$\mathcal{H}$olographic $\mathcal{N}$aturalness and $\mathcal{T}$opological $\mathcal{P}$hase $\mathcal{T}$ransitions

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We show that our Universe lives in a topological and non-perturbative vacuum state full of a large amount of hidden quantum hairs, the *hairons*. We will discuss and elaborate on theoretical evidences that the quantum hairs are related to the gravitational topological winding number in *vacuo*. Thus, hairons are originated from topological degrees of freedom, holographically stored in the de Sitter area. The hierarchy of the Planck scale over the Cosmological Constant (CC) is understood as an effect of a Topological Memory intrinsically stored in the space-time geometry. Any UV quantum destabilizations of the CC are re-interpreted as Topological Phase Transitions, related to the desapparence of a large ensamble of topological hairs. This process is entropically suppressed, as a tunneling probability from the N- to the 0-states. Therefore, the tiny CC in our Universe is a manifestation of the rich topological structure of the space-time. In this portrait, a tiny neutrino mass can be generated by quantum gravity anomalies and accommodated into a large N-vacuum state. We will re-interpret the CC stabilization from the point of view of Topological Quantum Computing. An exponential degeneracy of topological hairs non-locally protects the space-time memory from quantum fluctuations as in Topological Quantum Computers.

Introduction: topological aspects of the cosmological stabilization. In our recent works, we elaborated on the holographic stabilization of the Cosmological Constant (CC) from the large entropy content *in vacuo*. The CC problem is re-thought considering a holographic decoherence induced by a large ensamble of dynamical quantum hairs, dubbed hairons [1, 2][1]. All CC quantum instabilities are suppressed as

$$e^{-S} \sim e^{-N}\,,$$

where S is the Universe entropy and N is the number of hairons. We will elaborate on this picture and on the multitude of inter-related questions: *what are the hairons? How are they originated from? What is the mechanism behind the quantum information storage and memory in space-time?*

In this paper, we explore a curious and potentially insightful relation of **the N** with the gravitational instantons and the quantum gravity topological sector. We show that the entropic solution of the CC problem is also related to a *topological protection* against CC-Planckian mixings. Intriguingly, this corresponds to the Topological Phase Transitions interpolating vacuum states with different gravitational topological winding numbers.

Let us consider the topological sector of gravity:

$$\mathcal{S} = \int \mathcal{E}_G\,, \tag{1}$$

where

$$\mathcal{E}_G = dC_G\,, \quad C_G = \Gamma d\Gamma - \frac{3}{2}\Gamma\Gamma\Gamma\,,$$

$$\mathcal{E}_G = R\tilde{R}\,. \tag{2}$$

The $\mathcal{E}_G$ is a topological term; its integral provides for the winding integer N:

$$\mathcal{S}_N = N\,. \tag{3}$$

There is a multitude of deep relations among the topological action, the CC, the Planck scale and the winding number:

$$\mathcal{S}_N = \frac{1}{\alpha_G(\Lambda_N)} = \frac{M_{Pl}^2}{\Lambda_N} \sim N\,. \tag{4}$$

An Euclidean action as Eq.4 also corresponds to the entropy of the thermodynamical space-time system:

$$S = \log\Omega \sim N\,, \tag{5}$$

where Ω is the configuration space. This is leading us to a simple identification of the topological winding number and the entropic content of the space-time. In this sense, the space-time entropy is re-interpreted as a topological index.

Therefore, a so tiny CC, as the one observed in our Universe (around $\Lambda \sim 10^{-123} M_{Pl}^2$), would correspond to the

$$\bar{N} \sim 10^{123}$$

vacuum eigenvalue of the $|\bar{N}\rangle$ (see [1, 2]).

Rephrased in this way, the quantum stability issue of the CC appears as a problem of why the $|\bar{N}\rangle$ eigenstate would not spontaneously flow to the $|0\rangle$, in turn corresponding to the UV divergent case as $\Lambda = M_{Pl}^2/N \to \infty$ ($N \to 0$). To compute the quantum bubble diagrams would be re-thought, not only as an integral on the momenta, but also as a divergent series of the topological

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[1] See also Ref.[3] for further considerations on the relation among quantum gravity and the CC

winding number. Indeed, a particle with a momenta p would probe a number of hairons as $N \sim M_{Pl}^2/p^2$, i.e. the UV momentum corresponds to a IR topological number divergence.

Therefore, the spontaneous flow of the CC from the IR to the UV corresponds to the disapparence of N winding numbers, costing N-entropic units:

$$\langle 0|N\rangle \,. \tag{6}$$

This transition thermodinamically costs as an exponential suppression, i.e. as the Eq.1. This process is also interpreted as a quantum tunneling, related to an N-instanton. Thus, the quantum destabilization of the CC would cross several orders of magnitude, rendering it probabilistically forbidden as

$$\mathcal{P}(N \to 0) \sim e^{-\Lambda/M_{Pl}^2} \sim e^{-10^{123}} \,. \tag{7}$$

This is offering a simple re-interpretation of the CC problem. When the theoretical physicist calculates the vacuum to vacuum correlator,

$$\langle 0|\Phi(x)\Phi(x)|0\rangle \tag{8}$$

of any SM fields indicated here as Φ, he/she finds a disappointing result: quantum field theories typically predict quartic UV divergences. However, this correct result is typically misunderstood and misleadingly interpreted as an instability of the CC state. Indeed, a possible instability crossing from the IR to the UV domains is passing from a non-trivial correlator

$$\langle N|\Phi(x)\Phi(x)|0\rangle \,. \tag{9}$$

Rephrased in this way, quantum instabilities would work into the system for re-turning from a completely disorder to a fully ordered state. Indeed,

$$\langle N|\Phi(x)\Phi(x)|0\rangle = e^{-N}\langle 0|\Phi(x)\Phi(x)|0\rangle \,. \tag{10}$$

where

$$e^{-N} = e^{-S} = \frac{1}{\Omega(N)} \,,$$

and $\Omega(N)$ is the probability configuration volume.

Therefore, any quantum fluctuations cannot destabilize a maximal entropic state because of the large entropic barrier. The high entropy state can reach the UV domain only through an exponentially suppressed tunneling.

On the other hand this is naturally leading to the concept of an emerging CC as a thermal effect of a large number of hairons accumulated in vacuo:

$$T_{hairons} \sim \sqrt{\Lambda} \sim \frac{1}{\sqrt{N}} M_{Pl} \,. \tag{11}$$

It is certainly interesting that the number of hairons scales as the Topological winding number. Indeed, this is a signal that the entropic protection from CC-Planckian mixings is deeply related to the topological space-time properties. In other words, the $|0\rangle$ and the $|N\rangle$ have an inequivalent topological structure that can never be interpolated by a trivial geometric deformation on the space-time manifold. The $|N\rangle$ and the $|0\rangle$ can be transformed each others through a N-instanton with a multi-spherical topology. In other words, the dictionary among the N and the 0-vacuum states and the instanton topology is as follows:

$$\langle 0|N\rangle : \quad S_4 \to_{tunneling} \to S_2 \times S_2 \times .... \times S_2 = \prod_{n=0}^{N} (S_2)^n \,. \tag{12}$$

The $0-N$ quantum transition has a precise topological and geometrical sense, corresponding to a topological phase transition of the space-time. As we will see, this is related, in the real (non-euclidean) space-time, to gravitational topological defects puncturing the space-time boundary. In other words, the N labels the space-time defects. This also implies that the fundamental temperature in *vacuo*, providing for the CC, is emerging as a topological effect from a large N of gravitational defects. On the other hand, Eq.17 corresponds to a first order phase transition of the gravitational susceptibility in vacuo as

$$\langle R\tilde{R}\rangle_{N=0} \to \langle R\tilde{R}\rangle_{N} \,. \tag{13}$$

As we will see, this fact is leading to several potential breakthroughs towards our understanding of space-time memory and CC stabilization.

Dynamical Relaxation. Let us now promote the N to a dynamical field, as

$$N \to \varphi(x) \,, \tag{14}$$

where $\varphi(x)$ is the relaxon field. This corresponds to

$$\varphi(x) = \frac{M_{Pl}^2}{\Lambda(x)} \,, \tag{15}$$

where Λ is thought as dynamical. If $\varphi(x)$ evolves as a runaway field to the asymptotic infinite, then the CC will dynamically flow to the large-N-vacua. This mechanism is understood as Topological Phase Transitions among the different topological vacuum states labelled by the winding number. The relaxation mechanism can be specialized in a cosmological set-up. Indeed, Eq.15 in CC corresponds to the Hubble rate of the Universe. Considering a scalar field in cosmology, provoking the expansion of the Universe, it would be related to the Hubble rate as

$$\varphi = \frac{1}{H^2} = \frac{1}{\frac{1}{2}\dot{\phi} + V(\phi)} \,, \tag{16}$$

where the other ϕ field is a Dynamical DE scalaron. If the potential drives the scalar field ϕ to zero, then φ will run-away to a an asymptotic attractor point. This is interpreted as the dynamical generation of N-quanta, topologically stored in every Planckian volume and, therefore, appearing out with the space-time expansion. This process would be spontaneous since related to the entropic attractor

$$S=\varphi(t)=\frac{M_{Pl}^2}{\Lambda(t)}\,, \tag{17}$$

In other words, scenarios where the ϕ tends to relax to zero are probabilistically favored. Indeed, for maximizing the entropy, the φ field increases without any upper bound. Such a phenomena is interpreted as a dynamical proliferation in time of the number of hairons, corresponding to the CC screening effect [2].

What is the topological meaning of such a relaxation mechanism? The N winding number was promoted to a dynamical field; the fact that the φ increases up to infinity is interpreted as a spontaneous increasing of the topological complexity of the space-time. Such a phenomena corresponds to a spontaneous cascade of topological transmutations. This process asymptotically tends to accumulate an infinite number of gravitational defects in space-time. In other words, the φ can be interpreted as a topological order mean field.

Dark Energy and Neutrino mass. The Cosmological Relaxation mechanism can be related to the generation of neutrino masses from gravitational anomalies [7].

Indeed gravitational instantons induce anomalous terms that can dynamically break the $U_A(1)$ axial symmetry, generating a neutrino mass term from the neutrino current anomaly:

$$\partial_\mu J_5^\mu=\mathcal{E}_G\,. \tag{18}$$

If the vacuum state would be empty of the N-quanta, then, paradoxically, the anomalous term would be divergent

$$\langle 0|\,\mathcal{E}_G\,|0\rangle\sim \lim_{N\to 0}\Lambda_N\to\infty\,. \tag{19}$$

Reversing this argument, this may be a hint that we do not live in a trivial vacuum state, otherwise the neutrino would be affected by a new hierarchy problem. Fortunately, this is exactly the same issue just solved above for the CC. Therefore, in N-state of our Universe, corresponding to the observed Hubble rate, the lightest neutrino mass is

$$m_\nu^4\sim G_N^{-1}\Lambda_N\sim\frac{1}{N}M_{Pl}^4\sim 10^{-123}M_{Pl}^4\sim(1\,\mathrm{meV})^4\,. \tag{20}$$

In this sense, also the neutrino mass hierarchy is understood as a manifestation of the topological memory storage in space-time. On the other hand, the dynamical DE mechanism would relate the CC flow to zero to a dynamical neutrino mass. This also means that the neutrino mass is generated and stabilized as an environmental and thermal effect in the quantum criticality point.

Coherent state portrait. It exists a general duality among instantons and solitons (see for example Refs.[4]). The gravitational vacuum state storing hairons can be interpreted as a gravitational solitonic state, while the corresponding gravitational instantons as a tunneling of a soliton from nothing.

The soliton can be viewed as a coherent state of N-hairons, in turn related to the topological winding number. The solitonic quantum state can be expressed as a tensor product of coherent states [8, 9]:

$$|S\rangle=\prod_{\otimes k}|\alpha_k\rangle \tag{21}$$

where

$$|\alpha_k\rangle=e^{-\frac{1}{2}|\alpha_k|^2}\sum_{n_k=0}^{\infty}\frac{\alpha_k^{n_k}}{\sqrt{n_k!}}|n_k\rangle\,. \tag{22}$$

where $|n_k\rangle$ are number eigenstates of hairons with momentum k .

The mean occupation number is defined as

$$N=\int_k N_k=\int_k \alpha_k^*\alpha_k\,, \tag{23}$$

where

$$\hat{N}_k=\hat{a}_k a_k\to N_k=\langle S|\hat{a}_k^\dagger a_k|S\rangle\,. \tag{24}$$

An intrinsic energy scale of the soliton can be obtained as

$$E=\int_k |k|N_k\,. \tag{25}$$

Considering any hairons as Planckian, a Universe-size soliton corresponds to

$$E=\sqrt{N}M_{Pl}\,. \tag{26}$$

Within this picture, the enucleation of the hairon solitonic state [3] is

$$\langle 0|S\rangle=e^{-N}\,. \tag{27}$$

This transition amplitude is related to the solitonic synthesis from nothing, in turn dual to a gravitational instanton solution. Thus, the instantonic language can be rephrased in terms of the coherent state formalism.

Cosmological Phase Transitions and Criticality. Now, we move on the analogy among gravitational solitons

[2] An alternative axion-inspired approach can be found in Ref.[5]. Another self-adaptive holographic mechanism was suggested in Ref.[6].

[3] To be more accurate, there would also be a factor 1/2 in the exponential, essentially not important in our discussions.

with other known critical systems. As remarked by Kibble and Zurek, topological defects can appear in the early Universe or in condensed matter system around a critical temperature T_c [10–14]. An important example of this phenomena is the notorious Berezinskii-Kosterlitz-Thouless transition (BKT transition) [15]. It is related to a phase transition in the two dimensional XY models, in turn well describing a certain class of 2D systems in condensed matter – including a large class of thin disordered superconductors. Around a critical temperature T_c, the BKT transition is related to the the transition from a pair to an unpaired couple of vortices and anti-vortices. Indeed, in the 2D XY models, the (anti)vortices are topologically stable structures. The production of vortices becomes thermodynamically efficient around the T_c. In other words, for $T < T_c$, the bounded vortex-antivortex system has a lower energy and entropy than the two decouple ones.

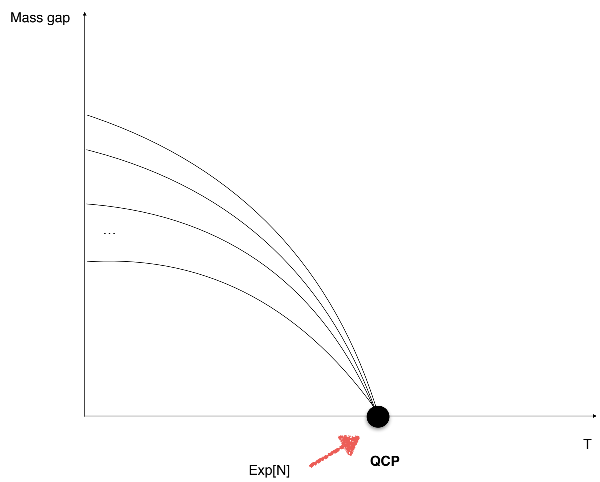

FIG. 1. The de Sitter space-time lives on a topological quantum critical point with an exponential degeneracy. The mass gap among the N-states flows to zero around the critical temperature $T_c \sim \sqrt{\Lambda}$, where Λ is the CC. The degeneracy corresponds to the hairon number N on the same ground state.

The critical temperature is understood as the saddle solution of the free-energy as

$$F = E - TS \rightarrow T_c\,. \tag{28}$$

Now, we will rise a similar question in the *arena* of quantum gravity. Is there any critical temperature related to a topological phase transition?

We will see that the answer is **yes** and it is exactly related to the Hawking temperature. Indeed, the minimal possible F for a thermodynamical system with a fixed internal energy – in the BH case its own mass – corresponds to the maximal entropy. In turn, the entropy is maximized if all the information is re-organized in a BH holographic state. The topological defects mentioned above are here replaced by gravitational solitons and instantons.

Indeed, in the case of BHs and de Sitter, the partition function is dominated by the euclidean action

$$\mathcal{S}_N = \beta \bar{E} = \frac{\beta^2}{L_{Pl}^2} = N \tag{29}$$

as follows

$$Z = \sum \exp(-\beta E) \simeq \exp(-N + O(1/\sqrt{N}))\,. \tag{30}$$

The free-energy is extremized on the saddle solution Eq.59, corresponding to

$$F = \langle E \rangle_{BH}/2 = T_c S \sim \beta_c \sim \sqrt{N} L_{Pl}\,, \tag{31}$$

where $\langle E \rangle \sim M_{BH}$ is the BH internal energy, corresponding to the BH mass (here we omit the numerical prefactors, as inessential for our discussions while $G_N = 1$). This is related to the Holographic entropy and temperature of the de Sitter space-time. This means that a Universe on a de Sitter phase lives on a state of criticality. On the other hand, Eq.31 does not correspond to a global minima, the F derivative is negative on the critical temperature

$$\frac{dF}{dT_c} = -\frac{1}{T_c^2}. \tag{32}$$

The fact that both the free-energy and its derivative diverge for $T_c \rightarrow 0$ can be interpreted as the evidence that the zero CC corresponds to a state of super quantum criticality and a first order phase transition. Indeed, Eq.32 is also related to the divergence of the heat capacity of de Sitter space-time.

