Classical Wheeler-DeWitt field and its Quantization

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Gravitating scalar field theory developed by re-interpreting the Wheeler-DeWitt equation as a classical field equation. Gravity and scalar fields are not coupled but unified into a single field. Although the classical theory is scalar, the corresponding quantum theory has spin-2. Whereas linearised quantum theory has scalar quantum.

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INTRODUCTION

The theory is developed by re-interpreting the Wheeler-DeWitt equation [1] as a classical field equation. Such an interpretation can be found in [2]. Unlike [2] this theory is not the third quantization. Φ is a gravitating scalar field defined only over 3-metric. 3-Ricci curvature scalar is an intrinsic property of this field just like mass is an intrinsic property of standard fields. Plane-wave limit of Φ (i.e., $e^{\pm iP^a q_a}$) gives a pure gravitational field. Quantization of full theory results in the quantum of spin-2. It does not have well-defined momentum. But quantization of linearized theory again gives scalar quantum and quantum has well-defined momentum.

The classical theory is discussed in the second section. This includes action formulation as well as Hamiltonian formulation. In section three, the theory applied to Schwarzschild spacetime and ordinary matter distribution. Quantum theory is done in the forth section. Linearized quantum theory is done in the fifth section. The sixth section summarises earlier sections. Possible implications, predictions are also discussed in this section.

WHEELER - DEWITT THEORY AS A CLASSICAL THEORY

Lagrangian formulation

The Wheeler-DeWitt equation which is considered to be describing the quantum nature of gravity [1] is re-interpreted as a classical field equation for Φ.

$$\left(\frac{\partial}{\partial q_{ab}}G_{abcd}\frac{\partial}{\partial q_{cd}} - \sqrt{q}R^{(3)}\right)\Phi = 0 \tag{1}$$

Since Φ is a functional of 3-metric alone, it satisfies diffeomorphism constraints.

$$D_b\frac{\partial \Phi}{\partial q_{bc}} = 0 \tag{2}$$

G_{abcd} is inverse DeWitt metric and its properties are derived in [1], Appendix A. It has signature $(-,+,+,+,+,+)$. Using co-ordinates (ζ, ζ^A) introduced by DeWitt (in 5.7, A23 and 5.10, [1]).

$$\zeta := \left(\frac{32}{3}\right)^{\frac{1}{2}}(\det \mathbf{q})^{\frac{1}{4}} \tag{3}$$

$$\bar{G}_{AB} := \mathrm{Tr}\left(\mathbf{q}^{-1}\frac{\partial \mathbf{q}}{\partial \zeta^A}\mathbf{q}^{-1}\frac{\partial \mathbf{q}}{\partial \zeta^B}\right) \tag{4}$$

$$\mathbf{q} := q_{ab}$$

in these co-ordinates DeWitt metric takes form

$$\begin{pmatrix} -1 & 0 \\ 0 & \frac{3}{32}\zeta^2\bar{G}_{AB}\end{pmatrix} \tag{5}$$

As analyzed by DeWitt, 5-dimensional manifold $\bar{M}$ is identified with $SL(3,\mathbb{R})/SO(3,\mathbb{R})$. Coordinates of this symmetric space are denoted by $\zeta^A = (r^1, r^2, x, y, z)$. r^1 and r^2 are two non-compact coordinates that can be thought of as radial co-ordinates on symmetric space. DeWitt has shown this manifold to be geodesically complete. Wheeler-DeWitt equation in these coordinates is re-written as (refer 5.20, [1])

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{32}{3\zeta^2}\frac{\partial}{\partial \zeta^A}\bar{G}^{AB}\frac{\partial}{\partial \zeta^B} - \frac{3}{32}\zeta^2 R^{(3)}\right)\Phi = 0 \tag{6}$$

The field is complex in general but I will be working with the real field only. The action functional for the corresponding field is assumed to be

$$\mathcal{A} = \int d\zeta \mathcal{D}\zeta \frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial \zeta}\right)^2 - \frac{32}{3\zeta^2}\frac{\partial \Phi}{\partial \zeta^A}\bar{G}^{AB}\frac{\partial \Phi}{\partial \zeta^B} + \frac{3\zeta^2}{32}R^{(3)}\Phi^2\right) \tag{7}$$

