Geometry and the Quantum: Basics

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Abstract

Motivated by the construction of spectral manifolds in noncommutative geometry, we introduce a higher degree Heisenberg commutation relation involving the Dirac operator and the Feynman slash of scalar fields. This commutation relation appears in two versions, one sided and two sided. It implies the quantization of the volume. In the one-sided case it implies that the manifold decomposes into a disconnected sum of spheres which will represent quanta of geometry. The two sided version in dimension 4 predicts the two algebras $M_2(\mathbb{H})$ and $M_4(\mathbb{C})$ which are the algebraic constituents of the Standard Model of particle physics. This taken together with the noncommutative algebra of functions allows one to reconstruct, using the spectral action, the Lagrangian of gravity coupled with the Standard Model. We show that any connected Riemannian Spin 4-manifold with quantized volume > 4 (in suitable units) appears as an irreducible representation of the two-sided commutation relations in dimension 4 and that these representations give a seductive model of the "particle picture" for a theory of quantum gravity in which both the Einstein geometric standpoint and the Standard Model emerge from Quantum Mechanics. Physical applications of this quantization scheme will follow in a separate publication.

Contents

1 Introduction

The goal of this paper is to reconcile Quantum Mechanics and General Relativity by showing that the latter naturally arises from a higher degree version of the Heisenberg commutation relations. One great virtue of the standard Hilbert space formalism of quantum mechanics is that it incorporates in a natural manner the essential “variability” which is the characteristic feature of the Quantum: repeating twice the same experiment will generally give different outcome, only the probability of such outcome is predicted, the

various possibilities form the spectrum of a self-adjoint operator in Hilbert space. We have discovered a geometric analogue of the Heisenberg commutation relations $[p, q] = i\hbar$. The role of the momentum p is played by the Dirac operator. It takes the role of a measuring rod and at an intuitive level it represents the inverse of the line element ds familiar in Riemannian geometry, in which only its square is specified in local coordinates. In more physical terms this inverse is the propagator for Euclidean Fermions and is akin to an infinitesimal as seen already in its symbolic representation in Feynman diagrams where it appears as a solid (very) short line .
The role of the position variable q was the most difficult to uncover. It has been known for quite some time that in order to encode a geometric space one can encode it by the algebra of functions (real or complex) acting in the same Hilbert space as the above line element, in short one is dealing with "spectral triples". Spectral for obvious reasons and triples because there are three ingredients: the algebra $\mathcal{A}$ of functions, the Hilbert space $\mathcal{H}$ and the above Dirac operator D. It is easy to explain why the algebra encodes a topological space. This follows because the points of the space are just the characters of the algebra, evaluating a function at a point $P \in X$ respects the algebraic operations of sum and product of functions. The fact that one can measure distances between points using the inverse line element D is in the line of the Kantorovich duality in the theory of optimal transport. It takes here a very simple form. Instead of looking for the shortest path from point P to point P' as in Riemannian Geometry, which only can treat path-wise connected spaces, one instead takes the supremum of $|f(P) - f(P')|$ where the function f is only constrained not to vary too fast, and this is expressed by asking that the norm of the commutator $[D, f]$ be ≤ 1. In the usual case where D is the Dirac operator the norm of $[D, f]$ is the supremum of the gradient of f so that the above control of the norm of the commutator $[D, f]$ means that f is a Lipschitz function with constant 1, and one recovers the usual geodesic distance. But a spectral triple has more information than just a topological space and a metric, as can be already guessed from the need of a spin structure to define the Dirac operator (due to Atiyah and Singer in that context) on a Riemannian manifold. This additional information is the needed extra choice involved in taking the square root of the Riemannian ds^2 in the operator theoretic framework. The general theory is K-homology and it naturally introduces decorations for a spectral triple such as a chirality operator γ in the case of even dimension and a charge conjugation operator J which is an antilinear isometry of $\mathcal{H}$ fulfilling commutation relations with

D and γ which depend upon the dimension only modulo 8. All this has been known for quite some time as well as the natural occurrence of gravity coupled to matter using the spectral action applied to the tensor product $\mathcal{A} \otimes A$ of the algebra $\mathcal{A}$ of functions by a finite dimensional algebra A corresponding to internal structure. In fact it was shown in [4] that one gets pretty close to zooming on the Standard Model of particle physics when running through the list of irreducible spectral triples for which the algebra A is finite dimensional. The algebra that is both conceptual and works for that purpose is

$$A = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

where $\mathbb{H}$ is the algebra of quaternions and M_k the matrices. However it is fair to say that even if the above algebra is one of the first in the list, it was not *uniquely* singled out by our classification and moreover presents the strange feature that the real dimensions of the two pieces are not the same, it is 16 for $M_2(\mathbb{H})$ and 32 for $M_4(\mathbb{C})$.

One of the byproducts of the present paper is a full understanding of this strange choice, as we shall see shortly.

Now what should one beg for in a quest of reconciling gravity with quantum mechanics? In our view such a reconciliation should not only produce gravity but it should also naturally produce the other known forces, and they should appear on the same footing as the gravitational force. This is asking a lot and, in the minds of many, the incorporation of matter in the Lagrangian of gravity has been seen as an unnecessary complication that can be postponed and hidden under the rug for a while. As we shall now explain this is hiding the message of the gauge sector which in its simplest algebraic understanding is encoded by the above algebra $A = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. The answer that we discovered is that the package formed of the 4-dimensional geometry together with the above algebra appears from a very simple idea: to encode the analogue of the position variable q in the same way as the Dirac operator encodes the components of the momenta, just using the Feynman slash. To be more precise we let $Y \in \mathcal{A} \otimes C_\kappa$ be of the Feynman slashed form $Y = Y^A \Gamma_A$, and fulfill the equations

$$Y^2 = \kappa, \qquad Y^* = \kappa Y \tag{1}$$

Here $\kappa = \pm 1$ and $C_\kappa \subset M_s(\mathbb{C})$, $s = 2^{n/2}$, is the real algebra generated by $n+1$ gamma matrices Γ_A, $1 \leq a \leq n+1$[1]

$$\Gamma_A \in C_\kappa, \quad \left\{\Gamma^A, \Gamma^B\right\} = 2\kappa\, \delta^{AB}, \; (\Gamma^A)^* = \kappa \Gamma^A$$

[1] It is $n+1$ and not n where Γ_{n+1} is up to normalization the product of the n others.

The one-sided higher analogue of the Heisenberg commutation relations is

$$\frac{1}{n!}\langle Y\,[D,Y]\cdots[D,Y]\rangle = \sqrt{\kappa}\,\gamma \quad (n \text{ terms } [D,Y]) \tag{2}$$

where the notation $\langle T\rangle$ means the *normalized* trace of $T = T_{ij}$ with respect to the above matrix algebra $M_s(\mathbb{C})$ ($1/s$ times the sum of the s diagonal terms T_{ii}). We shall show below in Theorem 1 that a solution of this equation exists for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ associated to a Spin compact Riemannian manifold M (and with the components $Y^A \in \mathcal{A}$) if and only if the manifold M breaks as the disjoint sum of spheres of unit volume. This breaking into disjoint connected components corresponds to the decomposition of the spectral triple into irreducible components and we view these irreducible pieces as quanta of geometry. The corresponding picture, with these disjoint quanta of Planck size is of course quite remote from the standard geometry and the next step is to show that connected geometries of arbitrarily large size are obtained by combining the two different kinds of geometric quanta. This is done by refining the one-sided equation (2) using the fundamental ingredient which is the real structure of spectral triples, and is the mathematical incarnation of charge conjugation in physics. It is encoded by an anti-unitary isometry J of the Hilbert space $\mathcal{H}$ fulfilling suitable commutation relations with D and γ and having the main property that it sends the algebra $\mathcal{A}$ into its commutant as encoded by the *order zero* condition : $[a, JbJ^{-1}] = 0$ for any $a, b \in \mathcal{A}$. This commutation relation allows one to view the Hilbert space $\mathcal{H}$ as a bimodule over the algebra $\mathcal{A}$ by making use of the additional representation $a \mapsto Ja^*J^{-1}$. This leads to refine the quantization condition by taking J into account as the two-sided equation[2]

$$\frac{1}{n!}\langle Z\,[D,Z]\cdots[D,Z]\rangle = \gamma \quad Z = 2EJEJ^{-1} - 1, \tag{3}$$

where E is the spectral projection for $\{1, i\} \subset \mathbb{C}$ of the double slash $Y = Y_+ \oplus Y_- \in C^\infty(M, C_+ \oplus C_-)$. More explicitly $E = \frac{1}{2}(1 + Y_+) \oplus \frac{1}{2}(1 + iY_-)$. It is the classification of finite geometries of [4] which suggested to use the direct sum $C_+ \oplus C_-$ of two Clifford algebras and the algebra $C^\infty(M, C_+ \oplus C_-)$. As we shall show below in Theorem 6 this condition still implies that the volume of M is quantized but no longer that M breaks into small disjoint

[2]The γ involved here commutes with the Clifford algebras and does not take into account an eventual $\mathbb{Z}/2$-grading γ_F of these algebras, yielding the full grading $\gamma \otimes \gamma_F$.

connected components. More precisely let M be a smooth connected oriented compact manifold of dimension n. Let α be the volume form (of unit volume) of the sphere S^n. One considers the (possibly empty) set $D(M)$ of pairs of smooth maps $\phi_\pm : M \to S^n$ such that the differential form[3]

$$\phi_+^\#(\alpha) + \phi_-^\#(\alpha) = \omega$$

does not vanish anywhere on M ($\omega(x) \neq 0 \ \forall x \in M$). One introduces an invariant $q(M) \subset \mathbb{Z}$ defined as the subset of $\mathbb{Z}$:

$$q(M) := \{\text{degree}(\phi_+) + \text{degree}(\phi_-) \mid (\phi_+, \phi_-) \in D(M)\} \subset \mathbb{Z}.$$

where degree(ϕ) is the topological degree of the smooth map ϕ. Then a solution of (3) exists if and only if the volume of M belongs to $q(M) \subset \mathbb{Z}$. We first check (Theorem 10) that $q(M)$ contains arbitrarily large numbers in the two relevant cases $M = S^4$ and $M = N \times S^1$ where N is an arbitrary connected compact oriented smooth three manifold. We then give the proof (Theorem 12) that the set $q(M)$ contains all integers $m \geq 5$ for any smooth connected compact spin 4-manifold, which shows that our approach encodes all the relevant geometries.
In the above formulation of the two-sided quantization equation the algebra $C^\infty(M, C_+ \oplus C_-)$ appears as a byproduct of the use of the Feynman slash. It is precisely at this point that the connection with our previous work on the noncommutative geometry (NCG) understanding of the Standard Model appears. Indeed as explained above we determined in [4] the algebra $A = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ as the right one to obtain the Standard Model coupled to gravity from the spectral action applied to the product space of a 4-manifold M by the finite space encoded by the algebra A. Thus the full algebra is the algebra $C^\infty(M, A)$ of A-valued functions on M. Now the remarkable fact is that in dimension 4 one has

$$C_+ = M_2(\mathbb{H}), \quad C_- = M_4(\mathbb{C}) \tag{4}$$

More precisely, the Clifford algebra Cliff$(+,+,+,+,+)$ is the direct sum of two copies of $M_2(\mathbb{H})$ and thus in an irreducible representation, only one copy of $M_2(\mathbb{H})$ survives and gives the algebra over $\mathbb{R}$ generated by the gamma matrices Γ^A. The Clifford algebra Cliff$(-,-,-,-,-)$ is $M_4(\mathbb{C})$ and it also

[3]We use the notation $\phi^\#(\alpha)$ for the pullback of the differential form α by the map ϕ rather than $\phi^*(\alpha)$ to avoid confusion with the adjoint of operators.

admits two irreducible representations (acting in a complex Hilbert space) according to the linearity or anti-linearity of the way $\mathbb{C}$ is acting. In both the algebra over $\mathbb{R}$ generated by the gamma matrices Γ^A is $M_4(\mathbb{C})$.

This fact clearly indicates that one is on the right track and in fact together with the above two-sided equation it unveils the following tentative "particle picture" of gravity coupled with matter, emerging naturally from the quantum world. First we now forget completely about the manifold M that was used above and take as our framework a fixed Hilbert space in which $C = C_+ \oplus C_-$ acts, as well as the grading γ, and the anti-unitary J all fulfilling suitable algebraic relations. So far there is no variability but the stage is set. Now one introduces two "variables" D and $Y = Y_+ \oplus Y_-$ both self-adjoint operators in Hilbert space. One assumes simple algebraic relations such as the commutation of C and JCJ^{-1}, of Y and JYJ^{-1}, the fact that $Y_\pm = \sum Y_A^\pm \Gamma_\pm^A$ with the Y_A commuting with C, and that $Y^2 = 1_+ \oplus (-1)_-$ and also that the commutator $[D, Y]$ is bounded and its square again commutes with both $C_\pm$ and the components Y^A, etc... One also assumes that the eigenvalues of the operator D grow as in dimension 4. One can then write the two-sided quantization equation (3) and show that solutions of this equation give an emergent geometry. The geometric space appears from the joint spectrum of the components $Y_A^\pm$. This would a priori yield an 8-dimensional space but the control of the commutators with D allows one to show that it is in fact a subspace of dimension 4 of the product of two 4-spheres. The fundamental fact that the leading term in the Weyl asymptotics of eigenvalues is *quantized* remains true in this generality due to already developed mathematical results on the Hochschild class of the Chern character in K-homology. Moreover the strong embedding theorem of Whitney shows that there is no a-priori obstruction to view the (Euclidean) space-time manifold as encoded in the 8-dimensional product of two 4-spheres. The action functional only uses the spectrum of D, it is the spectral action which, since its leading term is now quantized, will give gravity coupled to matter from its infinitesimal variation.

2 Geometric quanta and the one-sided equation

We recall that given a smooth compact oriented spin manifold M, the associated spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by the action in the Hilbert space $\mathcal{H} = L^2(M, S)$ of L^2-spinors of the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on M, and the Dirac operator D which in local coordinates is of the form

$$D = \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + \omega_\mu \right) \tag{5}$$

where $\gamma^\mu = e^\mu_a \gamma^a$ and ω_μ is the spin-connection.

2.1 One sided equation and spheres of unit volume

Theorem 1 *Let M be a spin Riemannian manifold of even dimension n and $(\mathcal{A}, \mathcal{H}, D)$ the associated spectral triple. Then a solution of the one-sided equation* (2) *exists if and only if M breaks as the disjoint sum of spheres of unit volume. On each of these irreducible components the unit volume condition is the only constraint on the Riemannian metric which is otherwise arbitrary for each component.*

Proof. We can assume that $\kappa = 1$ since the other case follows by multiplication by $i = \sqrt{-1}$. Equation (1) shows that a solution Y of the above equations gives a map $Y : M \to S^n$ from the manifold M to the n-sphere. Given n operators $T_j \in \mathcal{C}$ in an algebra $\mathcal{C}$ the multiple commutator

$$[T_1, \ldots, T_n] := \sum \epsilon(\sigma) T_{\sigma(1)} \cdots T_{\sigma(n)}$$

(where σ runs through all permutations of $\{1, \ldots, n\}$) is a multilinear totally antisymmetric function of the $T_j \in \mathcal{C}$. In particular, if the $T_i = a_i^j S_j$ are linear combinations of n elements $S_j \in \mathcal{C}$ one gets

$$[T_1, \ldots, T_n] = \mathrm{Det}(a_i^j)[S_1, \ldots, S_n] \tag{6}$$

Let us compute the left hand side of (2). The normalized trace of the product of $n+1$ Gamma matrices is the totally antisymmetric tensor

$$\langle \Gamma_A \Gamma_B \cdots \Gamma_L \rangle = i^{n/2} \epsilon_{AB\ldots L}, \quad A, B, \ldots, L \in \{1, \ldots, n+1\}$$

One has $[D,Y] = \gamma^\mu \frac{\partial Y^A}{\partial x^\mu}\Gamma_A = \nabla Y^A \Gamma_A$ where we let ∇f be the Clifford multiplication by the gradient of f. Thus one gets at any $x \in M$ the equality

$$\langle Y\,[D,Y]\cdots[D,Y]\rangle = i^{n/2}\epsilon_{AB\ldots L} Y^A \nabla Y^B \cdots \nabla Y^L \qquad (7)$$

For fixed A, and $x \in M$ the sum over the other indices

$$\epsilon_{AB\ldots L} Y^A \nabla Y^B \cdots \nabla Y^L = (-1)^A Y^A [\nabla Y^1, \nabla Y^2, \ldots, \nabla Y^{n+1}]$$

where all other indices are $\neq A$. At $x \in M$ one has $\nabla Y^j = \gamma^\mu \partial_\mu Y^j$ and by (6) the multi-commutator (with ∇Y^A missing) gives

$$[\nabla Y^1, \nabla Y^2, \ldots, \nabla Y^{n+1}] = \epsilon^{\mu\nu\ldots\lambda}\partial_\mu Y^1 \cdots \partial_\lambda Y^{n+1}[\gamma^1, \ldots, \gamma^n]$$

Since $\gamma^\mu = e^\mu_a \gamma_a$ and $i^{n/2}[\gamma_1, \ldots, \gamma_n] = n!\gamma$ one thus gets by (6),

$$\langle Y\,[D,Y]\cdots[D,Y]\rangle = n!\gamma \mathrm{Det}(e^\alpha_a)\,\omega \qquad (8)$$

where

$$\omega = \epsilon_{AB\ldots L} Y^A \partial_1 Y^B \cdots \partial_n Y^L$$

so that $\omega dx_1 \wedge \cdots \wedge dx_n$ is the pullback $Y^\#(\rho)$ by the map $Y : M \to S^n$ of the rotation invariant volume form ρ on the unit sphere S^n given by

$$\rho = \frac{1}{n!}\epsilon_{AB\ldots L} Y^A dY^B \wedge \cdots \wedge dY^L$$

Thus, using the inverse vierbein, the one-sided equation (2) is equivalent to

$$\det\left(e^a_\mu\right) dx_1 \wedge \cdots \wedge dx_n = Y^\#(\rho) \qquad (9)$$

This equation (9) implies that the Jacobian of the map $Y : M \to S^n$ cannot vanish anywhere, and hence that the map Y is a covering. Since the sphere S^n is simply connected for $n > 1$, this implies that on each connected component $M_j \subset M$ the restriction of the map Y to M_j is a diffeomorphism. Moreover equation (9) shows that the volume of each component M_j is the same as the volume $\int_{S^n}\rho$ of the sphere. Conversely it was shown in [8] that, for $n = 2, 4$, each Riemannian metric on S^n whose volume form is the same as for the unit sphere gives a solution to the above equation. In fact the above discussion gives a direct proof of this fact for all (even) n. Since all volume forms with same total volume are diffeomorphic ([17]) one gets the required result. ∎

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is then the direct sum of the irreducible spectral triples associated to the components. Moreover one can reconstruct the original algebra $\mathcal{A}$ as the algebra generated by the components Y^A of Y together with the commutant of the operators D, Y, Γ_A. This implies that a posteriori one recovers the algebra $\mathcal{A}$ just from the representation of the D, Y, Γ_A in Hilbert space. As mentioned above the operator theoretic equation (2) implies the integrality of the volume when the latter is expressed from the growth of the eigenvalues of the operator D. Theorem 1 gives a concrete realization of this quantization of the volume by interpreting the integer k as the number of geometric quantas forming the Riemannian geometry M. Each geometric quantum is a sphere of arbitrary shape and unit volume (in Planck units).

2.2 The degree and the index formula

In fact the proof of Theorem 1 gives a statement valid for any Y not necessarily fulfilling the one-sided equation (2). We use the non-commutative integral as the operator theoretic expression of the integration against the volume form $\det\left(e^a_\mu\right) dx_1 \wedge \cdots \wedge dx_n$ of the oriented Riemannian manifold M. The factor $2^{n/2+1}$ on the right comes from the factor 2 in $Y = 2e - 1$ and from the normalization (by $2^{-n/2}$) of the trace. The $⨍$ is taken in the Hilbert space of the canonical spectral triple of the Riemannian manifold.

Lemma 2 *For any $Y = Y^A \Gamma_A$, such that $Y^2 = 1$, $Y^* = Y$ one has*

$$⨍ \gamma \left\langle Y\,[D, Y]^n \right\rangle D^{-n} = 2^{n/2+1} \mathrm{degree}(Y) \tag{10}$$

Proof. This follows from (8) which implies that

$$\gamma \left\langle Y\,[D, Y] \cdots [D, Y] \right\rangle \det\left(e^a_\mu\right) dx_1 \wedge \cdots \wedge dx_n = n!\, Y^{\#}(\rho)$$

while for any scalar function f on M one has (see [7], Chapter IV,2,β, Proposition 5), with $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$ the volume of the unit sphere S^{n-1},

$$⨍ f D^{-n} = \frac{1}{n}(2\pi)^{-n} 2^{n/2} \Omega_n \int_M f \sqrt{g} dx^n$$

Thus the left hand side of (10) gives

$$⨍ \gamma \left\langle Y\,[D, Y]^n \right\rangle D^{-n} = \frac{1}{n}(2\pi)^{-n} 2^{n/2} \Omega_n n! \int_M Y^{\#}(\rho)$$

One has

$$\int_M Y^{\#}(\rho) = \mathrm{degree}(Y)\Omega_{n+1}$$

and

$$\frac{1}{n}(2\pi)^{-n} 2^{n/2} \Omega_n n! \Omega_{n+1} = 2^{n/2+1}.$$

using the Legendre duplication formula $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z)$. ■

3 Quantization of volume and the real structure J

We consider the two sided equation (3). The action of the algebra $C_+ \oplus C_-$ in the Hilbert space H splits H as a direct sum $H = H^{(+)} \oplus H^{(-)}$ of two subspaces corresponding to the range of the projections $1 \oplus 0 \in C_+ \oplus C_-$ and $0 \oplus 1 \in C_+ \oplus C_-$. The real structure J interchanges these two subspaces. The algebra C_+ acts in $H^{(+)}$ and the formula $x \mapsto Jx^* J^{-1}$ gives a right action of C_- in $H^{(+)}$. We let $Y' = iJY_- J^{-1}$ acting in $H^{(+)}$ and $\Gamma' = iJ\Gamma_- J^{-1}$ for the gamma matrices of C_-. This allows us to reduce to the following simplified situation occurring in $H^{(+)}$. We take M of dimension $n = 2m$ and consider two sets of gamma matrices Γ_A and Γ'_B which commute with each other. We consider two fields

$$Y = Y^A \Gamma_A, \ Y' = Y'^B \Gamma'_B \qquad A, B = 1, 2, \ldots, n+1 \tag{11}$$

The condition $Y^2 = 1 = Y'^2$ implies

$$Y^A Y^A = 1, \quad Y'^B Y'^B = 1 \tag{12}$$

Let $e = \frac{1}{2}(Y+1)$, $e' = \frac{1}{2}(Y'+1)$, $E = ee' = \frac{1}{2}(Z+1)$ then $Z = 2ee' - 1$ and thus

$$Z = \frac{1}{2}(Y+1)(Y'+1) - 1 \tag{13}$$

$$Z^2 = 4e^2 e'^2 - 4ee' + 1 = 1 \tag{14}$$

This means that $Z^2 = 1$ and we can use it to write the quantization condition in the form

$$\frac{1}{n!}\left\langle Z\left[D, Z\right]^n\right\rangle = \gamma \tag{15}$$

where $\langle\rangle$ is the normalized trace relative to the matrix algebra generated by all the gamma matrices Γ_A and Γ'_B.

3.1 The normalized traces

More precisely we let Mat_+ be the matrix algebra generated by all the gamma matrices Γ_A and Mat_- be the matrix algebra generated by all the gamma matrices Γ'_B. We define $\langle T\rangle_\pm$ as above as the *normalized* trace, which is 2^{-m} times the trace relative to the algebras $\mathrm{Mat}_\pm$ of an operator T in H. It is best expressed as an integral of the form

$$\langle T\rangle_\pm = \int_{\mathrm{Spin}_\pm} gTg^{-1}\, dg \tag{16}$$

where $\mathrm{Spin}_\pm \subset \mathrm{Mat}_\pm$ is the spin group and dg the Haar measure of total mass 1.

Lemma 3 *The conditional expectations $\langle T\rangle_\pm$ fulfill the following properties*

1. $\langle STU\rangle_+ = S\,\langle T\rangle_+ U$ *for any operators S, U commuting with* Mat_+ *(this holds similarly exchanging + and −)*
2. $\langle T\rangle = \left\langle \langle T\rangle_+\right\rangle_- = \left\langle \langle T\rangle_-\right\rangle_+$ *for any operator T.*
3. $\langle ST\rangle = \langle S\rangle_+ \langle T\rangle_-$ *for any operator S commuting with* Mat_- *and T commuting with* Mat_+*.*
4. $\langle ST\rangle = \langle S\rangle_- \langle T\rangle_+$ *for any operator S commuting with* Mat_+ *and T commuting with* Mat_-*.*

Proof. 1) follows from (16) since $gSTUg^{-1} = SgTg^{-1}U$ for S, U commuting with Mat_+ and $g \in \mathrm{Spin}_+$.
2) The representation of the product group $G = \mathrm{Spin}_+ \times \mathrm{Spin}_-$ given by $(g, g') \mapsto gg' \in \mathrm{Mat}_+\mathrm{Mat}_-$ is irreducible, and thus parallel to (16) one has

$$\langle T\rangle = \int_G gg'T(gg'^{-1}dgdg' = \left\langle \langle T\rangle_+\right\rangle_- = \left\langle \langle T\rangle_-\right\rangle_+ \tag{17}$$

using the fact that any g commutes with any g'.
3) This follows from (17) since one has

$$gg'ST(gg'^{-1} = gg'S(gg'^{-1}gg'T(gg'^{-1} = gSg^{-1}g'Tg'^{-1}$$

4) The proof is the same as for 3). ∎

3.2 Case of dimension 2

This is the simplest case, one has:

Lemma 4 *The condition* (15) *implies that the (*2*-dimensional) volume of* M *is quantized. If* M *is a smooth connected compact oriented* 2*-dimensional manifold with quantized volume there exists a solution of* (15)*.*

Proof. We shall compute the left hand side of (15) and show that

$$\langle Z\,[D,Z]\,[D,Z]\rangle = \frac{1}{2}\,\langle Y\,[D,Y]\,[D,Y]\rangle + \frac{1}{2}\,\langle Y'\,[D,Y']\,[D,Y']\rangle \tag{18}$$

Thus as above we see that (15) is equivalent to the quantization condition

$$\det\left(e^a_\mu\right) = \frac{1}{2}\epsilon^{\mu\nu}\epsilon_{ABC}Y^A\partial_\mu Y^B\partial_\nu Y^C + \frac{1}{2}\epsilon^{\mu\nu}\epsilon_{ABC}Y'^A\partial_\mu Y'^B\partial_\nu Y'^C \tag{19}$$

which gives the volume of M as the sum of the degrees of the two maps $Y : M \to S^2$ and $Y' : M \to S^2$. This shows that the volume is quantized (up to normalization). Conversely let M be a compact oriented 2-dimensional manifold with quantized volume. Choose two smooth maps $Y : M \to S^2$ and $Y' : M \to S^2$ such that when you add the pull back of the oriented volume form ω of S^2 by Y and Y' you get the volume form of M. This will be discussed in great details in §4. However, it is simple in dimension 2 mostly because, on a connected compact smooth manifold, all smooth nowhere-vanishing differential forms of top degree with the same integral are equivalent by a diffeomorphism ([17]). This solves equation (19). It remains to show (18). We use the properties

$$[D,e] = \left[D,e^2\right] = e\,[D,e] + [D,e]\,e$$

which can be written as

$$e\,[D,e] = [D,e]\,(1-e), \quad [D,e]\,e = (1-e)\,[D,e] \tag{20}$$

which imply

$$e\,[D,e]\,e = 0, \quad e\,[D,e]^2 = [D,e]^2\,e \tag{21}$$

Now with $Z = 2ee' - 1$ as above, one has

$$[D,Z] = 2\,[D,ee'] = 2\,[D,e]\,e' + 2e\,[D,e'] \tag{22}$$

Now $[D,e]$ commutes with e' because any element of Mat_+ (such as Γ^A) commutes with any element of Mat_- (such as Γ'^B) and for any scalar functions f,g one has $[[D,f],g]=0]$ so that $[D,Y^A]$ commutes with Y'^B. Similarly $[D,e']$ commutes with e (and e and e' commute) one thus gets

$$\begin{aligned}[D,Z]^2 &= 4\left([D,e]\,e' + e\,[D,e']\right)^2 \\ &= 4\Big([D,e]^2\,e' + e\,[D,e']^2 + [D,e]\,ee'\,[D,e'] + [D,e']\,ee'\,[D,e]\Big)\end{aligned} \tag{23}$$

One has

$$\begin{aligned}\frac{1}{4}Z\,[D,Z]^2 &= e'\,(2e-1)\,[D,e]^2 + e\,(2e'-1)\,[D,e']^2 \\ &\quad + (2e-1)\,[D,e]\,ee'\,[D,e'] + (2e'-1)\,[D,e']\,e'e\,[D,e]\end{aligned} \tag{24}$$

Using 4) of Lemma 3, one has

$$\left\langle e'\,(2e-1)\,[D,e]^2\right\rangle = \langle e'\rangle_-\left\langle (2e-1)\,[D,e]^2\right\rangle_+ = \frac{1}{2}\left\langle (2e-1)\,[D,e]^2\right\rangle$$

since $\left\langle (e-\frac{1}{2})\right\rangle_- = \frac{1}{2}\langle Y'\rangle_- = 0$. Similarly one has

$$\left\langle e\,(2e'-1)\,[D,e']^2\right\rangle = \frac{1}{2}\left\langle (2e'-1)\,[D,e']^2\right\rangle_- = \frac{1}{2}\left\langle (2e'-1)\,[D,e']^2\right\rangle$$

Moreover one has $\langle Y\,[D,Y]\rangle = 0$. This follows from the order one condition since one gets, using $Y^AY^A = 1$,

$$\langle Y\,[D,Y]\rangle = Y^A\left[D,Y^A\right] = \frac{1}{2}\left(Y^A\left[D,Y^A\right] + \left[D,Y^A\right]Y^A\right) = 0.$$

It implies that $\langle e\,[D,e]\rangle = 0$ since it is automatic that $\langle [D,Y]\rangle = 0$. We then get

$$\langle (2e-1)\,[D,e]\,ee'\,[D,e']\rangle = \langle (2e-1)\,[D,e]\,e\rangle_+\,\langle e'\,[D,e']\rangle_- = 0$$

and similarly for the other term. Thus we have shown that (18) holds. ■

3.3 The two sided equation in dimension 4

This calculation will now be done for the four dimensional case:

Lemma 5 *In the 4-dimensional case one has*

$$\left\langle Z\,[D,Z]^4\right\rangle = \frac{1}{2}\left\langle Y\,[D,Y]^4\right\rangle + \frac{1}{2}\left\langle Y'\,[D,Y']^4\right\rangle.$$

The condition 15 implies that the (4-dimensional) volume of M is quantized.

Proof. Now Γ_A and Γ'_A will have $A = 1, \cdots, 5$. We now compute, using (23)

$$\frac{1}{16}\,[D,Z]^4 = \Big([D,e]^2\,e' + e\,[D,e']^2 + [D,e]\,ee'\,[D,e'] + [D,e']\,ee'\,[D,e]\Big)^2$$

using (21) to show that the following 6 terms give 0,

$$(1)\times(4) = [D,e]^2\,e'\,[D,e']\,ee'\,[D,e] = 0, \quad \text{since} \;\; e'\,[D,e']\,e' = 0,$$

$$\begin{aligned}
(2)\times(3) &= e\,[D,e']^2\,[D,e]\,ee'\,[D,e'] = 0, \quad \text{since} \;\; e\,[D,e]\,e = 0,\\
(3)\times(1) &= [D,e]\,ee'\,[D,e']\,[D,e]^2\,e' = 0, \quad \text{since} \;\; e'\,[D,e']\,e' = 0,\\
(3)\times(3) &= [D,e]\,ee'\,[D,e']\,[D,e]\,ee'\,[D,e'] = 0, \quad \text{since} \;\; e'\,[D,e']\,e' = 0,\\
(4)\times(2) &= [D,e']\,ee'\,[D,e]\,e\,[D,e']^2 = 0, \quad \text{since} \;\; e\,[D,e]\,e = 0,\\
(4)\times(4) &= [D,e']\,ee'\,[D,e]\,[D,e']\,ee'\,[D,e] = 0, \quad \text{since} \;\; e'\,[D,e']\,e' = 0.
\end{aligned}$$

We thus get the remaining ten terms in the form

$$\begin{aligned}
\frac{1}{16}\,[D,Z]^4 &= \Big([D,e]^2\,e' + e\,[D,e']^2 + [D,e]\,ee'\,[D,e'] + [D,e']\,ee'\,[D,e]\Big)^2\\
&= [D,e]^4\,e' + [D,e]^2\,ee'\,[D,e']^2 + [D,e]^3\,ee'\,[D,e']\\
&+ [D,e']^2\,e'e\,[D,e]^2 + e\,[D,e']^4 + [D,e']^3\,ee'\,[D,e]\\
&+ [D,e]\,ee'\,[D,e']^3 + [D,e]\,ee'\,[D,e']^2\,e'e\,[D,e]\\
&+ [D,e']\,ee'\,[D,e]^3 + [D,e']\,ee'\,[D,e]^2\,e'e\,[D,e']
\end{aligned} \tag{25}$$

We multiply by $Z = 2ee' - 1$ on the left and treat the various terms as follows.

$$Z\,[D,e]^4\,e' = e'(2e-1)\,[D,e]^4$$

gives the contribution

$$\left\langle Z\,[D,e]^4\,e'\right\rangle = \langle e'\rangle\left\langle Y\,[D,e]^4\right\rangle = \frac{1}{32}\left\langle Y\,[D,Y]^4\right\rangle$$

The other quartic term

$$Ze\,[D,e']^4 = e(2e'-1)\,[D,e']^4$$

gives the contribution

$$\left\langle Ze\,[D,e']^4 \right\rangle = \frac{1}{32}\left\langle Y'\,[D,Y']^4 \right\rangle$$

For the cubic terms one has, using $e\,[D,e]^3\,e = e\,[D,e]\,e\,[D,e]^2 = 0$,

$$Z\,[D,e]^3\,ee'\,[D,e'] = -\,[D,e]^3\,ee'\,[D,e']$$

and it gives as above a vanishing contribution since $\langle e'\,[D,e'] \rangle = 0$ (and similarly for e). Similarly one has

$$Z\,[D,e']^3\,ee'\,[D,e] = -\,[D,e']^3\,ee'\,[D,e]$$

which gives a vanishing contribution, as well as

$$Z\,[D,e]\,ee'\,[D,e']^3 = -\,[D,e]\,ee'\,[D,e']^3$$

and

$$Z\,[D,e']\,ee'\,[D,e]^3 = -\,[D,e']\,ee'\,[D,e]^3\,.$$

We now take care of the remaining 4 quadratic terms. They are

$$\begin{aligned}&[D,e]^2\,ee'\,[D,e']^2 + [D,e']^2\,e'e\,[D,e]^2\\ &+ [D,e]\,ee'\,[D,e']^2\,e'e\,[D,e] + [D,e']\,ee'\,[D,e]^2\,e'e\,[D,e']\end{aligned}$$

One has, using the commutation of ee' with $[D,e]^2$

$$Z\,[D,e]^2\,ee'\,[D,e']^2 = [D,e]^2\,ee'\,[D,e']^2$$

so that the contributions of the two terms of the first line are

$$\left\langle e\,[D,e]^2 \right\rangle \left\langle e'\,[D,e']^2 \right\rangle + \left\langle e'\,[D,e']^2 \right\rangle \left\langle e\,[D,e]^2 \right\rangle \tag{26}$$

Now for the remaining terms one gets, using $e\,[D,e]\,e = 0$

$$Z\,[D,e]\,ee'\,[D,e']^2\,e'e\,[D,e] = -\,[D,e]\,ee'\,[D,e']^2\,e'e\,[D,e]$$

To compute the trace one uses the fact that $[D,e]\,e$ commutes with Mat_- and property 1) of Lemma 3 to get

$$\left\langle Z\,[D,e]\,ee'\,[D,e']^2\,e'e\,[D,e]\right\rangle_- = -\,[D,e]\,e\left\langle e'\,[D,e']^2\right\rangle_- e\,[D,e]$$

Next one has, using $\left\langle Y'\,[D,Y']^2\right\rangle = 0$ and $e' = \frac{1}{2}(Y'+1)$,

$$\left\langle e'\,[D,e']^2\right\rangle_- = \frac{1}{8}\left\langle [D,Y']^2\right\rangle \tag{27}$$

and this does not vanish but is a scalar function which is $\sum\left[D,Y'^A\right]^2$ and commutes with the other terms so that one gets after taking it across

$$\left\langle Z\,[D,e]\,ee'\,[D,e']^2\,e'e\,[D,e]\right\rangle = -\,\langle [D,e]\,e\,[D,e]\rangle\left\langle e'\,[D,e']^2\right\rangle$$

Next one has, using $\left\langle Y\,[D,Y]^2\right\rangle = 0$, and $Y\,[D,Y] + [D,Y]\,Y = 0$

$$\langle [D,e]\,e\,[D,e]\rangle = \frac{1}{8}\left\langle [D,Y]^2\right\rangle = \left\langle e\,[D,e]^2\right\rangle$$

which shows that

$$\left\langle Z\,[D,e]\,ee'\,[D,e']^2\,e'e\,[D,e]\right\rangle = -\left\langle e\,[D,e]^2\right\rangle\left\langle e'\,[D,e']^2\right\rangle$$

Note that to show that

$$\langle [D,e]\,e\,[D,e]\rangle = \left\langle e\,[D,e]^2\right\rangle$$

one can also use (by (20))

$$[D,e]\,e\,[D,e] = (1-e)\,[D,e]^2\,,\quad \left\langle (2e-1)\,[D,e]^2\right\rangle = \left\langle Y\,[D,e]^2\right\rangle = 0$$

Similarly one gets

$$\left\langle Z\,[D,e']\,ee'\,[D,e]^2\,e'e\,[D,e']\right\rangle = -\left\langle e'\,[D,e']^2\right\rangle\left\langle e\,[D,e]^2\right\rangle$$

Thus combining with (26), we get that the total contribution of the quadratic terms is 0.
Finally the second statement of Lemma 5 follows from Lemma 2. ■

3.4 Algebraic relations

It is important to make the list of the algebraic relations which have been used and do not follow from the definition of Y and Y'. Note first that for $Y = Y^A \Gamma_A$ with the hypothesis that the components Y^A belong to the commutant of the algebra generated by the Γ_B, one has

$$Y^2 = \pm 1 \implies [Y^A, Y^B] = 0, \ \forall A, B.$$

Indeed the matrices $\Gamma_A \Gamma_B$ for $A < B$, are linearly independent and the coefficient of $\Gamma_A \Gamma_B$ in the square Y^2 is $[Y^A, Y^B]$ which has to vanish. The similar statement holds for Y'. Moreover the commutation rule $[Y, Y'] = 0$ implies (and is equivalent to) the commutation of the components $[Y^A, Y'^B] = 0$, $\forall A, B$. Thus the components Y^A, Y'^B commute pairwise and generate a commutative involutive algebra $\mathcal{A}$ (since they are all self-adjoint). This corresponds to the order zero condition in the commutative case. We have also assumed the order one condition in the from $[[D, a], b] = 0$ for any $a, b \in \mathcal{A}$. But in fact we also made use of the commutation of the operator $\langle [D, Y]^2 \rangle$ with the elements of $\mathcal{A}$ and the $[D, a]$ for $a \in \mathcal{A}$ (and similarly for $\langle [D, Y']^2 \rangle$).

3.5 The Quantization Theorem

In the next theorem the algebraic relations between $Y_\pm$, D, J, $C_\pm$, γ are assumed to hold. We shall not detail these relations but they are exactly those discussed in §3.4 and which make the proof of Lemma 5 possible.
As in the introduction we adopt the following definitions. Let M be a connected smooth oriented compact manifold of dimension n. Let α be the volume form of the sphere S^n. One considers the (possibly empty) set $D(M)$ of pairs of smooth maps $\phi, \psi : M \to S^n$ such that the differential form

$$\phi^\#(\alpha) + \psi^\#(\alpha) = \omega$$

does not vanish anywhere on M ($\omega(x) \neq 0 \, \forall x \in M$). One defines an invariant which is the subset of $\mathbb{Z}$:

$$q(M) := \{\text{degree}(\phi) + \text{degree}(\psi) \mid (\phi, \psi) \in D(M)\} \subset \mathbb{Z}.$$

Theorem 6 *Let $n = 2$ or $n = 4$.*
(i) In any operator representation of the two sided equation (3) *in which the spectrum of D grows as in dimension n the volume (the leading term of the Weyl asymptotic formula) is quantized.*

(ii) Let M be a connected smooth compact oriented spin Riemannian manifold (of dimension $n = 2, 4$). Then a solution of (3) exists if and only if the volume of M is quantized [4] *to belong to the invariant $q(M) \subset \mathbb{Z}$.*

Proof. (i) By Lemma 5 one has, as in the two dimensional case that the left hand side of (15) is up to normalization,

$$L = \left\langle Y\,[D, Y]^4 \right\rangle + \left\langle Y'\,[D, Y']^4 \right\rangle \tag{28}$$

so that (15) implies that the volume of M is (up to sign) the sum of the degrees of the two maps. This is enough to give the proof in the case of the spectral triple of a manifold, and we shall see in Theorem 17 that it also holds in the abstract framework.
(ii) Using Lemma 5 the proof is the same as in the two dimensional case. Note that the connectedness hypothesis is crucial in order to apply the result of [17]. ∎

4 Differential geometry and the two sided equation

The invariant q_M makes sense in any dimension. For $n = 2, 3$, and any connected M, it contains all sufficiently large integers. The case $n = 4$ is more difficult but we shall prove below in Theorem 12 that it contains all integers $m > 4$ as soon as the connected 4-manifold M is a Spin manifold, an hypothesis which is automatic in our context.

4.1 Case of dimension $n < 4$

Lemma 7 *Let M be a compact connected smooth oriented manifold of dimension $n < 4$. Then for any differential form $\omega \in \Omega^n(M)$ which vanishes nowhere, agrees with the orientation, and fulfills the quantization $\int_M \omega \in \mathbb{Z}$, $|\int_M \omega| > 3$, one can find two smooth maps ϕ, ϕ' such that*

$$\phi^{\#}(\alpha) + \phi'^{\#}(\alpha) = \omega$$

where α is the volume form of the sphere of unit volume.

[4] up to normalization

Proof. By [1] as refined in [20], any Whitehead triangulation of M provides (after a barycentric subdivision) a ramified covering of the sphere S^n obtained by gluing two copies $\Delta^n_\pm$ of the standard simplex Δ^n along their boundary. One uses the labeling of the vertices of each n-simplex by $\{0, 1, \ldots, n\}$ where each vertex is labeled by the dimension of the face of which it is the barycenter. The bi-coloring corresponds to affecting each n-simplex of the triangulation with a sign depending on wether the orientation of the simplex agrees or not with the orientation given by the labeling of the vertices. One then gets a PL-map $M \to S^n$ by mapping each simplex with a $\pm$ sign to $\Delta^n_\pm$ respecting the labeling of the vertices. This gives a covering which is ramified only on the $(n-2)$-skeleton of $\Delta^n_\pm$. After smoothing one then gets a smooth map $\phi : M \to S^n$ whose Jacobian will be > 0 outside a subset K of dimension $n-2$ of M. Using the hypothesis $n < 4$ (which gives $(n-2)+(n-2) < n$), the set of orientation preserving diffeomorphisms $\psi \in \mathrm{Diff}^+(M)$ such that $\psi(K) \cap K = \emptyset$ is a dense subset of $\mathrm{Diff}(M)^+$, thus one finds $\psi \in \mathrm{Diff}^+(M)$ such that the Jacobian of ϕ and the Jacobian of $\phi' = \phi \circ \psi$ never vanish simultaneously. This shows that the differential form $\rho = \phi^\#(\alpha) + \phi'^\#(\alpha)$ does not vanish anywhere and by the result of [17] there exists an orientation preserving diffeomorphism of M which transforms this form into ω provided they have the same integral. But the integral of ρ is twice the integral of $\phi^\#(\alpha)$ which in turns is the degree of the map ϕ and thus the number of simplices of a given color. As performed the above construction only gives even numbers, since the integral of ρ is twice the degree of the map ϕ, but we shall see shortly in Lemma 9 that in fact the degree of the map ϕ is in $q(M)$ from a fairly general argument. ∎

4.2 Preliminaries in dimension 4

Let us first give simple examples in dimension 4 of varieties where one can obtain arbitrarily large quantized volumes.

First for the sphere S^4 itself one can construct by the same procedure as in the proof of Lemma 7 a smooth map $\phi : S^4 \to S^4$ whose Jacobian is ≥ 0 everywhere and whose degree is a given integer N. One can then simply take the sum $\omega = \phi^\#(\alpha) + \alpha$ which does not vanish and has integral $N + 1$.

Next, let us take $M = S^3 \times S^1$. Then one can construct by the same procedure as in the proof of Lemma 7 a smooth map $\phi : M \to S^4$ whose Jacobian is ≥ 0 everywhere and which vanishes only on a two dimensional subset $K \subset M$. Let $p : M \to S^3$ be the first projection using the product $M = S^3 \times S^1$.

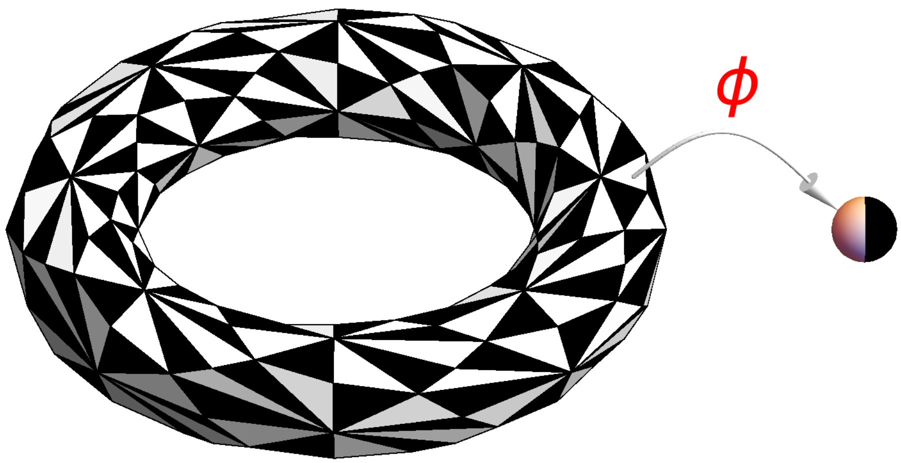

Figure 1: Triangulation of torus, the map ϕ maps white triangles to the white hemisphere (of the small sphere) and the black ones to the black hemisphere.

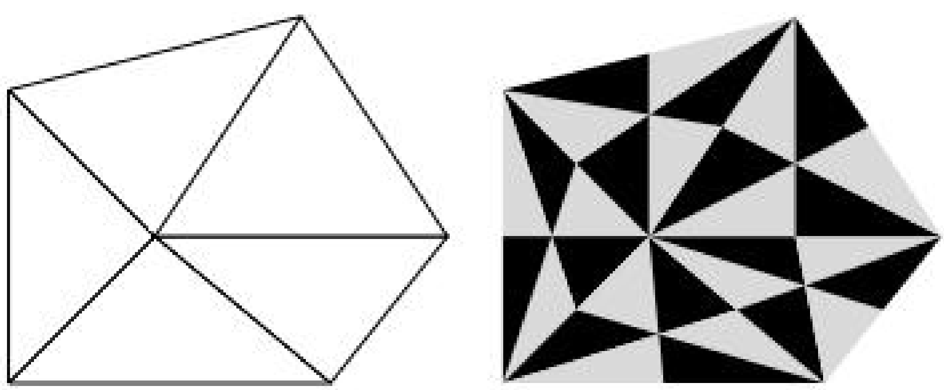

Figure 2: Barycentric subdivision.

Then $p(K)$ is a two dimensional subset of S^3 and hence there exists $x \in S^3$, $x \notin p(K)$. One can thus find a diffeomorphism $\psi \in \mathrm{Diff}^+(S^3)$ such that $\psi(p(K)) \cap p(K) = \emptyset$. Then the diffeomorphism $\psi' \in \mathrm{Diff}^+(M)$ which acts as $(x, y) \mapsto \psi'(x, y) = (\psi(x), y)$ is such that $\psi'(K) \cap K = \emptyset$. Thus it follows that the Jacobian of ϕ and the Jacobian of $\phi' = \phi \circ \psi'$ never vanish simultaneously and the proof of Lemma 7 applies. Note moreover that in this case $M = S^3 \times S^1$ is not simply connected and one gets smooth covers of arbitrary degree which can be combined with the maps (ϕ, ϕ').

4.3 Necessary condition

Jean-Claude Sikorav and Bruno Sevennec found the following obstruction which implies for instance that $D(\mathbb{C}P^2) = \emptyset$. In general

Lemma 8 *Let M be an oriented compact smooth 4-dimensional manifold, then, with w_2 the second Stiefel-Whitney class of the tangent bundle,*

$$D(M) \neq \emptyset \implies w_2^2 = 0$$

More generally if $D(M) \neq \emptyset$ and the dimension of M is arbitrary, the product of any two Stiefel-Whitney classes vanishes.

Proof. One has a cover of M by two open sets on which the tangent bundle is stably trivialized. Thus the product of any two Stiefel-Whitney classes vanishes. ■
Since a manifold is a Spin manifold if and only if $w_2 = 0$ this obstruction vanishes in our context.

4.4 Reduction to a single map

Here is a first lemma which reduces to properties of a single map.

Lemma 9 *Let $\phi : M \to S^4$ be a smooth map such that $\phi^{\#}(\alpha)(x) \geq 0$ $\forall x \in M$ and let $R = \{x \in M \mid \phi^{\#}(\alpha)(x) = 0\}$. Then there exists a map ϕ' such that $\phi^{\#}(\alpha) + \phi'^{\#}(\alpha)$ does not vanish anywhere if and only if there exists an immersion $f : V \to \mathbb{R}^4$ of a neighborhood V of R. Moreover if this condition is fulfilled one can choose ϕ' to be of degree 0.*

Proof. Let first ϕ' be such that $\phi^{\#}(\alpha) + \phi'^{\#}(\alpha)$ does not vanish anywhere. Then $\phi'^{\#}(\alpha)$ does not vanish on the closed set R and hence in a neighborhood $V \supset R$. Its restriction to V gives the desired immersion. Conversely let $f : V \to \mathbb{R}^4$ be an immersion of a neighborhood V of R. We can assume by changing the orientation of $\mathbb{R}^4$ for the various connected components of V that $f^{\#}(v) > 0$ where v is the standard volume form on $\mathbb{R}^4$. We first extend f to a smooth map $\tilde{f} : M \to \mathbb{R}^4$ by extending the coordinate functions. We then can assume that $f(M) \subset B_4 \subset \mathbb{R}^4$ where B_4 is the unit ball which we identify with the half sphere so that $B_4 \subset S^4$. We denote by $\beta = \alpha | B_4$ the restriction of α to B_4. We have $f^{\#}(\beta) > 0$ on V but not on M since the map $\tilde{f} : M \to S^4$ is of degree zero. Let $\rho > 0$ be a fixed volume form (nowhere vanishing) on M. Let $\epsilon > 0$ be such that

$$\phi^{\#}(\alpha)(x) \geq \epsilon \rho(x), \quad \forall x \notin V$$

For $y \in B_4$ and $0 < \lambda \leq 1$ we let λy be the rescaled element (using rescaling in $\mathbb{R}^4$). Then for λ small enough one has

$$|(\lambda \tilde{f})^{\#}(\alpha)(x)| \leq \frac{1}{2} \epsilon \rho(x), \quad \forall x \in M,$$

where the absolute value is on the ratio of $(\lambda \tilde{f})^{\#}(\alpha)$ with ρ. One then gets that with $\phi' = \lambda \tilde{f}$ one has

$$(\phi^{\#}(\alpha) + \phi'^{\#}(\alpha))(x) \neq 0, \quad \forall x \in M.$$

∎

4.5 Products $M = N \times S^1$

Let N be a smooth oriented compact three manifold. Then N is Spin, thus the condition $w_2^2 = 0$ is automatically fulfilled by $M = N \times S^1$. In fact:

Theorem 10 *Let N be a smooth oriented connected compact three manifold. Let $M = N \times S^1$, then the set $q(M)$ is non-empty, and contains all integers $m \geq r$ for some $r > 0$.*

Proof. Let $g : S^3 \times S^1 \to S^4$ be a ramified cover of degree m and singular set Σ_g. Let N be described as a ramified cover $f : N \to S^3$ ramified over a knot $K \subset S^3$ ([16], [13]). One may, using the two dimensionality of Σ_g, assume that

$$K \cap p_3(\Sigma_g) = \emptyset, \quad p_3 : S^3 \times S^1 \to S^3.$$

Let $h = f \times \mathrm{id} : N \times S^1 \to S^3 \times S^1$. Let $\Sigma_f \subset N$ be the singular set of f. one has $f(\Sigma_f) \subset K$ and thus, with $\Sigma_h \subset N \times S^1$ the singular set of h,

$$\Sigma_h = \Sigma_f \times S^1, \quad h(\Sigma_h) \cap \Sigma_g = \emptyset$$

since $h(\Sigma_h) = f(\Sigma_f) \times S^1 \subset K \times S^1$ is disjoint from Σ_g. Let then $\phi = g \circ h$. The singular set Σ_ϕ of ϕ is the union of Σ_h with $h^{-1}(\Sigma_g)$. This two closed sets are disjoint since $h(\Sigma_h) \cap \Sigma_g = \emptyset$. By Lemma 9 it is enough to find immersions in $\mathbb{R}^4$ of neighborhoods $V \supset \Sigma_h$ and $W \supset h^{-1}(\Sigma_g)$. By construction $\Sigma_h = \Sigma_f \times S^1$ is a union of tori with trivial normal bundle, since their normal bundle is the pullback by the projection of the normal bundle to Σ_f which is a union of circles. This gives the required immersion $V \to \mathbb{R}^4$. Moreover the restriction of h to a suitable neighborhood W of $h^{-1}(\Sigma_g)$ is a smooth covering of an open set of $S^3 \times S^1$. On each of the components W_j of this covering, the local situation is the same as for the inclusion of Σ_g in $S^3 \times S^1$. Thus one gets the required immersion $W \to \mathbb{R}^4$. This shows that the hypothesis of Lemma 9 is fulfilled and one gets that $D(M) \neq \emptyset$ and that

$$\text{degree}(f) + \text{degree}(g) \in q(M)$$

■

Remark 11 *Here is a variant, due to Jean-Claude Sikorav, of the above proof, also using Lemma 9. The* 4*-manifold* $M = N \times S^1$ *is parallelizable since any oriented* 3*-manifold is parallelizable (see for instance [13] for a direct proof), and by [18] Theorem 5, there is an immersion* $\psi : M \setminus \{p\} \to \mathbb{R}^4$ *of the complement of a single point* $p \in M$ *so that it is easy to verify the hypothesis of Lemma 9 and show that for any ramified cover* $\phi : M \to S^4$ *one has degree*$(\phi) \in q(M)$*.*

4.6 Spin manifolds

Theorem 12 *Let* M *be a smooth connected oriented compact spin* 4*-manifold. Then the set* $q(M)$ *contains all integers* $m \geq 5$*.*

Proof. We proceed as in the proof of Lemma 7 and get from any Whitehead triangulation of M (after a barycentric subdivision) a ramified covering γ of the sphere S^4 obtained by gluing two copies $\Delta^4_\pm$ of the standard simplex Δ^4 along their boundary. Let then V be a neighborhood of the 2-skeleton of the triangulation which retracts on the 2-skeleton. Then the restriction of the tangent bundle of M to V is trivial since the spin hypothesis allows one to view TM as induced from a Spin(4) principal bundle while the classifying space $B\mathrm{Spin}(4)$ is 3-connected. Thus the extension by Poenaru [18], Theorem 5, (see also [19]), of the Hirsch-Smale immersion theory ([14], [22]) to the case of codimension zero yields an immersion $V \to \mathbb{R}^4$. After smoothing γ while keeping its singular set inside V one gets that the hypothesis of Lemma 9 is fulfilled and this gives that $m \in q(M)$ where $2m$ is the number of simplices of the triangulation. For the finer result involving the small values of m one can use the theorem[5] of M. Iori and R. Piergallini [15], which gives a smooth ramified cover $\phi : M \to S^4$ of any degree $m \geq 5$ whose singular set $R \subset M$ is a disjoint union of smooth surfaces $S_j \subset M$. As above, when M is a Spin manifold, the condition of Lemma 9 is fulfilled so that $m \in q(M)$. Indeed as above, this shows that there exists an immersion of a neighborhood of each S_j in $\mathbb{R}^4$. Thus $q(M)$ contains any integer $m \geq 5$ for any Spin 4-manifold. ∎

Remark 13 *In fact in the above proof one needs to use immersion theory only when S_j is non-orientable. If S_j is orientable, then by Whitney's theorem ([23], §6.b)) the Euler class $\chi(\nu)$ of the normal bundle of $\phi(S_j) \subset S^4$ is $\chi(\nu) = 0$, while one has the proportionality with the Euler class of the normal bundle ν' of $S_j \subset M$. Thus $\chi(\nu') = 0$ and it follows that there is an embedding of a tubular neighborhood of S_j in $\mathbb{R}^4$.*

Remark 14 *As a countercheck it is important to note why the above proof does not apply in the case of $\mathbb{C}P^2$ seen as a double cover of the 4-sphere which is the quotient of $\mathbb{C}P^2$ by complex conjugation and gives a ramified cover with ramification on $\mathbb{R}P^2$. It is an exercice for the reader to compute directly the second Stiefel-Whitney class of the tangent space of $\mathbb{C}P^2$ restricted to the submanifold $\mathbb{R}P^2$ and check that it does not vanish.*

Corollary 15 *Let M be a smooth compact connected oriented spin Riemannian 4-manifold with quantized [6] volume ≥ 5. Then there exists an irreducible*

[5] This theorem is stated in the PL category but, as confirmed to us by R. Piergallini, it holds (for any $m \geq 5$) in the smooth category due to general results PL=Smooth in 4-dimensions.

[6] up to normalization

representation of the two-sided quantization relation such that the canonical spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *of* M *appears as follows, where* $\{Y^A, Y'^B\}''$ *is the double commutant of the components* Y^A, Y'^B*,*
- *Algebra :* $\mathcal{A} = \{f \in \{Y^A, Y'^B\}'' \mid f\mathcal{D} \subset \mathcal{D}\}, \quad \mathcal{D} = \cap_k \text{Domain} D^k$.
- *Hilbert space:* $\mathcal{H} = \prod E_A E'_B H$, $E_A = \frac{1}{2}(1 + \Gamma^A)$, $E'_B = \frac{1}{2}(1 + \Gamma'^B)$.
- *Operator: The operator is the restriction of* D *to* $\mathcal{H}$*.*

Proof. By Theorem 12 combined with Theorem 6, a solution of (3) exists for the spectral triple of M. Let ϕ, ϕ' be the corresponding maps $M \to S^4$. By a general position argument ([10], Chapter III, Corollary 3.3) one can assume that the map $(\phi, \phi') : M \to S^4 \times S^4$ is transverse to itself, without spoiling the fact that $\phi^{\#}(\alpha) + \phi'^{\#}(\alpha)$ does not vanish. The existence of self-intersections of $M \subset S^4 \times S^4$ prevents the components Y^A, Y'^B from generating the algebra of smooth functions on M but what remains true is that the double commutant $\{Y^A, Y'^B\}''$ is the same as the double commutant of $C^\infty(M)$ since the double points form a finite set. One then concludes that, with $\mathcal{D} = \cap_k \text{Domain} D^k$ one has

$$C^\infty(M) = \{f \in \{Y^A, Y'^B\}'' \mid f\mathcal{D} \subset \mathcal{D}\}$$

and it follows that the representation of the two-sided quantization relation is irreducible. The formulas for the Hilbert space and the operator are straightforward. ∎

5 A tentative particle picture in Quantum Gravity

One of the basic conceptual ingredients of Quantum Field Theory is the notion of particle which Wigner formulated as irreducible representations of the Poincaré group. When dealing with general relativity we shall see that (in the Euclidean = imaginary time formulation) there is a natural corresponding particle picture in which the irreducible representations of the two-sided higher Heisenberg relation play the role of "particles". Thus the role of the Poincaré group is now played by the algebra of relations existing between the line element and the slash of scalar fields.
We shall first explain why it is natural from the point of view of differential geometry also, to consider the two sets of Γ-matrices and then take the

operators Y and Y' as being the correct variables for a first shot at a theory of quantum gravity. Once we have the Y and Y' we can use them and get a map $(Y, Y') : M \to S^n \times S^n$ from the manifold M to the product of two n-spheres. The first question which comes in this respect is if, given a compact n-dimensional manifold M one can find a map $(Y, Y') : M \to S^n \times S^n$ which embeds M as a submanifold of $S^n \times S^n$. Fortunately this is a known result, the strong embedding theorem of Whitney, [24], which asserts that any smooth real n-dimensional manifold (required also to be Hausdorff and second-countable) can be smoothly embedded in the real 2n-space. Of course $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \subset S^n \times S^n$ so that one gets the required embedding. This result shows that there is no restriction by viewing the pair (Y, Y') as the correct "coordinate" variables. Thus we simply view Y and Y' as operators in Hilbert space and we shall write algebraic relations which they fulfill relative to the two Clifford algebras C_κ, $\kappa = \pm 1$ and to the self-adjoint operator D. We should also involve the J and the γ. The metric dimension will be governed by the growth of the spectrum of D.
The next questions are: assuming that we now no-longer use a base manifold M,

A: Why is it true that the joint spectrum of the Y^A and Y'^B is of dimension n while one has $2n$ variables.

B: Why is it true that the non-commutative integrals

$$\int\!\!\!\!\!\!- \gamma \left\langle Y [D, Y]^n \right\rangle D^{-n}, \quad \int\!\!\!\!\!\!- \gamma \left\langle Y' [D, Y']^n \right\rangle D^{-n}, \quad \int\!\!\!\!\!\!- D^{-n}$$

remain quantized.

5.1 Why is the joint spectrum of dimension 4

The reason why A holds in the case of classical manifolds is that in that case the joint spectrum of the Y^A and Y'^B is the subset of $S^n \times S^n$ which is the image of the manifold M by the map $x \in M \mapsto (Y(x), Y'(x))$ and thus its dimension is at most n.
The reason why A holds in general is because of the assumed boundedness of the commutators $[D, Y]$ and $[D, Y']$ together with the commutativity $[Y, Y'] = 0$ (order zero condition) and the fact that the spectrum of D grows like in dimension n.

5.2 Why is the volume quantized

The reason why B holds in the case of classical manifolds is that this is a winding number, as shown in Lemma 2.
The reason why B holds in the general case is that all the lower components of the operator theoretic Chern character of the idempotent $e = \frac{1}{2}(1+Y)$ vanish and this allows one to apply the operator theoretic index formula which in that case gives (up to suitable normalization)

$$2^{-n/2-1} \fint \gamma \left\langle Y\,[D,Y]^n \right\rangle D^{-n} = \mathrm{Index}\,(D_e)$$

This follows from the local index formula of [9] but in fact one does not need the technical hypothesis of [9] since, when the lower components of the operator theoretic Chern character all vanish, one can use the non-local index formula in cyclic cohomology and the determination in [7] Theorem 8, IV.2.γ of the Hochschild class of the index cyclic cocycle.
To be more precise one introduces the following trace operation, given an algebra $\mathcal{A}$ over $\mathbb{R}$ (not assumed commutative) and the algebra $M_n(\mathcal{A})$ of matrices of elements of $\mathcal{A}$, one defines

$$\mathrm{tr} : M_n(\mathcal{A}) \otimes M_n(\mathcal{A}) \otimes \cdots \otimes M_n(\mathcal{A}) \to \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$$

by the rule, using $M_n(\mathcal{A}) = M_n(\mathbb{R}) \otimes \mathcal{A}$

$$\mathrm{tr}\left((a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)\right) = \mathrm{Trace}(\mu_0 \cdots \mu_m) a_0 \otimes a_1 \otimes \cdots \otimes a_m$$

where Trace is the ordinary trace of matrices. Let us denote by ι_k the operation which inserts a 1 in a tensor at the k-th place. So for instance

$$\iota_0(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_m$$

One has $\mathrm{tr} \circ \iota_k = \iota_k \circ \mathrm{tr}$ since (taking $k = 0$)

$$\mathrm{tr} \circ \iota_0\left((a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)\right) =$$

$$= \mathrm{tr}\left((1 \otimes 1) \otimes (a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)\right)$$

$$= \mathrm{Trace}(1\mu_0 \cdots \mu_m) 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_m =$$

$$= \iota_0\left(\mathrm{tr}\left((a_0 \otimes \mu_0) \otimes (a_1 \otimes \mu_1) \otimes \cdots \otimes (a_m \otimes \mu_m)\right)\right)$$

The components of the Chern character of an idempotent $e \in M_s(\mathcal{A})$ are then given up to normalization by

$$\mathrm{Ch}_m(e) := \mathrm{tr}\,((2e-1) \otimes e \otimes e \otimes \cdots \otimes e) \in \mathcal{A} \otimes \mathcal{A} \otimes \ldots \otimes \mathcal{A} \tag{29}$$

with m even and equal to the number of terms e in the right hand side. Now the main point in our context is the following general fact

Lemma 16 *Let $\mathcal{A}$ be an algebra (over $\mathbb{R}$) and $Y = \sum Y^A \Gamma_A$ with $Y^A \in \mathcal{A}$ and $\Gamma_A \in C_+ \subset M_w(\mathbb{C})$ as above, $n+1$ gamma matrices. Assume that $Y^2 = 1$. Then for any even integer $m < n$ one has $\mathrm{Ch}_m(e) = 0$ where $e = \frac{1}{2}(1+Y)$.*

Proof. This follows since the trace of a product of $m+1$ gamma matrices is always 0. ∎
It follows that the component $\mathrm{Ch}_n(e)$ is a Hochschild cycle and that for any cyclic n-cocycle ϕ_n the pairing $< \phi_n, e >$ is the same as $< I(\phi_n), \mathrm{Ch}_n(e) >$ where $I(\phi_n)$ is the Hochschild class of ϕ_n. This applies to the cyclic n-cocycle ϕ_n which is the Chern character ϕ_n in K-homology of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with grading γ where $\mathcal{A}$ is the algebra generated by the components Y^A of Y and Y'^A of Y'. By [7] Theorem 8, IV.2.γ, (see also [11] Theorem 10.32 and [2] for recent optimal results), the Hochschild class of ϕ_n is given, up to a normalization factor, by the Hochschild n-cocycle:

$$\tau(a_0, a_1, \ldots, a_n) = ⨍ \gamma a_0 [D, a_1] \cdots [D, a_n] D^{-n}, \quad \forall a_j \in \mathcal{A}.$$

Thus one gets that, by the index formula, for any idempotent $e \in M_s(A)$

$$< \tau, \mathrm{Ch}_n(e) > = < \phi_n, e > = \mathrm{Index}\,(D_e) \in \mathbb{Z}$$

Now by (29) for $m = n$ and the fact that D commutes with the two Clifford algebras $C_\pm$, one gets, with $Y = 2e - 1$ as above, the formula

$$< \tau, \mathrm{Ch}_n(e) > = ⨍ \gamma \left\langle Y\, [D, Y]^n \right\rangle D^{-n}$$

The same applies to Y' and we get

Theorem 17 *The quantization equation implies that (up to normalization)*

$$⨍ D^{-n} \in \mathbb{N}$$

Proof. One has, from the two sided equation,

$$\frac{1}{n!}\left\langle Z\,[D,Z]^n\right\rangle = \gamma$$

so that

$$⨍ D^{-n} = ⨍ \gamma\gamma D^{-n} = \frac{1}{n!}⨍ \gamma\left\langle Z\,[D,Z]^n\right\rangle D^{-n}$$

and using (9)

$$⨍ \gamma\left\langle Z\,[D,Z]^n\right\rangle D^{-n} = \frac{1}{2}⨍ \gamma\left\langle Y\,[D,Y]^n\right\rangle D^{-n} + \frac{1}{2}⨍ \gamma\left\langle Y'\,[D,Y']^n\right\rangle D^{-n}$$

which gives the required result after a suitable choice of normalization since both terms on the right hand side give indices of Fredholm operators. ■

6 Conclusions

In this paper we have uncovered a higher analogue of the Heisenberg commutation relation whose irreducible representations provide a tentative picture for quanta of geometry. We have shown that 4-dimensional Spin geometries with quantized volume give such irreducible representations of the two-sided relation involving the Dirac operator and the Feynman slash of scalar fields and the two possibilities for the Clifford algebras which provide the gamma matrices with which the scalar fields are contracted. These instantonic fields provide maps Y, Y' from the four-dimensional manifold M_4 to S^4. The intuitive picture using the two maps from M_4 to S^4 is that the four-manifold is built out of a very large number of the two kinds of spheres of Planckian volume. The volume of space-time is quantized in terms of the sum of the two winding numbers of the two maps. More suggestively the Euclidean space-time history unfolds to macroscopic dimension from the product of two 4-spheres of Planckian volume as a butterfly unfolds from its chrysalis. Moreover, amazingly, in dimension 4 the algebras of Clifford valued functions which appear naturally from the Feynman slash of scalar fields coincide exactly with the algebras that were singled out in our algebraic understanding of the standard model using noncommutative geometry thus yielding the natural guess that the spectral action will give the unification of gravity with the Standard Model (more precisely of its asymptotically free extension as a Pati-Salam model as explained in [5]).

Having established the mathematical foundation for the quantization of geometry, we shall present consequences and physical applications of these results in a forthcoming publication [6].

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References

[1] J. Alexander, *Note on Riemann spaces.* Bull. Amer. Math. Soc. 26 (1920), no. 8, 370-372.

[2] A.L. Carey, A. Rennie, F. Sukochev, D. Zanin, *Universal measurability and the Hochschild class of the Chern character* arXiv: 1401.1860.

[3] A. H. Chamseddine and A. Connes, *The uncanny precision of the spectral action* , Comm. Math. Phys. **293**, 867-897.

[4] A. H. Chamseddine and A. Connes, *Why the Standard Model* , J. Geom. Phys. **58**, (2008) 38-47.

[5] A. H. Chamseddine, A. Connes, W. van Suijlekom, *Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification* , JHEP **11** (2013) 132.

[6] A. H. Chamseddine, Alain Connes and V. Mukhanov, in preparation.

[7] A. Connes, *Noncommutative Geometry.* Academic Press, 1994.

[8] A. Connes, *A short Survey of Noncommutative Geometry*, J. Math. Phys. 41 (2000), no. 6, 3832-3866.

[9] A. Connes, H. Moscovici *The local index formula in noncommutative geometry*. Geom. Funct. Anal. 5 (1995), no. 2, 174-243.

[10] M. Golubitsky, V. Guillemin, *Stable mappings and their singularities.* Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973.

[11] J. Gracia-Bondia, J. Varilly, H. Figueroa, *Elements of noncommutative geometry*. Birkhauser Advanced Texts: Basler Lehrbacher. [Birkhauser Advanced Texts: Basel Textbooks] Birkhauser Boston, Inc., Boston, MA, 2001.

[12] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology,* volumes 1-3, and in particular pages 347-351 volume 2 (sphere maps).

[13] H. Hilden, J. Montesinos, T. Thickstun, *Closed oriented 3-manifolds as 3-fold branched coverings of S^3 of special type*. Pacific J. Math. 65 (1976), no. 1, 65-76.

[14] M. Hirsch, *Immersions of manifolds*. Transactions A.M.S. 93 (1959), 242-276.

[15] M. Iori, R. Piergallini, 4-*manifolds as covers of the* 4-*sphere branched over non-singular surfaces*. Geometry and Topology, 6 (2002) 393-401.

[16] J. Montesinos, *A representation of closed orientable 3-manifolds as 3-fold branched coverings of S^3*. Bull. Amer. Math. Soc. 80 (1974), 845-846.

[17] J. Moser, *On the volume elements on a manifold*. Trans. Amer. Math. Soc. 120 1965 286-294.

[18] V. Poenaru, *Sur la théorie des immersions*. Topology 1 (1962) 81-100.

[19] A. Phillips, *Submersions of open manifolds*. Topology 6 (1967) 171-206.

[20] A. Ramirez, *Sobre un teorema de Alexander* An. Inst. Mat. Univ. Nac. Autonoma Mexico, 15, No 1 (1975) 77-81.

[21] B. Sevennec, *Janus immersions in the product of two n-spheres.*

[22] S. Smale, *The classification of immersions of spheres in Euclidean spaces.* Ann. Math. 69 (1959), 327-344

[23] H. Whitney, *On the topology of differentiable manifolds.* Lectures in Topology, pp. 101-141. University of Michigan Press, Ann Arbor, Mich., 1941.

[24] H. Whitney, *The self-intersection of a smooth* n*-manifold in* $2n$*-space*, Annals of Math. 45 Vol 2 (1944) 220-246.