In analogy with the phase transition theory and critical phenomena, one would imagine that this corresponds to a divergence of the correlation length

$$\zeta \sim (T - T_c)^{-\nu}, \tag{33}$$

where ν is a critical exponent. This is certainly self-consistent with the (A)dS/CFT approach, as the system is around a conformal critical point. We will see later that this fact has a precise topological entanglement entropy interpretation. Such a phenomena is related to the emergence of topological order around the critical temperature T_c. From the macroscopic space-time portrait, this corresponds to a high degeneracy of the ground state. Microscopically, this is related to a long-range quantum entanglement. The Black Hole is viewed as a highly degenerate and fully entangled N-state as typically happening in topological systems [4].

[4] Similar ideas were proposed in the Gravitational Bose-Einstein condensate approach, where the degeneracy is understood as a large number of gapless Bogoliubov modes [21–23]. It is certainly possible that our approach would be a dual portrait capturing the same quantum critical phenomena from the topological prospective.

In the sense of topological phase transitions, the entropy assumes the role of a topological index of the space-time complexity: the entropy is related to the space-time Euler characteristic or the genus, in turn related to the topological winding number. Indeed, the entropy phase transition is intimately related to the jump of the gravitational susceptibility

$$\langle R\tilde{R}\rangle_{N=0} \to \langle R\tilde{R}\rangle_{M=N}\,.$$

How we will see in next sections, this fact has an interpretation in a BH portrait from topological quantum computing.

In the case of de Sitter, a dynamical system with the $\varphi \to \infty$ would correspond to a walking critical temperature as $T_c \to 0$. When the criticality is reached, this can dynamically walk to increase the degeneracy in vacuo.

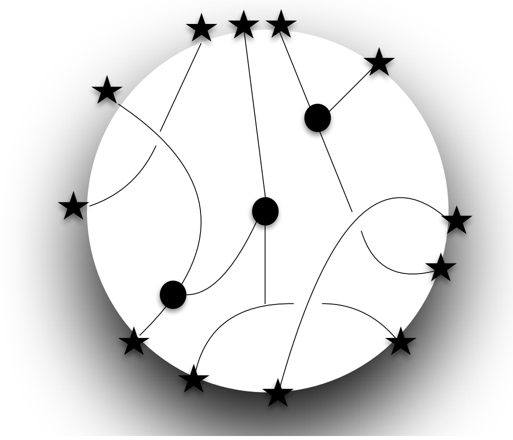

FIG. 2. A space-like section of the dS_3 bulk is shown. Gravitational Wilson lines or Semiclassical Wormholes intersect the horizon providing topological hairs (black stars). They can have braids or fusions (black dots).

From the scattering processes, the BH topological phase transition is dynamically reached from localizing a transplanckian CM energy into an impact parameter of $b \leq R_S = 2G_N\,E_{CM} \sim T_c^{-1}$, where R_S is the Schwarzschild radius (here we preferred to re-insert G_N). Aspects of scatterings are considered in another paper in preparation.

Hairons as Punctures on the Horizon. We now elaborate on the idea that holographic hairons correspond to bulk defects. A possibility, inspired by the Loop Quantum gravity approach (see Refs.[16, 17]) is that hairons are topological punctures on the BH or de Sitter horizons generated by Gravitational Wilson lines in the Bulk geometry. From the semiclassical approach, conversely, this can be related to gravitational quantum tunnels puncturing the BH horizon [18–20]. A third vision of the phenomena is in terms of gravitational coherent states ramifying in the bulk by means of the duality among instantons and coherent wave functions.

The action would correspond to a topological sector coupled with the puncture contribution:

$$N = \mathcal{S}_E = \int_{\Delta/\{\prod_{p=1}^{N}\{p\}\}} (\Gamma d\Gamma + \frac{2}{3}\Gamma^2)\,. \qquad (34)$$

where Δ is the continuous space-time Horizon and $\prod_p\{p\}$ is the ensamble of points puncturing the horizon.

Now, an interesting question is as follows. Let us suppose to have a quantum state of hairons as

$$|h_1, ..., h_N\rangle\,. \qquad (35)$$

What is the hairon spin-statistics? Are they bosons or fermions? What happens to the Eq.35 under permutational operations acting on hairon fields?

Naively one would think that, since we are in $3+1$ dimension, the permutation of a couple of hairons would just give a ± 1 corresponding to boson or fermion statistics. However, we will show a simple argument to convince the reader that this is not what is happening in our case: the hairons have an *anyon statistics* as happening in 2D topological materials [24–28, 32] [5].

To fix our idea, let us start from comparing the hairon picture with what happens to the case of two normal point-like particles and hairons. In 3D, one path γ_2, encircling the first particle can be always continuously deformable to another path γ_1 that does not encircle the second particle. In other words, the path can be deformed in such a way to pass just behind the second particle. On the other hand, the γ_1 loop can be contracted to just a point. This corresponds to the condition, on the wave function of the system, as

$$|\psi(\gamma_2)\rangle = |\psi(\gamma_1)\rangle = |\psi(0)\rangle\,. \qquad (36)$$

Eq.36 means to relate the wave function of the double circling path to the one with no any circling as

$$|\psi(\gamma_2)\rangle = R^2|\psi(0)\rangle,\;\; R^2 = 1\,. \qquad (37)$$

Thus, we can have only to cases, corresponding to bosons and fermions:

$$R_{B=+1,F=-1} = \pm 1\,.$$

However, for hairons, this is expected to be not true: a path circulating around one hairon cannot be contracted without crossing the gravitational Wilson lines or the gravitational instantons having the hairon punctures as

[5] It is possible that such an effect may be testable considering topological scatterings of Standard Model particles on space-time anyons. These effects may percolate into effective tiny violations of the Spin Statistics in the SM sector, testable in underground experiments with high precision [29–31].

edge states. Therefore, in our case, R would have a more general form [6] as

$$R = e^{iT}\,. \tag{38}$$

In general, the R transformation can be related to abelian or non-abelian structures, in turn corresponding to T as just a phase number or a matrix. In the case of quantum gravity, we would expect a non-abelian nature of anyonic hairons, for two motivations: i) from the algebraic prospective, gravity is more similar to non-abelian Yang-Mills theories rather than QED: if reformulated as a gauge theory, it would correspond to a $SO(3,1)$ local group; ii) non-abelian anyons are related to an exponential degeneracy of quanta populating the lowest energy level, compatible with the criticality condition envisaged in the case of Black Holes as well as for their exponentially enhanced memory storage e^N. In a de Sitter space-time, the energy levels (to not be confused with the CC levels) correspond to

$$E_N^2 = N M_{Pl}^2 \tag{39}$$

with a degeneracy of

$$S_N \sim e^N\,.$$

The next level is at $N-1$, therefore the level splitting is

$$\Delta^2_{N-N-1} = M^2_{Pl}\,. \tag{40}$$

The degeneracy of any next level is scaling as the N-number. By the way, this also offers a Fermi Golden Rule explanation of why transitions from the N-level to the zero one are exponentially disfavored as dressed by the density state factor

$$\rho_1/\rho_N \sim e^{-N}\,. \tag{41}$$

It is worth to remark that the R-transformation does not change the ground state Eq.39 of the system, compatible with the exponential degeneracy.

Elaborating on the duality among gravitational instantons and gravitational Wilson lines, puncturing the horizon, we wish to propose a possible vision of how black holes as well as the de Sitter Universe may store quantum informations. The punctures on the Horizon can correspond to several different world-lines. As we said before, punctures may be re-interpreted as our hairons. The puncture corresponds to a sort of gravitational defect in the bulk geometry, very much in analogy with the Ahranov-Bohm effect. Therefore, the BH can elaborate and store information in a large number of possibilties provided by the different *braids* on the hairon world-lines. This picture is inspired by topological quantum computers, tipically storing qu-bits with several different possibilities of braiding the anyons (see Ref.[33] for a complete review on these subjects). The idea is that information infalling inside a black hole is computed as a series of braidings. Hairons are non-abelian anyons that obey to non-Abelian braiding statistics. In space-time, Quantum information is stored in states with multiple hairons, which have a topological degeneracy and braids.

Let us consider the system of N non-abelian anyons in the ground degenerate state: $|\Psi_n(z_1, z_2, ..., z_N)\rangle$, where z_j are the hairon coordinates and $n = 1, ..., D$ labels for a D-dimensional protected subspace. We can consider the γ-path around the z_j, winding one anyon around another. Let us consider the following transformation on the hairon state:

$$|\Psi_n(z_1, z_2, ..., z_N)\rangle \to \sum_{m=1}^{D} W_{nm}(\gamma)|\Psi(z_1, z_2, ..., z_M)\rangle\,, \tag{42}$$

where

$$W(\gamma) = \mathbf{P}\exp\oint_\gamma \mathbf{\Gamma}\cdot d\mathbf{z}\,, \tag{43}$$

where $\mathbf{P}$ represents the path ordering. W is nothing but a Gravitational Wilson line.

The W transformation can be viewed as

$$W = F^{-1}R^2F\,, \tag{44}$$

where F is describing the Fusion among hairons into a new hairon. At this point of the discussion, this equivalence may be not clear, but we wish to shine light on these aspects in the following. In our portrait, the hairon evolutions are limited by these three rules: i) they can be created or annihilated, in pairwise fashion; ii) hairons can be fused to generate other hairons; iii) hairons can be exchanged as in Eq.42. The F(usion) in a collective non-local property of the hairons, as intrinsically non-local is the nature of the Wilson-lines.

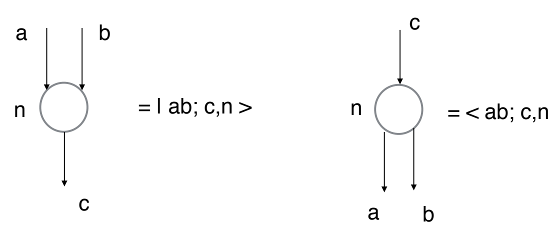

FIG. 3. A diagramatic representation of the anyon states.

In the following, we will elaborate on the topological quantum operations.

First of all, the hairons, or gravitational anyons, can interact to form a new one; the non-abelian interaction of them is related to a non-abelian product as

$$a \otimes b = \sum_c N^c_{ab} c \tag{45}$$

[6] Let us remark that R may also be no-unitary, as well as T non-hermitian if we consider non-abelian non-unitary groups as $SO(3,1)$ or $SU(1,1)$

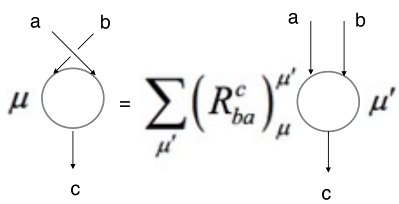

FIG. 4. A diagrammatic representation of the F(usion) operation is displayed.

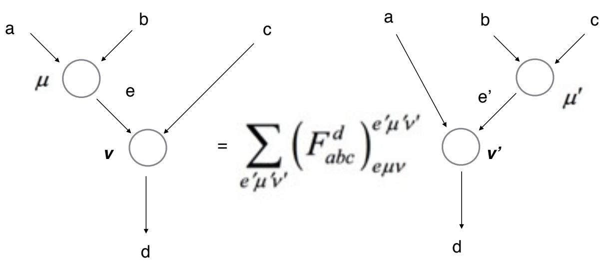

FIG. 5. A diagrammatic representation of the R operation is displayed.

where

$$a \otimes b = b \otimes a, \quad a \otimes 1 = a\,. \tag{46}$$

The fusion processes correspond to a vector space denoted as N_{ab}^{c}, dubbed topological Hilbert spaces (see Fig.3). A complete orthogonal basis describing the anyon system reads as

$$\{|a, b, c, \mu\rangle;\ \mu = 1, ..., N_{ab}^{c}\} \tag{47}$$

The R and F transformations acting on the hairon diagrams have simple pictorial representations as in Fig. 4-5. The two diagrams correspond to the state transitions:

$$|(ab)c; ec; d\rangle = \sum_{f} (F_{abc}^{d})_{ef} |a(bc); af; d\rangle\,, \tag{48}$$

$$|(ba)c; ec; d\rangle = \sum R_{ab}^{f} \delta_{e,f} |(ab)c; ec; d\rangle\,, \tag{49}$$

where δ is the Kronecker delta function and f spans all possible out-results of a, b.

This is suggesting an information computation through hairon/anyon exchanges. A topological computation can be initiated as a sequence of hairons and then, performing a sequence of particle exchanges, the hairon worldlines will trace out braids! In these processes also fusions are possible and in principle the final state can end with a complete annihilation process, but, indeed, with an entropic cost.

Such a computational system can also be dually reinterpreted as a neural network computation in the bulk, where hairons are the neural centers interconnected by gravitational synapsis extending on the bulk geometry (see Ref.[34] for recent attempts on similar research directions).

The computational operations of the gravitational anyons form a braid group related to the N-punctures on the horizon as $\mathcal{B}_N$ (See for example Ref.[35]). As we mentioned, any braid is related to F and R operations as

$$B = F^{-1} R F \in \mathcal{B}_N \tag{50}$$

The generators of $\mathcal{B}_N$ can be viewed as clockwise interchanges of the ith with the $i+1$th lines. Let us denote this generator as σ_i. The inverse operation σ_i^{-1} corresponds to a counter-clockwise rotation of ith and $i+1$th. The generators satisfy the following algebraic conditions:

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \text{if } |i-j| = 1\,, \tag{51}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i-j| > 1\,. \tag{52}$$

These relations are nothing but related to the notorious Yang-Baxter equations.

This can be defined as

$$\mathcal{B}_N = \{\sigma_1, ..., \sigma_N | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i\}\,. \tag{53}$$

The Braids provide for Logical gates as a basis for space-time quantum computation.

The abelianization of this group is a homomorphism mapping every σ_i to 1:

$$h:\, \mathcal{B}_N \to \mathbb{Z} \tag{54}$$

which also related with the winding number, in turn contributing to the space-time entropy.

Now, we wish to visualize the BH topological computation in terms of the quantum computing language. In doing it, we need to identify the information computing in terms of the ($|0\rangle$,$|1\rangle$) qu-bit basis.

Motivated by the multi-instantonic picture, we consider pairs of gravitational flux and anti-flux units as $|\alpha, \alpha^{-1}\rangle$. This can be visualized as the two edge points of a Wilson line puncturing the horizon. Let us consider two fluxon pairs $|\alpha, \alpha^{-1}\rangle$ and $|\beta, \beta^{-1}\rangle$. We can realize a basic gate by winding counterclockwise the $|\alpha, \alpha^{-1}\rangle$ around the $|\beta, \beta^{-1}\rangle$. This operation transforms the first fluxon state as

$$|\alpha, \alpha^{-1}\rangle \to |\beta\alpha\beta^{-1}, \beta\alpha^{-1}\beta^{-1}\rangle\,. \tag{55}$$

By means of this definition, we can identify a computational basis as follows:

$$|0\rangle = |\alpha, \alpha^{-1}\rangle, \ \ |1\rangle = |\beta\alpha\beta^{-1}, \beta\alpha^{-1}\beta^{-1}\rangle\,. \tag{56}$$

This is not the only possible basis choice, but it is certainly a simple one. Indeed the single isolated α appears as identical to the $\beta\alpha\beta^{-1}$ as non-locally next to each others. In this case, the states in Eq.56 do not have any

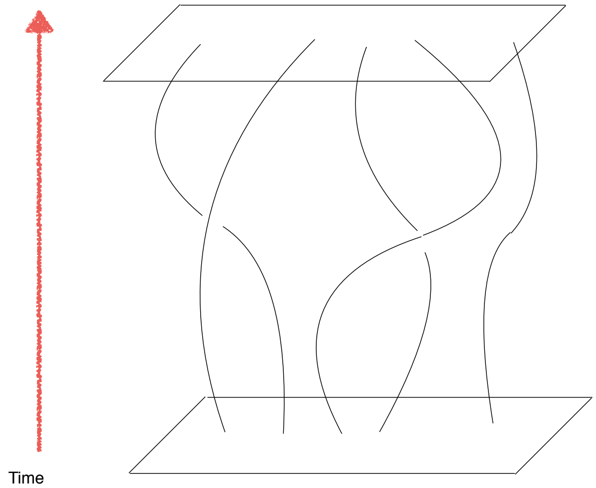

FIG. 6. The topological quantum computing of space-time through gravitational anyonic braids.

quantum superposition. In other words, quantum memory is protected by decoherence induced by the environment since non-locally stored as a phase effect induced by the fluxon-antifluxon entanglement.

An initial $|\bar{N}\rangle$ state corresponds to a sequence of qubits; for example as

$$|0,1,0,1,1,...,\rangle \equiv |\alpha,\alpha^{-1}\rangle \otimes |\beta\alpha\beta^{-1},\beta\alpha^{-1}\beta^{-1}\rangle \otimes ... \otimes ... \quad (57)$$

The dimensionality of this state is related to the total transition amplitude mediating the complete annihilation of all fluxon pairs into nothing. This amplitude is interpolated by the many Fusion processes as dictated by Eq.45:

$$N^{\beta}_{\alpha\alpha\alpha...\alpha} = \sum_{\{\beta_i\}} N^{\beta_1}_{\alpha\alpha} N^{\beta_2}_{\alpha\beta_1} N^{\beta_3}_{\alpha\beta_2} ... N^{\beta}_{\alpha\beta_N} \quad (58)$$

$$= \langle\beta|(N_\alpha)^N|\alpha\rangle \,,$$

where $|\alpha\rangle$ corresponds to $|\bar{N}\rangle$ and $|\beta\rangle$ to $|0\rangle$ topological states.

The N_α matrix has eigenstates and eigenvalues and it can be diagonalized by means of them as

$$N_\alpha = |v\rangle D_\alpha \langle v| + ... + \quad (59)$$

where

$$|v\rangle = \frac{1}{\mathcal{D}}|D_\alpha\rangle, \quad |\mathcal{D}|^2 = \sum_\alpha D^2_\alpha \,. \quad (60)$$

The D_α are quantum dimensions controlling the single annihilations of qu-bites in vacuo. Therefore,

$$N^{\beta}_{\alpha\alpha\alpha...\alpha} \sim D^N_\alpha/\mathcal{D}^2 + ... \,. \quad (61)$$

This is related to the single annihilation probabilities as

$$p(\alpha\bar{\alpha} \to 0) = (D_\alpha/\mathcal{D})^2 \sim N^{-2} \,, \quad (62)$$

And an annihilation of N pairs correspond to

$$p(N\,pairs \to 0) < e^{-N} \,, \quad (63)$$

where we used the Stirling approximation, having included an identical particle factor $N!$. This provides a topological computing interpretation of why the CC state with a large N cannot flow to the UV nothing state $|0\rangle$.

The lower dimensional case: punctures and gravitational instantons. It is instructive to consider the $2+1$ gravitational case. In this contest, we will show a correspondence of gravitational instantons with the entropy content of de Sitter space-time (see also Ref.[20] elaborating on this case).

Let us start considering a Wick rotation of the de Sitter space-time as

$$ds^2_E = \Big(1 - \frac{r^2}{l^2}\Big)dt^2_E + \Big(1 - \frac{r^2}{l^2}\Big)^{-1} dr^2 + r^2 d\theta^2 \,, \quad (64)$$

where $\beta = 2\pi l$ and $0 \leq t_E \leq \beta$. Eq.64 can be rewritten as a metrico of a S_3 sphere:

$$ds^2_E = \sin^2\rho\, dt^2_E + l^2 d\rho^2 + l^2\cos^2\rho\, d\theta^2 \,, \quad (65)$$

where $r = l\cos\rho$. The physics of a $2+1$ quantum gravity sector can be captured by a double Chern-Simons gauge theory with a gauge group $SU_+(2)\times SU_-(2)$. Each $\pm$ chiral sector has a connection which is a linear combination of the spin connection ω^a and the triad e^a:

$$A^a_\pm = \omega^a \pm \frac{1}{l}e^a \,. \quad (66)$$

The de-Sitter background can be recast from a class of non-trivial connections characterized by two real parameters γ, β:

$$A^3_\pm = \gamma\cos\rho\Big(d\theta \mp \frac{\beta}{l}dx^0\Big) \,, \quad (67)$$

$$A^2_\pm = \pm d\rho \,, \quad (68)$$

$$A^1_\pm = -\gamma\sin\rho\Big(d\theta \mp \frac{\beta}{l}dx^0\Big) \,. \quad (69)$$

The dS_3 corresponds to the $\gamma = 1$ case. On the other hand, the ratio of the two parameters is fixed as a requirement that any no conical singularities of holonomies are present at the horizon (on $\rho = 0$):

$$\beta = 2\pi l\gamma^{-1} \,. \quad (70)$$

In the next, we will focus on the $+$ chirality, having in mind a specularity of the the negative sector. The

Euclidean action implementing the boundary condition reads as

$$S_E[A,\beta] = S_B + S_{\rho=\pi/2} + S_{\rho=0}\,, \tag{71}$$

$$S_B = \frac{N}{4\pi}\int_{\mathcal{M}} \epsilon^{kl}\mathrm{Tr}(iA_k\partial_0 A_l - A_0 F_{kl})d^3x_E\,, \tag{72}$$

$$S_{\rho=\pi/2} = -\frac{N\beta}{4\pi l}\int_{\rho=\pi/2}\mathrm{Tr}(A_\phi)^2 dx_E^0 d\theta, \tag{73}$$

$$S_{\rho=0} = -\frac{N}{2}\int_{\rho=0} A_\phi^{(3)} dx_E^0 d\theta\,, \tag{74}$$

compatible with

$$A_0^a|_{\rho=0} = -2\pi\delta_3^a\,. \tag{75}$$

The euclidean action contains a term on the bulk with a topology $\mathcal{M} = \Sigma \times S_1$, with S_1 is the compactified euclidean time with periodicity dictated by $\tau = \beta^{-1}$. Above, we defined the topological winding number N emerging out as

$$N \sim l^2/L_{Pl}^2\,. \tag{76}$$

The other two terms are on the $\rho = 0, \pi/2$ boundaries.

The corresponding partition function has a saddle semiclassical solution on the special class of connections considered:

$$Z_A(\beta) = \frac{1}{\mathcal{N}}\sum_\gamma \mathrm{Exp}\Big(-\frac{1}{16G}\beta\gamma^2 + \frac{1}{4G}\pi\gamma l\Big)\,, \tag{77}$$

where $\mathcal{N}$ is the normalization factor.

In other words, the partition function is dominated by the sum over γ, at β fixed. The saddle point of Eq.77 lies on

$$\gamma\beta = 2\pi l\,, \tag{78}$$

which corresponds to the classical γ-point avoiding for conical singularities at the horizon. Indeed, the γ parametrizes the deficit angle of a conical singularity located at the extreme point $\rho = \pi/2$ $(r = 0)$.

Eq.77 is interpreted as associating energy levels and degeneracy factors labelled by γ:

$$E_\gamma = \frac{\gamma^2}{16G}\,, \tag{79}$$

$$\rho(\gamma) = \exp\Big(\frac{\pi\gamma l}{4G}\Big)\,. \tag{80}$$

For $\gamma = 1$, the de Sitter entropy is recast as

$$S_{=}\frac{\pi l}{4G} \sim N_+ + N_- \sim N \tag{81}$$

having consider the sum on the $\pm$-chiralities. The entropy is related to the degeneracy factor in Eq.80.

This result can be compared with the semiclassical limit of the exact result, computed in terms of spin-network representations

$$Z_A(\beta) = \sum_{2s}^{N} d^{-1} q^{s(s+1)/N}\frac{\sinh[2\pi(s+1/2)]}{\sinh\pi} f(q,N,s)\,, \tag{82}$$

$$f(q,N,s) = \sum_{n=-\infty}^{+\infty} q^{Nn^2+(2s+1)n}e^{2\pi Nn}\,, \tag{83}$$

$$d = \prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{i\theta})(1-q^n e^{-i\theta})\,, \tag{84}$$

$$q = e^{-\beta/l},\quad \theta = -2\pi i\,.$$

From the semiclassical identification we obtain

$$n = \frac{\gamma}{2} - \frac{1}{2N} - \frac{s}{N}\,, \tag{85}$$

where n, s are discrete numbers. This means that γ is quantized.

This result offers a series of reinterpretations as a hint towards a confirmation of the topological hairon portrait:

(i) the euclidean dS-metric is re-obtained as a the continuous limit of a discrete series of conic singularities [7]. The conic singularities are saddle solutions related to a class of non-trivial connections, i.e. to gravitational instantons. In other words, the Euclidean dS Hubble horizon is effectively emerging as a sum on a large number of horizonless geometries.

(ii) The dS-entropy is recast and it is proportional to the instantonic topological number. The degeneracy factor is exponentially growing as the topological winding number. On the other hand, the CC is quantized as the inverse of the topological number.

(iii) There is a correspondence among spin-network representations and gravitational instantons, sustaining the intepretation of the puncture-instanton correspondence. This can also suggest that the space-time topological computation is related to the spin-network dynamics in the bulk in the deep non-perturbative quantum gravity regime.

It is worth to remark that in the special case of $2+1$ quantum gravity on de Sitter all these considerations can be related to the Schwarzian quantum mechanical model on the circular time boundary. In turn the Schwarzian

[7] This fact was envisaged in our previous works in Refs. [36–38].

model is related to the IR conformal limit of the Sachdev-Ye-Kitaev model (SYK) [40]. This leads to a dual reinterpretation of hairon fields as SYK fields non-locally coupled through a gaussian set of random matrices.

Entanglement entropy and Topological order. The topological phase transitions, the emergence of a topological order around the space-time quantum criticality, the long-range entanglement, the exponential degeneracy of the ground state, the exotic anyon statistics of punctures, the entropic holographic scaling...All these analogies of space-time and topolological materials point out towards the definition of a Topological entanglement entropy of the BH and de Sitter geometries. In a topologically ordered two-dimensional disk, *Kitaev, Preskill, Levin and Wen* found that the von Neuman entanglement entropy holographically scales as

$$S(\rho) = \alpha L - \gamma + ... \qquad (86)$$

where L is the boundary length, in the limit of $L \to \infty$ [41, 42]. Eq.90 provides a measure of the entanglement of the interior and exterior degrees of freedom related to the density operator as

$$S(\rho) = -\mathrm{tr}\rho \log \rho \,. \qquad (87)$$

The α coefficient depends on particular aspects of short wave length modes close to the boundary. On the other hand, γ is interpreted as a universal constant dependent by the feature of the ground state entanglement. Such a term corresponds to the so dubbed topological entanglement entropy, depending by the total dimension of the system as

$$\gamma = \log \mathcal{D} \sim \log N \,. \qquad (88)$$

In our case, we may conjecture that this formula may be generalized to a higher dimensional case as

$$S(\rho) = \alpha A - \gamma + ... \sim N - \log N + ... \,, \qquad (89)$$

by means of the topological order on the boundary area A. Indeed, in this case, the γ scales as the log of a configuration space $\mathcal{D}$ that is as N. By means of the thermodynamical definition of entropy, we arrive to

$$S = -\frac{\partial F}{\partial T} = \frac{\partial}{\partial T}(T \log Z) \sim N - \log N \,. \qquad (90)$$

This relates the topological entanglement entropy to the winding number as

$$N \sim e^{\gamma} \,. \qquad (91)$$

The high proliferation of N states is connected to a positive topological entanglement entropy. This is another way to visualize that a changing of BH entropy corresponds to an alteration of the topological order in vacuo.

Final remarks. In this paper, we explored the connections among the $\mathcal{HN}$ paradigm and gravitational topological properties of the vacuum state. We showed that quantum hairs stored in space-time are intimately related to the topological winding number. The CC corresponds to a maximal entropic vacuum state, where the information storage is critically enhanced. Within this picture, the de-Sitter space-time lives in a critical conformal phase, where the critical temperature is exactly related to the CC. Around the critical temperature, there is a Topological Phase Transition of the space-time geometry. Indeed, the CC is stabilized by the maximal entropic and topological content of the Universe vacuum state: a IR to UV destabilization is exponentially suppressed. Then, we elaborated on the topological meaning of hairon fields. We realized that they can be interpreted as punctures on the space-time boundary from gravitational Wilson lines or Wormholes crossing the dS-bulk. The hairon spin statistics is neither fermionic nor bosonic: hairons are gravitational non-abelian anyons. This is caused by the topological complexity of the bulk as a Wilson line network. Such a topological picture is potentially insightful for our understanding of the space-time and the Black Hole information storage. It explains why the space-time has an exponentially enhanced memory and a area law, symptomatically suggesting that a large number of quantum hairs are in a degenerate entangled ground state. As in topological quantum computers, the effect of any fluctuations or noise, potentially creating or destroying any new qu-bits, is exponentially suppressed. This offers a quantum information portrait of the CC stabilization mechanism: the gravitational vacuum topology provides a cosmological defense of the CC memory. Any CC-Planck scale mixings, corresponding to a massive attack to the Universe Memory, are highly suppressed as a non-local topological protection of the hairon/anyon entangled state.

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