$\mathcal{D}\zeta$ is a measure over 5D manifold. If co-ordinates chosen as mentioned above then, an invariant measure can be taken as $\frac{dr^1 dr^2}{r^1 r^2}d\theta^1 d\theta^2 d\theta^3$. Invariance of an action functional under $\Phi \to \Phi + \delta\Phi$ gives (6). Invariance of an action under variation of $\zeta^\mu := (\zeta, \zeta^A)$ give conserved quantities (can also be found in [2])

$$T^\mu_\nu := \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi}{\partial \zeta^\nu}\right)}\frac{\partial \Phi}{\partial \zeta^\mu} - \mathcal{L}\delta^\nu_\mu \tag{8}$$

The geodesic distance between two 3-metrics can be written in the compact form (refer (A66), [1])

$$ds^2 = -d\zeta^2 + \zeta^2 d\bar{s}^2 \tag{9}$$

ds is a length element on the 6D manifold and $d\bar{s}$ is a length element on the 5D manifold (refer [1]).

Hamiltonian formulation

Momentum conjugate to Φ is defined as

$$\Pi := \frac{\partial \mathcal{L}}{\partial \frac{\partial \Phi}{\partial \zeta}} = \frac{\partial \Phi}{\partial \zeta} \tag{10}$$

The Hamiltonian is obtained by Legendre transformation

$$H = \int \mathcal{D}\zeta \frac{1}{2} \left(\Pi^2 + \frac{32}{3\zeta^2} \frac{\partial \Phi}{\partial \zeta^A} \bar{G}^{AB} \frac{\partial \Phi}{\partial \zeta^B} - \frac{3\zeta^2}{32} R^{(3)} \Phi^2 \right) \tag{11}$$

The Hamiltonian gives ζ evolution and equations of motion are govern by

$$\frac{\partial \Phi}{\partial \zeta} = \{\Phi, H\} \qquad \frac{\partial \Pi}{\partial \zeta} = \{\Pi, H\} \tag{12}$$

The situation for $R^{(3)} < 0$ is mathematically equivalent to $m^2 < 0$ coupling of the scalar field. In order for the Hamiltonian to be bounded from below, introduce $\lambda \Phi^4$ term with $\lambda > 0$. $\Phi = 0$ is not a maxima. The field roll down to either of the minima $\Phi_{\text{vac}} = \pm \sqrt{\frac{-\frac{3}{32}\zeta^2 R^{(3)}}{2\lambda}}$. For $R^{(3)} < 0$ quadratic coupling remains positive definite. Hamiltonian constraints corresponds tothis situation are $G_{abcd} P^{ab} P^{cd} - \sqrt{q} R^{(3)} + \lambda$.

- Discussion:

Baierlein, Sharp and Wheeler ([4],[5]) have shown in the classical theory that if the intrinsic geometry is given on any two hypersurfaces then, except in certain singular cases, total geometry of the entire space-time manifold is determined. DeWitt([1]) has shown that we hit *frontier* at $\zeta = 0$ and cannot be avoided by co-ordinate transformation. It can be mapped into singularities in some cases such as the big bang singularity, black hole singularity. These singularities of general relativity are mapped into classical Wheeler-DeWitt field theory. In the ADM theory, we get N, N^a and q_{ab} on solving equations of motion. But in the classical Wheeler-Dewitt theory, we get $\Phi(\zeta, \zeta^A)$ (or equivalently $\Phi(q_{ab})$) which is defined over 6D manifold (G^{abcd}, M). This manifold is discussed in [1]. In general, Φ can be a complex scalar field which can be written as a linear combination of two independent real scalar fields ($\Phi = \Phi_1 + i\Phi_2$). Such a theory has 2 degrees of freedom.

APPLICATIONS

The theory developed in the earlier section is applied to isotropic spacetimes. $N^a = 0$ is assumed for simplicity.

Static geometry

In absence of matter and static spacetime $K_{ab}K^{ab} - K^2 = 0$ and $R^{(3)} = 0$. The first condition gives

$$\frac{\partial^2}{\partial \zeta^2} \Phi = 0 \tag{13}$$

Assuming $\Phi|_{\zeta=1} = 0$.

$$\Phi = \Phi_0 (\zeta - 1) \tag{14}$$

Schwarschild solutions satisfy $R^{(3)} = 0$. Here, it is written in isotropic radial co-ordinates

$$g_{\mu\nu} dx^\mu dx^\nu = \alpha^2 dt^2 - f(\bar{r}) \left(d\bar{r}^2 + d\Omega^2 \right) \tag{15}$$

with $f(\bar{r}) = \left(1 + \frac{m}{2\bar{r}}\right)^4$ and $\alpha^2 = \frac{1 - \frac{m}{2\bar{r}}}{1 + \frac{m}{2\bar{r}}}$. In asymptotically flat limit

$$\Phi \propto \frac{m}{\bar{r}} \tag{16}$$

This is a gravitational potential due to particle of mass m at distance r. (13) also applicable when matter exists. In that case only functional form of ζ changes depending on solution to $R^{(3)} = -(8\pi G) T$.

Dynamical geometry

Asume $R^{(3)} = \frac{\varepsilon^2}{\zeta^2}$. Wheeler-DeWitt equation when extrinsic curvature is non-zero takes form

$$\left(\frac{\partial^2}{\partial \zeta^2} - \varepsilon^2 \right) \Phi = 0 \tag{17}$$

The field is oscillatory for $\varepsilon^2 < 0$. It is hyperbolic and essentially self coupled for $\varepsilon^2 > 0$. i.e., we add $\lambda\Phi^4$ term to make the Hamiltonian positive definite. Φ roll down from $\zeta^2 R^{(3)} = \varepsilon^2$ to $\zeta^2 R^{(3)} = -2\varepsilon^2$.

$$E = \frac{1}{2} \left(\frac{\partial \Phi}{\partial \zeta} \right)^2 - \frac{\varepsilon^2}{2} \Phi^2 + \lambda \Phi^4 \tag{18}$$

E is conserved quantity and can be interpreted as an energy of the Wheeler-DeWitt field Φ.

Appearance ε^2 does not necessarily tell if it is due to intrinsic curvature of 3 dimensional space itself or it is because of existence of ordinary scalar matter. Both are on equal footing concerning the extrinsic curvature term. On other hand, for $\varepsilon^2 < 0$ the field satisfying (17) gives $E = \frac{1}{2}|\varepsilon|^2$. That's precisely the matter energy differing just by factor $\frac{1}{2}$.

- Interpretation:

Φ is a gravitating scalar field. For isotropic static space, the intrinsic velocity of the field $\frac{\partial\Phi}{\partial\zeta}$ remains constant. The functional form of ζ obtained by solving 3-Ricci scalar shows that in an asymptotically flat spacetime Φ is a Newtonian gravitational field. For ordinary matter, Φ is oscillatory and phase space trajectories are elliptical. When curvature is positive, Φ is self-coupled. In this case topology of surface changes from $R^{(3)} = \frac{\varepsilon^2}{\zeta^2}$ to $R^{(3)} = -2\frac{\varepsilon^2}{\zeta^2}$. Φ accelerates in presence of an external source of gravity J (can be modeled by adding coupling term $J\Phi$). The theory shows both features gravity as well as scalar matter. So it is termed as a gravitating scalar field.

WHEELER - DEWITT FIELD: QUANTUM THEORY

The quantum nature of the relativistic field is known. Under canonical theories of gravity, loop quantum gravity successfully quantizes pure gravity. But this is a theory of gravitating scalar field. Therefore, it is natural to ask what is the quantum naure of the theory.

Usually, scalar field theory is quantized by going into the Fourier space of configuration space. Here, that procedure of quantization is extremely difficult because of nonlinearities of the theory. Instead, quantization is possible in the configuration space (ζ, ζ^A). Solution ω to Riccati equation (33) (which can be thought of as an effective intrinsic curvature term) allows us to write the Hamiltonian as a collection of harmonic oscillators. Define five sets of vector-valued annihilation and creation operators

$$a^A := \frac{1}{\sqrt{2}}\left(\Pi - i\sqrt{\frac{32}{3\zeta^2}}\bar{A}^{AC}\frac{\partial\Phi}{\partial\zeta^C} - i\omega\Phi\right)\hat{\zeta}^A \qquad (19)$$
$$a^{\dagger A} := \frac{1}{\sqrt{2}}\left(\Pi + i\sqrt{\frac{32}{3\zeta^2}}\bar{A}^{AB}\frac{\partial\Phi}{\partial\zeta^B} + i\omega\Phi\right)\hat{\zeta}^A$$

ω is a solution to (33). $\hat{\zeta}^A$ is 5-dimensional unit directed along A. The metric $\bar{G}^{AB}$ is positive definite and the space is symmetric. Therefore square root $\bar{A}^{AB}$ exists and is real. The non-trivial commutation relations satisfied by creation and annihilation operators are

$$\begin{aligned}&[a^A(\zeta,\zeta^C), a^{\dagger B}(\zeta,\zeta'^C)]\\ &= \begin{cases} i\sqrt{\frac{32}{3\zeta^2}}\bar{A}^{AC}\frac{\partial}{\partial\zeta^C}[\Pi,\Phi] + i\omega[\Pi,\Phi] & \text{if } A = B\\ 0 & \text{otherwise}\end{cases}\\ &= \left(\omega - \frac{\alpha(\zeta^A)}{\zeta}\right)\delta\left(\zeta^C,\zeta'^C\right)\delta^{AB}\end{aligned} \qquad (20)$$

$\alpha(\zeta^A) = \sqrt{\frac{32}{3\zeta^2}}\bar{A}^{AB}\frac{1}{\zeta^B}$. Here, $\frac{\partial}{\partial x}\delta(x) = -\frac{1}{x}\delta(x)$ is used. Operation between creation and annihilation operators is dot product in the 5 dimensional superspace and not a tensor product.

$$H = \int \mathcal{D}\zeta\left(\sum_A a^{\dagger A}a^A + \frac{D}{2}\left(\omega - \frac{\alpha(\zeta^A)}{\zeta}\right)\delta(0)\right) \qquad (21)$$

Refer Appendix for calculation. D is number of dimensions which can vary from 0 to 5. The second term is interpreted as a vaccum energy. The Hamiltonian can be thought of collection of Infinitely many harmonic oscillators. In general, the adjoint of an annihilation operator a is

$$\frac{1}{\sqrt{2}}\left(\Pi - i\sqrt{\frac{32}{3\zeta^2}}\bar{A}^{AC}\frac{\partial\Phi}{\partial\zeta^C} - i\omega^\star\Phi\right) \qquad (22)$$

Choosing real ω ensures self-adjointness of the Hamiltonian. In a region $\omega < \frac{\alpha(\zeta^A)}{\zeta}$, role of creation and annihilation operator get reversed. Ground state $u_0(\zeta,\zeta^A)$ is the one which gets annihilated by annihilation operator.

$$a^A u_0 = 0 \quad \text{for } \omega > \frac{\alpha(\zeta^A)}{\zeta} \qquad (23)$$
$$a^{A\dagger}u_0 = 0 \quad \text{for } \omega < \frac{\alpha(\zeta^A)}{\zeta}\Phi$$

and other eigen states are obtained by repeatedly applying creation operator. State functional $u(\zeta,\zeta^A)$ is five component object.

- Interpretation:

The Hamiltonian is a collection of five pairs of creation and annihilation operators. A quantum state is a five-component object indicating intrinsic property, i.e., spin. Each component of the spinor is a square-integrable function. The generators of $SL(3,\mathbb{R})/SO(3,\mathbb{R})$ form a spin $S=2$ under $SO(3)$ spin subalgebra. The theory remains quantum even when intrinsic curvature of 3-dimensional space is zero.
In general, the theory has two domains indicated by (23). Both have evolution opposite with respect to each other. Riccati equation appears because of non-linearity of the theory. Representation is known in (ζ,ζ^A) space and not in its fourier space where Riccati equation would have become algebraic equation. In general, the momentum which is identified by T^0_k components of (8) does not share Hamilton's eigenstates. Therefore the quantum of particular energy does not have well-defined momentum.

LINEARIZED QUANTUM THEORY

Unlike full theory, linearized theory allows standard quantization. Assume $(\zeta,\zeta^A) = (1+t, 1+x^A)$ (with $x^A \approx 0$) and $\bar{G}^{AB} \propto \delta^{AB}$

$$\Phi := \int \mathcal{D}k\,\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}e^{i\vec{k}.\vec{x}} + a^\dagger_{\vec{k}}e^{-i\vec{k}.\vec{x}}\right) \qquad (24)$$

$$\Pi := \int \mathcal{D}k\,(-i)\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(a_{\vec{k}}e^{ik.x} - a^\dagger_{\vec{k}}e^{-ik.x}\right) \qquad (25)$$

with $\omega_{\vec{k}}$ satisfying

$$\omega_{\vec{k}}^2 = \frac{32}{3}|\vec{k}|^2 - \varepsilon^2 \tag{26}$$

Creation and annihilation operators satisfy following commutaion relation

$$\left[a_{\vec{k}}, a_{\vec{k}'}\right] = \left[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}\right] = 0 \tag{27}$$

$$\left[a_{\vec{k}}, a^\dagger_{\vec{k}'}\right] = V\delta^{(D)}\left(\vec{k} - \vec{k}'\right) \tag{28}$$

V is a volume of phase space. For 2D dimensional phase space $V = (2\pi)^D$. Then Hamiltonian operator becomes

$$H = \int \mathcal{D}k\, \omega_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + \tfrac{1}{2}(2\pi)^D \delta(0)\right) \tag{29}$$

$$\vec{\mathbf{P}} = \int \mathcal{D}k\, \vec{k} a^\dagger_{\vec{k}} a_{\vec{k}} \tag{30}$$

The quantum of a linearized theory is scalar. It does not have two sectors with opposite time evolution. $|n, k^A\rangle$ are eigenstates of this Hamiltonian describing quantum in n-th state with momentum k^A. Unlike the full theory, eigenstates of Hamiltonian are eigenstates of momentum as well. Therefore this quantum of particular energy also has well-defined momentum.

DISCUSSION

Classical Wheeler-DeWitt field theory has both features gravity as well as scalar matter. Positive 3-Ricci curvature scalar is field theoretically unstable. Field roll down to get negative intrinsic curvature scalar. The quantum theory of corresponding scalar field theory happens to describe quantum of spin 2. But linearized theory describe scalar quantum. The quantum of spin-2 does not have well-defined momentum but the scalar quantum has well-defined momentum.

The quantum theory suggests that gravity is more fundamental and a scalar field arises out of gravity. This can happen either by forming a bound state of pair of gravitons or gravitons annihilating into the scalar particle. If Higgs is a bound state of gravitons, it would have a discrete energy spectrum. From which it is possible to infer gravitons. $\zeta^2 R^{(3)}$ has units of m^2 because $R^{(3)}$ is on equal footing with matter density. But whether it can be interpreted as mass or it is a generalization of mass is not clear yet. This needs further investigation. But such work can give a geometric perspective to Higgs mechanism.

Appendix

$$\tfrac{1}{2}\sum_A \left({a^\dagger}^A a^A + a^A {a^\dagger}^A\right) = \tfrac{1}{2}\left(\Pi^2 + \tfrac{32}{3\zeta^2}\tfrac{\partial\Phi}{\partial\zeta^A}\bar{G}^{AB}\tfrac{\partial\Phi}{\partial\zeta^B} + \omega^2\right)$$
$$+\omega\sqrt{\tfrac{32}{3\zeta^2}}\sum_A \bar{A}^{AB}\tfrac{\partial\Phi}{\partial\zeta^B}\Phi$$

Notice that

$$\tfrac{\partial}{\partial\zeta^B}\left(\omega\sqrt{\tfrac{32}{3\zeta^2}}\sum_A \bar{A}^{AB}\tfrac{1}{2}\Phi^2\right) - \tfrac{\partial}{\partial\zeta^B}\left(\omega\sqrt{\tfrac{32}{3\zeta^2}}\sum_A \bar{A}^{AB}\right)\tfrac{1}{2}\Phi^2$$
$$= \omega\sqrt{\tfrac{32}{3\zeta^2}}\sum_A \bar{A}^{AB}\tfrac{\partial}{\partial\zeta^B}\left(\tfrac{1}{2}\Phi^2\right)$$

The first term on the left hand side is a surface term. Assuming this term to vanish on the surface (after integrating over five dimensional manifold, we get

$$-\tfrac{\partial}{\partial\zeta^B}\left(\omega\sqrt{\tfrac{32}{3\zeta^2}}\sum_A \bar{A}^{AB}\right)\tfrac{1}{2}\Phi^2 \tag{31}$$

$$= \omega\sqrt{\tfrac{32}{3\zeta^2}}\sum_A \bar{A}^{AB}\tfrac{\partial}{\partial\zeta^B}\left(\tfrac{1}{2}\Phi^2\right) \tag{32}$$

ω is chosen such that it satisfy the following Riccati equation

$$\omega^2 - \frac{\partial}{\partial\zeta^B}\left(\omega\sqrt{\frac{32}{3\zeta^2}}\sum_A \bar{A}^{AB}\right) = -\frac{3}{32}\zeta^2 R^{(3)} \tag{33}$$

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