

The Spacetime Picture in Quantum Gravity

Alejandro Ascárate

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Abstract

We propose an approach which, by combining insights from Loop Quantum Gravity (LQG), Topos theory, Non-commutative Geometry *à la* Connes, and spacetime relationalism, provides fertile ground for the search of an adequate spacetime picture in Quantum Gravity. With this approach, we obtain a novel way of deducing the quantization of the possible values for the area of a surface. One gets the same area values than those from the area operator in standard LQG, but our approach makes a further prediction: some smaller values and sub-divisions are also allowed. In addition, the *area* arises as a noncommutative *distance* between two noncommutative points, and thus they should be interpreted as irreducible string-like objects at the physical level (where the area interpretation for the noncommutative distance holds).

Introduction

One of the reasons why General Relativity (GR) is so compelling, and its formalism so *intuitive* to handle (once we get used to the revolutionary changes it makes to our intuitive, pre-relativistic notions of space and time), is that the theory is based on a *spacetime picture formulation*. By this we mean that the main object of the theory is the pair

$$(M, g_{ab}),$$

where M is a 4-dimensional differentiable manifold (with the standard topological requirements) and g_{ab} a smooth lorentzian metric on it [1]. While the manifold M represents *physical spacetime*, the metric g_{ab} allows us to calculate quantities which represent such things as physical distances, times, and volumes in it. Space and time are physical notions which unlike (say) quarks, we *directly experience*, and this is what gives GR its intuitive appeal. This is also useful since it tends to be easier to develop a theory when we have a clear picture of what it describes at the physical level.

In the case of Quantum Gravity (QG), however, it's not at all clear what the thing that this theory supposedly describes actually is. Of course, we actually know that it should describe the quantum behavior of gravity and spacetime

geometry, since GR describes the classical behaviour of these notions, and QG is supposedly a quantization of the first. But, so far (and leaving aside the obvious fact that actually we don't even have a working theory of QG) no *clear* picture of these quantum phenomena is available, that is, one which can have an appeal and *physico-mathematical role* similar to that of the spacetime picture of classical GR, i.e. we lack a picture of *quantum spacetime*.

That doesn't mean that we don't have candidates or proposals for a QG theory—we have plenty of them. And what makes things even more unclear is that the various proposals look very different from each other. The proposals of interest here will be three¹:

- i) Loop Quantum Gravity (LQG) [2, 6];
- ii) Non-commutative Geometry *à la* Connes (NCG) [4];
- iii) the Topos Approach to Quantum Theory [3].

The task is to find an analogue in QG of the spacetime pair (M, g_{ab}) of classical GR. But to do this, we need to carefully study what exactly spacetime is at the deep physical level, in order to know how to proceed. The key that allows us to reach a *convergence* of these approaches is the philosophical view in which spacetime emerges from matter in a *relational* way [5, 6]. For the analogue of the spatial manifold $\Sigma \subset M$, for example, we will propose a framework in which “*quantum 3-dimensional space*” is the *spectral presheaf* $\underline{\Sigma}$ (a concept taken from iii)) based on the quantum kinematical algebra of Algebraic Quantum Gravity (AQG), which is an alternative to LQG (albeit closely related to it) that doesn't assume a background differentiable manifold and has also a better behaved classical limit.

1. Algebraic Reformulation of Classical Space(-time)

We will only deal with physical 3-space for now, since time requires more elaborated considerations. We assume $\mathcal{A}$ to be a *commutative* C^* -algebra (with unit). We know that, by using the *Gelfand transform*, there's a compact topological space Σ such that $\mathcal{A}$ gets represented as the algebra $C(\Sigma)$ of complex-valued continuous functions on Σ (note that applying the transform to $C(\Sigma)$ itself will give this very same algebra). Thus, if $\mathcal{A}$ was the space algebra, then we get the space picture, i.e. the space Σ , by this process. Our aim here will be to try to replicate this process in some way for the quantum case. That is, we take the space algebra $\mathcal{A}$ as the central object and the space picture as something secondary or derived from it. More precisely, the space Σ is given,

¹Actually, only i) is a theory of QG (in the sense of quantizing GR): ii) and iii) are abstract mathematical theories whose aim is to study geometrical properties of non-commutative spaces. They are sometimes mentioned as QG candidates because it's assumed that their techniques will be useful in the mathematical formulation of QG (indeed, that's one of the motivations of the authors of those theories). We, of course, agree with that view, and actually our aim here will be to put it into practice.

in the commutative case, by the Gelfand spectrum, that is, the set of algebraic states $\omega : \mathcal{A} \rightarrow \mathbb{C}$ which are also algebraic homomorphisms between $\mathcal{A}$ and $\mathbb{C}$; it turns out that these also comprise the set of all possible pure states. Thus, we want to study the space algebra $\mathcal{A}$ and its pure states ω , which represent the space “points”. In the standard approach, the structure of Σ is simply assumed to be that of a compact topological space, which in turns leads to a commutative structure for the algebra. Here we focus only on the algebra $\mathcal{A}$ and its pure states ω , where the set of the latter only gives a compact topological space Σ if $\mathcal{A}$ is commutative, which is precisely what we *won't* take for granted.

Now, both in classical GR and in LQG, a commutative structure is given to the algebra of physical 3-space. Of course, this is well suited for the former, but it seems problematic for the latter, since geometric properties like area are discretized. Thus, if we follow that approach, we risk giving space an a priori structure and also risk tangling into contradictions between assumptions and conclusions. We don't want to assume any a priori structure on space, in particular about the commutativity of the space algebra. Thus, we adopt a “space algebra first” approach rather than the usual “space points first” one. This is something natural to do if we adopt the so-called algebraic approach to quantum theories, in which one reconstructs the standard formalism of quantum theory from an abstract quantum “phase space” algebra [7]. Recall from what we said that one can introduce a structure on the “points space” and obtain from it the structure of the algebra of functions, or one can introduce a structure on the latter and from there try to deduce the one of the points space, in particular by taking the pure states on the algebra as the points; in the commutative case, the Gelfand transform links the two approaches and makes them equivalent, but if the space algebra is non-commutative (as we suspect), then the manifold disappears (and, obviously, the Gelfand equivalence too) and therefore a space algebra first approach is more appropriate, since the algebra is what survives even in the non-commutative case.

Definition 1.1: The (kinematical) phase space of GR is defined as

$$X = \{ [h_{ab}, \pi^{ab}] \mid h_{ab}, \pi^{ab} \in C^\infty(\Sigma) \text{ (as fields on } \Sigma) \},$$

where h_{ab} and π^{ab} are, respectively, a smooth riemannian metric on a spacelike Cauchy hypersurface Σ in a compact and boundaryless spacetime M , foliated by Σ as usual, and the conjugate momentum tensor density. ■

Definition 1.2:² The subset $\mathcal{F} \subset C(X)$ consists of the phase space functionals of the form

$$F_f([h_{ab}, \pi^{ab}]) \doteq \int_\Sigma f \epsilon(h_{ab}), \forall [h_{ab}, \pi^{ab}] \in X,$$

where $f \in C^\infty(\Sigma)$ and $\epsilon(h_{ab})$ is the volume element of h_{ab} , i.e. $\epsilon(h_{ab}) = \sqrt{h} d^3x$ ($h = \det(h_{ab})$). ■

²We will not discuss the topology of the phase space here, but the functionals defined here should be continuous under any reasonable topology on it, since the determinant of h_{ab} is just a linear combination of products of its components.

Proposition 1.1: The assignment $f \mapsto F_f$ is injective.

Proof: since the phase space point $[h_{ab}, \pi^{ab}]$ is composed by arbitrary smooth functions, $\sqrt{h}$ behaves as an arbitrary smooth function when one varies $[h_{ab}, \pi^{ab}]$ over *all* of phase space, and this makes the assignment $f \mapsto F_f$ injective, since

$$\begin{aligned} \int_{\Sigma} f \sqrt{h} d^3x &= \int_{\Sigma} f' \sqrt{h} d^3x, \forall h \in C^\infty(\Sigma) \\ \implies \int_{\Sigma} (f - f') \sqrt{h} d^3x &= 0, \forall h \in C^\infty(\Sigma) \\ \implies f - f' &= 0 \text{ almost everywhere} \end{aligned}$$

so by continuity $f - f'$ vanishes identically i.e. $f = f'$. $\square$

Definition 1.3: We now define a mapping $R : \mathcal{F} \times X \rightarrow C^\infty(\Sigma) \times \text{Obj}(\text{Hil})$ (where $\text{Obj}(\text{Hil})$ is the collection of objects in the category Hil of Hilbert spaces) by

$$(F_f, [h_{ab}, \pi^{ab}]) \mapsto R(F_f, [h_{ab}, \pi^{ab}]) \doteq (f, L^2(\mathcal{S}, \epsilon(h_{ab})),$$

(where $\mathcal{S}$ is the $C^\infty(\Sigma)$ -module of smooth spinor fields in Σ , which from now on we assume allows a spin structure). $\blacksquare$

Definition 1.4: For *fixed* $[h_{ab}, \pi^{ab}] \in X$ and *variable* $F_f \in \mathcal{F}$, we denote the first component of $R(F_f, [h_{ab}, \pi^{ab}])$ as $R_h(F_f)$. $\blacksquare$

Definition 1.5: We make $\mathcal{F}$ into an (unital) algebra $(\mathcal{F}, \cdot_{sp.})$, where the product $\cdot_{sp.}$ in $\mathcal{F}$ is defined as

$$\begin{aligned} (F_{f_1} \cdot_{sp} F_{f_2})([h_{ab}, \pi^{ab}]) &\doteq F_{f_1 f_2}([h_{ab}, \pi^{ab}]), \forall [h_{ab}, \pi^{ab}] \in X \\ &= \int_{\Sigma} f_1 f_2 \epsilon(h_{ab}). \blacksquare \end{aligned}$$

All of the previous constructions were introduced in order to see the algebra $C^\infty(\Sigma)$ as coming from a subset of phase space functionals once a point in X is taken. Indeed, $(\mathcal{F}, \cdot_{sp})$ is carried by R_h to the pointwise product of space functions (i.e. the product of $C^\infty(\Sigma)$). We want this because the process of canonical quantization only gives us the quantum phase space algebra, so we must adapt everything to this and formulate our concepts accordingly.

Corollary 1.1: The map R_h is a *bijection* between $\mathcal{F}$ and $C^\infty(\Sigma)$ (by Proposition 1.1), and is a faithful algebra representation of $\mathcal{F}$ into³ $\mathcal{B}(L^2(\mathcal{S}, \epsilon(h_{ab})))$ by multiplication operators, i.e. $R_h(F_f)(\psi)(x) \doteq f(x)\psi(x)$, $\forall \psi \in L^2(\mathcal{S}, \epsilon(h_{ab}))$. $\square$

We call R_h the “*relational representation*”, since *it gives a way of obtaining the algebra of physical 3-space purely from Field properties* (represented here

³ $\mathcal{B}(\mathcal{H})$ is the algebra of all the *bounded* operators that act on the Hilbert space $\mathcal{H}$, with the operator composition $\circ$ as algebraic product.

by the phase space functionals). This is precisely the essence of the spacetime relationist viewpoint, according to which *points of the spacetime manifold don't have an existence of their own, but rather acquire physical meaning only in the context of a particular fixed solution*, where they can be identified in terms of the values the different fields take on them in that particular solution (in other words: only once we fix a solution to the gravitational field equations i.e. a metric, can we label events with physically meaningful coordinates, such as distances and proper times). Since we live in a particular solution and not in phase space (which is more of a conceptual construct of ours), there is no problem with this, and *the dependency of our "relational representation" on a fixed metric is thus completely expected*.

Switching to a relational and algebraic frame of mind, we can take this representation as the actual way of defining how to build space from the phase space algebra of the gravitational field.

Before continuing, we recall some notions from NCG [4].

Definition 1.5: In a Hilbert space $\mathcal{H}$,

1. a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is a sub $*$ -algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ together with a self-adjoint operator D (possibly unbounded, so it's defined on a dense domain $\text{Dom } D$) of compact resolvent (that is, $(D^2 + 1)^{-1}$ is compact) and such that $[D, a] \in \mathcal{B}(\mathcal{H})$, $\forall a \in \mathcal{A}$ (for the commutator to be defined we need, of course, that $a(\text{Dom } D) \subseteq \text{Dom } D$);
2. the *purely algebraic distance formula* for a spectral triple is given by:

$$\mathcal{A}(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)|, \forall a \in \mathcal{A} \setminus \{ [a, D] \leq 1 \} \},$$

for any two *states* φ, ψ (normalized positive linear functionals on $\mathcal{A}$);

3. the spectral triple is *even* if there's a self-adjoint unitary operator Γ on $\mathcal{H}$ such $a\Gamma = \Gamma a$, $\forall a \in \mathcal{A}$, $\Gamma(\text{Dom } D) = \text{Dom } D$, and $D\Gamma = -\Gamma D$; if no such operator is given, then the spectral triple is *odd*;
4. the spectral triple is *real* if there's an antiunitary operator $J : \mathcal{H} \rightarrow \mathcal{H}$ such that $J(\text{Dom } D) \subset \text{Dom } D$ and $[a, Jb^*J^{-1}] = 0$, $\forall a, b \in \mathcal{A}$; it possesses a *real structure* if, in addition, $J^2 = \pm 1$, $JDJ^{-1} = \pm D$, and $J\Gamma = \pm\Gamma J$ (in the even case);
5. a real spectral triple is of *first order* if $[[D, a], Jb^*J^{-1}] = 0$, $\forall a, b \in \mathcal{A}$;
6. for $1 < p < \infty$, the operator ideal $\mathcal{L}^{p+}(\mathcal{H})$ is defined as⁴

$$\mathcal{L}^{p+}(\mathcal{H}) \doteq \left\{ T \in \mathcal{K}(\mathcal{H}) \setminus \sigma_N(T) = O(N^{(p-1)/p}), N \rightarrow \infty \right\};$$

⁴Where $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ is the space of compact operators, and $\sigma_N(T) \doteq \sum_{k=0}^{N-1} s_k(T)$, with $s_k(T)$ being the k -th eigenvalue (in decreasing order and counted with multiplicity) of the compact positive operator $|T| \doteq (T^*T)^{1/2}$.

if for a *positive* operator $T \in \mathcal{L}^1(\mathcal{H})$ the sequence $\left\{ \frac{\sigma_N(T)}{\ln N} \right\}_{N \in \mathbb{N}}$ is *convergent*, then its *Dixmier trace* is defined as

$$\mathrm{tr}^+(T) \doteq \lim_{N \rightarrow \infty} \frac{\sigma_N(T)}{\ln N};$$

the number $\dim_{\mathrm{Spec.}}(\mathcal{A}, \mathcal{H}, D) = n \in \mathbb{N}$ is the *spectral dimension* of a spectral triple if, for $\ker D = \{0\}$, we have $|D|^{-1} \in \mathcal{L}^{n+}(\mathcal{H})$ and $0 < \mathrm{tr}^+(|D|^{-n}) < \infty$; a spectral triple whose Hilbert space $\mathcal{H}$ is of *finite* dimension is considered as having spectral dimension $n = 0$ (for example, spectral triples that describe the geometry of a set with a *finite* number of points, and the Euclidean distances among them, are of this type); the non-commutative integral $\int$ of algebra elements a , is defined by⁵:

$$\int a \doteq \alpha_n \mathrm{tr}^+(a |D|^{-n});$$

7. a spectral triple with a real structure is of⁶ *KO-dimension* $\dim_{KO}(\mathcal{A}, \mathcal{H}, D; J, \Gamma) = 2 \pmod{8}$ if $J^2 = -1$, $JDJ^{-1} = +D$, and $J\Gamma = -\Gamma J$ (for the rest of possible *KO-dimensions* $\dim_{KO}(\mathcal{A}, \mathcal{H}, D; J, \Gamma) = m \pmod{8}$, $m \in \mathbb{N}$, see the table of signs in the cited references). ■

Example 1.1: The canonical example of a spectral triple is the usual commutative geometry of an n -dimensional smooth compact boundaryless M with Riemannian metric g_{ab} : $(C^\infty(M), L^2(\mathcal{S}, \epsilon(g_{ab})), \mathcal{D}_g)$, where $C^\infty(M)$ acts by multiplication operators, and $\mathcal{D}_g$ is the *Dirac differential operator*, which, for an orthonormal basis $\{e_\alpha\}_{\alpha=1, \dots, n}$ of the tangent spaces of M , is given *locally* by

$$\mathcal{D}_g \psi = -i \sum_{\alpha=1}^n \gamma^\alpha \nabla_{e_\alpha}^\mathcal{S} \psi, \quad \psi \in \mathcal{S},$$

where $\nabla_{e_\alpha}^\mathcal{S}$ is the spin connection and $\gamma^\alpha \in M_{2^m}(\mathbb{C})$, $\alpha = 1, \dots, n$ (with $n = 2m$ or $n = 2m + 1$, $m \in \mathbb{N}_0$), are the generators of the action of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^n)$ on $\mathbb{C}^{2^m}$ (and then $\{\gamma^\alpha, \gamma^\beta\}_{M_{2^m}(\mathbb{C})} = 2\delta^{\alpha\beta} I_{M_{2^m}(\mathbb{C})}$, $\alpha, \beta = 1, \dots, n$); we mention an important result which states that, if $\Delta^\mathcal{S}$ is the spinor Laplacian and s the scalar curvature, the following formula holds: $\mathcal{D}_g^2 = \Delta^\mathcal{S} + \frac{1}{4}s$. For pure states φ_p, φ_q , $p, q \in M$, one gets $\mathcal{D}(\varphi_p, \varphi_q) = d(q, p)$, where d is the usual distance induced by the metric g_{ab} , and hence g_{ab} is completely characterized by the purely algebraic and functional analytic information of the spectral triple. One also gets that $\dim_{\mathrm{Spec.}}(\mathcal{A}, \mathcal{H}, D) = \dim M = n$ and that $\int a_f = \int_M f \epsilon(g_{ab})$. With J given by the usual charge conjugation operator and Γ given by the $\mathbb{Z}_2$ -grading of the Clifford algebra, we get a real structure such that the first order differential operator $\mathcal{D}_g$ is also first order in the spectral triple sense, and such that $\dim_{KO}(\mathcal{A}, \mathcal{H}, D; J, \Gamma) = \dim M \pmod{8}$. The celebrated *reconstruction theorem of NCG* states that any abstract *commutative* spectral triple $(\mathcal{A}, \mathcal{H}, D)$ that satisfies several regularity assumptions (see references, [4]b) in particular)

⁵See the references for the definition when a is not positive.

⁶The *KO* comes from *KO*-homology.

is always such that $(\mathcal{A}, \mathcal{H}, D) \cong (C^\infty(M), L^2(\mathcal{S}, \epsilon(g_{ab})), \mathcal{D}_g)$ for some *unique* $(M, g_{ab}, \mathcal{S})$. A very special sub-case of commutative geometries are the (spectral) 0-dimensional triples corresponding to sets with a *finite* number of points $\{p_1, \dots, p_N\}$ (say, 2), and the Euclidean distances among them (d_{12}), where the algebra is given by the *diagonal* $N \times N$ matrices (2×2), with the diagonal corresponding to the values $f(p_1), f(p_2), \dots, f(p_N)$, of functions on the set of points, and with Dirac operator (in the case of 2 points) given by $D = \begin{pmatrix} 0 & d_{12}^{-1} \\ d_{12}^{-1} & 0 \end{pmatrix}$, which gives $\not{D}(\varphi_1, \psi_2) = d_{12}$; and tr^+ can be replaced by the ordinary trace tr on matrices (thus, e.g. $\not{D}(\varphi_1, \psi_2) = f I \doteq \frac{1}{N} \text{tr}(|D|^{-1}) = d_{12}$). Further examples will be given along the way as they are needed. ■

Proposition 1.2: If one starts with a subset $\mathcal{A}_\Sigma$ (with a commutative product $\cdot_{sp}$) of the phase space algebra $\mathcal{A}_X$ (i.e. $\mathcal{A}_\Sigma \subset \mathcal{A}_X$, but only as a set and *not* as a subalgebra⁷), which correspond, respectively, to $\mathcal{F}$ and $C(X)$ but *seen as abstract algebras*, then there exist a *purely algebraic* (i.e. which *doesn't* make use of the commutative manifold structure of space for its definition) faithful representation $(\tilde{R}_h[\mathcal{A}_\Sigma], \mathcal{H}_{\tilde{R}_h})$ of $\mathcal{A}_\Sigma$ and isomorphisms $\pi_X : \mathcal{A}_\Sigma \rightarrow \mathcal{F}$ and $\pi_\Sigma : \tilde{R}_h[\mathcal{A}_\Sigma] \rightarrow C^\infty(\Sigma)$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_\Sigma & \xrightarrow{\pi_X} & \mathcal{F} \\ \downarrow \tilde{R}_h & & \downarrow R_h \\ \tilde{R}_h[\mathcal{A}_\Sigma] & \xrightarrow{\pi_\Sigma} & C^\infty(\Sigma) \end{array}$$

i.e. such that

$$R_h = \pi_\Sigma \circ \tilde{R}_h \circ \pi_X^{-1}.$$

Proof: consider a hypothetical faithful representation $\tilde{R}_h$ of $\mathcal{A}_\Sigma$ on a Hilbert space $\mathcal{H}_{\tilde{R}_h}$ (i.e. $\tilde{R}_h[\mathcal{A}_\Sigma] \subset \mathcal{B}(\mathcal{H}_{\tilde{R}_h})$, both as set and as an algebra) and form an *abstract commutative spectral triple* (satisfying the regularity assumptions)

$$(\tilde{R}_h[\mathcal{A}_\Sigma], \mathcal{H}_{\tilde{R}_h}, D_h).$$

Then we can realize those operators as *smooth* functions on a manifold Σ via the Gelfand isomorphism π_Σ of the reconstruction theorem of NCG, which states that, for a *unique* metric h_{ab} and spin structure,

$$(\tilde{R}_h[\mathcal{A}_\Sigma], \mathcal{H}_{\tilde{R}_h}, D_h) \cong (C^\infty(\Sigma), L^2(\mathcal{S}, \epsilon(h_{ab})), \mathcal{D}_h),$$

where $\pi_\Sigma[\tilde{R}_h[\mathcal{A}_\Sigma]] = C^\infty(\Sigma)$. Next, one builds the phase space X of Definition 1.1 for this Σ . Finally, by the previous uniqueness, we can now recognize that the resulting R_h (for the metric h_{ab}) between the *phase space functions* in $\mathcal{F}$ obtained via the Gelfand isomorphism⁸ π_X (which acts on the whole of $\mathcal{A}_X$, i.e.

⁷This convention will be maintained whenever the symbol $\subset$ is used, unless some other sense is explicitly stated.

⁸Note that it also acts as an algebra isomorphism in the product $\cdot_{sp}$.

$\pi_X [\mathcal{A}_X] = C(X) \supset \mathcal{F} = \pi_X [\mathcal{A}_\Sigma]$) on one hand, and *space functions* on the other, is the concrete version of the initial $\tilde{R}_h$, i.e. the latter must be unique, and can therefore simply be defined as $\tilde{R}_h \doteq \pi_X \circ R_h \circ \pi_\Sigma^{-1}$ (with π_X, π_Σ^{-1} defined just using the concrete, manifold algebras), and the diagram commutes. $\square$

It's also useful to draw the previous diagram in the following alternative way, which emphasizes the role of picking a solution h in the construction:

$$\begin{array}{ccc} \mathcal{A}_\Sigma & \xrightarrow{\pi_X} & C(X) \\ \downarrow \tilde{R}_h & & \downarrow R_h \\ \mathcal{B}(\mathcal{H}_{\tilde{R}_h}) & \xrightarrow{\pi_\Sigma} & \mathcal{B}(L^2(S_n, \epsilon(h_{ab}))) \end{array}$$

We understand that the prime advantage of the present approach (where a representation of a subset of the phase space algebra with certain products is taken as the definition of the space algebra that relationally arises from the field, a general construction which obviously applies to both the classical and quantum cases) is that it provides a *common abstract procedure to build the space algebra in both the classical and quantum regimes*. Thus, the general process described here is *free from ad-hoc structures and only uses the structures that are common to both classical and quantum formalisms, namely, the abstract phase space algebra and its products*⁹. Note that

in the general, non-commutative case, only the left hand side of the previous diagram survives.

In this way, *all the geometrical information of the space is now contained in the spectral triple.*

2. Quantum (or Non-Commutative) Relational Space(-time)

2.1. General Idea

Now, in the case of QG, the phase space algebra is quantized by fusing its product (given by the pointwise product of phase space functions and which gives the probability structure of the theory) with the non-commutative Poisson bracket product, i.e. we get a non-commutative algebra whose only product has two different physical interpretations —that's what we mean by “fusing”. The result is a quantum, non-commutative (abstract) algebra $\hat{\mathcal{A}}_{QG}$ which gives rise to a non-commutative probability theory.

⁹but not their specifics, since they can be commutative or non-commutative and in any combination; in general, the relation between both algebras is given by the theory in consideration, which can be either classical field theory or quantum physics. In this sense, what we have exhibited is only a reconstruction of space and not a general duality between algebras and theories; this is why in Proposition 1.2 above we assumed the concrete commutative algebras as a given and only checked for the overall *consistency* once the relational construction is applied.

Hypothesis 2.1: We make the hypothesis that $\cdot_{sp}$ also gets fused with the other products¹⁰. ■

Definition 2.1: Consider the phase space of GR, $(X, C(X), \{\cdot, \cdot\}_{GR})$, and a background independent Poisson sub-algebra $\mathcal{A}_{GR}$ of phase space functionals there. The basic $\star$ -algebra, $\hat{\mathcal{A}}_{QG}$, in QG will be the algebra freely generated (with field $\mathbb{C}$) by the classical algebra¹¹ $\mathcal{A}_{GR}$ and with the relations imposed by the ideals $\mathcal{I}_{LP}$ (where $\star$ is the involution, L is from linear and P from i multiplied by the Poisson brackets. This process simply imposes the familiar Dirac commutation relations of canonical quantization to $\hat{\mathcal{A}}_{QG}$, that is: $[\hat{a}, \hat{a}']_{\hat{\mathcal{A}}_{QG}} \doteq \hat{a} \star_{\hat{\mathcal{A}}_{QG}} \hat{a}' - \hat{a}' \star_{\hat{\mathcal{A}}_{QG}} \hat{a} = i \widehat{\{a, a'\}_{GR}}$, where $\star_{\hat{\mathcal{A}}_{QG}}$ is the quantum probability algebraic product of $\hat{\mathcal{A}}_{QG}$ [2]). That is, $\hat{\mathcal{A}}_{QG}$ is the quotient:

$$\hat{\mathcal{A}}_{QG} \doteq \frac{\mathcal{A}_{GR}}{\mathcal{I}_{LP}}. \blacksquare$$

(Note that the subset of classical properties $\mathcal{A}_{GR}$ forms a Lie algebra $\mathfrak{A}_{GR} \doteq (\mathcal{A}_{GR}, \{\cdot, \cdot\}_{GR})$ with respect to the Poisson bracket, and then the obtained quantum algebra, by using this quantization process, is its *complex universal enveloping algebra* $U(\mathfrak{A}_{GR})$, which, unlike the Lie algebra, is an *associative* algebra. Also note that, due to the i in the Poisson brackets, the canonical embedding map of $\mathfrak{A}_{GR}$ into $U(\mathfrak{A}_{GR})$ is actually of the form $a \in \mathfrak{A}_{GR} \mapsto (-i\hat{a}) \in U(\mathfrak{A}_{GR})$).

The above construction defines a *quantization map* $\hat{\cdot}: \mathcal{A}_{GR} \longrightarrow \hat{\mathcal{A}}_{QG}$, $a \mapsto \hat{a}$, that embeds the elements of the classical algebra into the quantum one (although, of course, the quantum probability algebraic relations in the latter are now those determined by the Poisson bracket and not the pointwise product of $C(X)$; this map shouldn't be confused with the previous canonical embedding map of $\mathfrak{A}_{GR}$ into $U(\mathfrak{A}_{GR})$).

The next task is to identify a *subalgebra*

$$\hat{\mathcal{A}}_{SP} \subset \hat{\mathcal{A}}_{QG}$$

that plays a role equivalent to the one of $\mathcal{A}_{\Sigma}$ in the classical case of Proposition 1.2. The following definitions are suggested by that proposition and the “fusing of the products” assumption of Hypothesis 2.1:

Definition 2.2: If we write f_S for an element in $\mathcal{F}$ with $S \doteq \text{supp } f_S \subseteq \Sigma$, we can see $\mathcal{F}$ as the result of the application of an *indexation map* $I_{\mathcal{F}}: I \longrightarrow$

¹⁰In this way, it would seem that, in QG, one must go a step further regarding the scheme of canonical quantization, since we also have here the deformation of the space product $\cdot_{sp}$. It seems legitimate that this step is only necessary in QG since, in the other quantum field theories, there's always a background classical gravitational field, a metric, whose purpose is to give rise to the (classical) space of the theory (and, since we are at it, to simplify the concepts and the math), whereas in QG all fields vary and get quantized, and since in QG a deformed phase space product coexisting with a commutative sp product is something that could easily lead to internal contradictions, the most sensible hypothesis seems to be the one we just made.

¹¹Which is just the complex vector space generated by the basis $\{e_S\}$, where S runs over all the possible ordered, finite sequences of elements from $\mathcal{A}_{GR}$ (e.g. $S = (a_1, a_2, \dots, a_k)$, $k > 0$) and the algebra product is given at the basis level by $e_S \star e_T \doteq e_{(S,T)}$.

$\mathcal{F}, S \mapsto \tilde{I}_{\mathcal{F}}(S) \doteq f_S$, where I is the collection of all the supports S . We now *replace* I by a subset $I_{Dis} \subset I$ which can be *at most countably infinite*, and compute the Poisson brackets in $\mathcal{F}_{Dis} \doteq \tilde{I}_{\mathcal{F}}[I_{Dis}]$: the subset I_{Dis} should be selected in a way that makes $(\mathcal{F}_{Dis}, \{\cdot, \cdot\}_{GR})$ a Poisson subalgebra and, in particular, one whose structure constants *don't* depend on the differentiable manifold details of Σ nor on the details of the functions f_S varying over it, *but only on I_{Dis} , as an indexing set*. With this set up, we choose $\mathcal{A}_{GR}$ such that $\mathcal{F}_{Dis} \subset \mathcal{A}_{GR}$ (so that $\hat{\mathcal{F}}_{Dis} \subset \hat{\mathcal{A}}_{QG}$) and define:

$$\hat{\mathcal{A}}_{sp.} \doteq \hat{\mathcal{F}}_{Dis}. \blacksquare$$

(The interpretation of I_{Dis} , $\mathcal{F}_{Dis}$ will become clear in the concrete implementations of the next sections.)

Assume for the time being that we have that subalgebra. