right) \right]^{-\frac{1}{2}}$$
$$\exp \left[-i \frac{2\,\xi}{\alpha\,\hbar\,\sqrt{7}} \left(\tan^{-1} \left(\frac{1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\,\alpha\,p - 1}{\sqrt{7}} \right) \right) \right]. \tag{143}$$

This is nothing but the generalized momentum-space eigenstate of the position operator in the presence of both a minimal length and a maximal momentum.

The scalar product of formal position eigenstates can be given as [114]

$$\langle \phi_{\xi'} | \phi_\xi \rangle = \int_{-p_{pl}}^{+p_{pl}} \frac{dp}{(1 - \alpha\,p_0 + 2\,\alpha^2\,p_0^2)} \phi_{\xi'}^*(p)\,\phi_\xi(p),$$
$$= \frac{\alpha\,\sqrt{7}}{2}\,\rho_0 \, \exp \left(-i \frac{2\,(\xi - \xi')}{\alpha\,\hbar\,\sqrt{7}} \tan^{-1} \left(\frac{1}{\sqrt{7}} \right) \right)$$
$$\int_{-p_{pl}}^{+p_{pl}} \frac{\exp \left[-i \frac{2\,(\xi - \xi')}{\alpha\,\hbar\,\sqrt{7}} \tan^{-1} \left(\frac{4\,\alpha\,p - 1}{\sqrt{7}} \right) \right]}{(1 - \alpha\,p_0 + 2\,\alpha^2\,p_0^2)} \, dp, \tag{144}$$

where $\rho_0 = \left[\tan^{-1} \left(\frac{4\,\alpha\,p_{pl} - 1}{\sqrt{7}} \right) + \tan^{-1} \left(\frac{4\,\alpha\,p_{pl} + 1}{\sqrt{7}} \right) \right]^{-1}$. Then,

$$\langle \phi_{\xi'} | \phi_\xi \rangle = \Omega \left(\exp \left\{ -i \left[\frac{2(\xi - \xi')}{\alpha\hbar\sqrt{7}} \tan^{-1} \left(\frac{4\alpha p_{pl} - 1}{\sqrt{7}} \right) - \frac{\pi}{2} \right] \right\} - \right.$$
$$\left. \exp \left\{ i \left[\frac{2(\xi - \xi')}{\alpha\hbar\sqrt{7}} \tan^{-1} \left(\frac{4\alpha p_{pl} + 1}{\sqrt{7}} \right) + \frac{\pi}{2} \right] \right\} \right), \tag{145}$$

where Ω is defined as

$$\Omega = \frac{\rho_0 \hbar \alpha \sqrt{7}}{2 (\xi - \xi')} \exp\left[-i \frac{2 (\xi - \xi')}{\alpha \hbar \sqrt{7}} \tan^{-1}\left(\frac{1}{\sqrt{7}}\right)\right]. \tag{146}$$

Fig. 4 compares the behavior of $\langle \phi_{\xi'} | \phi_\xi \rangle$ with $\xi - \xi'$ using the two GUP approaches [29, 114]. The red curve shows a much more oscillating behavior. It is obvious that the scalar product used in the GUP approach [114] causes a more broadening relative to the one of Ref. [29].
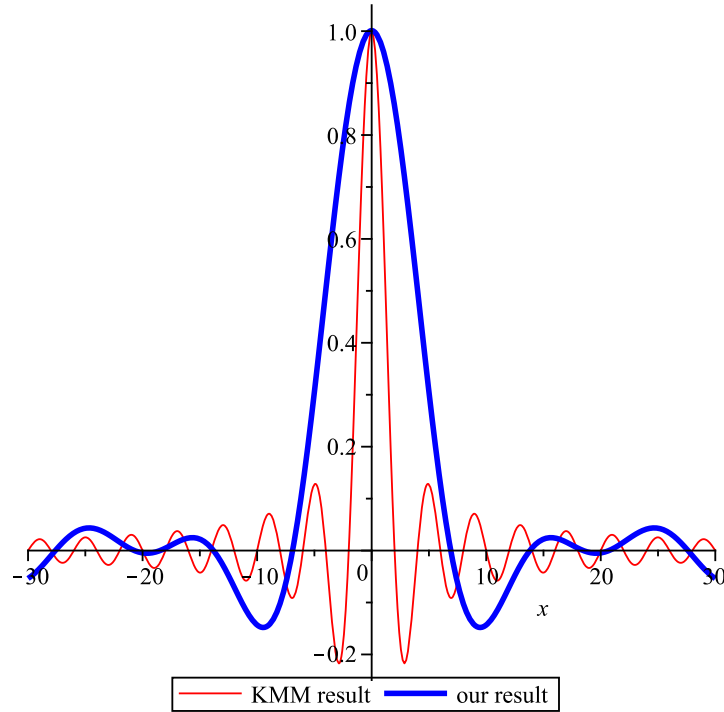
Fig. 4: The dependence of $\langle \phi_{\xi'} | \phi_\xi \rangle$ on $\xi - \xi'$ as calculated in two GUP approaches [29, 114]. The graph taken from Ref. [114], to which "our result" refers.

For these formal position eigenvectors, the expectation value of energy reads

$$\left\langle \phi_\xi \left| \frac{\mathbf{p}^2}{2m} \right| \phi_\xi \right\rangle = \int_{-p_{pl}}^{+p_{pl}} \frac{\phi_{\xi'}^*(p) \frac{p^2}{2m} \phi_\xi(p)}{(1 - \alpha p + 2\alpha^2 p^2)} \, dp. \tag{147}$$

Thus

$$\left\langle \phi_\xi \left| \frac{\mathbf{p}^2}{2m} \right| \phi_\xi \right\rangle = \frac{\alpha \sqrt{7} \rho_0}{4m} \int_{-p_{pl}}^{+p_{pl}} \frac{\phi_{\xi'}^*(p) \frac{p^2}{2m} \phi_\xi(p)}{(1 - \alpha p + 2\alpha^2 p^2)} \, p^2 \, dp, \tag{148}$$

and we get

$$\left\langle \frac{\mathbf{p}^2}{2\,m} \right\rangle \;=\; \left[\frac{\sqrt{7}\rho P_{pl}}{4\,m} + \frac{\sqrt{7}\,\rho}{32\,m\,\alpha} \, \ln\left(\frac{1 - \alpha\, p_{pl} + 2\,\alpha^2\, p_{pl}^2}{1 + \alpha\, p_{pl} + 2\,\alpha^2\, p_{pl}^2} \right) - \frac{3}{16\,m\,\alpha} \right]. \tag{149}$$

As shown in previous sections, there is no divergence in the energy spectrum as calculated in framework of KMM GUP approach [29], especially in the presence of both minimal length and maximal momentum. It turns out that the expectation values of the energy as calculated by the GUP approach suggested by Ali *et al.* [63, 75, 113] are no longer divergent [114]. Furthermore, the expectation values of energy are not lying within the domain of P, which physically means that they have infinite momentum uncertainty.

2. Maximal localization states

The explicit calculation of the states $|\phi_\zeta^{ml}\rangle$ of a maximum localization around the position ζ requires that [29]

$$\left\langle \phi_\zeta^{ml} \left| \hat{X} \right| \phi_\zeta^{ml} \right\rangle \;=\; \zeta. \tag{150}$$

As in section IV B 2 and using Eqs. (131) and (132) and the differential equation in momentum space, Eq. (102),

$$\left[\left(i\hbar(1 - \alpha p + 2\alpha^2 p^2)\, \partial_p - \langle \mathbf{X} \rangle \right) + i\hbar \frac{1 + 2\alpha^2(\Delta p)^2 + 2\alpha^2\langle p \rangle^2 - \alpha\langle p \rangle}{2(\Delta p)^2}(p - \langle p \rangle) \right] \phi(p) \;\approx\; 0. \tag{151}$$

The minimal position uncertainty can be deduced from the solution of this differential equation taking into account that $\langle \mathbf{X} \rangle = \zeta$, $\langle p \rangle = 0$ and $\Delta p = \alpha/2$, which are corresponding to the states of absolutely maximal localization and critical momentum uncertainty. By normalization where the Planck momentum is of the order of magnitude as $P_{pl} = \alpha/2$, then $\eta = \frac{4\alpha p_{pl} - 1}{\sqrt{7}} = \frac{3}{\sqrt{7}}$. Therefore, the momentum-space wavefunctions $\phi_\zeta^{ml}(p)$ of the states which are maximally localized around the position $\langle \mathbf{X} \rangle = \zeta$ read [114]

$$\phi_\zeta^{ml}(p) \;=\; \frac{\sqrt{6\alpha}\left[\sqrt{8}\, e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}\left(\frac{\eta}{3}\right)} \right]^{-\frac{1}{2}}}{(1 + \alpha p + 2\alpha^2 p^2)^{\frac{3}{4}}}$$

$$e^{\frac{-\eta}{2} \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right)} \, e^{-i\frac{2\zeta}{\alpha\hbar\sqrt{7}}\left(\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right)\right)}. \tag{152}$$

The difference between this result and that obtained in framework of KMM GUP [29] is to be originated in the presence of the first order of the momentum, Eq. (131), which

implies the existence of a maximal momentum. The maximal localization states are now the proper physical states of the finite energy [114].

$$\left\langle \phi_\zeta^{ml} \left| \frac{\hat{\mathbf{P}}^2}{2\,m} \right| \phi_\zeta^{ml} \right\rangle = \frac{2\sqrt{\beta}}{\pi} \int_{-p_{pl}}^{+p_{pl}} \frac{\phi_\zeta^{ml*}(p)\,\frac{p^2}{2m}\,\phi_\zeta^{ml}(p)}{(1 - \alpha p + 2\alpha^2 p^2)}\,dp \approx \frac{1}{32\,m\,\alpha^2}. \tag{153}$$

3. *Quasiposition wavefunctions transformation*

The concept of quasiposition wavefunction means the projecting arbitrary states on maximally localized states in order to obtain the probability amplitude for the particle being maximally localized around a concrete position [29, 114]. The transformation of a state's wavefunction in the momentum wavefunction representation into its quasiposition wavefunction [114] would be

$$\phi(\zeta) = A \int_{-p_{pl}}^{+p_{pl}} \frac{e^{\frac{-\eta}{2}\tan^{-1}\left(\frac{4\,\alpha\,p-1}{\sqrt{7}}\right)}}{(1 + \alpha\,p + 2\alpha^2\,(p^2))^{\frac{7}{4}}}\,e^{i\,H\,\zeta}, \tag{154}$$

where the modified wavenumbers read

$$A = \sqrt{6\alpha}\,\left[\sqrt{8}\,e^{\eta\tan^{-1}(\eta)} - e^{-\eta\tan^{-1}\left(\frac{\eta}{3}\right)} \right]^{-\frac{1}{2}}, \tag{155}$$

$$H = \frac{2}{\alpha\hbar\sqrt{7}}\,\left[\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\,\alpha\,p-1}{\sqrt{7}}\right) \right]. \tag{156}$$

Then,

$$\lambda(p) = \frac{\pi\,\alpha\,\hbar\,\sqrt{7}}{\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\,\alpha\,p-1}{\sqrt{7}}\right)}, \tag{157}$$

which would be the modified wavelength for the quasiposition wavefunction representation of physical states. Since α is not vanishing and p is limited to the Planck momentum, there is no wavelength smaller than

$$\lambda_0 = \lambda(p_{pl}) = \frac{\pi\,\alpha\,\hbar\,\sqrt{7}}{\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\,\alpha\,p_{pl}-1}{\sqrt{7}}\right)}. \tag{158}$$

By implementing the relation between energy and momentum, $E = p^2/2m$, we get

$$E(\lambda) = \frac{2}{m\alpha^2}\,\left(\frac{\tan\left(\frac{\hbar\pi\alpha\sqrt{7}}{\lambda}\right)}{\tan\left(\frac{\hbar\pi\alpha\sqrt{7}}{\lambda}\right) + \sqrt{7}} \right)^2, \tag{159}$$

$$E(\lambda_0) = \frac{P_{pl}^2}{2\,m}, \tag{160}$$

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which apparently agrees with ordinary QM. These expressions obviously do not diverge. The importance of this result is its difference to the KMM [29], where the quasiposition wavefunctions in contrast to ordinary QM had no longer arbitrarily fine *ripples*. This is because the energy of the short wavelength modes were divergent. Here, similar to ordinary QM, those wavefunctions can have arbitrarily fine *ripples*, because there is no longer divergence in the energy at λ_0. This is an important result from the new GUP approach, which contains both minimal length and maximal momentum.

VI. APPLICATIONS OF QUADRATIC GUP APPROACH

A. Early Universe: Friedmann equations

At very short distances, the holographic principle for gravity is assumed to relate the gravitational quantum theory to QFT. The entropy of black hole is related to the area of the apparent horizon [24, 115]. The covariant entropy bound in the Friedmann-Lemaitre-Robertson-Walker (FLRW) matric is found indicating to a holographic nature in terms of temperature and entropy [116]. The cosmological boundary can be chosen as the cosmological apparent horizon instead of the event horizon of a black hole. In light of this, we recall that the statistical (informational) entropy of a black hole can be calculated using the brick wall method [117]. In order to avoid the divergence near the event horizon, a cutoff parameter would be utilized. Since the degrees of freedom would be dominant near the horizon, the brick wall method is usually replaced by a thin-layer model making the calculation of entropy possible [118–125]. The entropy of FLRW Universe is given by time-dependent metric. In calculating the entropy of various black holes [126–136, 139], GUP approach can be utilized. Furthermore, the effect of GUP on the reheating phase after the inflation of the Universe has been studied [137].

Therefore, the influence of GUP on the thermodynamics of the FLRW Universe provides a deep understanding of the QG corrections to the dynamics of the FLRW Universe [139]. For instance, the entropy of the apparent horizon of the FLRW Universe gets a correction if one takes into consideration the effect of GUP [138].

1. Some basic features of FLRW Universe

In FLRW Universe, the standard $(n+1)$-dimensional metric reads

$$ds^2 = h_{ab} \, dx^a \, dx^b + r^2 \, d\Omega_{n-1}^2, \tag{161}$$

where $x^a = (t, r)$ and $h_{ab} = \texttt{diag}(-1, a^2/(1-kr^2))$ where $\vec{r} = a(t)r$ and $x^0 = t$. $d\Omega_{n-1}^2$ is the line element of an $(n+1)$-dimensional unit sphere. $a(t)$ and k are scale factor and curvature parameter $k = -1, 0, +1$, respectively. Then, the radius of the apparent horizon is given by

$$R_A = \left(H^2 + \frac{k}{a^2} \right)^{-1/2}. \tag{162}$$

It is obvious that the time evolution of the scale factor strongly depends on the background equation of state. Seeking for simplicity, we utilize [141]

$$a(t) = t^{2/3\bar{k}}, \tag{163}$$

where t is the cosmic time, $\bar{k} = 1 - (b\,c)^2/(1 - c^2)$ and b and c are free and dimensionless parameters. The Hubble parameter $H = \dot{a}/a$ and radius of the apparent horizon, respectively, read

$$H(t) = \frac{2}{3} \frac{1}{\bar{k}} \frac{1}{a^{3\bar{k}/2}}, \tag{164}$$

$$R_A = \left(H \sqrt{1 + \left(\frac{3}{2}\bar{k}\right)^{4/3\bar{k}} H^{4/3\bar{k}-2} k} \right)^{-1/2}. \tag{165}$$

The dot represents derivative with respect to the cosmic time t. From the metric given in Eq. (161) and the Einstein in non-viscous background equations, we get

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{166}$$

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p), \tag{167}$$

where Λ is the cosmological parameter. Then, the total energy density ρ and temperature T inside the sphere of radius R_A can be evaluated as follows.

$$\rho = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right) + 1} \frac{n(n-1)}{16\pi G} R_A^{n-1}, \tag{168}$$

$$T = \frac{R_A}{2\pi} H^2 \left| 1 + \frac{1}{2H^2}\left(\dot{H} + \frac{k}{a^2}\right) \right|, \tag{169}$$

where n gives the dimension of the Universe and p stands for the pressure. From Eq. (162) and (166), it is obvious that the inverse radius of the apparent horizon is to be determined by the energy-momentum tensor, i.e. matter and cosmological constant Λ. Taking into consideration the viscous nature of the background geometry makes the treatment of thermodynamics of FLRW considerably complicated [142–150]. For completeness, we give the cross section of particle production

$$\sigma = \frac{1}{M_p^2} \left[\frac{\rho}{M_p} \left(\frac{8\Gamma\left(\frac{n}{2}\right)}{n-2} \right) \right]^{2/(n-2)}, \tag{170}$$

where Γ is the gamma function. The continuity equation, time evolution of energy density, will be given in Eq. (177).

We consider a $(n+1)$-dimensional FLRW Universe, the metric field equation, Eq. (161), is given by [138]

$$ds^2 = -dt^2 + a^2 \left(\frac{dr^2}{1-k\,r^2} + r^2\,d\Omega_{n-1}^2 \right), \tag{171}$$

where $d\Omega_{n-1}^2$ denotes the line element of an $(n-1)$-dimensional unit sphere. In FLRW spacetime, there is a dynamical apparent horizon, which is a marginally trapped surface with vanishing expansion [138]. Using the notion $\tilde{r} = a\,r$, the radius of the apparent horizon can be written as

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}, \tag{172}$$

If we suppose that the apparent horizon has an associated entropy S and a temperature T[138], then

$$S = \frac{A}{4\,G}, \tag{173}$$

$$T = \frac{1}{2\,\pi\,\tilde{r}_A}, \tag{174}$$

where A is the apparent horizon area $A = n\Omega_n \tilde{r}_A^{n-1}$ with $\Omega_n = \pi^{n/2}/\Gamma(n/2+1)$ being the volume of an n-dimensional unit sphere [138].

The Friedmann equations, Eqs. (166) and (167) reads [139, 170]

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}\,(\rho + p), \tag{175}$$

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\,\rho. \tag{176}$$

When taken into account the first law of thermodynamics, $dE = TdS$ [139, 170] where S is the entropy of the system. In order to get Eq.(176), one should use the continuity equation of the perfect fluid [139, 170]

$$\dot{\rho} + n\,H(\rho + p) = 0. \tag{177}$$

The energy density ρ is related to the pressure of the cosmic fluid, $p = \omega\,\rho$, i.e., the equation of state. The implementation of viscous equations of state in early Universe was analysed, systematically [143–148, 150].

We begin with the GUP approach given in Ref. [29]

$$\Delta x \, \Delta p \geq 1 + \alpha^2 \, l_p^2 \, \Delta \, p^2. \tag{178}$$

After some simple manipulations, we get the momentum uncertainty

$$\Delta p \geq \frac{1}{\Delta x} \left[\frac{\Delta x^2}{2 \, \alpha^2 \, l_p^2} - \frac{\Delta x^2}{2 \, \alpha^2 \, l_p^2} \sqrt{1 - \frac{4 \, \alpha^2 \, l_p^2}{(\Delta x)^2}} \right] = \frac{1}{\delta x} \, f_G(\Delta x^2), \tag{179}$$

where

$$f_G(\Delta x^2) = \frac{\Delta x^2}{2 \, \alpha^2 \, l_p^2} - \frac{\Delta x^2}{2 \, \alpha^2 \, l_p^2} \sqrt{1 - \frac{4 \, \alpha^2 \, l_p^2}{(\Delta x)^2}}. \tag{180}$$

We can identify the energy of absorbed or emitted particle as uncertainty of the momentum [151],

$$dE \simeq \Delta p. \tag{181}$$

From quantum properties of absorbed or emitted particle, the Heisenberg uncertainty principle $\delta p \geq \hbar/\Delta x$ can be implemented. In natural units $c = \hbar = k_B = 1$, we find that the increase or decrease in the area of the apparent horizon can be expressed as

$$dA = \frac{4 \, G}{T} \, dE \simeq \frac{4 \, G}{T} \frac{1}{\Delta x}. \tag{182}$$

When the GUP effect, Eq. (179), is taken into consideration, the change in the apparent horizon area can be modified as

$$dA_G = \frac{4 \, G}{T} dE \simeq \frac{4 \, G}{T} \frac{1}{\delta x} f_G(\delta x^2). \tag{183}$$

Using Eq. (182), we get

$$dA_G = f_G(\delta x^2) dA. \tag{184}$$

The position uncertainty δx of absorbed or emitted particle can be chosen as the particle's Compton length, which is equivalent the inverse of Hawking temperature [280],

$$\delta x \simeq 2 \, \tilde{r}_A = 2 \left(\frac{A}{n \, \Omega_n} \right)^{\frac{1}{n-1}}. \tag{185}$$

Thus, the departure function $f_G(\delta x^2)$ can be re-expressed in terms of A [139],

$$f_G(A) = \frac{2}{\alpha^2 \, l_p^2} \left(\frac{A}{n \, \Omega_n} \right)^{\frac{2}{n-1}} \left(1 - \sqrt{1 - \alpha^2 \, l_p^2 (\frac{n \, \Omega_n}{A})^{\frac{2}{n-1}}} \right). \tag{186}$$

Hereafter, we use $f_G(A)$ to represent the departure function $f_G(\delta x^2)$. At $\alpha = 0$, the Taylor series of $f_G(A)$ gives

$$f_G(A) = 1 + \frac{\alpha^2 \, l_p^2}{4} \left(\frac{n \, \Omega_n}{A} \right)^{\frac{2}{n-1}} + \frac{(\alpha^2 \, l_p^2)^2}{8} \left(\frac{n \, \Omega_n}{A} \right)^{\frac{4}{n-1}} + \sum_{d=3} c_d (\alpha l_p)^{2d} \left(\frac{n \Omega_n}{A} \right)^{\frac{2d}{n-1}}, \quad (187)$$

where c_d is a constant. If Eq. (187) is substituted in Eq.(184) and then integrated, we get a modified area A_G. Also, we get the correction to the entropy-area relation by using $S_G = A_G/4G$ [138]. But integrating Eq.(184) might be complicated and dimension dependent. Therefore, as anticipated in Ref. [139], the discussions should be divided into three cases.

For $n = 3$

the departure function reads [139]

$$f_G(A) = 1 + \pi \, \alpha^2 \, l_p^2 \, \frac{1}{A} + 2 \left(\pi \, \alpha^2 \, l_p^2 \right)^2 \frac{1}{A^2} + \sum_{d=3} c_d \left(4 \, \pi \, \alpha^2 \, l_p^2 \right)^{2d} \frac{1}{A^d}. \quad (188)$$

By substituting Eq. (188) in Eq. (184) and integrating, we obtain the modified relation of the apparent horizon area [139].

$$A_G = A + \pi \, \alpha^2 \, l_p^2 \, \ln(A) - 2 \left(\pi \, \alpha^2 \, l_p^2 \right)^2 \frac{1}{A} - \sum_{d=3} \frac{c_d \left(4 \, \pi \, \alpha^2 \, l_p^2 \right)^{2d}}{d-1} \frac{1}{A^{d-1}} + C, \quad (189)$$

where C is the integral constant. By making use of Bekenstein-Hawking area law [138], $S = A/4G$, we obtain an expression for the entropy of the apparent horizon under the effect of GUP. Accordingly, the modified entropy is given as [139]

$$S_G = \frac{A}{4G} + \frac{\pi \, \alpha^2 \, l_p^2}{4 \, G} \ln \frac{A}{4G} - 2 \left(\frac{\pi \, \alpha^2 \, l_p^2}{4G} \right)^2 \left(\frac{A}{4G} \right)^{-1} - \sum_{d=3} \frac{c_d \left(\frac{16 \, \pi^2 \, \alpha^4 \, l_p^4}{4G} \right)^d}{d-1} \left(\frac{A}{4G} \right)^{1-d} + K, (190)$$

where K stands for a constant. This result shows that the correction to the entropy due to GUP gives an opposite contribution to the area entropy. But when starting with a modified entropy-area relation it was shown recently [140] that the first law of thermodynamics can produce a modified Friedmann equation [139].

Now, we come back to the main results of the GUP approach which was presented in Ref. [140] and apply it to the case of the modified entropy-area relation, Eq. (190). Suppose that the apparent horizon has an entropy $S_G(A)$ and applying the

first law of thermodynamics to the apparent horizon of FLRW Universe, we obtain the corresponding Friedmann equations

$$\left(\dot{H} - \frac{k}{a}\right) S_G'(A) = -\pi\,(\rho + p), \tag{191}$$

$$\frac{8\,\pi\,G}{3}\rho = -\frac{\pi}{G}\int S_G'(A)\left(\frac{4G}{A}\right)^2 dA, \tag{192}$$

where a prime stands for the derivative with respect to A. Eq. (191) and (192) are nothing but the modified first and second Friedmann equation corresponding to the modified apparent horizon entropy $S_G(A)$. From Eq. (184), we get obtain

$$S_G'(A) = \frac{f_G(A)}{4G}. \tag{193}$$

Substituting Eqs. (188) and (193) in the modified Friedmann equations, (191) and (192), we obtain the modified Friedmann equations according to the GUP approach [139],

$$\left(\dot{H} - \frac{k}{a}\right)\left[1 + \pi\,\alpha^2\,l_p^2\,\frac{1}{A} + 2\left(\pi\,\alpha^2\,l_p^2\right)^2\frac{1}{A^2} + \sum_{d=3} c_d\left(4\,\pi\,\alpha^2\,l_p^2\right)^{2d}\frac{1}{A^d}\right] = -4\,\pi\,G(\rho + p), \tag{194}$$

$$\frac{8\,\pi\,G}{3}\rho = 4\,\pi\left[\frac{1}{A} + \frac{1}{2}\alpha^2\,l_p^2\,\frac{1}{A^2} + \frac{2}{3}\left(\pi\,\alpha^2\,l_p^2\right)^2\frac{1}{A^3} + \sum_{d=3}\frac{c_d}{d+1}\left(4\,\pi\,\alpha^2\,l_p^2\right)^{2d}\frac{1}{A^{d+1}}\right]. \tag{195}$$

For odd $n > 3$

by substituting Eq. (187) in Eq. (184) and then integrating [139]

$$A_G = A + \sum_{d=1}^{d=\frac{n-3}{2}} c_d\,(\alpha\,l_p)^{2d}\,\frac{n-1}{n-2d-1}A\left(\frac{n\,\Omega_n}{A}\right)^{\frac{2d}{n-1}}$$

$$+\; c_{\frac{n-1}{2}}\,(\alpha\,l_p)^{n-1}\,n\,\Omega_n\,\ln(A) + \sum_{d=\frac{n+1}{2}} c_d\,(\alpha\,l_p)^{2d}\,\frac{n-1}{n-2d-1}A\left(\frac{n\,\Omega_n}{A}\right)^{\frac{2d}{n-1}}. \tag{196}$$

By implementing Bekenstein-Hawking area law [138], $S = A/4G$, and when taking into account the effect of GUP [139], we obtain an expression for the entropy of the apparent horizon

$$S_G = \frac{A}{4G} + \sum_{d=1}^{d=\frac{n-3}{2}} c_d(\alpha\,l_p)^{2d}\frac{n-1}{n-2d-1}\frac{A}{4G}\left(\frac{n\Omega_n}{A}\right)^{\frac{2d}{n-1}}$$

$$+\; \sum_{d=\frac{n+1}{2}} c_d\,(\alpha\,l_p)^{2d}\,\frac{n-1}{n-2d-1}\,\frac{A}{4G}\left(\frac{n\,\Omega_n}{A}\right)^{\frac{2d}{n-1}} + const. \tag{197}$$

In order to obtain the modified Friedmann equations from the modified entropy-area relation (197) in $(n + 1)$-dimensional FLRW spacetime, we have to generalize the approach given in Ref. [140] to an $(n + 1)$-dimensional FLRW Universe. The original approach [140] is only valid in an $(3 + 1)$-dimensional FLRW Universe. The first law of thermodynamics on the apparent horizon $dE = TdS$ leads to [139].

$$A(\rho + p)H\tilde{r}_A dt = \frac{1}{2\pi\tilde{r}_A} dS_G, \tag{198}$$

where $A(\rho + p)H\tilde{r}_A dt = dE$ is the amount of energy having crossed the apparent horizon. With some simple manipulations, we can obtain the Friedmann equations in $(n + 1)$-dimensional FLRW Universe [139],

$$\left(\dot{H} - \frac{k}{a^2}\right) f_G(A) = -\frac{8\pi G}{n - 1}(\rho + p), \tag{199}$$

$$\frac{8\pi G}{n}\rho = -\int f_G(A)\left(\frac{A}{n\Omega_n}\right)^{\frac{-2}{n-1}} \frac{dA}{A}, \tag{200}$$

Substituting Eq. (187) in Eqs. (199) and (200), we obtain the modified Friedmann equations in $(n + 1)$-dimensional FRW spacetime according to the GUP approach [138, 139]

$$\left(\dot{H} - \frac{k}{a^2}\right)\left[1 + \frac{\alpha^2 l_p^2}{4}\left(\frac{n\Omega_n}{A}\right)^{\frac{2}{n-1}} + \frac{(\alpha^2 l_p^2)^2}{8}\left(\frac{n\Omega_n}{A}\right)^{\frac{4}{n-1}} + \sum_{d=3} c_d(\alpha\, l_p)^{2d}\left(\frac{n\Omega_n}{A}\right)^{\frac{2d}{n-1}}\right] =$$
$$-\frac{8\pi G}{n - 1}(\rho + p), \quad (201)$$

and

$$\frac{16\pi G}{n(n-1)}\rho = \left(\frac{n\,\Omega_n}{A}\right)^{\frac{2}{n-1}} + \sum_{d=1} \frac{c_d}{d+1}(\alpha\, l_p)^{2d}\left(\frac{n\,\Omega_n}{A}\right)^{\frac{2d+2}{n-1}}. \tag{202}$$

We note here that these equations are independent on whether n is an odd or even number. For $n = 3$, Eqs. (201) and (202) will be reduced to Eqs. (194) and (195), respectively.

For even $n > 3$

we obtain an expression for the modified entropy of the apparent horizon [139]

$$S_G = \frac{A}{4G} + \frac{\alpha^2\, l_p^2}{4}\frac{n-1}{n-3}\frac{A}{4G}\left(\frac{n\,\Omega_n}{A}\right)^{\frac{2}{n-1}}$$
$$+ \sum_{d=2} c_d\,(\alpha\, l_p)^{2d}\frac{n-1}{n-2d-1}\frac{A}{4G}\left(\frac{n\,\Omega_n}{A}\right)^{\frac{2d}{n-1}}. \tag{203}$$

When d is an even number, then the logarithmic term does not exist in the correction to the entropy of the apparent horizon of FLRW spacetime. This implies that the logarithmic correction term in the entropy of the apparent horizon is dimension-dependent. Since the modified Friedmann equations (201) and (202) in an $(n+1)$-dimensional FLRW Universe is not relevant to that whether n is an even or odd number [139], the modified Friedmann equations from modified entropy, Eq. (203), are given by Eqs. (201) and (202).

3. Entropic corrections and modified Friedmann equations

The entropic corrections in modified Friedmann equations appear in two types:

Logarithmic-type corrections:

We start with the corrected entropy-area relation [153–155]

$$S = \frac{A}{4G} + \alpha \, \ln \frac{A}{4G} + \beta \, \frac{4G}{A}, \tag{204}$$

where the Newton's constant $G = L_p^2$ and the area $A = 4\pi \widetilde{r}_A^2$. The relevant effective area of the holographic surface is defined as [153–155]

$$\widetilde{A} = A + 4\,\alpha\,L_p^2 \, \ln\left(\frac{A}{4\,L_p^2}\right) + \frac{16\,\beta\,L_p^4}{A}. \tag{205}$$

Then, the increase in the effective volume can be calculated as

$$\frac{d\widetilde{V}}{d\,t} = \frac{\widetilde{r}_A}{2} \frac{d\,\widetilde{A}}{d\,t} = 4\,\pi\,\widetilde{r}_A^2\,\dot{\widetilde{r}}_A \left(1 + \frac{\alpha\,L_p^2}{\pi\,\widetilde{r}_A^2} - \frac{\beta\,L_p^4}{\pi^2\,\widetilde{r}_A^4}\right). \tag{206}$$

From the fact that

$$\frac{d\widetilde{V}}{dt} = -2\,\pi\,\widetilde{r}_A^5 \, \frac{d}{dt}\frac{1}{\widetilde{r}_A^2} + \frac{\alpha\,L_p^2}{2\,\pi\,\widetilde{r}_A^4} - \frac{\beta\,L_p^4}{3\,\pi^2\,\widetilde{r}_A^6}, \tag{207}$$

we propose that the effective degrees of freedom (at apparent horizon) are

$$\widetilde{N}_{\text{sur}} = \frac{4\,\pi\,\widetilde{r}_A^2}{L_p^2}\left(1 + \frac{\alpha L_p^2}{2\pi\widetilde{r}_A^2} - \frac{\beta L_p^4}{3\pi^2\widetilde{r}_A^4}\right). \tag{208}$$

According to the equipartition law [155], we have

$$N_{bulk} = \frac{2}{T}|E_{Komar}| = -2\,(\rho + 3p)\,\frac{V}{T}, \tag{209}$$

where the expression of the Komar energy [155] has been inserted. The appearance of a minus sign is due to the fact that we are considering the accelerating phase. The Hawking temperature [155] $T = 1/2\pi \tilde{r}_A$ and the cosmic volume $V = 4\pi \tilde{r}_A^3/3$, then [155]

$$N_{bulk} = -\frac{16\,\pi^2}{3}(\rho + 3p)\,\tilde{r}_A^4 = \frac{16\,\pi^2}{3}\left(\frac{\dot{\rho}}{H} + 2\,\rho\right)\tilde{r}_A^4. \tag{210}$$

Using the continuity expression, Eq. (177) and substituting Eqs. (206), (208), and (210) in the expansion law $d\tilde{V}/dt = L_p^2 H\tilde{r}_A(\tilde{N}_{sur} - N_{bulk})$ [155], we get

$$4\,\pi\,\tilde{r}_A^3\,\frac{\dot{a}}{a}\left(1 + \frac{\alpha\,L_p^2}{2\,\pi\,\tilde{r}_A^2} - \frac{\beta\,L_p^4}{3\,\pi^2\,\tilde{r}_A^4}\right) - 4\,\pi\,\tilde{r}_A^2\dot{\tilde{r}}_A\left(1 + \frac{\alpha\,L_p^2}{\pi\,\tilde{r}_A^2} - \frac{\beta\,L_p^4}{\pi^2\,\tilde{r}_A^4}\right) =$$
$$\frac{16\,\pi^2\,L_p^2}{3}\frac{1}{a}(\dot{\rho}\,a + 2\,\rho\,\dot{a})\tilde{r}_A^5. \tag{211}$$

Multiplying both sides by $a^2/2\pi\tilde{r}_A^5$, and then integrate both sides and approximating the integration constant to vanish, we get the modified Friedmann equation [155]

$$\left(H^2 + \frac{k}{a^2}\right) + \frac{\alpha L_p^2}{2\pi}\left(H^2 + \frac{k}{a^2}\right)^2 - \frac{\beta L_p^4}{3\pi^2}\left(H^2 + \frac{k}{a^2}\right)^3 = \frac{8\,\pi\,L_p^2}{3}\rho. \tag{212}$$

Power-law corrections:

The entropy with power-law corrections [152] can be deduced as follows.

$$S = \frac{A}{4\,L_p^2}\left(1 - K_\alpha\,A^{1-\frac{\alpha}{2}}\right). \tag{213}$$

From the definition $K_\alpha = \alpha(4\pi)^{\frac{\alpha}{2}-1}/(4-\alpha)r_c^{2-\alpha}$, where r_c is the crossover scale, the effective degrees of freedom at the apparent horizon read [155]

$$\tilde{N}_{sur} = \frac{4\,\pi\,\tilde{r}_A^2}{L_p^2}\left[1 - \left(\frac{r_c}{\tilde{r}_A}\right)^{\alpha-2} + C\tilde{r}_A^2\right]. \tag{214}$$

In the limit $\alpha \longrightarrow 0$ and with the constant constant $C = 1/r_c^2$, no entropic correction should appear. The exactly-modified Friedmann equation has been derived [155–157]

$$H^2 + \frac{k}{a^2} - \frac{1}{r_c^2}\left[r_c^\alpha\left(H^2 + \frac{k}{a^2}\right)^{\frac{\alpha}{2}} - 1\right] = \frac{8\,\pi\,L_p^2}{3}\rho. \tag{215}$$

The influence of GUP on the thermodynamics of the FLRW Universe shows that the GUP contributes with some corrections to the entropy-area relation at the apparent horizon of the FLRW Universe as well as to the Friedmann equations. The later imply that the GUP affects the dynamics of the FLRW Universe. The leading logarithmic correction term exists only for odd number in one-dimensional FLRW spacetime. This term gives a positive contribution to the entropy of the apparent horizon. For even number in one-dimensional FLRW spacetime, the logarithmic correction term disappears from the entropy. The expansion of the Universe is attributed to the difference between the degrees of freedom on a holographic surface and the one in the bulk. The idea taken from the modification of holographic screen in both ways "power-law corrections" or "logarithmic corrections" implies an additional term due the introduction of the minimal length to the entropy-area relation which will be modify the Friedmann equations.

B. Inflationary parameters

1. Hybrid inflation and black hole production

In a scenario of semi-classical black hole combining hybrid inflation [168] and characterized by the hybrid inflation model, the inflation fields (ϕ, ψ) are governed by the inflation potential,

$$V(\phi, \psi) = \frac{1}{2}\, m^2\, \phi^2 + \frac{1}{2}\, \gamma\, \phi^2\, \psi^2 + \left(M^2 - \frac{\sqrt{\lambda}}{2}\, \psi^2 \right)^2,$$ (216)

where M be the mass of the black hole. There are two conditions on ϕ:

- When ϕ executes a *"slow-roll"* down [169], then the potential is responsible for more than 60 *e*-folds expansion while ψ remains zero.

- But if ϕ is reduced to a critical value, $\phi_c = \sqrt{2M^2\sqrt{\lambda}/\gamma}$, the phase transition which results in a *"rapid-fall"* [169] of the energy density of the ψ field, ends the inflation. The latter lasts only for a few *e*-folds.

The equations of motion (EoM) for these fields read [169].

$$\ddot{\phi} + 3\,H\,\dot{\phi} + \left(m^2 + \gamma\,\psi^2\right) = 0, \tag{217}$$

$$\ddot{\psi} + 3\,H\,\dot{\psi} + \left(\lambda\,\psi^2 + \gamma\,\phi^2 - 2\,\sqrt{\lambda}M^2\right) = 0. \tag{218}$$

The Hubble parameter can be taken into consideration from the Friedmann equation [169]

$$H^2 = \frac{8\,\pi}{3\,m_p^2}\left(\frac{1}{2}\,\dot{\phi}^2 + \frac{1}{2}\,\dot{\psi}^2 + V(\phi,\psi)\right). \tag{219}$$

The solution for the ψ field in the small ϕ regime [169] measured backwards from the end of the inflation is given as

$$\psi\left(N(t)\right) = \psi_e\,e^{-\kappa\,N(t)}, \tag{220}$$

where $\kappa = -3/2 + \sqrt{9/2 + 2\sqrt{\lambda}\,M^2/H_*^2}$ is the angular momentum and $N(t) = H_*(t_e - t)$ is the number of e-folds existing from t_e to t with $H_* = \sqrt{8\pi/3}\,M^2/m_p$.

It has been shown that a large number of small black holes can be produced during the second stage of the inflation [169]. The quantum fluctuations of ψ induce variations to take place in the second inflation stage, i.e, $\delta t = \psi/\dot{\psi}$ [169]

$$\delta N = H_*\,\frac{\psi}{\delta\dot{\psi}}. \tag{221}$$

The curvature contrasts related to the number of e-folds are given as [169]

$$\delta = \frac{\delta\rho}{\rho} = \frac{2 + 2\,\omega}{5 + 3\,\omega}\,\delta N, \tag{222}$$

where $\omega = p/\rho$ being the EoS. The quantity $\delta N = 1/\kappa$ was defined as [169]

$$\delta \approx \frac{2 + 2\,\omega}{5 + 3\,\omega}\frac{1}{\kappa}. \tag{223}$$

The probability of a region of mass m [231, 232]

$$P(m) \sim \delta(m)\,\exp\left(-\frac{\omega^2}{2\,\delta^2}\right), \tag{224}$$

with an initial density contrast $\delta(m) \equiv \delta\rho/\rho|_m$. It was assumed that the Universe was inflated $\exp(N_c)$ times during the second stage of the inflation era [158]

$$\exp(N_c) \sim \left(\frac{2\,m_p}{\kappa\,H_*}\right)^{1/\kappa}. \tag{225}$$

If the second stage of inflation is short, i.e. $N_c \sim \mathcal{O}(1)$, then the energy direct after the inflation may still be dominated by the oscillations of ψ with $p = 0$. The scale factor of the

Universe after the inflation would grow as $(t\,H_*)^{2/3}$ [169]. When the scale $(t\,H_*)^{2/3}\,H_*^{-1}\,e^{N_c}$ becames comparable to the particle horizon t or $t \sim (t\,H_*)^{2/3}\,H_*^{-1}\,e^{N_c}$ [169], then

$$t \sim t_h = H_*^{-1}\,\exp\left(3\,N_c\right). \tag{226}$$

At this time, if the density contrast was $\delta \sim 1$, then a black hole with size $r_s \sim H_*^{-1}\,e^{3\,N_c}$ would be formed with an initial mass [169]

$$\mu_i \sim \frac{m_p}{H_*}\,\exp\left(3\,N_c\right) \equiv \alpha\,\frac{m_p}{H_*}\left(\frac{2\,m_p}{\kappa\,H_*}\right)^{3/\kappa}. \tag{227}$$

A dimensionless parameter α is introduced to account for the dynamic range of the gravitational collapse [169]. Since H_* depends on M, while s on M and λ, the initial mass of the black hole depends only on the mass and the coupling in the ψ-sector of the hybrid inflation [169].

2. Randall-Sundrum II model on inflationary dynamics

The scalar field ϕ which drives the inflation has the energy density and the pressure [110]

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{228}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{229}$$

where $V(\phi)$ is the inflation potential. The calculation of the inflationary scalar density perturbations in the presence of the minimal length was preformed [110]. The slow-roll regime has [159]

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \tag{230}$$

$$3\,H\,\dot{\phi} \approx -V'(\phi). \tag{231}$$

A fundamental energy scale ϵ in order of the Planck energy, which adds to the correction [160] seems to define the conformal time

$$\tau = -\frac{1}{a\,H}, \tag{232}$$

and the comoving momentum, κ, which is related to the physical momentum, p,

$$\kappa = a\,p = -\frac{p}{\tau\,H}. \tag{233}$$

The conformal time is given as $\tau_0 = -\epsilon/H\kappa$.

Using the quadratic GUP [29], we can change the comoving momentum before τ_0 from κ to $\kappa(1 + \beta\kappa^2)$ [161]. This modifies the dispersion relation which was supported by the loop quantum gravity and non-commutative geometry [162]. The equation governing the evolution of perturbations in the inflation reads [161]

$$\mu_\kappa'' + \mu_\kappa \left(\kappa^2 - \frac{a''}{a} \right) = 0, \tag{234}$$

where μ is related to the scalar field $\mu = a\,\delta\phi$. The scalar spectral index in the presence of the minimal length cutoff is given as [161],

$$
\begin{aligned}
n_s &= \frac{d\,\ln(\Re_s)}{d\,\ln(\kappa)(1 + \beta\,\kappa^2)} + 1 = \frac{(1 + \beta\,\kappa^2)}{(1 + 3\,\beta\,\kappa^2)} \frac{d\,\ln(Re_s)}{d\,\ln(\kappa)} + 1 \\
&\approx (1 - 2\,\beta\,\kappa^2) \frac{d\,\ln(\Re_s)}{d\,\ln(\kappa)} + 1,
\end{aligned}
\tag{235}
$$

where $\Re_s$ is the amplitude of the scalar perturbation. The change in the Hubble parameter due to the GUP will be realized using the slow-roll parameters [159, 160]. At the horizon crossing epoch, we have [159]

$$\frac{d}{d\,\kappa} H = -\frac{\epsilon H}{\kappa}. \tag{236}$$

When κ is replaced by $\kappa(1 + \beta\,\kappa^2)$, then we get

$$H \approx \kappa^{-\epsilon} \exp\left(-\beta\,\epsilon\,\kappa^2 \right). \tag{237}$$

By using Eq. (234), then the tensorial density fluctuations are given as [160]

$$\Re_t(\kappa) = \left(\frac{H}{2\,\pi} \right)^2 \left[1 - \frac{H}{\epsilon_p} \sin\left(\frac{2\,\epsilon_p}{H} \right) \right], \tag{238}$$

where the second term on the right hand side is a direct contribution from the quantum gravity effect and ϵ_p being the Planck energy. But for scalar density fluctuations, one should multiply the tensorial density fluctuations by an extra term $(H/\dot{\phi})^2$ [110]

$$\Re_s = \left(\frac{H}{\dot{\phi}} \right)^2 \left(\frac{H}{2\,\pi} \right)^2 \left[1 - \frac{H}{\epsilon_p} \sin\left(\frac{2\,\epsilon_p}{H} \right) \right], \tag{239}$$

where H was given in Eq. (237). The variation of β, which is essentially a fixed quantity related to the minimal length, means a control on the strength of the quantum gravity effect. Then, the ratio tensor-to-scalar reads [161]

$$\frac{\Re_t}{\Re_s} = \left(\frac{\dot{\phi}}{H} \right)^2 = \left(\frac{16\,\pi\,\sqrt{\epsilon}V(\phi)}{M_4\,H} \right)^2, \tag{240}$$

where M_4 is 4-dimensional (fundamental) Planck scale. The difference between tensor-to-scalar ratio in standard and modified case is shown in Fig. (5) [161]. It is obvious that the ration increases linearly with the incorporation of the quantum gravity effects.
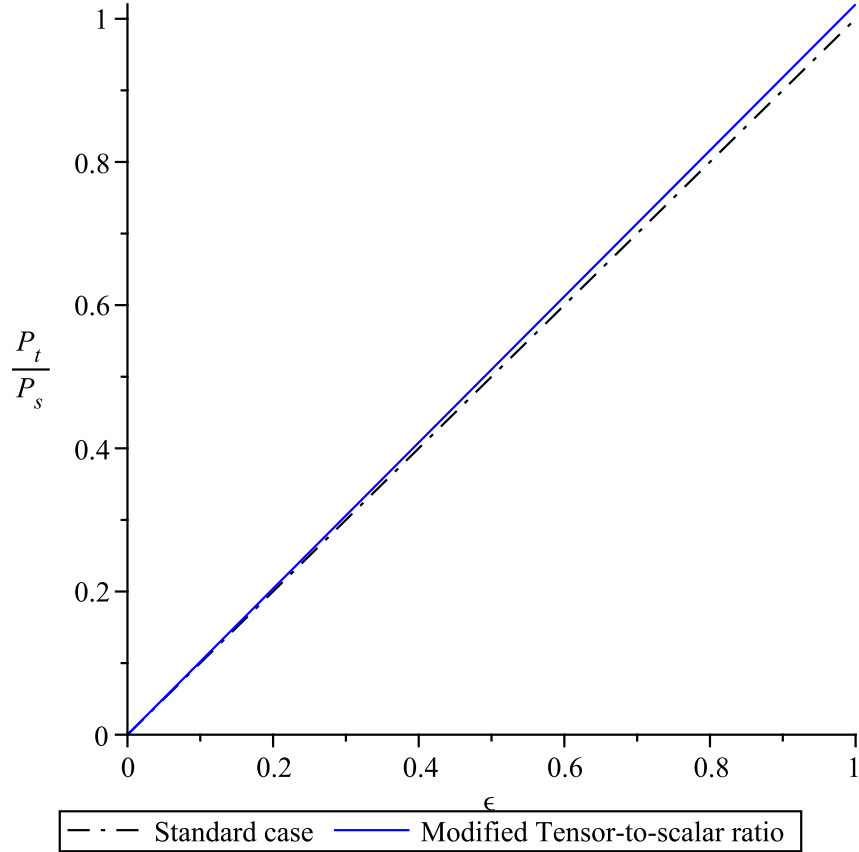
Fig. 5: The difference between tensor-to-scalar ratio in standard and GUP-modified inflation is given in dependence on the energy ϵ at fixed κ and $\beta = 10^{-2}$. The graph taken from [161].

3. conclusion

By studying the effect of GUP on the inflationary dynamics of both the standard 4D theory and the Randall-Sundrum II braneworld setup, it was shown that in the presence of the strong quantum gravity effects the spectral index is not scale invariant [110]. In this sense, any deviation from the scale invariance of the spectral index essentially contains a footprint of these high energy effects [110]. There is an oscillatory behavior in the κ-dependence of the density fluctuations which essentially can be detected in the CMB spectrum [161] as

a trace of these effects. Another possible signature may be some imprints on the cosmic microwave background (CMB) fluctuations due to the thermodynamics of primordial black hole (PBH)-CMB interactions [110].

C. Black hole thermodynamics

1. Black hole entropy and GUP approaches

The thermodynamics of the system of interest is easily accessible through the partition function

$$Z = \frac{1}{2\pi\hbar} \int \exp\left[-\beta H(x,p)\right] \, dx \, dp. \tag{241}$$

The deformation of generalized commutation relations reads

$$[X_i, \, P_j] = i\hbar \, f_{ij}(X, P),$$
$$[P_i, \, P_j] = i\hbar \, h_{ij}(X, P),$$
$$[X_i, \, X_j] = i\hbar \, g_{ij}(X, P), \tag{242}$$

where operators X_i and P_j are coordinates and momentum variables, respectively. The deformation functions f_{ij}, g_{ij} and h_{ij} possessing properties like bilineary, Libniz rules and Jacobi identity [163]. The given relations can be reduced to the deformed Poisson brackets

$$\{X_i, \, P_j\} = f_{ij}(X, P),$$
$$\{P_i, \, P_j\} = h_{ij}(X, P),$$
$$\{X_i, \, X_j\} = g_{ij}(X, P). \tag{243}$$

These relations are anti-symmetric and also bilinear, besides, they obey the Libniz rules and the Jacobi identity [99, 164]. Then, the partition function, Eq. (241), for deformed case can be interpreted in terms of X and P [263]

$$Z_{deformed} = \frac{1}{2\pi\hbar} \int \exp\left[-\beta H(x,p)\right] \frac{dX \, dP}{J}. \tag{244}$$

According to the quadratic GUP approach [29], $g(X, P) = h(X, P) = 0$ and $\{X, P\} = f(X, P) = 1 + \sigma P^2$. Therefore, the partition function of quantum black hole becomes

$$Z_{GUP} = \frac{1}{2\pi\hbar} \int dX \exp\left[-\beta V(\mathbf{X})\right] \int dP \frac{\exp\left(-\frac{\beta c \ell_p}{2\hbar} P^2\right)}{1 + \sigma P^2}. \tag{245}$$

The corrected partition function reads [163]

$$Z_Q^{GUP} = \frac{\ell_p}{\hbar} \sqrt{\frac{\pi}{3 \beta E_p \sigma}} \, \exp\left[-\left(\frac{\beta^2 E_p{}^2}{16 \pi} + \frac{c \ell_p \beta}{2 \hbar \sigma} \right) \right] \, \Gamma\left(\frac{1}{2} \frac{c \ell_p \beta}{2 \hbar \sigma} \right). \tag{246}$$

For the case $c\ell_p\beta/2\hbar\sigma \gg 1$, the corrected partition function leads to [163]

$$Z_Q^{GUP} = \sqrt{\frac{2\pi}{3}} \frac{1}{\beta E_p} \, \exp\left(-\frac{\beta^2 E_p{}^2}{16\pi} - \frac{\hbar \sigma}{c \ell_p \beta} \right). \tag{247}$$

Similar to the non-deformed case, it was defined that [163]

$$\overline{E} = -\frac{\partial \ln(Z_Q^{GUP})}{\partial \beta} = \frac{E_p{}^2}{8\pi}\beta + \frac{1}{\beta} - \frac{\hbar}{c \ell_p \beta^2}\sigma = Mc^2. \tag{248}$$

In framework of GUP, the temperature of quantum black hole β can be given in term of Hawking temperature β_H

$$\beta = \beta_H \left[1 - \frac{1}{\beta_H M c^2} + \frac{M E_p}{(\beta_H M c^2 - 1)(\beta_H M c^2 - 2)}\sigma \right]. \tag{249}$$

By using the obtained temperature, the entropy is accounted for [163]

$$\begin{aligned}
\frac{S_{GUP}}{k} &= \frac{A_s}{4\ell_p^2}\left[1 - \frac{1}{\beta_H M c^2} \right]^2 + \frac{A_s}{4\,\ell_p^2}\left[1 - \frac{1}{\beta_H M c^2} \right]\frac{M E_p}{(\beta_H M c^2 - 2)(\beta_H M c^2 - 1)}\sigma \\
&\quad -\frac{1}{2}\ln\left(\frac{A_s}{4\ell_p^2}\left[1 - \frac{1}{\beta_H M c^2} \right]^2 + \frac{A_s}{4\,\ell_p^2}\left[1 - \frac{1}{\beta_H M c^2} \right]\frac{M E_p}{(\beta_H M c^2 - 2)(\beta_H M c^2 - 1)}\sigma \right) \\
&\quad -\frac{2 E_p \sigma}{c^2}\left[\beta_H\left(1 - \frac{1}{\beta_H M c^2} \right) \right]^{-1} - \frac{1}{2}\ln(24) + 1 \; .
\end{aligned} \tag{250}$$

Then, the definition of GUP to the Hawking-Bekenstein entropy obviously reads [163],

$$\frac{S_{BH}^{GUP}}{k} = \frac{S_{BH}}{k}\left(1 + \frac{E_p^3}{8\pi M^2 c^6}\sigma \right), \tag{251}$$

which leads to

$$\frac{S^{GUP}}{k} = \frac{S_{BH}^{GUP}}{k} - \frac{1}{2}\ln\left(\frac{S_{BH}^{GUP}}{k} \right) - 2 M c^2 \left(\frac{S_{BH}^{GUP}}{S_{BH}} - 1 \right) + \mathcal{O}\left(S_{BH}^{GUP-1} \right), \tag{252}$$

This correction [163] is similar to ones derived from other methods [165, 166, 280]. Furthermore, it was shown that this result has the same form as that of the non-deformed case, the logarithmic correction to the entropy appears with a $-1/2$ factor [163].

2. *Black hole remnant*

As a result of GUP there should exist a Planck size at the end of the black hole evaporation [169]. GUP may prevent the black hole from complete evaporating, i.e., there should exist a black hole remnant with Planck mass. The stability of such a Planck size containing the remnant mass may be further protected by super-symmetry [169] in form of an extremal black hole. The uncertainty relation in position is given as

$$\Delta x \geq \frac{\hbar}{\Delta p} + \zeta^2 \ell_p^2 \frac{\Delta p}{\hbar}, \tag{253}$$

where ℓ_p is Planck length and ζ is a factor originated in the String theory [29].

In the vicinity of the black hole surface, there is an intrinsic uncertainty in the position, which is approximately equal to the Schwarzschild radius [167], $\Delta x \approx r_s = 2GM_{BH}/c^2$. Under the GUP effect, the emitted photon from the black hole is characterized by the temperature, which is related to the Hawking temperature [167], $T_H = \hbar c^3/(8\pi G M_{BH})$. The modified black hole temperature is given by

$$T_H = \frac{\mu \, m_p \, c^2}{4 \, \pi \, \zeta^2} \left[1 \mp \sqrt{1 - \frac{\zeta^2}{\mu^2}} \right], \tag{254}$$

where $\mu = M_{BH}/m_p$ is the mass of the black hole M_{BH} normalized to the Planck mass m_p. Note that the temperature becomes complex and unphysical when the mass becomes less than $\zeta \, m_p$ and the Schwarzschild radius becomes less than $2\zeta\ell_p$. The minimum length allowed by the GUP approach is given at $\mu \, \zeta$ [169]. The Hawking temperature T_H is finite but its slope is infinite, which is corresponding to vanishing heat capacity. The black hole evaporation is then going to stop. In Stefan-Boltzmann law, the rate of evaporation reads

$$\dot{\mu} = -\frac{16 \, g}{t_{ch}} \frac{\mu^6}{\xi^8} \left[1 - \sqrt{1 - \xi^2/\mu^2} \right]^4, \tag{255}$$

where $t_{ch} = 60(16)^2 \pi \, t_p \approx 4.8 \times 10^4 \, t_p$ is a characteristic time for BH evaporation, and $t_p = (\hbar \, G/c^5)^{1/2} \approx 0.54 \times 10^{-43}$ s is the Planck time. Note that the energy output given by Eq. (255) is finite at the end point, where $\mu = \xi$, i.e. $d\mu/dt \,|_{\mu=\xi} = -16 \, g/(\xi^2 \, t_{ch})$. Thus, the black hole with an initial mass μ_i evaporates till it leaves a concrete remnant in a time given as

$$\begin{aligned}
\tau &= \frac{t_{ch}}{16 \, g} \left[\frac{8}{3}\mu_i^3 + \frac{8}{3}(\mu_i^2 - \xi^2)^{3/2} - 4\,\xi^2(\mu_i^2 - \xi^2)^{1/2} - 8\,\xi^2 \, \mu_i + 4\,\xi^3 \, \cos^{-1}\frac{\xi}{\mu_i} + \frac{19}{3}\xi^3 - \frac{\xi^4}{\mu_i} \right] \\
&\approx \frac{\mu_i^3}{3 \, g} \, t_{ch}, \tag{256}
\end{aligned}$$

where $\mu_i \gg 1$. The evaporation time in this limit agrees with the standard Hawking picture.

3. Conclusion

We have determined the thermodynamic properties of black hole by introducing the relevant Bekenstein-Hawking entropy in the GUP framework and concluded that again the logarithmic correction of the entropy appears with a pre-factor $1/2$. Furthermore, the value of the entropy diminishes. This can be comprehended from the fact that the GUP reduces the available physical states in the black hole remnant [169]. Since H_* depends on M while s on M and λ, the initial black hole mass depends on the mass and coupling in the sector of hybrid inflation [169].

D. Compact stellar objects

1. Compact stars and Tolman-Oppenheimer-Volkoff equation

The configuration of a spherically symmetric static star composed of perfect fluids is determined by the Tolman-Oppenheimer-Volkoff (TOV) equation [251, 252]

$$\frac{dP}{dr} = -\left(\rho + \frac{P}{c^2}\right) \frac{G\,m(r) + 4\,\pi\,G\,r^3\,\frac{P}{c^2}}{r\left[r - 2\,G\,\frac{m(r)}{c^2}\right]}, \tag{257}$$

with

$$\frac{dm(r)}{dr} = 4\,\pi\,r^2\,\rho(r), \tag{258}$$

where P and ρ are respectively the pressure and the macroscopic energy density measured in proper coordinates.

The equation of state (EoS) and appropriate boundary conditions, Eqs. (257) and (258) can be supplied to determine $P(r)$, $m(r)$ and $\rho(r)$. If the pressure and the gravitational potential remain small forever, i.e. $P(r) \ll \rho\,c^2$, $2Gm(r)/c^2r \ll 1$, then the TOV equation reduces to the fundamental equation of Newtonian gravity

$$\frac{dP}{dr} = -\rho(r)\,\frac{G\,m(r)}{r^2}, \tag{259}$$

which is very well suitable to describe the low density compact stars. For compact stars like neutron stars, GR plays an important role [295]. An ideal neutron star would be

the simplest model, in which the nuclear interactions are ignored and the pressure of cold degenerate neutrons contends against the gravitational collapse [252]. Various types of EoS are introduced to represent strongly interacting components and nuclear interactions. The QG effects are studied in various models [253–257, 259]. These would be very interesting, when addressing cold but large density and high pressure.

In Fermi stars, the statistics of the ideal gases based on GUP has been discussed by many authors [260–264]. A system composed of ultra-relativistic Fermi gas was studied under the effects of GUP [264] at zero temperature. Furthermore, the proper particle number, energy density and pressure have been determined [264]

$$\frac{N}{V} = \frac{8\pi}{(hc)^3} E_H^3 f(\kappa), \tag{260}$$

$$\rho = \frac{8\pi}{c^2(hc)^3} E_H^4 h(\kappa), \tag{261}$$

$$P = \frac{8\pi}{(hc)^3} E_H^4 g(\kappa), \tag{262}$$

where $E_H = c/\sqrt{\beta} = M_p c^2/\sqrt{\beta_0}$ denotes the Hagedorn energy [264] and $\kappa = \varepsilon_F \sqrt{\beta/c^2} = \varepsilon_F/E_H$. Moreover

$$h(\kappa) \equiv \frac{1}{4} \frac{\kappa^4}{(1+\kappa^2)^2}, \tag{263}$$

$$f(\kappa) \equiv \frac{1}{8} \left[\frac{\kappa(\kappa^2 - 1)}{(1+\kappa^2)^2} + \tan^{-1}(\kappa) \right], \tag{264}$$

$$g(\kappa) \equiv \kappa f(\kappa) - h(\kappa). \tag{265}$$

These functions are related to the presence of the GUP corresponding to quadratic of momenta. When κ increases, the proper pressure blows up, while the proper energy density and the proper number density both are bounded [266]. This is a manifestation of the minimal length. Based on the precision measurement of Lamb shift, an upper bound of β_0 is given by $\beta_0 < 10^{36}$ [74]. A relatively rough but stronger restriction was estimated [258]. However, a better bound is gained from simple electroweak consideration $\beta_0 < 10^{34}$.

For $\beta_0 = 10^{34}$, Eqs. (261) and (262) can be estimated as

$$\rho = 5.24 \times 10^{95} \frac{h(\kappa)}{\beta_0^2} \sim 10^{27} h(\kappa) \text{ kg} \cdot \text{m}^{-3}, \tag{266}$$

$$P = 4.73 \times 10^{112} \frac{1}{\beta_0^2} g(\kappa) \sim 10^{44} g(\kappa) \text{ Pascals.} \tag{267}$$

We can compare these with the normal nuclear density and pressure. The highest pressure recorded under laboratory controlled conditions [266]

$$\rho_n = 2.7 \times 10^{17} \text{ kg} \cdot \text{m}^{-3}, \tag{268}$$

$$P_n \sim 10^{34} \text{ Pascals}. \tag{269}$$

In vicinity of nuclear matter at equilibrium density, it is found that the quantum gravitational effects are not important. However, for density higher than the normal nuclear one, it is of great interest to investigate the cores of the compact stars, where QG plays a main role by considering the degeneracy pressure regardless of the interaction correction. On the other hand, several accurate masses determinations of neutron stars are available from radio binary pulsars [266].

By applying the Newtonian limit, Eq. (259), with a uniform density, two configurations of the compact stars have been addressed [264].

- The star is almost composed of ultra-relativistic particles.

- The major contribution to the mass is coming from non-relativistic cold nuclei.

However, in order to discuss the core of ultra-compact stars, one has to use TOV Eqs. (257) and (258). By setting $r = r_0 \tilde{r}$, $m = m_0 \tilde{m}$, $P = P_0 \tilde{P}$ and

$$\rho = \frac{m_0}{4 \pi r_0^3} \tilde{\rho} \equiv \rho_0 \tilde{\rho},$$
$$P_0 = \rho_0 c^2, \tag{270}$$
$$\frac{G m_0}{c^2 r_0} \equiv 1,$$

the TOV Eqs (257) and (258) are reduced to the following dimensionless ones

$$\frac{d\tilde{P}}{d\tilde{r}} = -(\tilde{\rho} + \tilde{P}) \frac{\tilde{m} + \tilde{r}^3 \tilde{P}}{\tilde{r}(\tilde{r} - 2\tilde{m})}, \tag{271}$$

$$\frac{d\tilde{m}}{d\tilde{r}} = \tilde{r}^2 \tilde{\rho}. \tag{272}$$

In the vicinity of vanishing r, EoS are given by Eqs. (260), (261) and (262). At $\kappa \to 0$, it is straightforward to recover $P = \rho/3c^2$. Defining $r = r_0 \tilde{r}$, $m = m_0 \tilde{m}$ with

$$r_0^{-2} \equiv \frac{4 \pi G}{c^4} \frac{8 \pi}{(h c)^3} E_H^4, \tag{273}$$

$$m_0 \equiv 4 \pi r_0^3 \frac{8 \pi}{c^2 (h c)^3} E_H^4 = 1.93 \times 10^{-8} \beta_0 \text{ kg}, \tag{274}$$

$$P_0 = \rho_0 c^2 = \frac{8 \pi}{(h c)^3} E_H^4, \tag{275}$$

65

where r_0 is the minimum radius [264] and ρ_0 is defined in the last expression

$$r_0 = \sqrt{\frac{\pi}{4}} \beta_0 \, l_p = \sqrt{\frac{\pi}{4}} \sqrt{\beta_0} \, \Delta_{\min} = 1.43 \times 10^{-35} \, \beta_0 \ \text{m}. \tag{276}$$

When QG is not introduced to a compact system which is almost composed of ultra-relativistic fermions, then EoS reads $\tilde{P} = \tilde{\rho}/3$ and an exact solution was suggested [265]

$$\frac{2\tilde{m}(\tilde{r})}{\tilde{r}} = \frac{3}{7}, \tag{277}$$

$$\tilde{P}(\tilde{r}) = \frac{\tilde{r}^{-2}}{14}. \tag{278}$$

This means that on the surface of the compact star the pressure is not vanishing. Apparently, this result does not meet the physical boundary conditions. However, one has to take into consideration that this is an analytic solution describing the central region of compact stars with divergent pressure in the core [252]. For compact stars without introducing the quantum gravity effect, this solution turns to be universal [251, 252, 265]. Also, we note that the length scale r_0 in Eq. (270) is uncertain. Thus, r, m, ρ and P can be any size.

From Eq. (278), the pressure gets divergent in the core. Therefore, the influences from QG should be included in the discussion. Obviously, near the surface, the particles are non-relativistic, while in the region around the core, the particles are ultra-relativistic [252]. This determines the EoS and the boundary conditions.

In the vicinity of vanishing r, EoS is given by Eqs. (260), (261) and (262). In the limit $\kappa \to 0$, it is straightforward to recover that $P = \rho/3c^2$. By using the definitions [266], $r = r_0 \tilde{r}$, $m = m_0 \tilde{m}$ with

$$r_0^{-2} \equiv \frac{4\pi G}{c^4} \frac{8\pi}{(hc)^3} E_H^4, \tag{279}$$

$$m_0 \equiv 4\pi r_0^3 \frac{8\pi}{c^2(hc)^3} E_H^4 = 1.93 \times 10^{-8} \beta_0 \ \ (\text{kg}), \tag{280}$$

$$P_0 = \rho_0 \, c^2, \qquad \rho_0 = \frac{8\pi}{c^2(hc)^3} E_H^4, \tag{281}$$

where r_0 is the minimum radius [264],

$$r_0 = \sqrt{\frac{\pi}{4}} \beta_0 \, l_p = \sqrt{\frac{\pi}{4}} \sqrt{\beta_0} \, \Delta_{\min} = 1.43 \times 10^{-35} \, \beta_0 \ \text{m}. \tag{282}$$

The proper length r_0 in Eq. (279) apprears also Eqs. (257), (258), (261) and (262). From previous expressions, it is likely that the system can not have an arbitrary scale. This is

entirely determined by β_0. By substituting the above expressions for P and ρ, i.e., Eqs. (261) and (262) in Eq. (271) and (272), we get [266]

$$\frac{d\tilde{m}(\tilde{r})}{d\tilde{r}} = \tilde{r}^2 h(\kappa), \tag{283}$$

$$\frac{d\kappa(\tilde{r})}{d\tilde{r}} = \frac{-\kappa(\tilde{r})\,[\tilde{m}(\tilde{r}) + \tilde{r}^3\,g(\kappa)]}{\tilde{r}[\tilde{r} - 2\tilde{m}(\tilde{r})]}. \tag{284}$$

Since the density is likely regular in the center, then $m(0) = 0$ is apparently a boundary condition.

2. Conclusion

By using TOV equation and EoS of zero temperature ultra-relativistic Fermi gas based on GUP and the quantum gravitational effects on the compact stars, it was shown that $2m(r)/r$ varies with r [266]. The QG plays an important role in the region $r \sim 10^3 r_0$, where $r_0 \sim \beta_0 l_p$, near the center of compact stars. It is found that the metric components are $g_{tt} \sim r^4$ [266] and $g_{rr} = [1 - r^2/(6r_0^2)]^{-1}$ [266]. All these effects are different from those obtained from classical gravity. They can be applied to neutron stars or even denser ones like quark stars.

E. Saleker-Wigner inequalities

1. Saleker-Wigner inequalities and Heisenberg uncertainty principle

Based on HUP at the event horizon, the scale R_{g} uses the conventional derivation of the Hawking lifetime to determine the black hole temperature. Assuming that the black hole is a black body allows us the use of Stefan-Boltzmann law in calculating the lifetime of the black hole (complete evaporation) [298, 299]. According to HUP

$$\Delta p \sim \hbar/\Delta x. \tag{285}$$

If a clock of mass M has the quantum position uncertainty Δx, then its momentum uncertainty reads $\hbar\,\Delta x^{-1}$. The clock should have an accuracy τ and be able to measure time intervals up to a maximum T. After a time t, the position uncertainty of the clock will grow to [233]

$$\Delta x' = \Delta x + \hbar t M^{-1}\Delta x^{-1}. \tag{286}$$

If the effects of the mass are neglected, then the minimum position uncertainty is given as $\Delta x = \sqrt{\hbar t / M}$. In order to keep the clock accurate over the total running time T, its linear spread λ must be limited

$$\lambda \geq 2\sqrt{\hbar T / M}. \tag{287}$$

The same order of magnitude of the position uncertainty means that the size of the clock must be larger than the uncertainty in its position. This is the Salecker-Wigner first clock inequality [300]. To read out time within an accuracy τ, the quantum position uncertainty must not be larger than the minimum wavelength of the quanta striking it in order to read the time. That is $\Delta x' \leq c\tau$. The use of a signal with nonzero rest mass would give a more rigorous limit. This condition gives a bound on the minimum mass of the clock

$$M \geq \frac{4\hbar}{c^2 \tau} \left(\frac{T}{\tau} \right). \tag{288}$$

This is the Salecker-Wigner second clock inequality [300]. Obviously, this inequality is more limited than that imposed by Heisenberg energy-time uncertainty principle because it requires that a clock still show proper time even after being read. The quantum uncertainty in its position must not introduce significant inaccuracies in its measurement of time over the total running time [233].

To derive Salecker-Wigner clock inequalities, Eqs. (287) and (288), unsqueezed, unentangled, and Gaussian wave packets without any detailed phase information should be assumed. The black hole can be seen as analogue clocks not as digital quantum clocks [301]. Assuming that the minimum clock size is the Schwarzschild radius $R_{\mathrm{g}} = 2GM/c^2$, then the maximum running time of the black hole is [56]

$$T \sim \frac{G^2 M^3}{\hbar c^4} = \frac{M^3}{m_{\mathrm{p}}^3} t_{\mathrm{p}}, \tag{289}$$

where t_{p} and m_{p} are the Planck time and mass, respectively. The maximum running time of a black hole is the Hawking lifetime [232]. If the black hole evaporation, Eq. (289), is not given, there should be a maximum lifetime for a black hole state. Compared with the conventional method, the application of the Salecker-Wigner inequality, Eq. (287), to the event horizon scale predicts the Hawking lifetime, Eq. (289). This is valid even without the assumption that the black hole should be a black body radiator.

It intends to obtain modified clock inequalities based on GUP which as discussed takes into account some properties of the black holes, not including the gravity. The modified black hole lifetime can be found [233].

2. Modified Salecker-Wigner inequalities

Salecker-Wigner inequalities are based on the Heisenberg position-momentum uncertainty principle $p \sim \hbar/\Delta x$ [300]. But, as discussed in previous section, if the quantum theory is combined with some basic concepts of QG, the Heisenberg position-momentum uncertainty principle is likely modified and so do Salecker-Wigner inequalities. Using HUP and some properties of the black holes, Scardigli had shown how GUP can be derived from a *gedanken* experiment [37, 66]

$$\Delta x \geq \frac{\hbar}{\Delta p} + l_{\mathrm{p}}^2 \frac{\Delta p}{\hbar}, \tag{290}$$

where $l_{\mathrm{p}}^2 = \sqrt{G\hbar/c^3}$ is the Planck distance. It is obvious that the GUP approach, Eq. (290), can be written in a general form as $\Delta x \geq \hbar(1/\Delta p + \beta \Delta p)$, where β is a constant [234]. Accordingly, a modified black hole lifetime was obtained [233] by using a conventional method [299].

$$
\begin{aligned}
T_{\mathrm{ACS}} = \frac{1}{16} \Big\{ & \frac{8}{3} \Big(\frac{M}{m_{\mathrm{p}}} \Big)^3 - 8 \frac{M}{m_{\mathrm{p}}} - \frac{m_{\mathrm{p}}}{M} + \frac{8}{3} \Big[\Big(\frac{M}{m_{\mathrm{p}}} \Big)^2 - 1 \Big]^{3/2} \\
& -4 \sqrt{\Big(\frac{M}{m_{\mathrm{p}}} \Big)^2 - 1} + 4 \arccos \Big(\frac{m_{\mathrm{p}}}{M} \Big) + \frac{19}{3} \Big\} t_{\mathrm{ch}},
\end{aligned} \tag{291}
$$

where the subscript stands for Adler-Chen-Santiago [299] $t_{\mathrm{ch}} = 16^2 \times 60\pi t_{\mathrm{p}}$. In deriving this expression, Adler, Chen and Santiago [299] assumed that the black hole is a black body radiator and the dispersion relation $E = pc$ holds. But, if the uncertainty principle is modified, the dispersion relation may also be modified [235].

Because the space-time fluctuation will be significant when the measured length scale approaches the Planck distance, it is reasonable to expect that the linear spread of a clock must not be less than the Planck distance. In fact, the GUP approach, Eq. (290), implies a minimum length, $2l_{\mathrm{p}}$, which can be considered as a limit on the linear spread of a clock [300]. From Eq. (290), if a clock of mass M has the quantum position uncertainty Δx, then its momentum uncertainty will be $\Delta p \sim \frac{\Delta x \hbar}{2l_{\mathrm{p}}^2} \left[1 - \sqrt{1 - 4l_{\mathrm{p}}^2/\Delta x^2} \right]$ [299]. Following the steps

to derive the Salecker-Wigner inequalities as given in Ref. [233], then

$$\lambda \geq 2\, l_{\mathrm{p}} \sqrt{1 + \frac{\hbar T}{M l_{\mathrm{p}}^2}}, \tag{292}$$

is stronger than the limit given in Eq. (287). This can reproduce the limit given in Eq. (287) for $\hbar T \gg M l_{\mathrm{p}}^2$. Here, we also require that the position uncertainty created by the measurement of time must not be larger than the minimum wavelength of the quanta used to read the clock. Then, Salecker-Wigner second inequality, Eq. (288), is modified [233]

$$M \geq \frac{4\hbar T}{c^2 \tau^2} \frac{1}{1 - 4 t_{\mathrm{p}}^2/\tau^2}. \tag{293}$$

This inequality relates the mass, total running time, accuracy of the clock, and the Planck time with each others, and may links together our concepts of gravity and quantum uncertainty. Obviously, it gives a limit to the accuracy of the clock $\tau > 2 t_{\mathrm{p}}$. Like Salecker-Wigner inequalities, Eqs. (287) and (288), Eqs. (292) and (293) are valid for single analogue clocks, but not for digital quantum ones. The maximum running time of the black hole is also modified [233].

$$T_{\mathrm{MB}} \sim \frac{M R_{\mathrm{g}}^2}{4\hbar} \left(1 - 4 l_{\mathrm{p}}^2 / R_{\mathrm{g}}^2\right) = \frac{M^3}{m_{\mathrm{p}}^3} \left(1 - m_{\mathrm{p}}^2 / M^2\right) \, t_{\mathrm{p}}. \tag{294}$$

Obviously, Eq. (294) contains a term $M\, t_{\mathrm{p}}/m_{\mathrm{p}}$, which distinguishes it from the Hawking lifetime, Eq. (289). This expression holds for $M \geq m_{\mathrm{p}}$. Using the GUP approach, Eq. (290), Adler, Chen and Santiago [299] found that the thermal radiation of the black hole stops at the Planck distance, and the black hole becomes an inert remnant possessing only gravitational interaction. This is consistent with the results obtained in modified clock inequalities background [233]. Aside from the factor $16^2 \times 60\pi$, the first two terms of Adler-Chen-Santiago (ACS) lifetime T_{ACS} are consistent with the modified black hole lifetime T_{MB}. The comparison between the Hawking lifetime T_{H} and the modified black hole lifetime T_{MB} and ACS lifetime T_{ACS} are presented in Fig. 6.

The minimum interval that the black hole can be used to measure the travel time of photons across the black hole is given as [56, 301]

$$\tau \sim 2\, G \frac{M}{c^3} = \frac{R_{\mathrm{g}}}{c}. \tag{295}$$

Thus, the black hole can be viewed as an information-processing system, in which the number
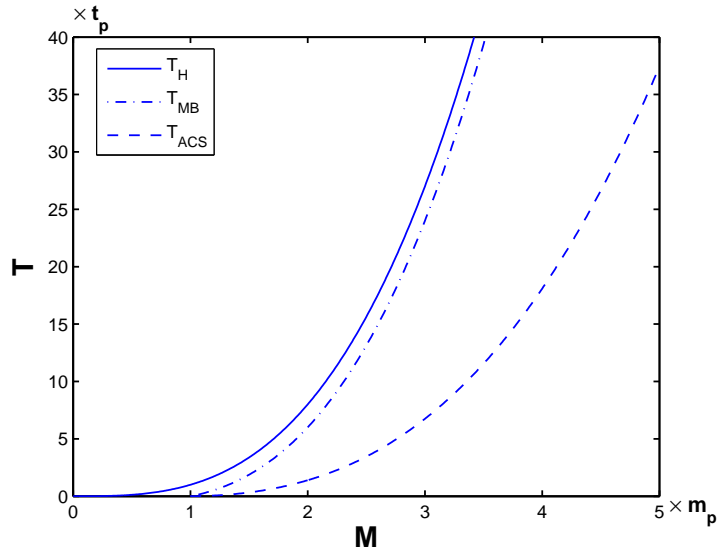
Fig. 6: Comparison between Hawking lifetime T_{H}, modified clock inequality lifetime T_{MB}, and Adler-Chen-Santiago lifetime T_{ACS}, aside from a numerical factor $16^2 \times 60\pi$. The graph taken from [233].

of computational steps reads [233]

$$N \equiv \frac{T_{\mathrm{MB}}}{\tau} \sim \frac{M^2}{m_{\mathrm{p}}^2} \left(1 - \frac{m_{\mathrm{p}}^2}{M^2}\right). \tag{296}$$

The identification of the black hole entropy [24, 115] or the holographic principle [249, 250] gives the number of bits, which are required to specify the information content of the black hole at the event horizon area in Planck units.

3. Conclusion

The modified clock inequalities, which give bounds on the size and the accuracy of the analogue clock must be lager than two times the Planck distance l_{p} and time t_{p}, respectively. A modified black hole lifetime $T_{\mathrm{MB}} \sim \frac{M^3}{m_{\mathrm{p}}^3} t_{\mathrm{p}} (1 - m_{\mathrm{p}}^2/M^2)$ is obtained, which is apparently different from Hawking lifetime. Obviously, this gives a natural limit to the mass of the black holes [232]. By viewing a black hole as an information-processing system [233], the number of bits required to specify the information content of the black hole at the event horizon area in Planck units reads $N \sim M^2/m_{\mathrm{p}}^2 (1 - m_{\mathrm{p}}^2/M^2)$.

F. Entropic Nature of the gravitational force

1. Newton's law of entropic nature

Based on holographic principle, Verlinde revised the nature of the gravitational force [241]. Assuming that [241]

- In the vicinity of the surface Ω, the change of the surface entropy is proportional to Δx and the change of the radial distance of the mass m from the surface, i.e.

$$\Delta S_\Omega = 2\pi k_B \frac{\Delta x}{\lambda_m}. \tag{297}$$

- A force F arises from the generic form of the thermodynamic EoS,

$$F \Delta x = T \Delta S_\Omega. \tag{298}$$

- On the surface Ω, N bits of information are stored i.e.,

$$N = \frac{A_\Omega}{\ell_P^2}, \tag{299}$$

  where A_Ω is the area of Ω and ℓ_P is the Planck length.

- The surface Ω is in thermal equilibrium at the temperature T. All bytes are equally likely and the energy of Ω is equipartitioned among them, i.e.

$$U_\Omega = \frac{1}{2} N \, k_B \, T = M \, c^2, \tag{300}$$

  where M is the rest mass of the source. As a result, one can derive the Newton's law [297].

$$F = G \frac{M \, m}{r^2} \tag{301}$$

2. Non-commutative geometry implying a modification in Newton's law

From Verlinde's procedure [241], section VI F 1, modifications of Newton's law can be derived. The entropy is related to the description of gravity with the underlying microstructure of a quantum spacetime [241]. Therefore, the entropy can be determined [297],

$$\Delta S_\Omega = k_B \, \Delta A \left(\frac{c^3}{4 \, \hbar \, G} + \frac{\partial s}{\partial A} \right). \tag{302}$$

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Here $s(A)$ is a function of the area. The mind of the non-commutative geometry has a specific tool for the description of the microscopic structure of a quantum system [297]. We start with a revision of Verlinde's assumptions [297]. The non-commutative geometry encodes spacetime microscopic degrees of freedom by means of a new uncertainty relation among coordinates [297].

$$\Delta x^{\mu} \Delta x^{\nu} \geq \theta. \tag{303}$$

The parameter θ has the dimension of a length squared and emerges as a natural ultraviolet cutoff from the geometry, when the coordinate operators fail to commute [297].

$$[x^{\mu}, x^{\nu}] = i \Theta^{\mu\nu}, \tag{304}$$

where $\theta = |\Theta^{\mu\nu}|$. Because of the presence of an uncertainty on Ω, there exists a fundamental unit ΔS_{θ}, which is perceived at the displacement $\Delta x_{min} \propto \lambda_m$. Therefore, the change of entropy reads [297]

$$\Delta S_{\Omega} = \Delta S_{\theta} \left(\frac{\Delta x}{\Delta x_{min}} \right), \tag{305}$$

where for later convenience, we set $\Delta x_{min} = \alpha^2 \lambda_m / (8\pi)$. At the surface Ω, the fundamental unit of the surface can be determined from the microscopic theory. This coincides with θ. Therefore, the number of bits reads $N = A_{\Omega}/\theta$. For the non-commutative geometry, the Planck scale and also the GUP parameter α will introduce a correction to the change of the entropy [297],

$$\Delta S_{\theta} = k_B \theta \left(\frac{c^3}{4\hbar G} + \frac{\partial s}{\partial A} \right). \tag{306}$$

The temperature will be given as

$$T = \frac{M}{r^2} \frac{\theta c^2}{2\pi k_B}. \tag{307}$$

The combination of these equations into the entropic nature of the gravitational force, Eq. (298), will be implying a correction in the Newton's law with a positive correction, where the derivation of the entropy to the area will have a positive value [297].

$$F = \frac{Mm}{r^2} \left(\frac{4c^3\theta^2}{\hbar\alpha^2} \right) \left[\frac{c^3}{4\hbar G} + \frac{\partial s}{\partial A} \right]. \tag{308}$$

The entropic force, Eq. (308), coincides with the Newton's law to first term [297], if $\theta = \alpha \ell_P^2$. As a result, the modified Newton's law [297] reads

$$F = \frac{GMm}{r^2} \left[1 + 4\ell_P^2 \frac{\partial s}{\partial A} \right]. \tag{309}$$

3. Conclusion

Verlinde considered the gravitation force having an entropic nature. This would mean that other theory would be allowed to deal with the gravity such as the thermodynamic mechanics [241]. The introduction of the non-cmmutative geometry will imply change in the entropy as a function of the area A at the surface Ω. The introduction of Planck scale to the α parameter, the GUP parameter, implies a correction to the Newton's law of the universal gravitational. The appearance of linear term, which reflects the effect of linear GUP approach, implies another modification in the number of bits and in the temperature of the black hole. All these lead to change in the energy of the system to also change in the gravitational force with a minus correction and inversely proportionality to the cubic of the apparent radius of the black hole.

G. Measurement of time intervals

In absence of a theory of QG, the law of gravitation at short distances remains unknown, but at large distances relative to the Planck length, GR would be a good approximation [238]. To measure the time intervals, one must have a clock located at a distance x from the observer. The observer obtains his information by looking at the clock. Therefore, the clock must emit at least one photon toward the observer. There are three sources for uncertainty in this process [238]:

- The clock's accuracy Δt.

- The time taken by the photon to reach the observer has uncertainty due to the uncertainty of the metric caused by the clock's energy uncertainty ΔE.

- The size of the clock is another source. Accordingly, the uncertainty in the distance that the photon should have to travel in order to reach the observer is $2R$, where R is the clock's radius. Therefore, this error would contribute with $2R/c$ to the total uncertainty.

1. Uncertainty in time at the shortest distance x_c

At $R \leq x_c$, the shortest length reads $x_c = \alpha\sqrt{G\hbar/c^3}$. This refers to GR as a good approximation to QG. One might assume that $F(R)$ and thus $F(x_c) = \frac{2}{c}x_c$ and $F(R) > 0$ for $x_c > R > 0$ [238]. At distances larger than x_c, the Schwarzschild solution would be utilized, $r > x_c$ [238]. At this distance, the metric equation reads

$$ds^2 = -c^2 dt^2 \left(1 - \frac{2GE}{c^4 r}\right) + \frac{dr^2}{1 - \frac{2GE}{c^4 r}}, \tag{310}$$

where E is the energy of the clock. Then, the speed of light is given as

$$v = \frac{dr}{dt} = c - \frac{2GE}{c^3 r}. \tag{311}$$

The time taken by the photon to reach the observer from distance x_c is given as

$$T = \int_{x_c}^{x} \frac{dr}{v} = \frac{1}{c}(x - x_c) + \frac{2}{c^5}GE \log\left(\frac{c^4 x - 2GE}{c^4 x_c - 2GE}\right). \tag{312}$$

Notice that $2GE/c^4 < x_c$, otherwise the photon will be locked at the clock's black hole. We use

$$\log\left(\frac{x - a}{y - a}\right) > \log\left(\frac{x}{y}\right), \qquad (x > y > a > 0), \tag{313}$$

in order to obtain

$$\Delta T > 2\frac{\Delta E}{c^5}G \log\left(\frac{x}{x_c}\right). \tag{314}$$

We use the uncertainty inequality [239], $\Delta t \Delta E \geq \hbar$ and $F(R) > 0$ [238] which leads to

$$\Delta T_{tot}(\Delta E) > \frac{\hbar}{\Delta E} + \frac{2\Delta EG}{c^5} \log\left(\frac{x}{x_c}\right), \tag{315}$$

where ΔT_{tot} is the error in the whole process.

Expression equation (315) implies that there exists a minimum time uncertainty [238]

$$\Delta T_{min} = 2\sqrt{\frac{2}{c^5}G\hbar}\sqrt{\log\left(\frac{x}{x_c}\right)}, \tag{316}$$

at an energy [238],

$$\Delta E = \sqrt{\frac{\hbar c^5}{2G \log\left(\frac{x}{x_c}\right)}}. \tag{317}$$

As mentioned above $\Delta E < c^4 x_c/(2G)$, the relation given in Eq. (316) is satisfied only for $x > e^{2/\alpha^2} x_c$.

If $x_c < x < e^{2/\alpha^2} x_c$, then the minimum uncertainty on time reads [238]

$$\Delta T_{min} = \frac{x_c}{c} \left[\frac{2}{\alpha^2} + \log\left(\frac{x}{x_c}\right) \right],$$

(318)

which is corresponding to the energy

$$\Delta E = \frac{x_c c^4}{2G}.$$

(319)

2. Uncertainty in time at the largest distance x_c

Suppose that $R > x_c$, then GR can be used inside the clock. The time takes the photon to reach the observer from distance R is given as [238]

$$T = \int_{x_c}^{x} \frac{dr}{v} = \frac{1}{c}(x - R) + \frac{2}{c^5} GE \log\left(\frac{c^4 x - 2GE}{c^4 R - 2GE}\right).$$

(320)

Thus [238]

$$\Delta T > \frac{2}{c^5} \Delta E \log\left(\frac{x}{R}\right),$$

(321)

$$\Delta T_{tot}(\Delta E, R) = \Delta t + \Delta T + 2\frac{R}{c} > \frac{\hbar}{\Delta E} + \frac{2}{c^5} \Delta E G \log\left[\frac{x}{R}\right] + 2\frac{R}{c},$$

(322)

where $R > 2\Delta E G/c^3$. Otherwise, the photon will be locked at the clock's black hole. Therefore, the departure function becomes [238],

$$f(R) = \frac{\Delta E\, G}{c^3} \log\left[\frac{x}{R}\right] + R.$$

(323)

This is an increasing function. In order to measure time as accurate as possible, we should use a clock with $R = x_c$ [238]. The total uncertainty in time reads [238]

$$\Delta T_{tot}(\Delta E) > \frac{\hbar}{\Delta E} + \frac{2\Delta EG \log\left[\frac{x}{x_c}\right]}{c^5} + \frac{2}{c} x_c > \frac{\hbar}{\Delta E} + \frac{2\Delta E\, G\, \log\left[\frac{x}{x_c}\right]}{c^5}.$$

(324)

3. Conclusion

For particles added to the system of interest, there should be an increase in the uncertainty of the metric, even without decreasing Δt. Thus the total error gets larger. The possibility of finding a measurable maximal energy and a minimal time interval is estimated in different quantum aspects. First, we find that the quadratic generalized uncertainty

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principle (GUP) approach gives non-physical results. The resulting maximal energy ΔE violates the conservation of energy. The minimal time interval Δt shows that the direction of the arrow of time is backward. Furthermore, Itzhaki [238] summarized that the measured uncertainty would represent a basic property of the Nature.

VII. APPLICATIONS OF LINEAR GUP APPROACH

A. Inflationary parameters

The study of linear GUP effects on the inflationary era is an essential ingredient to many investigations [68]. Some of these have been elaborated in sections VI A 1 and VI A 2. In doing this, we start from the number density arising from the quantum states in early Universe. Then, we calculate the free energy and entropy density. The idea of calculating thermodynamic quantities from the quantum nature of physical systems dates back to a about one decade [171–176], where the entropy arising from mixing of the quantum states of degenerate quarks in a very simple hadronic model has been estimated and applied to different physical systems.

1. Inflation parameters and linear GUP approach

As discussed in previous sections, the linear GUP approach [63, 113] predicts a maximum observable momentum and a minimal measurable length. Furthermore, the standard commutation relations are conjectured to be changed. In order to relate this with the inflation era, we define ϕ as the scaler field deriving the inflation in the early Universe [68]. The pressure and energy density, are given in Eq. (228) and (229), respectively.

The main potential slow-roll parameters [177] are given as

$$\epsilon = \frac{M_p{}^2}{2} \left(\frac{\acute{V}(\phi)}{V(\phi)} \right)^2, \tag{325}$$

$$\eta = M_p{}^2 \frac{\acute{\acute{V}}(\phi)}{V(\phi)}, \tag{326}$$

where $M_p = m_p/\sqrt{8\pi}$ is the four dimensional reduced Planck mass. The slow-roll approximations guarantee that the quantities in Eq. (325) and (326) are much smaller than unity. These conditions are supposed to ensure an inflationary phase, in which the expansion of the universe is accelerating, where the conformal time reads

$$\tau = -\frac{1}{a\,H}. \tag{327}$$

To distinguish it from the curvature parameter k [68], the wave number is denoted by j. Here, j is assumed to give the *comoving* momentum. Then momentum modification leads

to $j \longrightarrow j(1 - \alpha j)$. Accordingly, the modification in the comic scale a reads

$$a = \frac{j(1 - \alpha j)}{P}. \tag{328}$$

To avoid the divergence near the event horizon, a cutoff parameter can be utilized. The scalar spectral index is given by

$$n_s = \frac{d \ln p_s}{d \ln j(1 - \alpha j)} + 1 = \frac{(1 - 2\alpha j)}{(1 - \alpha j)} \frac{d \ln p_s}{d \ln j} + 1 \simeq (1 - \alpha j) \frac{d \ln p_s}{d \ln j} + 1. \tag{329}$$

where p_s is the amplitude of the scalar density perturbation, i.e., the scalar density fluctuations. Due to the modified commutators, a change in the Hubble parameter H is likely expected. This can be realized using slow-roll parameters. In the standard case, the spectral index can be expressed in these quantities as follows [178].

$$n_s = 1 + 2\eta - 6\epsilon, \tag{330}$$

where η and ϵ are given in Eqs (325) and (326). Finally the *"running"* of the spectral index is given by

$$n_r = \frac{d n_s}{d \ln j} = 16\epsilon\eta - 24\epsilon^2 - 2\zeta, \tag{331}$$

where

$$\zeta = M_p \frac{\acute{V}(\phi)\,\acute{\acute{V}}(\phi)}{V^2(\phi)}, \tag{332}$$

is another slow-roll parameter. At the horizon crossing epoch, the derivative of H with respect to j leads to [177, 179] $dH/dj = -\epsilon\,H/j$. Thus, with the momentum modification, we get an approximative expression for H as a function of the modified momentum

$$H \simeq j^{-\epsilon}\,\exp(\epsilon\,\alpha\,j). \tag{333}$$

So far, we can conclude that the linear GUP approach seems to enhance the Hubble parameter so that $H(\alpha = 0)/H(\alpha \neq 0) < 1$.

2. *Tensorial and scalar density fluctuations in the inflation era*

One of the main consequences of the inflation is the generation of primordial cosmological perturbations [180] and the production of long wavelength gravitational waves (tensor

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perturbations) [68]. Therefore, the tensorial density perturbations (gravitational waves) produced during the inflation era seem to serve as an important tool helping in distinguishing between different inflationary models [181]. These perturbations typically give a much smaller contribution to the cosmic microwave background (CMB) radiation anisotropy than the inflationary adiabatic scalar perturbations [182].

The tensorial and scalar density fluctuations, respectively, are given as [68]

$$
\begin{aligned}
p_t &= \left(\frac{H}{2\pi}\right)^2 \left[1 - \frac{H}{\Lambda} \sin\left(\frac{2\Lambda}{H}\right)\right] \\
&= \left(\frac{k^{-\epsilon}e^{\epsilon\alpha k}}{2\pi}\right)^2 \left[1 - \frac{k^{\epsilon-1}e^{-\epsilon\alpha k}}{a} \sin\left(\frac{2}{ak^{1-\epsilon}e^{\epsilon\alpha k}}\right)\right],
\end{aligned}
\tag{334}
$$

$$
\begin{aligned}
p_s &= \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \left[1 - \frac{H}{\Lambda} \sin\left(\frac{2\Lambda}{H}\right)\right] \\
&= \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{k^{-\epsilon}e^{\epsilon\alpha k}}{2\pi}\right)^2 \left[1 - \frac{k^{\epsilon-1}e^{-\epsilon\alpha k}}{a} \sin\left(\frac{2}{ak^{1-\epsilon}e^{\epsilon\alpha k}}\right)\right].
\end{aligned}
\tag{335}
$$

Then, the ratio tensor-to-scalar fluctuations, p_t/p_s, reads [178, 181, 183]

$$
\frac{p_t}{p_s} = \left(\frac{\dot{\phi}}{H}\right)^2 = \left(\frac{16\pi\sqrt{\epsilon}\,V}{M_4\,k^{-\epsilon}e^{\epsilon\alpha k}}\right)^2.
\tag{336}
$$

In the standard case, i.e. without GUP, this ratio is assumed to linearly depend on the inflation slow-roll parameters [178] $p_t/p_s = \mathcal{O}(\epsilon)$. An exact dependence shall be given in Eq. (341).

It is apparent that Eq. (333) gives an estimation for H in terms of the wave number j. To estimate $\dot{\phi}$, we start with the equation of motion for the scalar field, i.e., the Klein-Gordon equation [68],

$$
\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V(\phi) = 0.
\tag{337}
$$

The $\dot{\phi}$-term has the same role as that of the friction term in classical mechanics. In order to get inflation from a scalar field, we assume that Eq. (337) is valid for a very flat potential leading to neglecting its acceleration, i.e, neglecting the first term. Some inflationary models introduce the slow-roll parameter $\eta_H = -\ddot{\phi}/H\dot{\phi} = -\ddot{H}/2H\dot{H}$. Therefore, the requirement to neglect $\ddot{\phi}$ is equivalent to guarantee that $\eta_H << 1$.

$$
\dot{\phi} = -\frac{1}{3H}\partial_\phi V(\phi),
\tag{338}
$$

where the potential itself is model dependent, for example, $V(\phi) = M_p \exp[-\sqrt{2/H_0 p}\,\phi]$ [184]. According to the model presented in Ref. [68],

$$\dot{\phi} = \left(\frac{\sqrt{2\epsilon}\,V}{M_p\,H}\right)^2. \tag{339}$$

Then, the tensor-to-scalar fluctuations read [68]

$$\frac{p_t}{p_s} = \left[\frac{\sqrt{2}\,V}{M_p}\,\frac{\sqrt{\epsilon}}{j^{-2\epsilon}\,\exp\left(2\,\epsilon\,\alpha\,j\right)}\right]^2. \tag{340}$$

Fig. 7: The tensorial density fluctuations p_t is given in dependence on the wave number j (left panel) and on the slow-roll parameter ϵ (right panel). The GUP parameter α is kept constant, $\alpha = 10^{-2}$ GeV^{-1} (lower bound). It is assumed the $\sqrt{2}\,V/M_p$ remains constant, (nearly unity). These two assumptions set the physical scale. The graphs taken from Ref. [68].
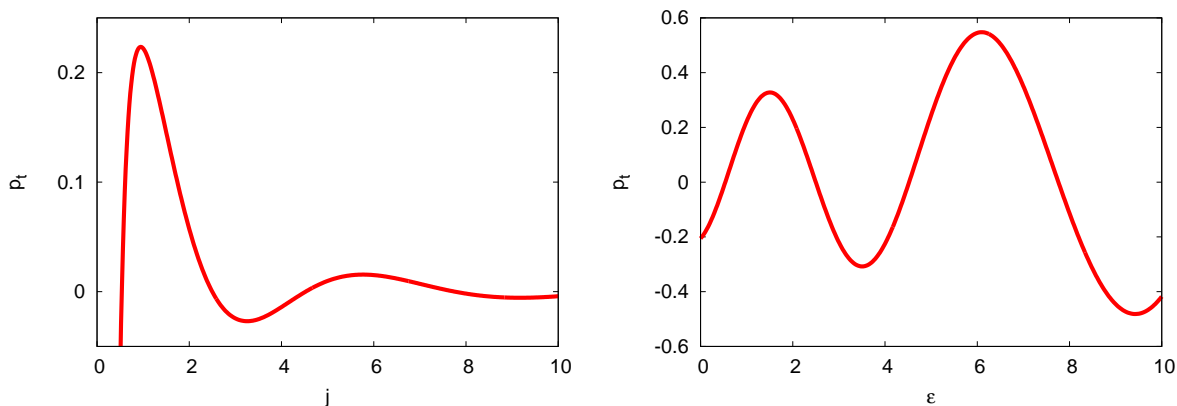
Fig. 7 gives the tensorial density fluctuations p_t in dependence on the wave number j (left panel) and on the slow-roll parameter ϵ (right panel). In both graphs, the GUP parameter α is kept constant, $\alpha = 10^{-2}$ GeV^{-1}, i.e., the upper bound is utilized. Also, it is assumed that the potential is nearly of the order of the reduced mass M_p, i.e. $\sqrt{2}V/M_p \sim 1$. It is obvious that p_t diverges to negative values at low j. Increasing j brings p_t to positive values. After reaching a maximum value, it decreases almost exponentially and simultaneously oscillates around the abscissa. The amplitude of oscillation drastically decreases with increasing j. The right-hand panel shows that $p_t(\epsilon)$ oscillates around the abscissa. Here, the amplitude of the oscillation raises with increasing ϵ. The oscillation can be detected essentially in the CMB spectrum quantizing the primordial residuals of the quantum gravity effects.

Fig. 8 refers to nearly the same behavior as that of the dependence of scalar density fluctuations p_s on the wave number j and ϵ. It is apparent that p_s diverges to negative
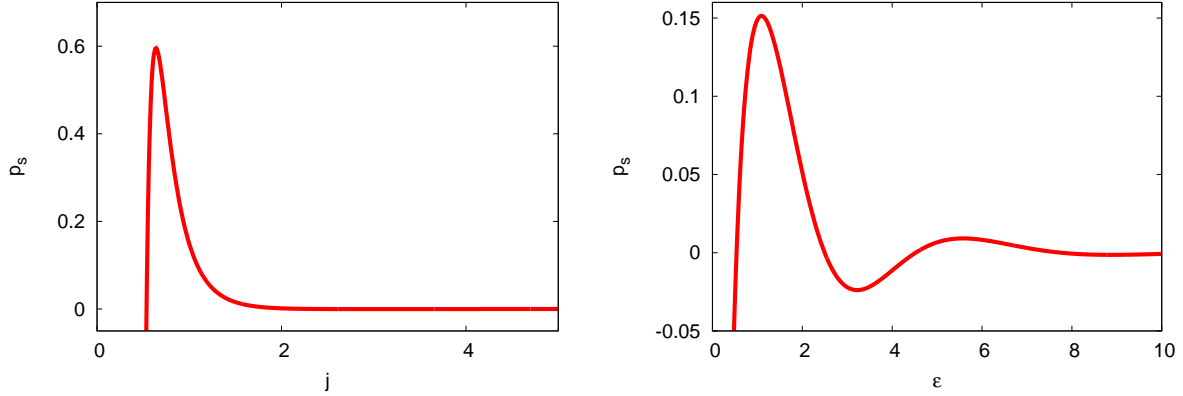
Fig. 8: The scalar density fluctuations p_s is given in dependence on j (left panel) and on slow-roll parameter ϵ (right panel). α and $\sqrt{2}V/M_p$ have the same values as in Fig. 7. They set the physical scale. The graphs taken from Ref. [68].

value at low j. Increasing j brings p_s to positive values. But after reaching a maximum value, it decreases almost exponentially. Nevertheless its values keep their positive sign. The oscillation of $p_s(\epsilon)$ is also observed. Here, $p_s(\epsilon)$ behaves almost similar to $p_t(k)$. After reaching a maximum value, it almost exponentially decreases and simultaneously oscillates around the abscissa. The amplitude of oscillation drastically decreases with increasing ϵ.

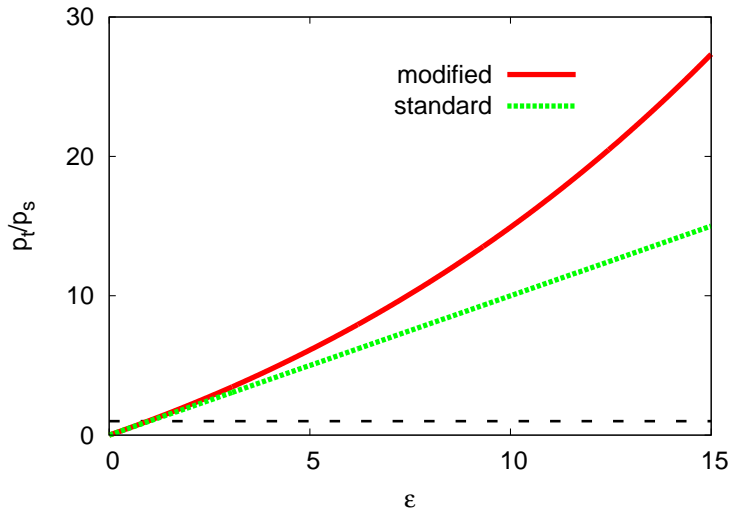**Fig.** 9: The dependence of the ratio p_t/p_s on the slow-roll parameter ϵ is given in "standard" and "modified" cases. The GUP parameter α (in "modified" case) and $\sqrt{2}V/M_p$ have the same values as in Fig. 7 and therefore the physical scale is defined. The horizontal dashed line represents constant ratio p_t/p_s. The graph taken from Ref. [68].

Fig. 9 gives the ratio p_t/p_s in dependence on ϵ in two cases. The first case, the "standard" one, is given by solid curve. The second case, the "modified" case, is given by dashed curve. The latter is characterized by finite α, while in the earlier case, α vanishes. Compared to the "standard" case, there is a considerable increase in the values of p_t/p_s with raising ϵ. For the "modified" case, i.e., upper bound of $\alpha = 10^{-2}$ GeV^{-1}, the best fit results in an exponential function [68]

$$\frac{p_t}{p_s} = \mu \, \epsilon^\nu, \tag{341}$$

where $\mu = 0.875 \pm 0.023$ and $\nu = 1.217 \pm 0.014$. All these quantities are given in natural units. For the "standard" case, the results can be fitted by

$$\frac{p_t}{p_s} = \epsilon. \tag{342}$$

The difference between Eqs. (341) and (342) is stemming from the factor in the denominator, which reflects the correction due to the GUP approach.

3. Scalar spectral index and linear GUP approach

As discussed above, the CMB results and other astrophysical observations strongly make constrains on the standard cosmological parameters such as H, baryon density n_b and even the age of the Universe [185, 186]. It turns to be necessary to have constrains on the power spectrum of the primordial fluctuations [187]. This is achievable through the spectral index. From Eq. (329), the scalar spectral index at $\sqrt{2}V/M = 1$ reads [68]

$$
\begin{aligned}
n_s = {} & 1 + \Big\{ 4e^{-6j\alpha\epsilon} j^{6\epsilon} \pi^2 (1 - j\alpha) \\
& \epsilon \left[-\frac{3}{2\pi^2} e^{6j\alpha\epsilon} j^{-6\epsilon} \left(1 - \frac{e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \sin\left(\frac{2e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \right) \right) + \right. \\
& \frac{3}{2\pi^2} e^{6j\alpha\epsilon} j^{1-6\epsilon} \alpha \left(1 - \frac{e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \sin\left(\frac{2e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \right) \right) + \\
& \frac{1}{4\pi^2 \epsilon} e^{6j\alpha\epsilon} j^{-6\epsilon} \left(-\frac{1}{a} e^{-j\alpha\epsilon} j^{-1+\epsilon} \left(\frac{2e^{-j\alpha\epsilon} j^{-1+\epsilon}(-1+\epsilon)}{a} - \frac{2e^{-j\alpha\epsilon} j^{\epsilon}\alpha\epsilon}{a} \right) \cos\left(\frac{2e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \right) - \\
& \left. \frac{e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} (-1+\epsilon) \sin\left(\frac{2e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \right) + \frac{e^{-j\alpha\epsilon} j^{\epsilon}\alpha\epsilon}{a} \sin\left(\frac{2e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \right) \right) \right] \Big\} \Big/ \\
& \left[1 - \frac{e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \sin\left(\frac{2e^{-j\alpha\epsilon} j^{-1+\epsilon}}{a} \right) \right].
\end{aligned}
\tag{343}
$$

The *"running"* of the spectral index n_s is defined by Eq. (331). The results of $n_r = d\,n_s/d\ln j$ are depicted in the right panel of Fig. 10. Early analysis of the Wilkinson Microwave Anisotropy Probe (WMAP) data [188, 189] indicates that $n_r = -0.03\pm0.018$. As noticed in Ref. [189], such an analysis may require modification, as the statistical significance seems to be questionable. On the other hand, it is indicated that the spectral index quantity $n_s - 1$ seems to run from positive values on the long length scales to negative values on the short length ones. This is also noticed in left-hand panel of Fig. 10, where n_s vs. ω is drawn. Such a coincident observation can be seen as an obvious evidence that our model agrees well with WMAP. Recent WMAP analysis shows that $n_s = 0.97 \pm 0.017$ [190]. The importance of such agreement would be the firm prediction of the inflationary cosmology through the consistency relation between scalar and tensor spectra. The physics at the Planck scale is conjectured to modify the consistency relation, considerably. It also leads to the running of the spectral index. For modes which are larger than the current horizon, the tensor spectral index is positive [191].

4. Consequences on later eras of the cosmological history
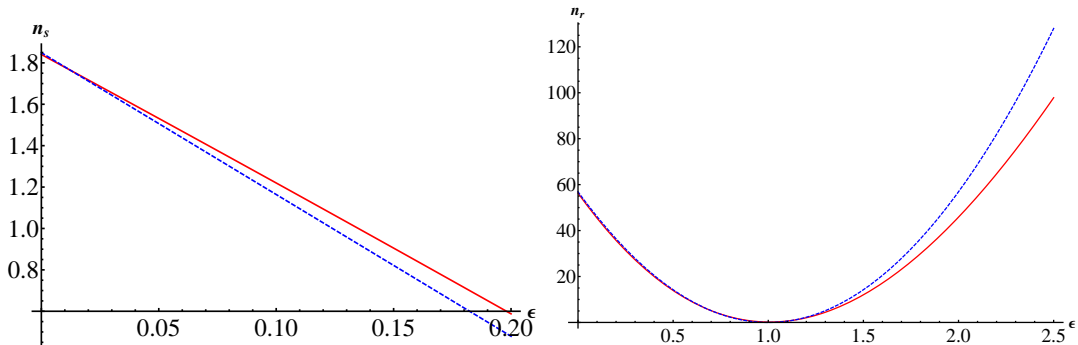
Fig. 10: Left-hand panel: the spectral index n_s is given in dependence on ϵ, where j and a are kept constant (equal 1). The *"running"* of n_s is shown in the right-hand panel. The solid curves represent the results from the modified momentum $j \to j(1 - \alpha j)$, i.e. applying the GUP approach. The dashed curves represent the standard case (unchanged momentum), i.e. $\alpha = 0$. All these quantities are given in natural units. The graphs taken from Ref. [68].

In describing the primordial power spectrum, almost all inflation models utilize three independent parameters:

- the amplitude of the scalar fluctuations,

- the tensor-to-scalar ratio n_r and

- the scalar spectral index n_s.

All of these parameters are observationally measurable. They allow the connection between the high-energy physics and the observational cosmology, in particular CMB.

The dependence of tensor-to-scalar, p_t/p_s, on ϵ is drawn in Fig. 9. The "modified" momentum characterized by finite α and reflecting the quantum gravity effects, shows a considerable increase with raising ϵ. Accordingly, the best fit was given in Eq. (342). The "standard" case can be fitted by

$$\left. \frac{p_t}{p_s} \right|_s = \epsilon. \tag{344}$$

The relation between Eqs. (342) and (344) can be given as

$$\left. \frac{p_t}{p_s} \right|_{qc} = \left(\frac{\mu}{\left. \frac{p_t}{p_s} \right|_s} \right)^{\nu}, \tag{345}$$

where the values of the fitting parameters μ and ν were given in Eq. (341).

The dependence of n_s on ϵ is presented in the left-hand panel of Fig. 10, while the dependence of its "running", Eq. (331), is illustrated in the right-hand panel. Including quantum gravity effects keeps the linear dependence of $n_s(\epsilon)$ unchanged, but makes it slower than in the standard case of unchanged momentum. Increasing ϵ leads to an increase in the difference between modified and unmodified momentum. The running n_s is not affected by quantum gravity at $\epsilon < 1$. At higher ϵ values, n_r in modified momentum gets slower than the one in standard case.

The spectral index n_s describes the initial density ripples in the Universe. If n_s is small, the ripples with longer wavelengths are strong, and vice versa. This has the effect of raising the CMB power spectrum on one side and lowering it on the other side. n_s is like a fingerprint of the very beginning of the universe in that first trillionth of a second after the Big Bang called inflation. The way of distributing matter during the initial expansion reflects the nature of the energy field controlling the inflation. The current observations of n_s are in agreement with the inflation prediction of a nearly scale-invariant power spectrum, corresponding to a slowly rolling inflation field and a slowly varying Hubble parameter during

inflation. Based on Eq. (333), the GUP approach seems to enhance the Hubble parameter so that $H(\alpha = 0) < H(\alpha \neq 0)$.

5. *Conclusions*

An evaluation for the tensorial and scalar density fluctuations in the inflation era is introduced. The tensorial p_t and scalar density fluctuations p_s are given in dependence on the wave number j and on the slow-roll parameter ϵ. For a systematic comparison, the GUP parameter α is kept constant, $\alpha = 10^{-2}$ GeV^{-1}]. Also, it is assumed the $\sqrt{2}V/M_p \sim 1$. We conclude that p_t diverges to negative value at low j. Increasing j brings p_t to positive values. After reaching a maximum value, it almost exponentially decreases and simultaneously oscillates around the abscissa. The amplitude of oscillation drastically decreases with increasing j. Also, $p_t(\epsilon)$ is found to oscillate around the abscissa. Here, the amplitude of the oscillation raises with increasing ϵ. The oscillation can be detected essentially in the CMB spectrum quantizing the primordial residuals of the quantum gravity effects.

The running of spectral scalar index n_s, which is defined by scalar index, is utilized to shed light on its scaling. The WMAP data indicates that the spectral index quantity $n_s - 1$ seems to run from positive values on long length scales to negative values on short length scales [192]. This behavior was confirmed in Ref. [68]. The importance of such agreement would be the firm prediction of inflationary cosmology through the consistency relation between scalar and tensor spectra. The Planck scale physics is conjectured to modify the consistency relation considerably. It also leads to the running of the spectral index. For modes that are larger than the current horizon, the tensor spectral index is positive.

B. Lorentz invariance violation

The combination of HUP and the finiteness of the speed of light c would lead to creation and annihilation processes, especially when studying the Compton wavelength of the particle of interest [36, 37]. Another consequence of the space-time foamy structure at small scales is the Lorentz invariance violation (LIV), which is originated in the proposal that Lorentz invariance (LI) may represent an approximate symmetry of the Nature (dates back to about four decades) [193]. A self-consistent framework for analyzing possible violation

of LI was suggested by Coleman and Glashow [194, 195]. In gamma ray bursts (GRB), the energy dependent time offsets are investigated in different energy bands assuming standard cosmological model [196]. A kind of weak indication for the redshift dependence of the time delays suggestive of LIV has been found. A comprehensive review on the main theoretical motivations and observational constraints on Planck scale suppressed Lorentz invariance violation is given in Ref. [197] and the references therein. Recently, the Planck scale itself turns to be accessible in quantum optics [73].

The modified dispersion relationship likely leads to further predictions which apparently have feasibility in experiments, such as an energy dependent speed of light. The gamma ray observations [198] might imply that the speed of light was faster in the very early Universe, when the average energy was comparable to Planck scale [206]. As pointed out by Moffat [207], and Albrecht and Magueijo [208], such an effect could provide an alternative solution to the horizon problem and other problems addressed by inflation. Such modified dispersion relations may also lead to corrections to the predictions of inflationary cosmology, observable in future high precision measurements of the CMB spectrum. Finally, a modified dispersion relation may lead to an explanation of the dark energy in terms of energy trapped very high momentum and low-energy quanta, as pointed out by Mersini *et al.* [209].

The linear GUP approach assumes that the momentum of a particle with mass M having distant origin and an energy scale comparable to the Planck scale would be a subject of a tiny modification [63, 75, 113] so that the comoving momenta can be given as [69]

$$p_\nu = p_\nu \left(1 - \alpha\, p_0 + 2\,\alpha^2\, p_0^2\right), \tag{346}$$

$$p_\nu^2 = p_\nu^2 \left(1 - 2\,\alpha p_0 + 10\,\alpha^2\, p_0^2\right), \tag{347}$$

where p_0 is the momentum at low energy. The parameter $\alpha = \alpha_0/(c\, M_{pl}) = \alpha_0 l_{pl}/\hbar$ [63, 75, 113], where α_0 is dimensionless parameter of order one. Then in comoving frame, the dispersion relation reads

$$E_\nu^2 = p_\nu^2\, c^2 \left(1 - 2\,\alpha\, p_0\right) + M_\nu^2\, c^4. \tag{348}$$

When taking into consideration a linear dependence of p on α and ignoring the higher orders of α, then the Hamiltonian is

$$\mathbf{H} = \left(p_\nu^2\, c^2 - 2\,\alpha\, p_\nu^3\, c^2 + M_\nu^2\, c^4\right)^{1/2}. \tag{349}$$

There are several experimental and theoretical developments [199, 200] showing threshold anomalies in ultra high-energy cosmic ray protons [201, 202] and possible TeV photons [203, 204].

1. Comoving velocity and time of arrival

The derivative of Eq. (349) with respect to the momentum results in a comoving time-dependent velocity, i.e., Hamilton equation,

$$v(t) = \frac{1}{a(t)} \left(P_{\nu_0}^2 c^2 - 3\alpha P_{\nu_0}^2 c^2 \right) \left(P_{\nu_0}^2 c^2 - 2\alpha P_0^3 + M_\nu^2 c^4 \right)^{-1/2}, \tag{350}$$

$$= \frac{c}{a(t)} \left(1 - 2\alpha p_0 - \frac{M_\nu^2 c^2}{2p_\nu^2} + \alpha p_0 \left[\frac{M_\nu^2 c^2}{p_\nu^2} - \frac{M_\nu^2 c^4}{p_\nu^2 c^2 + M_\nu^2 c^4} + \frac{M_\nu^2 c^4}{p_\nu^2 c^2 + M_\nu^2 c^4} \frac{M_\nu^2 c^2}{2p_\nu^2} \right] \right). \tag{351}$$

The comoving momentum is related to the physical one through $p_\nu = p_{\nu_0}(t_0)/a(t)$ and the scale factor a is related to the redshift z,

$$a(z) = \frac{1}{1+z}. \tag{352}$$

In the relativistic limit, $p \gg M$, the fourth and fifth terms in Eq. (351) simply cancel each other. Then [69]

$$v(z) = c(1+z) \left[1 - 2\alpha(1+z)p_{\nu_0} - \frac{M_\nu^2 c^2}{2(1+z)^2 p_{\nu_0}^2} + \alpha \frac{M_\nu^4 c^4}{2(1+z)^3 p_{\nu_0}^3} \right], \tag{353}$$

in which p_0 is treated as a comoving momentum. Thus, the relative change in the relative velocity can be deduced [69]

$$\frac{\Delta v(z)}{c} = \alpha \left(-2(1+z)^2 p_{\nu_0} + \frac{M_\nu^4 c^4}{2(1+z)^2 p_{\nu_0}^3} \right) - \frac{M_\nu^2 c^2}{2(1+z)p_{\nu_0}^2}. \tag{354}$$

The comoving redshift-dependent distance travelled by the particle of interest is defined as

$$r(z) = \int_0^z \frac{v(z)}{(1+z)H(z)} dz, \tag{355}$$

where $H(z)$ is the Hubble parameter depending on z. From Eqs. (353) and (355), the time of flight is given as [69]

$$t_\nu = \int_0^z \left[1 - 2\alpha(1+z)p_{\nu_0} - \frac{M_\nu^2 c^2}{2(1+z)^2 p_{\nu_0}^2} + \alpha \frac{M_\nu^4 c^4}{2(1+z)^3 p_{\nu_0}^3} \right] \frac{dz}{H(z)}, \tag{356}$$

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which counts for the well-known time of flight of a prompt low-energetic photon (first term). In other words, the time of flight is invariant in Lorentz symmetry. Furthermore, it is apparent that Eq. (356) contains a time of flight delay

$$\Delta t_\nu = \int_0^z \left[2\alpha \left((1+z)\, p_{\nu_0} - \frac{M_\nu^4 c^4}{4\,(1+z)^3\, p_{\nu_0}^3} \right) + \frac{M_\nu^2\, c^2}{2\,(1+z)^2 p_{\nu_0}^2} \right] \frac{dz}{H(z)}. \qquad (357)$$

Obviously, the first and second terms are due to LIV effects stemming from GUP (both have α parameter). The third term gives the effects of the particle mass on the time of flight delay. Furthermore, the second term alone seems to contain a mixed effects from LIV (GUP) and rest mass.

In order to determine Δt_ν, Eq. (357), it is essential to find out observational results and/or reliable theoretical model for the redshift-dependence of the Hubble parameter H,

$$H(z) = \frac{1}{a(z)} \left(\frac{da(z)}{dz} \frac{dz}{dt} \right) = -\frac{1}{1+z} \frac{dz}{dt}. \qquad (358)$$

This expression can be deduced from Eq. (352). In general, the expansion rate of the Universe varies with the cosmological time [142–145, 147–150]. It depends on the background matter/radiation and its dynamics [143]. The cosmological constant reflecting among others the dark matter content seems to affect the temporal evolution of H [144]. Fortunately, the redshift z itself can be measured with a high accuracy through measuring the spectroscopic redshifts of galaxies having certain uncertainties ($\sigma_z \leq 0.001$). Based on this, a differential measurement of time at a given redshift interval automatically provides a direct and clean measurement of $H(z)$ [210–212]. These measurements can be used to derive constraints on essential cosmological parameters [213]. The measurements of the expansion rate and their constrains in evaluating the integrals given in Eq. (357) are implemented [69]. We give an example on the applications of these results on the early-type galaxies.

2. Applications on early-type galaxies

Out of a large sample of early-type galaxies (about 11000) extracted from several spectroscopic surveys spanning over $\sim 8 \times 10^9$ years of cosmic look-back time, i.e., $0.15 < z < 1.42$ [211], most massive, red elliptical galaxies, passively evolving and without signature of ongoing star formation are picked up and used as standard cosmic chronometers [213]. The differential age evolution turns to be accessible, which gives an estimation for the cosmic

time and can directly probe $H(z)$. A list of new measurements of $H(z)$ with $5 - 12\%$ uncertainty is introduced in Ref. [211]. Fig. 11 shows these observations as estimated in the BC03 model [214]. They are combined with CMB data and can be used to set constrains on possible deviations from the standard (minimal) flat ΛCDM model [212]. The right-hand panel shows a data set taken from MS model [215]. It is obvious that the results are model-dependent [69].
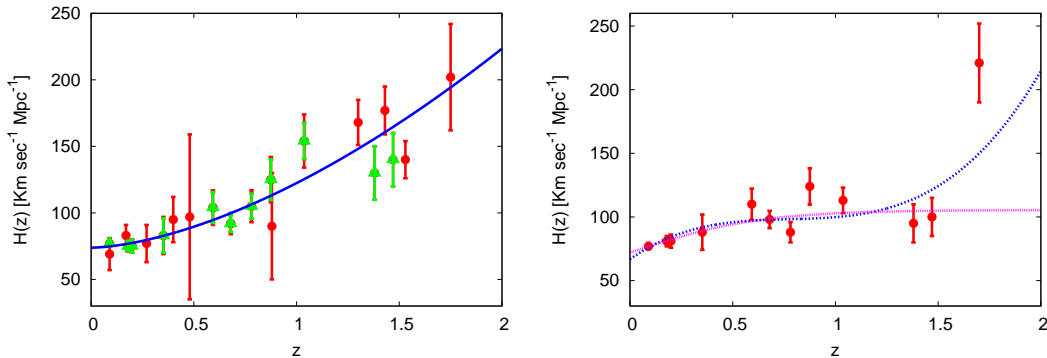

Fig. 11: Left-hand panel: the Hubble parameter H calculated in BC03 model (open triangle) and in combination with CMB data, that gives constrains of the possible deviations from the standard (minimal) flat ΛCDM model (solid circles), is given in dependence on the redshift z. The results from MS model are drawn in the right-hand panel. The curves represent the fitting parameters (see text for details). The graphs taken from Ref. [69].

The observational measurements have been fitted as follows [69]. For the results obtained from BC03 model [214] and using a combination with CMB data and setting constrains on possible deviations from the standard (minimal) flat ΛCDM model [212], the expression [69]

$$H(z) = \beta_1 + \gamma_1 z + \delta_1 z^2, \tag{359}$$

where $\beta_1 = 72.68 \pm 3.03$, $\gamma_1 = 19.14 \pm 5.4$ and $\delta_1 = 29.71 \pm 6.44$, fits well with the observations. The solid curve in left-hand panel of Fig. 11 represents the results from this expression. For the MS model [215] measurements, we suggest two expressions [69]:

$$H(z) = \beta_2 + \gamma_2 z + \delta_2 z^2 + \epsilon_2 z^3, \tag{360}$$

$$H(z) = \beta_3 + \gamma_3 \tanh(\delta_3 z), \tag{361}$$

where $\beta_2 = 66.78 \pm 8.19$, $\gamma_2 = 113.27 \pm 7.5$, $\delta_2 = -140.72 \pm 12.6$, $\epsilon_2 = 60.61 \pm 5.48$, $\beta_3 = 71.94 \pm 4.35$, $\gamma_3 = 33.51 \pm 7.94$ and $\delta_3 = 1.6 \pm 0.1$. The results of Eq. (360) are given

by dashed curve in the right-hand panel of Fig. 11. Eq. (361) is drawn as dotted curve, where the largest point is excluded while remaining points build up the ensemble used in the fitting [69]. It is obvious that the implementation of Eq. (360), which is obviously a rational function, in Eq. (357) results in a non-analytic integral. On the other hand, implementing Eq. (361) in Eq. (357) makes the second and third integrals non-solvable, while the first term is.

It is apparent that Eq. (359) simplifies the integrals given in Eq. (357). Accordingly, there are two types of LIV contributions to the time of flight delay:

1. Finite GUP parameter α appears in two terms as follows

$$2\alpha\, p_{\nu_0} \int_0^z (1+z)\, \frac{dz}{H(z)} = \frac{\alpha}{\gamma} p_{\nu_0} \left[\ln\left[\beta_1 + z(\gamma_1 + \delta_1 z)\right] - \frac{2(\gamma_1 - 2\delta_1)}{A} \mathrm{atan}\left(\frac{\gamma_1 + 2\delta_1 z}{A}\right) \right], \quad (362)$$

$$
\begin{aligned}
-\alpha\, \frac{M_\nu^4 c^4}{2\, p_{\nu_0}^3} \int_0^z \frac{1}{(1+z)^3} \frac{dz}{H(z)} = & \frac{-\alpha}{(\beta_1 - \gamma_1 + \delta_1)^3} \frac{M_\nu^4 c^4}{4\, p_{\nu_0}^3} \left[\frac{2(\gamma_1 - 2\delta_1)(\beta_1 - \gamma_1 + \delta_1)}{1+z} \right. \\
& + \left[3\gamma_1\delta_1 - \gamma_1^2 + \delta(\beta_1 - 3\delta_1) \right] \ln\left(\beta_1 + z(\gamma_1 + \delta_1 z)\right) \\
& - \frac{(\beta_1 - \gamma_1 + \delta_1)^2}{(1+z)^2} + 2\left[\gamma_1^2 - 3\gamma_1\delta_1 + \delta_1(3\delta_1 - \beta_1)\right] \ln(1+z) \\
& \left. - \frac{2(\gamma_1 - 2\delta_1)}{A}\left[\gamma_1^2 - \gamma_1\delta_1 + \delta_1(\delta_1 - 3\beta_1)\right] \mathrm{atan}\left(\frac{\gamma_1 + 2\delta_1 z}{A}\right) \right],
\end{aligned}
\quad (363)
$$

where $A = (4\beta_1\delta_1 - \gamma_1^2)^{1/2}$.

2. Furthermore, Eq. (359) gives an exclusive estimation for the mass contribution to the time of flight delay,

$$
\begin{aligned}
\frac{M_\nu^2 c^2}{2\, p_{\nu_0}^2} \int_0^z \frac{1}{(1+z)^2} \frac{dz}{H(z)} = & \frac{1}{(\beta_1 - \gamma_1 + \delta_1)^2} \frac{M_\nu^2 c^2}{2\, p_{\nu_0}^2} \left\{ \frac{\gamma_1^2 - 2\gamma_1\delta_1 + \delta_1(\delta_1 - \beta_1)}{A} \mathrm{atan}\left(\frac{\gamma_1 + 2\delta_1 z}{A}\right) \right. \\
& \left. - \frac{\beta_1 - \gamma_1 + \delta_1}{1+z} - \frac{1}{2}(\gamma_1 - 2\delta_1) \ln\left[\frac{(1+z)^2}{\beta_1 + z(\gamma_1 + \delta_1 z)} \right] \right\}.
\end{aligned}
\quad (364)
$$

3. *Conclusions*

With varying the redshift, the relative change in the speed of massive muon neutrino and its time of flight delays is calculated. The redshift depends on the temporal evolution of H, which can be estimated from a large sample of early-type galaxies extracted from several spectroscopic surveys spanning over $\sim 8 \times 10^9$ years of cosmic lookback, most massive, red elliptical galaxies, passively evolving and without signature of ongoing star formation are

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picked up and used as standard cosmic chronometers giving a cosmic time directly probe for $H(z)$. The measurements according to BC03 model and in combination with CMB data constraining the possible deviations from the standard (minimal) flat ΛCDM model are used to estimate the z-dependence of the Hubble parameter. The measurements based on MS model are used to show that the results are model-dependent.

C. Black hole thermodynamics

The finding that black holes should have well-defined entropy and temperature represented one of the greatest achievements in recent astrophysics [24, 115]. In statistical physics and thermodynamics, the thermal evolution of entropy relates the number of thermal macrostates to that of microstates of the system of interest in thermal medium. In GR, the BH entropy is a pure geometric quantity so that when comparing BH with a thermodynamic system, we find an important difference. Whether BH has interior degrees of freedom corresponding to its entropy, the Bekenstein-Hawking entropy delivered an answer to this and characterized the statistical meaning [24, 115]. Counting the microstates was proposed by Medved and Vagenas [216], that this presumably lies within the framework of QG. For example, the String theory [217] and the loop quantum gravity [218] succeeded in presenting an statistical explanation formulated in an entropy-area law. The proportionality relating BH entropy with area was derived from classical thermodynamics, as well [219].

1. Number of quantum states, entropy and free energy

In brick wall model, the entropy can be calculated as follows.

$$
S_0 = \beta^2 \left. \frac{\partial F_0}{\partial \beta} \right|_{\beta=\beta_H} = \frac{\beta^2}{\pi} \int_{r_++\epsilon}^{L} dr \, \frac{1}{\sqrt{f}} \int_{m\sqrt{f}}^{\infty} d\omega \, \left. \frac{\omega e^{\beta\omega} \left(\frac{\omega^2}{f} - m^2 \right)^{1/2}}{\left(e^{\beta\omega} - 1 \right)^2} \right|_{\beta=\beta_H} , \quad (365)
$$

where β is the inverse temperature, F_0 is the free energy and L and ϵ are infrared and ultraviolet regulators, respectively. β_H is the inverse Hawking temperature. In a zero-temperature quantum mechanical system around the black hole, the entropy reads

$$
S_0^{ext} \approx \ln \left(\frac{1}{2\Lambda\epsilon} \right) , \quad (366)
$$

which can be interpreted as the physical limit that Λ should be less than $1/(2\epsilon)$.

In natural units, the modified uncertainty relation

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \left[1 - 2\,\alpha \,\langle p \rangle + 4\,\alpha^2 \,\langle p^2 \rangle \right], \tag{367}$$

leads to a modification in the volume of phase cell in $(1+1)$-dimensions from 2π to $2\pi \left(1 - 2\,\alpha\,p + 4\,\alpha^2\,p^2 \right)$. The number of quantum states with energy less than ϵ is given as [220]

$$n_0(\omega) = \frac{1}{2\pi} \int dr \; dp_r = \frac{1}{\pi} \int_{r_+ + \epsilon}^{L} dr \; \frac{1}{\sqrt{f}} \left(\frac{\omega^2}{f} - m^2 \right)^{1/2}, \tag{368}$$

where m in the mass of the scalar field and ω is a parameter of the substitution of Klein-Gordon equation. The expression equation (368) will be changed to

$$\begin{aligned}
n_I(\omega) &= \frac{1}{2\pi} \int dr \; dp_r \; \frac{1}{1 - 2\,\alpha\,p + 4\,\alpha^2\,p^2} \\
&= \frac{1}{2\pi} \int dr \; \frac{1}{\sqrt{f}} \frac{\left(\frac{\omega^2}{f} - m^2 \right)^{1/2}}{1 - 2\,\alpha \left(\frac{\omega^2}{f} - m^2 \right)^{1/2} + 4\,\alpha^2 \left(\frac{\omega^2}{f} - m^2 \right)},
\end{aligned} \tag{369}$$

where r and f are estimated as follows. In Schwarzschild gauge, the metric and field tensors, respectively, read

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2, \tag{370}$$

$$F_{rt} = F_{rt}(r). \tag{371}$$

The function $f(r)$ in the static solution is defined as

$$f(r) = 1 - \frac{M}{\Lambda}e^{-2\Lambda r} + \frac{Q^2}{4\Lambda^2}e^{-4\Lambda r}, \tag{372}$$

where M is the mass of black hole and Q gives its charge. The outer event horizon has the radius

$$r_+ = \frac{1}{2\Lambda} \ln \left[\frac{M}{2\Lambda} + \sqrt{\left(\frac{M}{2\Lambda} \right)^2 - \left(\frac{Q}{2\Lambda} \right)^2} \right]. \tag{373}$$

In light of this, its derivative vanishes and the Klein-Gordon equation is reduced to

$$\frac{d^2R}{dr^2} + \frac{1}{f}\frac{df}{dr}\frac{dR}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - m^2 \right) R = 0, \tag{374}$$

where $\phi(r) = \exp(-i\omega t) R(r)$. Using WKB approximation, then $R \sim \exp(iS(r))$,

$$p_r^2 = \frac{1}{f}\left(\frac{\omega^2}{f} - m^2\right), \tag{375}$$

and $p_r = dS/dr$ and

$$p^2 = \frac{\omega^2}{f} - m^2. \tag{376}$$

At Hawking temperature, Eq. (369), can be used to derive the free energy [220]

$$F_0 = -\frac{1}{\pi}\int_{r_++\epsilon}^{L} dr \frac{1}{\sqrt{f}} \int_{m\sqrt{f}}^{\infty} \frac{\left(\frac{\omega^2}{f} - m^2\right)^{1/2}}{e^{\beta\omega} - 1} d\omega, \tag{377}$$

which turns is a subject of change

$$
\begin{aligned}
F_I &= -\int_{m\sqrt{f}}^{\infty} d\omega \frac{n_I(\omega)}{e^{\beta\omega} - 1} \\
&= -\frac{1}{\pi}\int dr \frac{1}{\sqrt{f}} \int_{m\sqrt{f}}^{\infty} \frac{\left(\frac{\omega^2}{f} - m^2\right)^{1/2}}{(e^{\beta\omega} - 1)\left[1 - 2\alpha\left(\frac{\omega^2}{f} - m^2\right)^{1/2} + 4\alpha^2\left(\frac{\omega^2}{f} - m^2\right)\right]} d\omega.
\end{aligned} \tag{378}
$$

2. Black hole entropy and linear GUP approach

Near the event horizon, i.e., in the range $(r_+, r_+ + \epsilon)$, $f \to 0$, the entropy can be deduced from Eq. (378)

$$S_0 = \frac{\beta^2}{\pi}\int_{r_++\epsilon}^{L} dr \frac{1}{\sqrt{f}} \int_{m\sqrt{f}}^{\infty} d\omega \left.\frac{\omega e^{\beta\omega}\left(\frac{\omega^2}{f} - m^2\right)^{1/2}}{(e^{\beta\omega} - 1)^2}\right|_{\beta=\beta_H}. \tag{379}$$

Once again, the entropy given in Eq. (379) will be changed to

$$
\begin{aligned}
S_I &= \frac{\beta^2}{\pi}\int dr \frac{1}{\sqrt{f}} \int_{m\sqrt{f}}^{\infty} \frac{\omega\left(\frac{\omega^2}{f} - m^2\right)^{1/2} e^{\beta\omega}}{e^{2\beta\omega-2}\left[1 - 2\alpha\left(\frac{\omega^2}{f} - m^2\right)^{1/2} + 4\alpha^2\left(\frac{\omega^2}{f} - m^2\right)\right]} d\omega \\
&= \frac{1}{\pi}\int_{r_+}^{r_++\epsilon} dr \frac{1}{\sqrt{f}} \int_0^{\infty} \frac{f^{-1/2}\beta^{-1}x^2}{(1 - e^{-x})(e^x - 1)\left[1 - 2\alpha\frac{x}{\beta\sqrt{f}} + 4\alpha^2\frac{x^2}{\beta^2 f}\right]} dx,
\end{aligned} \tag{380}
$$

where $x = \beta\omega$. We note that as $f \to 0$, then ω^2/f is the dominant term in the bracket containing $\omega^2/f - m^2$. We are interested in the thermodynamic contributions of just vicinity

near horizon $r_+, r_+ + \epsilon$, which corresponds to a proper distance of the order of the minimal length. The latter can be related to the GUP parameter α. So we have from Eq. (370)

$$\alpha = \int_{r_+}^{r_+ + \epsilon} \frac{dr}{\sqrt{f(r)}}, \tag{381}$$

which apparently sets a lower bound to α. Then, the entropy reads

$$S_I = \frac{1}{\pi\,\alpha} \int_{r_+}^{r_+ + \epsilon} \frac{dr}{\sqrt{f(r)}} \int_0^\infty \frac{a^2\,X^2}{\left(e^{\frac{a X}{2}} - e^{-\frac{a X}{2}}\right)^2 (1 - 2\,X + 4\,X^2)}\,dX, \tag{382}$$

where

$$x = \frac{\beta}{\alpha}\,\sqrt{f}\,X = a\,X. \tag{383}$$

Then

$$S_I = \frac{1}{\pi}\,\Sigma_I = \frac{1}{\pi} \int_0^\infty \frac{a^2\,X^2}{\left(e^{\frac{a X}{2}} - e^{-\frac{a X}{2}}\right)^2 (1 - 2\,X + 4\,X^2)}\,d\,X. \tag{384}$$

We note that as $r \to r_+$, $f \to 0$, then $a \to 0$ and

$$\lim_{a\to 0} \frac{a^2\,X^2}{\left(e^{a\,X/2} - e^{-a\,X/2}\right)^2} = 1. \tag{385}$$

Therefore,

$$\Sigma_I = \int_0^\infty \frac{d\,X}{1 - 2\,X + 4\,X^2} = \frac{2\,\pi}{3\sqrt{3}}, \tag{386}$$

and

$$S_I = \frac{1}{\pi}\,\Sigma_I = \frac{2}{3\,\sqrt{3}}. \tag{387}$$

So far, we can conclude that S_I is finite and does not depend on any other parameter. We notice that in contrast to the case of brick wall method, there is no divergence within the just vicinity near the horizon due to the effect of the generalized uncertainty relation on the quantum states.

3. *Linear GUP approach and entropy of Schwarzshild black hole*

In natural units, the line element in Schwarzschild black hole is given as

$$ds^2 = -\left(1 - 2\frac{M}{r}\right)d\,t^2 + \left(1 - 2\frac{M}{r}\right)^{-1} d\,r^2 + r^2\,d\Omega_2^2. \tag{388}$$

Then, Hawking radiation temperature T, horizon area A and entropy S, respectively, read

$$T = \frac{1}{4\pi r_H} = \frac{1}{8\pi M}, \tag{389}$$

$$A = 4\pi r_H^2 = 16\pi M^2, \tag{390}$$

$$S = \pi r_H^2 = 4\pi M^2, \tag{391}$$

where $r_H = 2M$ is the location of the black hole horizon. The increase (decrease) in the horizon area due to absorbing (radiating) a particle of energy dM can be expressed as

$$dA = 8\pi r_H \, dr_H = 32\pi M \, dM. \tag{392}$$

This particle is conjectured to satisfy Heisenberg's uncertainty relation $\Delta x_i \, \Delta p_j \geq \delta_{ij}$.

According to the linear GUP approach, the area and entropy, respectively, can be re-written as

$$A_{GUP} = A - 4\alpha \sqrt{\pi} \sqrt{A} + 8\pi \alpha^2 \ln\left(\sqrt{\frac{A}{\pi}} + 2\alpha\right), \tag{393}$$

$$S_{GUP} = S - 2\alpha \sqrt{\pi} \sqrt{S} + \alpha^2 \pi \ln S + C, \tag{394}$$

where $\alpha \ll \sqrt{A/\pi}$ and C is an arbitrary constant. We notice that the coefficient of $\ln S$ is also positive, but the entropy gets an additional term, $2\alpha \sqrt{\pi} \sqrt{S}$.

4. Linear GUP approach and energy density of Schwarzshild black hole

As given in sections VI A and VII A, the Friedmann equation (first law of thermodynamics) reads

$$\left(\dot{H} - \frac{k}{a^2}\right) S'_{GUP} = -4\pi G (\rho + p), \tag{395}$$

where the energy density is given as

$$\rho = -\frac{3}{8G} \int S'_{GUP}(A) \left(\frac{A}{4}\right)^{-2} dA. \tag{396}$$

Using Eq. (394), then

$$\left(\dot{H} - \frac{k}{a^2}\right)\left[1 - 2\alpha\left(\frac{\pi}{A}\right)^{1/2} + 4\alpha^2\left(\frac{\pi}{A}\right)\right] = -16\pi G (\rho + p). \tag{397}$$

The modified energy density

$$\rho_{GUP} = \frac{3}{8\pi G}\left[\left(\frac{\pi}{A}\right) - \frac{4}{3}\alpha\left(\frac{\pi}{A}\right)^{3/2} - 2\alpha^2\left(\frac{\pi}{A}\right)^2\right]$$

$$= \rho\left[1 - \frac{4}{3}\alpha\left(\frac{2}{3}\pi\rho\right)^{1/2} + \frac{4}{3}\pi\alpha^2\rho\right]. \tag{398}$$

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5. Conclusions

We show that the quantum correction of the geometric entropy of charged black hole has one great advantage. By doing this, one can avoid being biased in favor of a certain theory of QG. For example, the correction to the Bekenstein-Hawking entropy, which relates the entropy to the cross-sectional area of the BH horizon, includes a series of terms, where the coefficient of the leading-order correction, the logarithmic term, is suggested as a discriminator of prospective fundamental theories for QG. It is essential to suggest a method that fixes this, but it should not depend on the utilized models for QG. For instance, this might be the holographic principle.

The brick wall method is used to calculate the statistical (informational) entropy of black hole. In doing this, a cutoff parameter is assuming in order to avoid the divergence near the event horizon. Because the degrees of freedom are likely dominant near the horizon, it is assumed that the brick wall method should be replaced by a thin-layer model making the calculation of entropy possible. For instance, the entropy of FLRW Universe can be given by time-dependent metric. It is found that the black hole entropy is logarithmically related to the ultraviolet regulator ϵ, so that the physical entropy is limited to $\Lambda < 2\epsilon$.

When comparing black hole entropy with the one that counts for the microstates Ω, we can simply relate $A/4$ to $\ln \Omega$. This is valid as long as the gravity is sufficiently strong so that the horizon radius is much larger than the Compton wavelength. In order to apply the GUP approach, we start with the modified momentum and statistically derive expressions for area and entropy. Then, we apply the holographic principle. Based on the linear GUP approach, the black hole thermodynamics and entropy get substantial corrections. We found that the logarithmic divergence in the entropy-area relation turns to be positive. Furthermore we find that S gets an additional terms, such as $2\,\alpha\,\sqrt{\pi}\,\sqrt{S}$.

D. Compact stellar objects

For an isolated macroscopic body consisting of N non-interacting and ultra-relativistic particles, the background of the particles motion is assumed to be flat. Studying the ground state properties of a Fermi gas composed of N ultra-relativistic electrons shows that the state energy ϵ is entirely given by $c\,p$, i.e., temperature is much larger than the particle

rest mass. At low T, the vacuum effect of fermions can be neglected, i.e., the total particle number is conserved. The modified number of particle of Fermi gas can be given as [221]

$$N(p) = \frac{8\pi}{h^3} V \int_0^{p_F} \frac{p^2 dp}{(1 - \alpha p)^4}, \tag{399}$$

where p_F is the Fermi momentum. Therefore, Eq. (399) can be re-written in terms of Fermi energy ϵ_F

$$N(\epsilon_F) = \frac{8\pi}{3(h\,c)^3} V \frac{\epsilon_F^3}{\left(1 - \frac{\alpha}{c}\epsilon_F\right)^3}. \tag{400}$$

Introducing $\kappa = \epsilon_F/\epsilon_H$, which is equivalent to $\alpha\epsilon_F/c$, then

$$N(\kappa) = \frac{8\pi}{(h\,c)^3} V \epsilon_H^3 f(\kappa), \tag{401}$$

where $\epsilon_H = c/\alpha$ is the Hagedorn energy and

$$f(\kappa) = \frac{1}{3} \frac{\kappa^3}{(1 - \kappa)^3}. \tag{402}$$

The ground state energy can be calculated from

$$U_0(\epsilon) = \frac{8\pi}{(hc)^3} V \int_0^{\epsilon_F} \frac{\epsilon^3 d\epsilon}{\left(1 - \frac{\alpha}{c}\epsilon\right)^4}. \tag{403}$$

In terms of κ, the ground state energy and pressure, respectively, read [221]

$$U_0(\kappa) = \frac{8\pi}{(hc)^3} V \epsilon_H^4 g(\kappa), \tag{404}$$

$$P(\kappa) = \frac{N}{V}\epsilon_F - \frac{U_0}{V} = \frac{8\pi}{(h\,c)^3} \epsilon_H^4 h(k), \tag{405}$$

where

$$g(\kappa) = \ln(1 - \kappa) + \frac{\kappa}{(1 - \kappa)^3} - \frac{15}{6} \frac{\kappa^2}{(1 - \kappa)^3} + \frac{11}{6} \frac{\kappa^3}{(1 - \kappa)^3}, \tag{406}$$

$$h(\kappa) = \frac{1}{3} \frac{\kappa^4}{(1 - \kappa)^3} - \left[\ln(1 - \kappa) + \frac{\kappa}{(1 - \kappa)^3} - \frac{15}{6} \frac{\kappa^2}{(1 - \kappa)^3} + \frac{11}{6} \frac{\kappa^3}{(1 - \kappa)^3}\right]. \tag{407}$$

These two quantities are presented in Fig. 12. It is obvious that both diverge at $\kappa \to 1$. $g(\kappa)$ diverges much faster than $h(\kappa)$. This would mean that the validity of this approach is limited to the Fermi energy. It is bounded from above by a maximum energy bound c/α. This is completely consistent with the predicted maximum measurable momentum $1/\alpha$ in [63, 113].
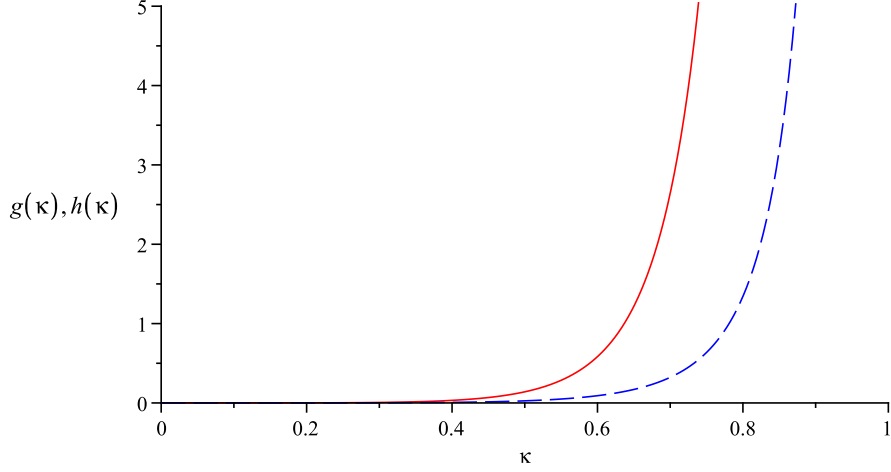
Fig. 12: The dimensionless quantities g (solid curve), Eq. (406), and h (dashed curve), Eq. (407), are presented as functions of κ. The graph taken from Ref. [221].

The Hagedorn energy (or equivalently temperature) is defined as $\epsilon_H = M_p c^2/\alpha_0$. It is a scale to set the limit of applying the GUP approach. Accordingly, Eqs. (404) and (405) can be re-written as [221]

$$U_0(\kappa) = \frac{8\pi V}{(hc)^3}\epsilon_H^4 \left[\frac{\kappa^4}{4} + \frac{4\kappa^5}{5}\right], \tag{408}$$

$$P(\kappa) = \frac{8\pi}{(hc)^3}\epsilon_H^4 \left[\frac{\kappa^4}{12} + \frac{\kappa^5}{5}\right], \tag{409}$$

where the condition $\kappa \ll 1$ has been implemented. The corrections to $U_0(\kappa)$ and $P(\kappa)$ would be given in terms of δ [221]

$$\frac{U_0(\kappa)}{V} = \frac{3^{4/3}}{4}\frac{hc}{(8\pi)^{\frac{1}{3}}}\left(\frac{N}{V}\right)^{4/3}\left(1 + \frac{16}{5}(3\pi^2)^{1/3}\delta\right), \tag{410}$$

$$P(\kappa) = \frac{3^{1/3}}{4}\frac{hc}{(8\pi)^{\frac{1}{3}}}\left(\frac{N}{V}\right)^{4/3}\left(1 + \frac{12}{5}(3\pi^2)^{1/3}\delta\right), \tag{411}$$

where κ can be given in terms of number density N/V,

$$\kappa^3\left(\frac{1}{3} + \kappa\right) = \frac{(hc)^3}{8\pi}\frac{1}{\epsilon_H^3}\frac{N}{V}, \tag{412}$$

which would lead to a relation between κ and δ

$$\kappa = (3\pi^2)^{\frac{1}{3}}\delta\left[1 - (3\pi^2)^{\frac{1}{3}}\delta\right] + \mathcal{O}(\delta^3) \tag{413}$$

It is obvious that the results are stemming for the framework modifying the dispersion relations, Eq. (348). The framework is based in GUP and modified measure of the momentum space, Eq. (399). It is therefore outside the effective field theory framework [222].

1. Compact stars with non-relativistic cold nuclei

The major contributions to the mass of white dwarfs are non-relativistic cold nuclei having mass $M = 2N/m_p$ [223]. The white dwarfs have two properties [224]:

- the electrons are described by relativistic dynamics and

- the electron gas is completely degenerate.

Therefore, the electron gas would be treated as a zero-temperature gas. Thus, $\epsilon_F = 2Nc^2/\alpha_0 m_p$ indicating that $\kappa \ll 1$ and Eq. (409) seems to reflect that the QG effects increase the degenerate pressure. Should this effects is confirmed, then the QG corrections to the mass of white dwarfs arise.

The electronic degeneracy pressure is supposed to resist the gravitational collapse and keep the electron gas at a given density. At equilibrium, the pressure reads

$$P_0(R) \;=\; \frac{\lambda}{4\pi} G \left(\frac{M}{R^2} \right)^2, \tag{414}$$

where $R^3 \equiv V$ and λ is free parameter of the order of unity. Nevertheless, its value depends on how the matter is distributed inside the white dwarf. From Eqs. (411) and (414), and by ignoring the constants (assign them to unity), the pressure can be expressed in terms of internal energy

$$\left(\frac{N}{V} \right)^{4/3} (1 + \delta) = \frac{GM^2}{R^4}. \tag{415}$$

By substituting $M = 2Nm_p$, the correction to the mass of the compact star reads [221]

$$M = M_0 \left(1 + \left(\frac{N}{V} \right)^{\frac{1}{3}} \alpha \, \hbar \right). \tag{416}$$

We set $M_0 = (hc/G)^{\frac{3}{2}} (2m_p)^{-2}$. For white dwarfs, M_0 approximately approaches the Chandrasekhar limit (about 1.44 $M_\odot$).

Apparently, Eq. (416) concludes that the quantum gravity correction seems to be proportional to the density number of the star [221]. For a white dwarf, in which the density number $N = 10^{36}$, the average distance $\bar{d} = 10^{-12}$, and the Fermi energy $\epsilon_F = 10^5$ eV [221].

In the present analysis, we set an upper bound to $\alpha_0 \times l_p$ leading to an intermediate between the Planck and the electroweak scale. We used the bound $\alpha \leq 10^{-2}$ GeV^{-1} (i.e.,

$\alpha_0 \leq 10^{17}$) depending on the derived bounds on the parameter α_0 in Ref. [75]. This bound was derived by calculating the effect of QG with non-relativistic heavy meson systems like charmonium [75], which may be a relevant example for the white dwarfs. The latter are mainly constituted of non-relativistic nuclei. Based on these values, the QG correction to mass of the white dwarf is given by [221]

$$M_{GUP} = M_0 \left(1 + 10^{-5}\right). \tag{417}$$

Two remarks are now in order [221].

- The correction seems to be more stringent than the one derived for compact stars with QG corrections [225]. The correction given in Ref. [225] is 10^{-10}.

- The QG corrections [221] is positive referring to resisting the collapse of the compact stars. It is obvious that this conclusion agrees with the result in [225].

2. *Compact stars with ultra-relativistic nuclei*

There are other configurations in which the star is almost composed of ultra-relativistic nuclei. In this case, the mass of the nuclei is compressed as $M = U_0/c^2$. The constituents of the white dwarfs are characterized by an ideal Fermi gas and total mass $M = U_0/c^2$. The electronic degeneracy pressure is assumed to resist the gravitational collapse. At equilibrium, the radius of the white dwarf is given by [221]

$$R = \frac{\lambda}{8\pi} R_S \frac{g(\kappa)}{h(\kappa)} = \frac{\lambda}{8\pi} R_S Q(\kappa), \tag{418}$$

where $Q(\kappa) = g(\kappa)/h(\kappa)$ and the parameter λ approximately equals unity. In the considered case, the Schwarzschild radius reads

$$R_S = 2G\frac{M}{c^2} = 2G\frac{U_0}{c^4}. \tag{419}$$

At $\lambda \approx 1$, the results are presented in Fig. 13 for the stringent value of the parameter α [75]. We observe that the radius approaches its minima as $\kappa \to 1$, and divers as $\kappa \to 0$. The number density N/V from Eq. (401) and mass density $M = U_0/c^2$ from Eq. (417) are presented also in Fig. 13. We observe that the number density, the mass density and the pressure approach their minima as $\kappa \to 0$, but they reach their maximum values as $\kappa \to 1$.
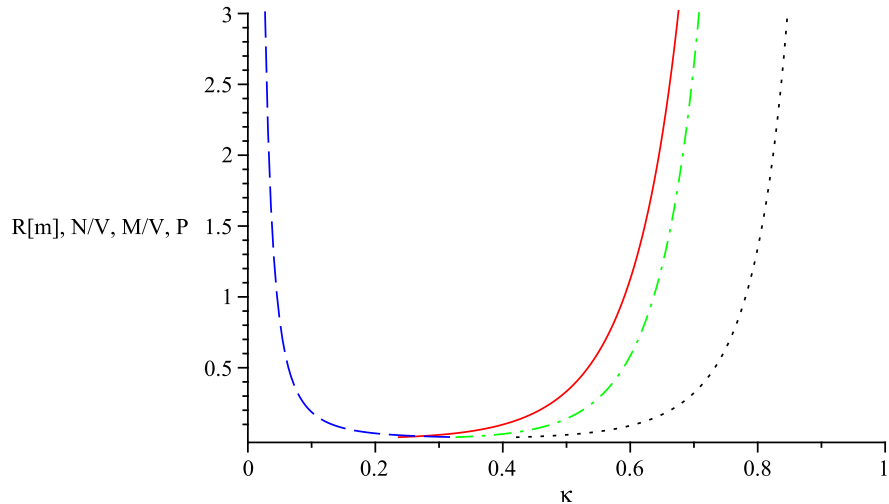
Fig. 13: The modified radius of white dwarf, Eq. (418), is given in dependence on κ (dashed curve) at $\alpha_0 \approx 10^{17}$ i.e $\alpha \approx 10^{-2}$ GeV^{-1} [75]. The modified normalized particle density in Fermi gas at vanishing temperature $N(\kappa)/V$ is given in dependence on κ (solid curve) at $\alpha = 10^{-2}$ GeV^{-1}. The normalized mass density $M(\kappa)/V$ is given as dash-dotted curve at $\alpha = 10^{-2}$ GeV^{-1}. The normalized pressure is given as dotted curve. The graph taken from Ref. [221].

Current observations indicate that white dwarfs have smaller radii than expected [226]. The behavior of R vs κ in Fig. 13 suggests that R is decreasing as $\kappa \to 1$. This offers a possible explanation for the smaller radii observations. Similar analysis has been done in the context of DSR [229] and modified dispersion relations [227, 228].

3. Conclusions

Effects of the linear GUP approach on the thermodynamic properties of the compact stars are investigated. Concretely, the impact on the Chandrasekhar limit andthe gravitational collapse is studied. It is concluded that the QG corrections would increase the Chandrasekhar limit and hence they resist the gravitational collapse. Furthermore, it is found that the radius of the compact star is decreasing as the energy increasing, which might be considered as a possible explanation for the smaller radii observations.

E.   Saleker-Wigner inequalities

The proposal that fundamental limits can be utilized in order to govern mass and size of the physical system to register time dates back to nearly six decades [230]. Salecker and Wigner were pioneers in suggesting the use of a quantum clock [20, 21] in measuring *distances* between events in space-time [21]. This quantum clock is given as constrains of smallest accuracy and maximum running time as a function of mass and position uncertainties. As introduced in section I, the Salecker-Wigner second constrain is more severe than HUP. The latter requires that only one single simultaneous measurement of both energy and time, for instance, can be accurate. The Salecker-Wigner constrains assume that the repeated measurements should not disturb the clock. On other hand, the clock itself should be able to accurately register time over its total running period.

Four decades later, Barrow applied Salecker-Wigner inequalities in order to describe the quantum constrains on black hole lifetime [56]. It is found that the BH running time should be correspondent to the Hawking lifetime, which is calculated under the assumption that the BH is a black body. It is found that the Schwarzschild radius of BH is correspondent to the constrains on Saelcker-Wigner size. Furthermore, the information processing power of a black hole is estimated by the emitted Hawking radiation [57].

1.   Salecker-Wigner inequalities and black hole evaporation

As anticipated in section VIE, the second Salecker-Wigner inequality is more severe than the standard Heisenberg energy-time uncertainty principle. This is simply because it requires that a quantum clock is able to show proper time even after the time was being read. In other words, the quantum uncertainty in its position does not produce a significant inaccuracy in its time measurement. This property is conjuncted to hold over long periods, i.e., coherent time intervals. The terminology *"coherence"* has to do with the correlation properties of the signal used in the measurement. The *"coherent time"* is defined as the time period within which the signal remains *"coherent"*

$$\tau_c = \frac{1}{\Delta \nu_c} \approx \frac{\lambda_c^2}{c \, \Delta \lambda_c},$$

$$(420)$$

where the subscript c refers to coherence.

From HUP, the momentum uncertainty in *single analogue* quantum clock of mass m is $\hbar/2\,\Delta\,x$, where $\Delta\,x$ is uncertainty in its quantum position. After time t, the clock position spread increases to

$$\Delta\,x' \;=\; \Delta\,x + \frac{\hbar\,t}{m}\,\frac{1}{2\,\Delta\,x}. \tag{421}$$

Assuming that the mass of quantum clock remains unchanged, then Eq. (421) leads to a minimum time spread

$$\Delta\,x \;\geq\; \sqrt{\hbar\,\frac{t_{max}}{2\,m}}, \tag{422}$$

where t_{max} is the total *"coherent"* time. Expression equation (421) is known as Salecker-Wigner first inequality.

In the case that the mass depends on the uncertainty in position, then the minimum time spread reads [230]

$$\Delta\,x \;\geq\; \frac{\hbar\,t_{max}\,m' - \sqrt{\hbar\,t_{max}\,[8\,m^2 + (m')^2\,\hbar\,t_{max}]}}{4\,m^2}, \tag{423}$$

where $m' = dm/d\Delta\,x$. The positive sign is evaluated as non-physical.

If the time measurements are repeated and have to remain reliable, then the position uncertainty which in turn must be caused by the repeated measurements, should be smaller than the minimum wavelength of the reading signals, i.e., $\Delta\,x \leq c\,T_{min}$. For an unsqueezed, unentangled and Gaussian signal, the minimum size can be give in minimum mass of the quantum clock. From Eq. (422), the mass-time inequality is given as [230]

$$m \;\geq\; \frac{\hbar}{2\,c^2}\,\frac{t_{max}}{t_{min}^2}, \tag{424}$$

which is known as Salecker-Wigner second inequality.

2. *Salecker-Wigner inequalities and linear GUP approach*

Assuming a black hole of a size comparable to the Schwarzschild radius, $r_s = 2\,G\,m/c^2$, then Salecker-Wigner first inequality, Eq. (421), can be applied on it. From Eq. (422), the maximum running time (lifetime) of black hole reads

$$t_{max} \;\leq\; 8\frac{G^2\,m^3}{\hbar\,c^4}, \tag{425}$$

$$\leq\; 8\,\frac{G}{c^3}\left(\frac{m^3}{M_p^2}\right), \tag{426}$$

where $M_p = \sqrt{c\hbar/G}$ is Planck mass. Here after, we refer to black hole mass as m. It should not be mixed with the normalized mass of black hole mentioned in previous sections. Obviously, these expressions are compatible with the Hawking lifetime [231]. Eqs. (425) and (426) give answers to the question, *"how does the life of a black hole run out?"* [230]. As discussed in the previous sections, the mass of black hole quantum clock is the only parameter that describes a reliable mechanism. It offers an alternative possibility not based on the assumption that black hole has to be a black body radiator [231].

At Planck scale, the space-time fluctuation becomes significant. Therefore, it is natural to set a bound to the linear spread of the quantum clock, Eq. (422). The natural bound is the Planck distance. As given in introduction, the GUP approach gives prediction for a minimal measurable length. Therefore, $\alpha_0 \, \ell_p$ would be taken as the smallest linear spread of the quantum clock.

At time t, the position uncertainty due to GUP becomes [230]

$$\Delta x' \;=\; \Delta x + \frac{2\Delta x + \frac{4}{3}\alpha_0 \, \ell_p \, \sqrt{\mu}}{4\,(1+\mu)\,\alpha_0^2 \, \ell_p^2 \, m}\, \hbar\, t \left[1 - \sqrt{1 - \frac{8\,(1+\mu)\,\alpha_0^2 \ell_p^2}{\left(2\Delta x + \frac{4}{3}\alpha_0 \ell_p \, \sqrt{\mu}\,\right)^2}}\,\right]. \qquad (427)$$

Then

$$\Delta x_{GUP} \;\geq\; \frac{1}{2}\left[-A_1 + \frac{\sqrt{2}\,(m\,A_2 + 2\,\hbar\,t)^2}{\sqrt{m\,(m\,A_2 + 2\,\hbar\,t)^2\,(m\,A_2 + 4\,\hbar\,t)}}\right], \qquad (428)$$

where

$$A_1 \;=\; \frac{4}{3}\alpha_0\ell_p\sqrt{\mu}, \qquad (429)$$

$$A_2 \;=\; 4(1+\mu)\alpha_0^2\ell_p^2. \qquad (430)$$

At $\alpha_0 = 0$, the Salecker-Wigner position uncertainty is recovered

$$\Delta x_{SW} \;\geq\; \sqrt{\hbar\frac{t}{2\,m}}. \qquad (431)$$

In Eqs. (428) and (431), the negative solutions are evaluated as non-physical. It is apparent that Eq. (431), in which GUP effects are excluded, is identical with the Salecker-Wigner first inequality, Eq. (422). The difference between Eq. (428) and Eq. (422) simply reads [230]

$$\Delta x_{GUP} - \Delta x_{SW} \;=\; \frac{1}{2}\left[-A_1 - \hbar^2\, t_{max}^2 \sqrt{\frac{2}{m\,\hbar^3\,t_{max}^3}}\right.$$

$$\left. + \;(m\,A_2 + 2\hbar\,t_{max})^2 \sqrt{\frac{2}{m(m\,A_2 + 2\hbar t_{max})^2(m\,A_2 + 4\hbar t)}}\right], \qquad (432)$$
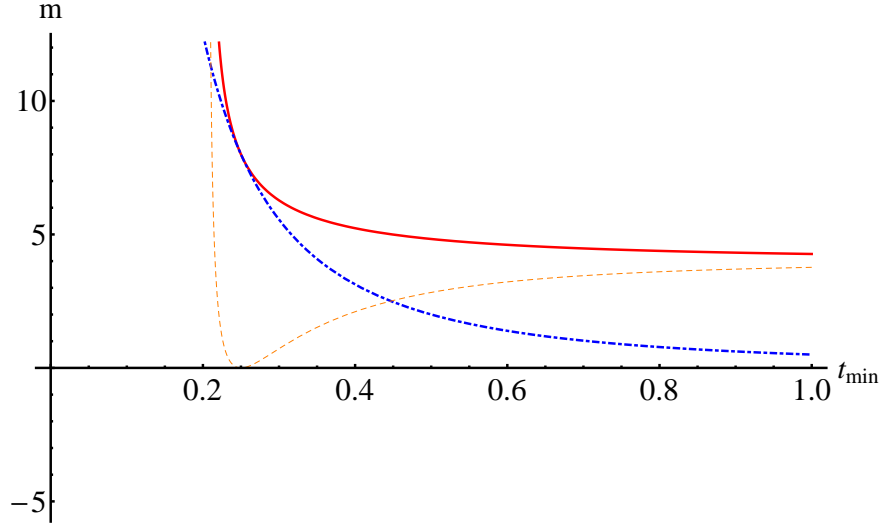
Fig. 14: The black hole mass is given in dependence on its lifetime with (solid line) and without (dash-dotted line) GUP and their difference (dashed line). The values of the variables A_1, A_2, $\hbar$, and c are taken unity. The graph is taken from Ref. [230].

which obviously vanishes at vanishing α_0.

Assuming that the quantum position uncertainty should not be larger that the minimum wavelength of measuring signal, so that in Eq. (428), it is assumed that $\Delta x_{GUP} \leq c\,t_{min}$ [230],

$$m_{GUP} \geq \frac{-\left[2\,\hbar\,t_{max}A_3 \pm 2\hbar\,t_{max}\,(A_1 + 2c\,t_{min})\,\sqrt{A_3}\right]}{A_2\,A_3}, \tag{433}$$

where $A_3 = A_1^2 - 2A_2 + 4c\,t_{min}\,(A_1 + c\,t_{min})$. The positive sign defines a non-physical solution, where

$$2\hbar\,t_{max}\,(A_1 + 2c\,t_{min})\,\sqrt{A_3} > 2\,\hbar\,t_{max}A_3, \tag{434}$$

implies that

$$\sqrt{A_3} < A_1 + 2c\,t_{min}. \tag{435}$$

At vanishing α_0, Eq. (433) goes back to the Salecker-Wigner second inequality, Eq. (424). At this scale, the inequality, Eq. (435), turns on an equality in t_{min}. The difference between Eq. (433) and Eq. (424) results in [230]

$$m_{GUP} - m_{SW} = \frac{1}{2}\left(-\frac{4\,\hbar\,t_{max}}{A_2} - \frac{\hbar t_{max}}{c^2\,t_{min}^2} - \frac{4\,\hbar^2\,t_{max}^2\,(A_1 + 2\,c\,t_{min})^2}{A_2\,\sqrt{\hbar^2\,t_{max}^2\,(A_1 + 2\,c\,t_{min})^2\,A_3}}\right). \tag{436}$$
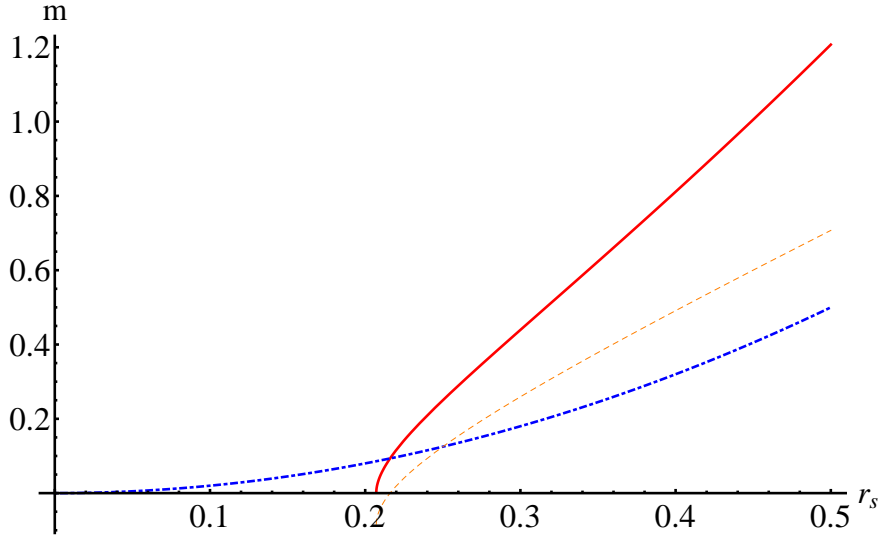
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Fig. 15: The black hole mass is given in dependence on r_s. The values of the variables A_1, A_2, $\hbar$, and c are taken unity. The graph taken from Ref. [230].

The modified black hole lifetime can be derived assuming that the spread of quantum clock has a minimum value, the Schwarzschild radius, r_s,

$$t_{GUP} = \frac{1}{16\,\hbar^2}\left[-\hbar\,m\,A_4 - \hbar\,m\,A_4\,(1 - 128\,A_2)^{1/2}\right], \tag{437}$$

where $A_4 = -4A_1^2 + 8A_2 - 16r_s A_1 - 16r_s^2$. The solution including negative sign is taken as physical. At $\alpha_0 = 0$, the modified black hole lifetime, Eq. (437), goes back to Salecker-Wigner inequality, Eq. (425). The difference between black hole lifetime in GUP approach and Salecker-Wigner inequality reads [230]

$$t_{GUP} - t_{SW} = 2\frac{m\,r_s^2}{\hbar} = 8\frac{G\,m}{c^3}\left(\frac{m}{M_p}\right)^2, \tag{438}$$

and depicted in Fig. 14.

3. Conclusions

Based on the assumption that the black hole is a radiator, a reliable estimation of its lifetime is introduced. To this end, another approach based the Salecker-Wigner inequalities was utilized [230]. The reasons are obvious. The Salecker-Wigner inequalities are assumed to be more severe than the Heisenberg energy-time uncertainty principle. The quantum clock is conjectured to show proper time even after the time was being read and the quantum

uncertainty in position does not produce a significant inaccuracy in the time measurement. This property is conjuncted to hold over long *"coherent"* time intervals.

At Planck scale, the smallest linear spread of the quantum clock is set to $\alpha_0 \ell_p$. Assuming the mass remains unchanged, the Salecker-Wigner first inequality is reproduced. When applying GUP approach, the resulting position uncertainty does not match with Salecker-Wigner first inequality. The difference depends on the maximum lifetime. Through Salecker-Wigner second inequality, the latter can be related to the minimum lifetime.

Assuming that the quantum position uncertainty is limited to the minimum wavelength of measuring signal, the Salecker-Wigner second inequality can be reproduced. The difference between black hole mass with and without GUP is not negligible. The modified black hole lifetime can be deduced if the spread of quantum clock is limited to a minimum value. The natural one is the Schwarzschild radius. Based on GUP, the resulting lifetime difference depends on black hole mass and α_0.

F. Minimal time measurement

About fifty years ago, Shapiro pointed out that the possible time delay resulting from the observation that light appearing to slow down as it passes through a gravitational potential could be measured within our solar system [69, 236, 237]. As given in section VII E 1, utilizing the fundamental limits governing mass and size of any physical system, Salecker and Wigner [20, 21] suggested that a minimum time interval can be even registered. In 1927, a hypothetical indivisible interval of time taken as a ratio between the diameter of the electron and the velocity of light, being equivalent to approximately $\sim 10^{-24}$ s, was proposed by Robert Levi [1].

Itzhaki considered the uncertainty principle and utilized the Schwarzschild solution in large scale in order to estimate the minimal measurable time interval [238]. He found that the uncertainty in time measurement depends on the distance separating the observer from the event, the clock accuracy and size, and the time taken by photon to reach the observer. Assuming distances, in which GR offers a good approach for QG, then the shortest distance $x_c = \beta \, (G\hbar/c^3)^{1/2}$, where β is an arbitrary parameter. The minimum error in the time

measurement is estimated as

$$\Delta t = \sqrt{\frac{8\,G\,\hbar}{c^5}\,\ln\left(\frac{x}{x_c}\right)} = 2\,\sqrt{2\,\ln\left(\frac{x}{x_c}\right)}\,t_{pl},\tag{439}$$

where $t_{pl} = \sqrt{G\,\hbar/c^5}$ is the Planck time. This expression is valid at distance $x > x_c\,\exp(2/\beta^2)$, where $x_c = \beta\,(G\,\hbar/c^3)^{1/2}$ is the shortest distance for which it is assumed that GR is a good approximation to QG. The corresponding minimum error in the energy is given by

$$\Delta E = \sqrt{\frac{\hbar\,c^5}{2\,G\,\ln\left(\frac{x}{x_c}\right)}} = \sqrt{\frac{1}{2\,\ln\left(\frac{x}{x_c}\right)}}\,\frac{\hbar}{t_{pl}}.\tag{440}$$

Then, the minimal time and maximal energy at $x_c < x < x_c\,\exp(2/\beta^2)$, respectively, read

$$\Delta t_{min} = \frac{x_c}{c}\left[\frac{2}{\beta^2} + \ln\left(\frac{x}{x_c}\right)\right],\tag{441}$$

$$\Delta E_{max} = \frac{c^4}{2\,G}\,x_c = \frac{\hbar\,x_c}{2\,c}\,\frac{1}{t_{pl}^2}.\tag{442}$$

1. *Linear GUP approach: uncertainty in time and minimum measurable time*

In linear GUP approach, the uncertainty in time reads

$$\Delta t \geq \frac{1}{2}\frac{\hbar}{\Delta E}\left[1 - 2\frac{\alpha}{c}\,\Delta E\right] = \frac{\hbar}{2\,\Delta E} - \frac{\alpha}{c}\,\hbar,\tag{443}$$

implying that the physical limits require $2\alpha\Delta E < c$. The minimum measurable time interval Δt_m is to be deduced under the condition that the derivative $d\,\Delta t/d\,\Delta E$ vanishes. Then,

$$-\frac{\hbar}{2\,(\Delta E)^2} = 0,\tag{444}$$

which leads to

$$\Delta E_{max} = \infty,\tag{445}$$

$$\Delta t_{min} = -\frac{\alpha}{c}\,\hbar = -\frac{\alpha_0}{M_{pl}\,c^2}\,\hbar,\tag{446}$$

where α is replaced by $\alpha_0/M_{pl}c$. It is obvious that the measurable maximal energy gets infinite while the measurable minimal time interval has a negative value. Both results are obviously non-physical. While ΔE apparently violates the conservation of energy, Δt shows that the direction of the arrow of time becomes opposite.

The uncertainty in time as estimated from Schwarzchild solution at distance x_c at which GR is a good approximation to QG is given as [238]

$$\Delta t \geq \frac{\hbar}{2\,\Delta E} + G\frac{\Delta E}{c^5}. \tag{447}$$

Then, the maximal measurable energy and minimal measurable time interval, respectively, read

$$\Delta E_{max} = c^2\sqrt{\frac{\hbar\,c}{2\,G}} = \frac{\hbar}{\sqrt{2}}\frac{1}{t_{pl}}, \tag{448}$$

$$\Delta t_{min} = \frac{1}{c^2}\sqrt{\frac{2\,G\,\hbar}{c}} = \sqrt{2}\,t_{pl}. \tag{449}$$

Both quantities are positive. It is obvious that both of them depend on the Schwarzshild radius, which is related to the black hole mass, $r_s = (2G/c^2)\,m$. It is worthwhile to note that both quantities are related to the Planck time t_{pl}. Accordingly, they are bounded.

3.   *Time uncertainty and minimum measurable time at distance larger than x_c*

When the photon travels a distance x larger than x_c, then the total uncertainty in time is estimated as

$$\Delta t_{total} \geq \frac{\hbar}{2\Delta E} + G\frac{\Delta E}{c^5} + 2G\frac{\Delta E}{c^5}\ln\left(\frac{x}{x_c}\right). \tag{450}$$

The maximal measurable energy and corresponding minimal measurable time interval, respectively, are given as

$$\Delta E_{max} = \sqrt{\frac{c^5\hbar}{2G[1 + 2\ln(\frac{x}{x_c})]}} = \sqrt{\frac{1}{2[1 + 2\ln(\frac{x}{x_c})]}}\,\hbar\,t_{pl}, \tag{451}$$

$$\Delta t_{min} = \sqrt{\frac{\hbar G(1 + 2\ln(\frac{x}{x_c}))}{2c^5}} + \sqrt{\frac{\hbar G}{2c^5(1 + 2\ln(\frac{x}{x_c}))}}\left[\left(1 + 2\ln\left(\frac{x}{x_c}\right)\right)\right]$$

$$= \sqrt{\frac{2\hbar G}{c^5}\left[1 + 2\ln\left(\frac{x}{x_c}\right)\right]} = \sqrt{2\left[1 + 2\ln\left(\frac{x}{x_c}\right)\right]}\,t_{pl}. \tag{452}$$

The resulting ΔE_{min} and Δt_{min} are finite and positive. Both quantities are related to t_{pl}.

4. Conclusions

The maximal measurable energy ΔE and minimal measurable time ΔT are related to t_{pl} and therefore both are accordingly bounded. The Itzhaki model used the most simple time measurement process. It was concluded that any particles that will be added must necessarily increase the uncertainty of the metric without decreasing the minimal measurable time. Furthermore, Itzhaki summarized that the measured uncertainty would represent a basic property of the Nature.

The possibility of finding measurable maximal energy and minimal time are estimated in different quantum aspects. First, we find that the linear GUP approach gives non-physical results. The resulting maximal energy ΔE violates the conservation of energy. The minimal time interval Δt shows that the direction of the arrow of time is backwards. So far, we conclude that the applicability of the linear GUP approach is accordingly limited or even altered.

G. Entropic nature of gravitational laws and Friedmann equations

In a one-dimensional chain as the Ising model [240], we assume that every single spin is positioned at a distance d apart from the two neighbourhoods. Then, the macroscopic state of such a chain can be defined by d. Depending on d, the entire chain would have various configurations so that if $d \to l$, the chain has much less configurations than if $d \ll l$, where l is the chain's length. From statistical point-of-view, the entropy is given by the number of microscopic states $S = k_B \ln \Omega$. Due to second law of thermodynamics, such a system tends to approach a state of maximal entropy. Accordingly, the chain in the macroscopic state d tends to go to a another state with a much higher entropy. The force that causes such a statistical tendency is defined as the entropic force. In light of this, the entropic force is a phenomenological mechanism deriving a system to approach maximum entropy, i.e., increasing the number of microscopic states which will be inhered in the phase space. There are various examples on the entropic force, for example polymer molecules and even the elasticity of rubber bands.

Verlinde has proposed that the gravity might not be a fundamental force and could be considered as an entropic force [241]. In light of this, we recall that the earliest idea about

gravity regarded as a non-fundamental interaction has been introduced by Sakharov [267], where the spacetime background was assumed to emerge as a mean field approximation of underlying microscopic degrees of freedom. Similar behavior is observed in hydrodynamics [268]. As discussed in earlier sections, the BH entropy is to be related to the area of the BH horizon, while the temperature is to be related to the surface gravity. Both entropy and temperature are assumed to be related to the BH mass [24, 115]. Thus, the connection between thermodynamics and geometry leads to the Einstein's equations of the gravitational field from the relations connecting heat, entropy and temperature [269]. The Einstein's equations themselves connect the energy-momentum tensor with the space geometry. Advocating the gravity as non-fundamental interaction leads to the assumption that the gravity would be explained as an entropic force caused by changes in the information associated with the positions of material bodies [241]. When combining the entropic force with the Unruh temperature [270], the second law of Newton is obtained. But when combining it with the holographic principle and using the equipartition law of energy, the Newton's law of gravitation is obtained. The modification on the entropic force due to corrections to the area law of entropy, which is derived from quantum effects and extra dimensions, was investigated [271].

1. *Newton's law of gravity and GUP approaches*

The non-commutative geometry which is considered as a completely Planck scale effect has been utilized to derive the modified Newton's law of gravity [272–275]. All these approaches implement the following scheme (chain):

- modified theory of gravity $\rightarrow$

- modified black hole entropy $\rightarrow$

- modified holographic surface entropy $\rightarrow$

- Newton's law corrections.

The same *flow chart* is followed in deriving the linear GUP approach [63, 113]. The results are compared with Randall-Sundrum model of extra dimension, which also predicts the modification of Newton's law of gravity at the Planck scale [276, 277], where we think there

may be some connection between GUP and extra dimension theories because they predicting similar physics at least for the case of Newton's law of gravity, which may be considered as a distinct result from the previous studies. The effect of GUP on the Newton's law of gravity was studied in Ref. [278].

2. Gravity as an entropic force

At temperature T, the entropic force F of a gravitational system is given as [241]

$$F\Delta x = T\Delta S, \tag{453}$$

where ΔS is the change in the entropy so that at a displacement Δx, each particle carries its own portion of entropy change. From the correspondence between the entropy change ΔS and the information about the boundary of the system and using Bekenstein's argument [24, 115], it is assumed that $\Delta S = 2\pi k_B$, where $\Delta x = \hbar/m$ and k_B is the Boltzmann constant.

$$\Delta S = 2\pi k_B \frac{mc}{\hbar} \Delta x, \tag{454}$$

where m is the mass of the elementary component.

The holographic principle assumes that *for any closed surface without worrying about its geometry inside, all physics can be represented by the degrees of freedom on this surface itself.* This implies that the QG can be described by a topological quantum field theory, for which all physical degrees of freedom can be projected onto the boundary [249]. The information about the holographic system is given by N bits forming an ideal gas. It is conjectured that N is proportional to the entropy of the holographic screen,

$$N = \frac{4S}{k_B}, \tag{455}$$

then according to Bekenstein's entropy-area relation [24, 115]

$$S = \frac{k_B c^3}{4G\hbar} A. \tag{456}$$

Therefore, one gets

$$N = \frac{Ac^3}{G\hbar} = \frac{4\pi r^2 c^3}{G\hbar}, \tag{457}$$

where r is the radius of the gravitational system and the area of the holographic screen $A = 4\pi r^2$ is implemented in deriving this equation. It is assumed that each bit emerges

outwards from the holographic screen, i.e., one dimension. Therefore each bit carries an energy equal to $k_B T/2$. By using the equipartition rule in calculating the energy of the system, one gets

$$E = \frac{1}{2} N k_B T = \frac{2\pi c^3 r^2}{G\hbar} k_B T = Mc^2. \tag{458}$$

By substituting Eq. (453) and Eq. (454) into Eq. (458), we get Newton's law of gravitation

$$F = G\frac{Mm}{r^2}. \tag{459}$$

3. Black hole horizon area and entropy

For a black hole absorbing a quantum particle with energy E and size R, the area of the black hole is supposed to increase by the amount [115].

$$\Delta A \geq 8\pi\, \ell_p^2\, E\, R, \tag{460}$$

The quantum particle itself implies the existence of a finite bound given by

$$\Delta A_{min} \geq 8\pi\, \ell_p^2\, E\, \Delta x. \tag{461}$$

By using Eq. (458) in the inequality expression (461), we obtain,

$$\Delta A_{min} \geq 8\pi\ell_p^2 \left[1 - \frac{2}{3}\alpha_0\ell_p\sqrt{\mu}\frac{1}{\Delta x}\right]. \tag{462}$$

According to the argument given in Refs. [279, 280], the length scale is chosen to be the inverse surface gravity

$$\Delta x = 2\, r_s, \tag{463}$$

where r_s is the is the Schwarzschild radius. This argument implies that

$$(\Delta x)^2 \sim \frac{A}{\pi}. \tag{464}$$

by substituting Eq. (464) in Eq. (462), we get [67]

$$\Delta A_{min} = \lambda\ell_p^2 \left[1 - \frac{2}{3}\alpha_0\,\ell_p\,\sqrt{\frac{\mu\,\pi}{A}}\right], \tag{465}$$

where parameter λ will be fixed later. According to Refs. [24, 115], the BH entropy is conjectured to depend on the horizon area. From the information theory [281], it has been

found that the minimal increase of entropy should be independent on the area. It is just one bit of information which is $b = \ln(2)$ [67]

$$\frac{dS}{dA} = \frac{\Delta S_{min}}{\Delta A_{min}} = \frac{b}{\lambda \ell_p^2 \left[1 - \frac{2}{3} \alpha_0 \, \ell_p \, \sqrt{\mu \, \pi \, A}\right]}, \tag{466}$$

where b is a free parameter. By expanding the last expression in orders of α and then integrating, we get the entropy

$$S = \frac{b}{\lambda \ell_p^2} \left[A + \frac{4}{3} \alpha_0 \, \ell_p \, \sqrt{\mu \, \pi \, A}\right]. \tag{467}$$

By using Hawking-Bekenstein assumption, $b/\lambda = 1/4$, so that

$$S = \frac{A}{4 \, \ell_p^2} + \frac{2}{3} \alpha_0 \, \sqrt{\pi \, \mu \, \frac{A}{4 \, \ell_p^2}}. \tag{468}$$

Although it was found in Ref. [282] that the power–law corrections to Bekenstein-Hawking entropy are ruled out based on arguments from Boltzmann–Einstein formula, it was found that the power-law corrections may explain the observed cosmic acceleration today [283].

We conclude that the entropy is directly related to the area and gets a correction due to the linear GUP approach. The temperature of the black hole is given as [67]

$$T = \frac{\kappa}{8\pi} \frac{dA}{dS} = \frac{\kappa}{8\pi} \left[1 - \frac{2}{3} \alpha_0 \, \ell_p \, \sqrt{\mu \, \frac{\pi}{A}}\right]. \tag{469}$$

Then, the temperature is not only proportional to the surface gravity but also depends on the black hole's area.

4. Linear GUP approach and entropic Newtonian laws

Using the corrected entropy given in Eq. (468), we find that the number of bits should also be corrected as follows.

$$N' = \frac{4S}{k_B} = \frac{A}{\ell_p^2} + \frac{4}{3} \alpha_0 \, \sqrt{\mu \, \pi \, \frac{A}{\ell_p^2}}. \tag{470}$$

By substituting Eq. (470) into Eq. (458) and by using Eq. (453), we get

$$E = F \, c^2 \left(\frac{r^2}{m \, G} + \frac{\alpha \, \sqrt{\mu} \, r}{3 \, m \, G}\right). \tag{471}$$

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It is apparent that Eq. (471) implies modifications in the Newtonian laws,

$$F = G\frac{M\,m}{r^2}\left(1 - \frac{\alpha\,\sqrt{\mu}}{3\,r}\right). \tag{472}$$

From the Newtonian second law,

$$m\,\ddot{r} = -G\frac{M\,m}{r^2}\left(1 - \frac{\alpha\,\sqrt{\mu}}{3r}\right). \tag{473}$$

where r is the apparent horizon radius

$$\ddot{r} = -\frac{4\pi G}{3}\,\alpha\,\rho\left(1 - \frac{\alpha\,\sqrt{\mu}}{3r}\right). \tag{474}$$

5. Entropic Newtonian laws and modifications in Friedmann equations

Multiplying both sides of Eq. (474) by $a\dot{a}$ results in [67]

$$\dot{a}\,\ddot{a} = -\frac{4\,\pi\,G}{3}a\,\dot{a}\,\rho\left(1 - \frac{\alpha\,\sqrt{\mu}}{3r}\right), \tag{475}$$

With the equations of state

$$p = \frac{1}{3}\,\rho, \tag{476}$$

$$\rho_0 = -3\,H\,(\rho + p) = -4\,H\,\rho, \tag{477}$$

and the relations

$$\frac{d}{dt}\,\dot{a}^2 = 2\,\dot{a}\,\ddot{a}, \tag{478}$$

$$\frac{d}{dt}\,(\rho\,a^2) = \rho_0\,a^2 + 2\,a\,\dot{a}\,\rho, \tag{479}$$

the integration of Eq. (475) leads to

$$\dot{a}^2 + C = \frac{8\,\pi\,G}{3}\,\rho\,a^2\left(1 - \frac{\alpha\,\sqrt{\mu}}{3\,r}\int\frac{d\,(\rho\,a^2)}{\rho\,a^3}\right), \tag{480}$$

where C is the integral constant, which as it will explained below, is nothing but the curvature constant, k. The energy density can be expressed as

$$\rho = \rho_0\,a^{-3(1-\omega)}, \tag{481}$$

where ω is the speed of sound, $\omega = p/\rho \equiv c_s^2$.

$$d(\rho_0\,a^{-3(1-\omega)}) = -3(1-\omega)\rho_0\,a^{-3\omega-2})\,da, \tag{482}$$

$$\rho\,a^3 = \rho_0\,a^{-3\omega}. \tag{483}$$

Accordingly,

$$\dot{a}^2 + C = \frac{8\pi G}{3} \rho \, a^2 \left(1 - \frac{\alpha \sqrt{\mu}(3\omega + 1)}{3\, r\, a}\right), \tag{484}$$

which can be rewritten as [67]

$$\left[H^2 + \frac{C}{a^2}\right] + \frac{\alpha \sqrt{\mu}}{3}(3\omega + 1)\left[H^2 + \frac{C}{a^2}\right]^{3/2} = \frac{8\pi G}{3}\, \rho, \tag{485}$$

where $r\,a$ represents the apparent horizon radius, $(H^2 + C/a^2)^{-1/2}$. Expression equation (485) is the modified Friedmann equation, where C is equivalent to the curvature constant. A detailed solution of H with respect to ρ is presented in Appendix A.

6. Conclusions

The expression equation (472) apparently states that the modification in the Newton's law of gravity seems to agree with the predictions of the Randall-Sundrum II model [276] which contains one uncompactified extra dimension and length scale Λ_R. The only difference is the sign. The modification in the Newton's gravitational potential on brane is given as [277]

$$V_{RS} = \begin{cases} -G\frac{mM}{r}\left(1 + \frac{4\Lambda_R}{3\pi r}\right), & r \ll \Lambda_R \\ \\ -G\frac{m\,M}{r}\left(1 + \frac{2\Lambda_R}{3r^2}\right), & r \gg \Lambda_R \end{cases}, \tag{486}$$

where r and Λ_R are radius and the characteristic length scale, respectively. It is clear that the gravitational potential is modified at short distance. We notice that Eq. (472) agrees with Eq. (486) when $r \ll \Lambda_R$. The only difference is the sign. This result shows that $\alpha \sim \Lambda_R$, which would help to set a new upper bound on the value of the GUP parameter α. This means that the proposed GUP approach [63, 113] is apparently able to predict the same physics as Randall-Sundrum II model. The latter assumes the existence of one extra dimension compactified on a circle whose upper and lower halves are identified. If the extra dimensions are accessible only to gravity and not to the standard model field, the bound on their size can be fixed by an experimental test of the Newton's law of gravity, which has only been led down to ~ 4 mm. This was the result, about ten years ago [284]. In recent gravitational experiments, it is found that the Newtonian gravitational force, the $1/r^2$-law,

seems to be maintained up to $\sim 0.13 - 0.16$ mm [285]. However, it is unknown whether this law is violated or not at sub-μm range. Further applications of this modifications have been discussed in [286] which could be the same for the GUP modification. This similarity between GUP and extra dimensions applications would assumes a new bounds on the GUP parameter α with respect to the extra dimension length scale Λ_R.

The modification in Eq. (474) has multiple consequences. We have worked out one of them. The Friedmann equations, Eq. (485), are derived. It is apparent that the entire modification is placed in the second term in lhs, which obviously depends on H, as well. The solution of H with respect to ρ is presented in Appendix A. It is obvious that the dependence of H on ρ is not monotonic. Reducing ρ, or increasing the cosmic time t, is accompanied with reducing H. Another behavior is characterized by certain value of ρ (or at concrete t). The Hubble parameter H increases with the further decrease in ρ. The rate strongly depends on geometry of the Universe, k.

H.   Thermodynamics of high-energy collisions

As discussed, the GUP approach apparently causes modifications in the fundamental commutator bracket between position and momentum operators. Then, it seems natural that this would result in considerable modifications in the Hamiltonian. Furthermore, this would affect a host of quantum phenomena, as well.

For a particle of mass M having a distant origin and an energy comparable to the Planck scale, the momentum would be a subject of a tiny modification and so that the dispersion relation can be expressed as in Eq. (348). Modified dispersion relations have be observed in DSR [227, 228]. Calculations based on these have been presented [229].

The phase space integral can be expressed as follows [71].

$$\sum_i \frac{V}{(2\pi)^3} \int_0^\infty d^3p \rightarrow \sum_i \frac{V}{2\pi^2} \int_0^\infty \frac{p^2 dp}{(1-\alpha p)^4}.$$ (487)

The partition function of an ensemble of N ideal (collision-free) constituents at vanishing

chemical potential reads

$$\ln z(T, V, \alpha) = \sum_i^N \pm \frac{V\, g_i}{2\pi^2} \int_0^\infty \frac{p^2}{(1 - \alpha\, p)^4}$$

$$\ln \left\{ 1 \pm \exp \left[-\frac{p\sqrt{(1 - 2\,\alpha\, p) + \left(\frac{m_i}{p}\right)^2}}{T} \right] \right\} dp, \qquad (488)$$

where $\pm$ stand for bosons and fermions, respectively. Equation (488) can be decomposed into

$$\ln z(T, V, \alpha) = \sum_i^N \pm \frac{V\, g_i}{2\,\pi^2} \int_0^\infty p^2\, \ln \left\{ 1 \pm \exp \left[-\frac{p\sqrt{(1 - 2\,\alpha\, p) + \left(\frac{m_i}{p}\right)^2}}{T} \right] \right\} dp, \qquad (489)$$

$$+ \sum_i^N \pm \frac{V\, g_i}{2\,\pi^2} \int_0^\infty p^2\, F(\alpha\, p)\, \ln \left\{ 1 \pm \exp \left[-\frac{p\sqrt{(1 - 2\,\alpha\, p) + \left(\frac{m_i}{p}\right)^2}}{T} \right] \right\} dp, (490)$$

where $F(\alpha\, p)$ is a series function.

The pressure is directly related to free energy of the system of interest; $p(T, V, \alpha) = T\partial \ln z(T, V, \alpha)/\partial V$. The number density reads

$$n(T, V, \alpha) = \sum_i^N \pm \frac{g_i}{2\,\pi^2} \int_0^\infty p^2\, \frac{\exp \left[-\frac{p\sqrt{(1 - 2\,\alpha\, p) + \left(\frac{m_i}{p}\right)^2}}{T} \right]}{1 \pm \exp \left[-\frac{p\sqrt{(1 - 2\,\alpha\, p) + \left(\frac{m_i}{p}\right)^2}}{T} \right]}\, dp, \qquad (491)$$

$$+ \sum_i^N \pm \frac{g_i}{2\,\pi^2} \int_0^\infty p^2\, F(\alpha\, p)\, \frac{\exp \left[-\frac{p\sqrt{(1 - 2\,\alpha\, p) + \left(\frac{m_i}{p}\right)^2}}{T} \right]}{1 \pm \exp \left[-\frac{p\sqrt{(1 - 2\,\alpha\, p) + \left(\frac{m_i}{p}\right)^2}}{T} \right]}\, dp. \qquad (492)$$

The simplest way to calculate the energy density is to multiply the number of quantum states $n(T, V, \alpha)$ by the energy of each state. Equations (490) and (492) take into account possible modifications in the phase space [287–289]. In equations (489) and (491), the phase space is apparently not a subject of modification, while the dispersion relation is.

1. Linear GUP approach at the QCD scale

The central question is whether the GUP approach is applicable at the level of QCD scale, ~ 1 GeV. If this would raise *havoc* with higher energy phenomena and would probably show up in high-precision measurements at low energy before they showed up in QGP studies, the experimental inferences of the QGP are barely better candidates to text this. The phenomenology should be well thought out, as it seems that if the dispersion relation, Eq. (348), were sufficiently modified to affect QGP observations, it would seem to alter other measurements in a more easily measured way.

Instead of modifying the dispersion relation, we may allow the phase space to be modified. In doing this, we start with the single-particle equilibrium distribution function [287, 289, 290]. The maximum number of *micro* states is given by solving

$$\frac{\partial}{\partial n_j}\left(S - \alpha N - \beta E\right) = \frac{\partial}{\partial n_j}\left(\ln N! + \ln \Pi_i^n g_i^{n_i} - \sum_i^n \ln n_i! - \alpha N - \beta E\right) = 0, \quad (493)$$

which means that only the terms having same subscript j remain finite. The coefficients α and β are Lagrange multipliers in entropy maximization. Each of these multipliers basically adds some unknown amount of each independent constraint to the function being optimized and ensures that the constraints are satisfied.

$$\frac{\partial}{\partial n_j}\ln \Pi_j g_j^{n_j} - \frac{\partial}{\partial n_j}\ln n_j! - \alpha - \beta\epsilon_j = 0. \quad (494)$$

Utilizing the Stirling approximation, then the occupation number,

$$n_j = g_j \exp\left(-\alpha - \beta\epsilon_j\right), \quad (495)$$

which apparently falls off exponentially with increasing ϵ, since, as will be shown below, $\gamma = \exp(-\alpha)$ is constant.

Then, the grand-canonical partition reads

$$Z_{gc}(T,V,\mu) = \mathbf{Tr}\left[\exp^{\frac{\mu\hat{\mathbf{b}}-\hat{\mathbf{H}}}{\mathbf{T}}-\alpha}\right], \quad (496)$$

$$f_{gc}(T,V,\mu) = \frac{\exp\left(\frac{-\hat{H}}{T} - \alpha\right)}{Z_{gc}(T,V,\mu)}. \quad (497)$$

With these assumptions, the dynamics of the partition function can be calculated as sum

over *single-particle partition* functions Z_{gc}^i of all hadrons and resonances.

$$
\begin{aligned}
\ln Z_{gc}(T, V, \mu) &= \sum_i \ln Z_{gc}^i(T, V, \mu) \\
&= \sum_i \pm \frac{g_i}{2\pi^2} V \int_0^\infty k^2 dk \ln \left(1 \pm \gamma \, \lambda_i \, e^{-\epsilon_i(k)/T}\right),
\end{aligned} \tag{498}
$$

where $\lambda_i = \exp(\mu_i/T)$ is the i-th particle fugacity and $\gamma = \exp(-\alpha)$ is the quark phase space occupation factor.

Furthermore, constraints on Lorentz invariance violation are very essential. If Lorentz invariance is instead deformed and the quantities in Eq. (488) transform under this deformed transformation, this necessarily leads to a modification of the addition law of momenta, as well. The definition of "temperature", the parameter β given in Eq. (494), involves taking an average and thus summing energies. Then, it becomes non-trivial, i.e., $\beta = 1/T$ would be also a subject of modification.

2. Conclusions

The works on this topics are still ongoing [71]. At QCD scale, which is accessible by means of high-energy experiments and lattice QCD simulations at high temperature. The GUP approach would be applicable. In this limit, modifications on the phase space, Lorentz invariance and even temperature should be utilized.

VIII. ALTERNATIVE APPROACHES TO GUP

In this section, we introduce other GUP approaches, which introduce higher order modifications. One approach gives prediction for the minimal length uncertainty, section VIII A. A second one predicts maximum momentum besides the minimal length uncertainty, section VIII B. An extensive comparison between three GUP approaches is elaborated in section VIII C.

A. Higher order GUP with minimal length uncertainty

Nouicer suggested a higher order GUP approach [65] which agrees with the GUP given in Eq. (75) to the leading order and predicts a minimal length uncertainty, as well. The Heisenberg algebra of the new GUP approach can be given by

$$[x,\, p] = i\,\hbar\,\exp\left(\beta\, p^2\right). \tag{499}$$

Apparently, this algebra can be fulfilled from the following representation of position and momentum operators:

$$X\,\psi(p) \;=\; i\,\hbar\,\exp\left(\beta\, p^2\right)\partial_p\,\psi(p), \tag{500}$$

$$P\,\psi(p) \;=\; p\,\psi(p). \tag{501}$$

These position and momentum operators are symmetric. Both imply modified completeness relation

$$\langle\phi|\psi\rangle \;=\; \int_{-\infty}^{\infty} dp\,\exp\left(-\beta\, p^2\right)\phi^*(p)\,\psi(p). \tag{502}$$

The scalar product of the momentum eigenstates changes to

$$\langle p|p^{'}\rangle \;=\; \exp\left(\beta\, p^2\right)\delta(p-p^{'}). \tag{503}$$

Also, the absolutely smallest uncertainty in position is given as

$$(\Delta\, x)_{min} \;=\; \sqrt{\frac{e}{2}}\,\hbar\,\sqrt{\beta}. \tag{504}$$

B. Higher order GUP with minimal length and maximal momentum uncertainty

Another higher order GUP* approach was proposed by Pedram [64] assuming n dimensions and implying both minimal length uncertainty and maximal observable momentum, i.e.,

$$[X_i,\, P_j] \;=\; \frac{i\,\hbar\,\delta_{ij}}{1-\beta\,p^2}, \tag{505}$$

where $p^2 = \sum_j^3 p_j\,p_j$. If the components of the momentum operator are assumed to commutate,

$$[P_i,\, P_j] \;=\; 0. \tag{506}$$

The Jacobi identity determines the commutation relations between the components of the position operator

$$[X_i,\, X_j] \;=\; \frac{2\,i\,\hbar\,\beta}{(1-\beta\,p^2)^2}\,(P_i\,X_j - P_j\,X_i), \tag{507}$$

which apparently results in a non-commutative geometric generalization of the position space. In order to fulfil these commutation relations, the position and momentum operators in the momentum space representation can be written as

$$X_i\,\phi(p) \;=\; \frac{i\,\hbar}{1-\beta\,p^2}\partial_{p_i}\phi(p), \tag{508}$$

$$P_j\,\phi(p) \;=\; p\,\phi. \tag{509}$$

In one dimension, the symmetricity condition of the position operator implies the following modified completeness relation with a domain varying from $-1/\sqrt{\beta}$ to $+1/\sqrt{\beta}$ [64] (apparently differs from KMM [29]):

$$\langle \phi|\psi\rangle \;=\; \int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}} dp(1-\beta p^2)\phi^*(p)\psi(p). \tag{510}$$

Furthermore, the scalar product of the momentum eigenstates will be changed to

$$\langle p|p'\rangle \;=\; \frac{\delta(p-p')}{(1-\beta p^2)}. \tag{511}$$

Also, the particle's momentum is bounded from above

$$P_{max} \;=\; \frac{1}{\sqrt{\beta}}. \tag{512}$$

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The presence of an upper bound on the momentum agrees with DSR theories [62, 108]. As we shall see, the physical observables such as energy and momentum are not only non-singular, but they are also bounded from above. The absolutely smallest uncertainty in position reads

$$(\Delta X)_{min} = \frac{3\sqrt{3}}{4} \hbar \sqrt{\beta}.$$ (513)

This new GUP* approach [64] gives an estimation for the minimal length uncertainty and the maximal observable momentum, simultaneously. It includes a quadratic term of momentum and apparently assures non-commutative geometry. The maximal observable momentum agrees with the one estimated in DSR [62, 108]. If the binomial theorem is applied on this GUP* approach, the GUP approach which was predicted in the String theory [46, 100], black hole physics [44, 66] can be reproduced.

It is worthwhile to notice that his new GUP* approach [64] does not agree with the commutators relation which was predicted in DSR [62, 108]. The latter contains a linear term of momentum that is responsible for the existence of maximal observable momentum.

C. Comparison between three GUP approaches

Tab. VIII C summarizes an extensive comparison between the GUP approaches of KMM [29], Ali, Das, Vagenas (ADV) [63, 113] and Pedram [64]. The minimum position uncertainty varies from $\hbar\,\alpha$ or $\hbar\,\sqrt{\beta}$ (both are equivalent) and $\sqrt{27}\,\hbar\,\alpha/4$, respectively. There is a maximum momentum uncertainty in ADV, although, it is wrongly called maximum momentum. The maximum momentum diverges in KMM, while it remains finite, $1/4\alpha$ and $1/\sqrt{\beta}$, respectively, in ADV and Pedram. The momentum operator and resulting geometry remain unchanged in all approaches. The position operator characterizes the different approaches. The maximum localised state slightly varies. The resulting energy (wavelength) related to quasiposition and wavefuction are very characteristic.

Comparsion	KMM [29]	ADV [63, 113]	Pedram [64]
Algebra $[x,\,p]$	$i\hbar\left(1+\beta p^2\right)$	$i\hbar\left(1-\alpha p+2\alpha^2 p^2\right)$	$\frac{i\hbar}{1-\beta p^2}$
$(\Delta x)_{min}$	$\hbar\sqrt{\beta}$	$\hbar\alpha$	$\frac{3\sqrt{3}}{4}\hbar\beta$
$(\Delta p)_{max}$	-	$\frac{M_{pl}c}{\alpha_0}$	-
P_{max}	Divergence	$\left(\frac{1}{4\alpha}\right)$	$\left(\frac{1}{\sqrt{\beta}}\right)$
$\mathbf{P}.\phi(p)$	$p\phi(p)$	$p\phi(p)$	$p\phi(p)$
$\mathbf{X}.\phi(p)$	$i\hbar\left(1+\beta p^2\right)\partial_p\phi(p)$	$i\hbar\left(1-\alpha p+2\alpha^2 p^2\right)\partial_p\phi(p)$	$\frac{i\hbar}{1-\beta p^2}\partial_p\phi(p)$
Geometry	$[x_i,x_j]\neq 0$	$[x_i,x_j]\neq 0$	$[x_i,x_j]\neq 0$
$\langle\frac{p^2}{2m}\rangle_{max-localize-state}$	$\frac{1}{2m\beta}$	$\frac{1}{32m\alpha^2}$	$\frac{3}{2m\beta}$
$(E(\lambda)$ or $\lambda(E))_{quasi-position}$	$\frac{1}{2m\beta}\left(\tan\frac{2\pi\hbar\sqrt{\beta}}{\lambda}\right)^2$	$\frac{2}{m\alpha^2}\left(\frac{\tan(\frac{\hbar\alpha\sqrt{7}}{\lambda})\pi}{\tan(\frac{\hbar\alpha\sqrt{7}}{\lambda})\pi+\sqrt{7}}\right)^2$	$\frac{2\pi\hbar}{(1-\frac{2}{3}m\beta E)\sqrt{2mE}}$
λ_0 of wavefuntion	$4\hbar\sqrt{\beta}$	$\frac{\pi\alpha\hbar\sqrt{7}}{\left(\tan^{-1}\frac{\eta}{3}+\tan^{-1}\frac{4\alpha p_{pl}-1}{\sqrt{7}}\right)}$	$3\pi\hbar\sqrt{\beta}$

Tab. I: A comparison between the main features of the GUP approaches that were proposed by KMM [29], ADV [63, 113] and Pedram [64].

IX. EQUIVALENCE PRINCIPLES AND KINETIC ENERGY

In this section, we review the argumentation against the GUP approaches. It seems that dogmatic concepts would derive others to stand against the implementation of GUP. But, any scientific discussion is concentrated on abstract argumentation. We start with the *"equivalence principle"*, which is one of the five principles forming the basis of GR, where the motion of a gravitational test particle in a gravitational field should be independent on the mass and composition of the test particle [291]. On the other hand, when taking into consideration the Strong (SEP) [66] and the Weak Equivalence Principle (WEP) [66], the gravitational field is coupled to almost everything [291].

The GUP effects on the equivalence principles shall be studied in section IX A. The universality of the gravitational redshif shall be discussed in section IX A 1. The law of reciprocal action shall be given in section IX A 2. This leads to study of the universality of the free fall, section IX A 3. Finally, section IX B is devoted to the kinetic energy of composite system.

A. GUP effects on equivalence principles

The Newtonian mechanics in a gravitational field apparently fulfils WEP effects [66, 302]. This is nothing but the equivalence of inertial and the gravitational masses effects. The QM apparently does not violate the equivalence principle effects. This can be shown from studying the Heisenberg equations of motion. For simplicity, let us consider one-dimensional motion with the Hamiltonian of a test particle. A macroscopic body considered as a point-like particle of mass m embedded a uniform gravitational field. The Hamiltonian is given by

$$H = \frac{p^2}{2m} - m\,g\,x. \tag{514}$$

The gravitational field is characterized by the acceleration g, which is directed along the x axis. Note that the inertial mass m (in the first term) is equal to the gravitational mass m (in the second one). Let us consider the classical limit using the correspondence between the commutator in QM and the Poisson bracket in classical mechanics

$$\{A,\,B\} \;=\; \frac{1}{i\hbar}[A,\,B]. \tag{515}$$

The Heisenberg equations of motion reads

$$\dot{x} \;=\; \{x,\,H\} = \{x,\,p\}\frac{\partial H}{\partial p} = \frac{p}{m}, \tag{516}$$

$$\dot{p} \;=\; -\{p,\,H\} = -\{x,\,p\}\frac{\partial H}{\partial p} = m\,g. \tag{517}$$

These two equations ensure that the momentum at the quantum level is given as $p = m\,\dot{x}$ and the acceleration $\ddot{x}$ is mass-independent as the case in the classical physics. It is obvious that the equivalence principle is preserved at the quantum level, where $\{x,\,p\}$ is unity.

According to KMM algebra, Eq. (76), the modified Heisenberg equations of motion read

$$\dot{x} \;=\; \{x,\,H\} = \{x,\,p\}\frac{\partial H}{\partial p} = \frac{p}{m}\left(1 + \beta\,p^2\right), \tag{518}$$

$$\dot{p} \;=\; -\{p,\,H\} = -\{x,\,p\}\frac{\partial H}{\partial p} = mg\left(1 + \beta\,p^2\right). \tag{519}$$

In deformed space, the trajectory of the point-like mass in the gravitational field depends on the particle's mass [291]. If we suppose that the deformation parameter is the same for all bodies, then the equivalence principle should be violated. The acceleration $\dot{x}$ is not mass-independent because of the mass-dependence through the momentum p. Therefore,

the equivalence principle is dynamically violated because of GUP [292, 294]. In other words, any added term of momentum to the Heisenberg relation leads to violating the equivalence principle [292, 294]. The predicted violations of the equivalence principle are compared to experimental observations for the universality [293] of the gravitational redshift, law of reciprocal action and universality of free fall.

The bounds derived for β, the GUP parameter as given in KMM, are tighter than those obtained from quantum mechanical predictions given in Ref. [74]. Keeping the same level of approximation, the modified geodesic equation is given as [294]

$$\frac{d^2\,x^i}{dt^2} \approx \frac{1}{2}\left(1 + 5\,\beta_m\right)\partial^i\,h_{00}, \tag{520}$$

where $\beta_m = \beta\,m^2/2$.

1. Universality of gravitational redshift

In the conventional case, $\beta_m = 0$, Eq. (520), and from the Newtonian equation of the gravitational potential at a distance r from a mass M [293, 294],

$$\frac{d^2x}{dt^2} = -\nabla\phi, \tag{521}$$

$$\phi = -G\frac{M}{r}. \tag{522}$$

As given in Ref. [295], $h_{00} = -2\phi \implies g_{00} = -(1 + 2\phi)$, In the present case, we have $(1 + 5\beta_m)h_{00} = -2\phi \implies h_{00} \approx -2\phi(1 - 5\beta_m) \implies g_{00} = -(1 + 2\phi(1 - 5\beta_m))$ [294].

A test of the universal influence of the gravitational field on clocks based on different physical principles requires clock comparison during their common transport through different gravitational potentials [293, 294]. There is a large variety of clocks which can be compared in [293]:

- light clocks (optical resonators),

- various atomic clocks,

- various molecular clocks,

- gravitational clocks based on the revolution of planets or binary systems,

- the Earth rotation,

- pulsar clocks based on the spin of stars

- clocks based on decay of particles.

In order to measure the gravitational redshift effect [295], one needs two observation points, say x_1, x_2 and consider a given atomic transition [293, 294].

The ratio of frequencies ν_2 and ν_1, where ν_2 refers to a light beam coming from x_2 and goes to x_1 and ν_1 refers to the end position, i.e., the position of observation, is given as [97, 293, 294]

$$\frac{\nu_2(x_2)}{\nu_1(x_1)} = \left(\frac{g_{00}(x_2)}{g_{00}(x_1)}\right)^{1/2} = \left(\frac{1 + 2(1 - 5\beta_m)\phi(x_2)}{1 + 2(1 - 5\beta_m)\phi(x_1)}\right)^{1/2} \approx 1 + (1 - 5\beta_m)\left(\phi(x_2) - \phi(x_1)\right). \quad (523)$$

On a phenomenological level, the comparison of two collocated clocks A and B was given in Ref. [295]

$$\frac{\nu_A(x_2)}{\nu_A(x_1)} = 1 + (1 - 5(\beta_m)_A)\left(\phi(x_2) - \phi(x_1)\right), \quad (524)$$

$$\frac{\nu_B(x_2)}{\nu_B(x_1)} = 1 + (1 - 5(\beta_m)_B)\left(\phi(x_2) - \phi(x_1)\right). \quad (525)$$

Combining these two equations leads to

$$\frac{\nu_A(x_2)}{\nu_B(x_2)} \approx (1 - 5\left[(\beta_m)_A - (\beta_m)_B\right])\left(\phi(x_2) - \phi(x_1)\right)\frac{\nu_A(x_1)}{\nu_B(x_1)}. \quad (526)$$

A mismatch of the frequency ratios will signal a violation of the equivalence principle. According to recent observational result, $|\alpha_{Hg} - \alpha_{Cs}| \leq 5 \times 10^{-6}$, where α_{Hg}, α_{Cs} stand for clock dependent parameters for Mercury and Cesium [293, 294]. In our case $\alpha_{Hg} = 5\beta m_{Hg}^2$ and $\alpha_{Cs} = 5\beta m_{Cs}^2$ [293, 294]. Conventionally, one considers $\beta = \beta_0/M_{pl}^2$ with $\beta_0 = 1$ [74], in which the mismatch turns to be

$$\frac{m_{Hg}^2 - m_{Cs}^2}{M_{pl}^2} \approx 10^{-32}. \quad (527)$$

This signal is very small. Another interpretation [74] is to consider an upper bound for

$$\beta_0 \leq \left(\frac{10^{-9}}{10^{-25}}\right)^2 \times 10^{-6} \approx 10^{26}. \quad (528)$$

Where β_0 be the upper bound of the GUP paramter. This is below the upper bound of $\beta_0 \leq 10^{34}$. On the other hand, this is compatible with the electroweak scale but much tighter than the bounds suggested in Ref. [74] from Lamb shift and Landau level measurements [74, 90], but also weaker than $\beta_0 \leq 10^{21}$ derived from scanning tunnelling microscope which is the current measurement [74].

2. Law of reciprocal action

A key model of the violation of the reciprocal action law is the estimation of the difference in active and passive gravitational masses [293, 294]. This is accessed through the motion of active and passive masses and their possible non-equality. The active mass m_A is the source of the gravitational field, where the gravitational potential $\Delta U = 4\pi m_a \delta(x)$, whereas the passive mass m_p reacts to it, $m_i \ddot{x} = m_p \nabla U(x)$. Here, m_i is the inertial mass and x the position of the particle. The equations of motion for a gravitationally bound of a two-body system are given as [293, 294]

$$m_{1i}\,\ddot{x}_1 = G\,m_{1p}\,m_{2a}\,\frac{x_2 - x_1}{|x_2 - x_1|^3}, \tag{529}$$

$$m_{2i}\,\ddot{x}_2 = G\,m_{2p}\,m_{1a}\,\frac{x_1 - x_2}{|x_1 - x_2|^3}, \tag{530}$$

where the indices 1 and 2 refers to first and second particles. The motion of the center-of-mass coordinate reads [293, 294].

$$X = \frac{m_{1i}\,x_1 + m_{2i}\,x_2}{m_{1i} + m_{2i}}, \tag{531}$$

$$\ddot{X} = G\,\frac{m_{1p}\,m_{2p}}{m_{1i} + m_{2i}}\,C_{21}\,\frac{x_2 - x_1}{|x_2 - x_1|^3}, \tag{532}$$

$$C_{21} = \frac{m_{2a}}{m_{2p}} - \frac{m_{1a}}{m_{1p}}. \tag{533}$$

Thus, if $C_{21} \neq 0$, then the active and passive masses are different and the center-of-mass shows a self-acceleration along the direction of x. This is a violation of Newtonian *actio-equals-reactio*-law [293, 294]. A limit has been derived by Lunar Laser Ranging (LLR) [293, 294, 296]. No self-acceleration of the moon has been observed yielding a limit of $|C_{Al-Fe}| \leq 7 \times 10^{-13}$ [293, 294, 296]. This provides a considerably tighter bound $\beta_0 \leq 10^{19}$ than the one provided by gravitational redshift (see above) and other earlier bounds [74].

3. Universality of free fall

According to GR, the neutral free particles follow the geodesic and hence the motion is independent of the nature of the neutral particle. Its validity is tested by measuring the so-called Eötvös parameter [293, 294].

$$\eta = \frac{(g_A - g_B)}{\frac{1}{2}(g_A + g_B)}, \tag{534}$$

where g_A and g_B are accelerations of two particles A and B, respectively, in the "same" gravitational field. A non-zero η signals violation of universality of free fall. But in the present case, the active mass gets different corrections for A and B and in turn the gravitational field perceived by them is not the same. In the field of mass M, the acceleration of A and B can be given as

$$g_A = (1 - 5(\beta_m)_A) g, \tag{535}$$

$$g_B = (1 - 5(\beta_m)_B) g. \tag{536}$$

Thus, we find that

$$\eta = \frac{(1 - 5(\beta_m)_A) - (1 - 5(\beta_m)_B)}{\frac{1}{2}[(1 - 5(\beta_m)_B) + (1 - 5(\beta_m)_B)]} \approx \beta_0 \frac{5m_B^2 - 5m_A^2}{M_{p\ell}^2}. \tag{537}$$

The torsion pendulum leads to $\eta \leq 2 \times 10^{-13}$ which yields once again $\beta \leq 10^{19}$ [74, 90]. It should be noted that the results will not hold for macroscopic bodies, due to $\beta_m \ll 1$ [293, 294].

So far, we conclude that the minimally extended point-particle-model satisfying GUP approach leads to a modified geodesic equation. At low energy and in the limit of weak gravity, as considered in previous sections, this effect translates into a modified gravitational potential. Furthermore, the correction depends on the test particle energy or its mass [294]. This leads to a violation in the equivalence principle. Results were predicted for the violation in the contexts of gravitational redshift, law of reciprocal action and universality of free fall. The comparison with experimental results predicts improved bounds for the GUP parameter [294].

B. Kinetic energy of composite system

We recall that *the kinetic energy has the additivity property and does not depend on composition of a body but only on its mass.* Then, we consider N particles with masses m_i and deformation parameters γ_i. It is equivalent to the situation when the macroscopic body is divided into N parts, which can be treated as point-like particles with the corresponding masses and deformation parameters [292]. We consider the case when each particle of the system moves with the same velocity as the whole system. The kinetic energy can be given as a function of velocity. From the relation between velocity and momentum, Eq. (518), in

the first approximation over γ, we find [292]

$$P = m\dot{X}\left(1 - \gamma m^2 \dot{X}^2\right). \tag{538}$$

Then, in the first order approximation of γ, the kinetic energy is given as function of velocity

$$K.E = \frac{1}{2}m\dot{X}^2 - \gamma m^3 \dot{X}^4. \tag{539}$$

At the quantum level, we show that the motion of the center-of-mass of a composite system in deformed space is governed by an effective parameter. In other words, the deformation parameter for a macroscopic body is given as

$$\gamma = \sum_j \mu_i^3 \gamma_i, \tag{540}$$

where $\mu_j = m_i / \sum_j m_i$ and γ_i are the masses and deformation parameters of particles of composite system (body).

C.   Conclusions

The presence of GUP effects implying some noise in GR, where the equivalence principle should be postulates of it. GUP introduces a mass term to the geodesic equation which violates the equivalence principle. In section IX, various observations in GR were studied. These should be estimated again the presence of these effects, and compared to the upper bounds of the GUP approach. The latter was estimated in these observations with the upper bounds [74, 90]. The scanning tunnelling microscope [74] appears differently [294]. This means that the violation of the equivalence principle does not support the idea of modification of the Heisenberg principle. The presence of GUP effect corrects the kinetic energy, which as known is independent on the composition of the system.

X. DISCUSSION

The quantum aspects of the gravitational fields can emerge in the limit, in which the different types of interactions, like strong weak and electromagnetism can be distinguished from each other. In the string theory, the particles are conjectured to have their origin in the fundamental strings. This fundamental scale is nothing but the string length, which is also supposed to be in order of the Planck length. The current researches of the quantum problems in the presence of gravitational field at very high energy near to the Planck scale implies new physical laws and even corrections of the spacetime of our Universe. The quantum field theory in curved background can be normalized by introducing a minimal observable length as an effective cutoff in ultraviolet domain. The string cannot probe distances smaller than its own length.

We review different approaches of GUP, that predict the existence of a minimal length uncertainty. The non-zero length expresses a non-zero state in the description of the Hilbert space and is able to fulfil the non-commutative geometry. These should have impacts on discreteness and quantization of space and on aspects related to the quantum field theory. The elicitation of the minimal length from various experiments, such as string theory, black hole physics and loop quantum gravity, imitates the quantum gravity. All of them predict corrections to the quadratic momentum in Heisenberg algebra. Many authors represent such algebra under modification in position operator which agrees with the Hilbert space representation and takes into consideration the states of space (eigenvectors) corresponding to the energy (eigenvalue). Others represent such modified algebra by modification in the linear momentum. This is motived by momentum modification at very high energy, which is supposed to fulfil the Hilbert space representation but also approves the idea of modified dispersion relation of the energy-momentom relation.

The Doubly Special Relativity should provide a GUP approach with an additional term reflecting the possibility to deduce information about the maximum measurable momentum. This new term and the one related to the minimal uncertainty on position are - in modified Heisenberg algebra - of first order of momentum. Some authors suggest a combination of all previously-proposed GUP-approaches in one concept, as anticipated in DSR and the String theory, black hole physics and Loop quantum gravity. Others prefer to revise the GUP of minimal length in order to overcome some obstacles. Another suggestion for GUP

dependent on Feynman propagator should display an exponential ultra-violet cutoff. All of these verify the predication of minimal length at very high energy, in spite of the different physical expression or the algebraic representation of Heisenberg principle. In summary, we have different GUP approaches with a lot of applications in various branches of physics.

An unambiguous experiment evidence to ensure these ideas is till missing. Some physicists prefer to deny due to their convention. Some have objections. Here we review both points-of-views. Value of the GUP parameter remains another puzzle to be verified. For example, the principles of GR developed by Einstein are seen as obstacles against the interpretation of the GUP approaches. It is thought that they violate the equivalence principle, for instance. In thermodynamics, the natural property of the kinetic energies of particles is assumed to be violated under consideration of these approaches. As a reason, symmetries can be broken in quantum field theory. The value of the Keplerian orbit and the correction of the continuity equation for some fields are no longer correct. In the present review, we have summarized all these proposals and discussed their difficulties and applications. We aimed to elucidate some of these proposals. On the other hand, from various *gedanken* experiments that have been designed to measure the area of the apparent horizon of a black hole in QG, the uncertainty relation is found preformed. The modified Heisenberg algebra, which was suggested to investigate GUP, introduces a relation between QG and Poincare algebra. Under the effect of GUP in an n-dimension space, it is found that even the gravitational constant G and the Newtonian law of gravity are subject of modifications. The interpretation of QM through a quantization model formulated in 8-dimensional manifold implies the existence of an upper limit in the accelerated particles. Nevertheless, GUP approaches given in forms of quadratic and linear terms of momenta assume that the momenta approach maximum value at very high energy (Planck scale).

The Heisenberg uncertainty principle expresses one of the fundamental properties of the quantum systems. Accordingly, there should be a fundamental limit of the accuracy with which certain pairs of physical observables, such as the position and momentum of particle, can be measured, simultaneously. In other words, the more precisely one observable is measured, the less precise the other one can be determined. In QM, the physical observables are described by operators acting on the Hilbert space of states.

Various examples can be mentioned to support the phenomena that uncertainty principle would be affected by QG. In context of polymer quantization, the commutation relations

are given in terms of the polymer mass scale. The standard commutation relations are conjectured to be changed or better expressed to be generalized at Planck length. Such modifications are supposed to play an essential role in the quantum gravitational corrections at very high energy. Accordingly, the standard uncertainty relation of QM should be replaced by a gravitational uncertainty relation having a minimal observable length of the order of the Planck length. On the other hand, the detectability of quantum space-time foam with gravitational wave interferometers has been addressed. The limited measurability of the smallest quantum distances has been criticized. An operative definition for the quantum distances and the elimination of the contributions from the total quantum uncertainty were given. Barrow applied Wigner inequalities in order to describe the quantum constrains on the black hole lifetime. It was found that the black hole running time should be correspondent to the Hawking lifetime, which is to be calculated under the assumption that the black hole is a black body. Therefore, the utilization of Stefan-Boltzmann law is eligible. It is found that the Schwarzschild radius of black hole is correspondent to the constrains on Wigner size. Furthermore, the information processing power of a black hole is estimated by the emitted Hawking radiation.

There are several observations supporting GUP approaches and offer a valuable possibility to study the influence of the minimal length on the properties of a wide range of physical systems, especially at quantum scale. The effects of linear GUP approach have been studied on compact stars, Newtonian law of gravity, inflationary parameters and thermodynamics of the early Universe, Lorentz invariance violation and measurable maximum energy and minimum time interval. It was observed that GUP can potentially explain the small observed violations of the weak equivalence principle in neutron interferometry experiments and also predicts a modified invariant phase space which is relevant to LT. It is suggested that GUP can be measured directly in Quantum Optics Lab.

For example, the experimental tests of Lorentz invariance become more accurate. A tiny Lorentz-violating term can be added to the conventional Lagrangian, then experiments should test Lorentz invariance by setting upper bounds to the coefficients of this term, where the velocity of light c should differ from the maximum attainable velocity of a material body. This small adjustment of the speed of light leads to focus of the modification of the energy-momentum relation and to add possible δv to the vacuum dispersion relation which could be sensitive to a type of candidates for the quantum gravity effect that has been recently consid-

ered in the particle physics literature. In additional to that, the possibility that the relation connecting energy and momentum in special relativity may be modified at Planck scale, because of the threshold anomalies of ultra-high energy cosmic ray (UHECR) is conventionally named as *Modified Dispersion Relations (MDRs)*. This can provide new and many sensitive tests for the special relativity. Successful searches would reveal a surprising connection between the particle physics and cosmology. The speed of light not limited to that but many searchers about the modification of the energy-momentum conservations laws of interaction such as pion photo-production by inelastic collisions of cosmic-ray nucleons with the cosmic microwave background and higher energy photon propagating in the intergalactic medium which can suffer inelastic impacts with photons in the Infra-Red background resulting in the production of an electron-positron pair.

The systematic study of the black hole radiation and the correction due to entropy/area relation gain the attention of theoretical physicists. For instance, there are nowadays many methods to calculate Hawking radiation. Nevertheless, all results show that the black hole radiation is very close to the black body spectrum. This conclusion raised a very difficult question whether the information is conserved in the black hole evaporation process? The black hole information paradox has been puzzled problem. The study of thermodynamic properties of black holes in space-times is therefore a very relevant and original task. For instance, based on recent observation of supernova, the cosmological constant may be positive. The possible corrections can be calculated by means of approaches to the quantum gravity. Through the comparison of the corrected results obtained from this alternative approaches, it can be shown that suitable choice of the expansion coefficients in the modified dispersion relations leads to the same results in the GUP approach.

The existence of minimal length and maximum momentum accuracy is preferred by various physical observations. Thought experiments have been designed to illustrate influence of the GUP approaches on the fundamental laws of physics, especially at Planck scale. The concern about the compatibility with equivalence principles, universality of gravitational redshift and free fall and reciprocal action law should be addressed. The value of the GUP parameters remains a puzzle to be verified. Furthermore, confronting GUP approaches to further applications would elaborate essential properties. The ultimate goal would be the empirical evidence that the same is indeed quantized and its fundamental is given by the minimal length accuracy. If the current technologies would not able to implement this pro-

posal, we are left with the empirical prove that the modifications of various physical systems can be estimated, accurately. To this destination, we should try to verify the given approaches, themselves. We believe that the compatibility with MDR would play the role of the Rosetta stone translating GUP in energy-momentum relations. The latter would have cosmological and astrophysical observations.

Acknowledgement

Appendix A: Solution of Eq. (485)

Equation (485) can be solved with respect to H. The first two real roots read

$$H = \pm \left(-A + \frac{D}{3\sqrt[3]{2}B^2} - \frac{2\sqrt[3]{2}C\rho}{D} + \frac{\sqrt[3]{2}}{3B^2D} + \frac{1}{3B^2} \right)^{1/2}, \tag{A1}$$

where $A = k/a^2$, $B = \alpha\sqrt{\mu}(3\omega+1)/3$ and $C = 8\pi G/3$. While

$$D = \sqrt[3]{27C^2\rho^2B^4 - 18C\rho B^2 + 3\sqrt{3}\sqrt{27B^8C^4\rho^4 - 4B^6C^3\rho^3} + 2} \tag{A2}$$

is real and strongly depends on ρ. H remains real as long as

$$\frac{D}{3\sqrt[3]{2}B^2} + \frac{\sqrt[3]{2}}{3B^2D} + \frac{1}{3B^2} > A + \frac{2\sqrt[3]{2}C\rho}{D}, \tag{A3}$$

which is apparently valid, because of the denominator B. Fig. fig:AppndxH shows the Hubble parameter H as a function of energy density ρ, positive root in Eq. (A1). The three curves represents the three values of the curvature parameter k, 1 (dotted curve), 0 (solid curve) and -1 (dashed curve). The region of discontinuity reflects rho-values, at which the square root, Eq. (A1), gets imaginary. In calculating these curves, we use $a = G = \alpha = 1$, $\omega = 1/3$ and $\mu = (2.82/\pi)^2$. It is apparent, that the dependence of H on ρ is not monotonic. Reducing ρ, which is corresponding to increasing the cosmic time t, is accompanied with reducing H, as well. Then, starting from a certain value of ρ (and indirectly of t), H increases

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with the further decrease in ρ. In other words, the rate of expansion reduces. Then, then rate rapidly increases. The rate strongly depends on geometry of the Universe, k.
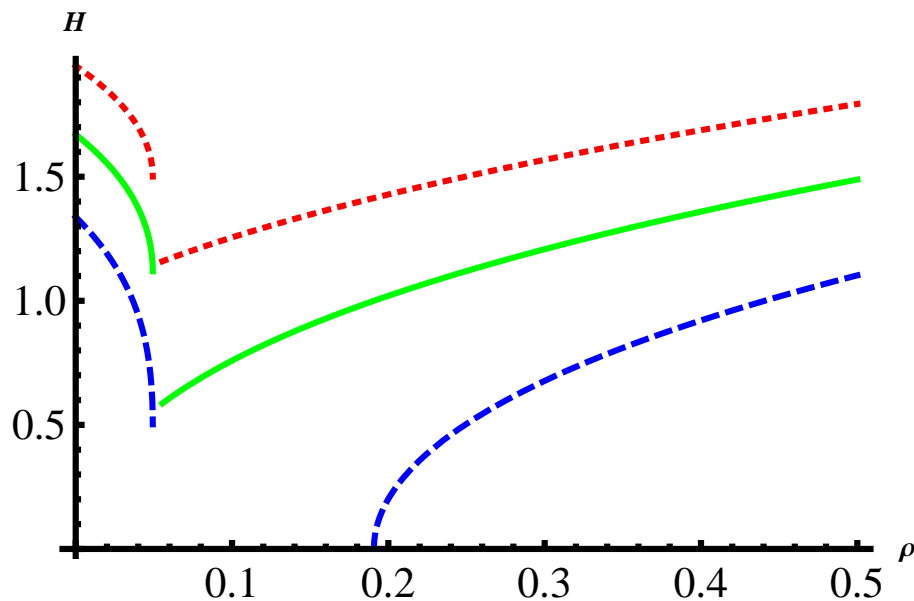

Fig. 16: The Hubble parameter H is given as a function of the energy density ρ. The three curves represents the three values of curvature parameter k, 1 (dotted), 0 (solid) and -1 (dashed) from top to bottom. The discontinuity reflects the region, in which the square root gets imaginary.

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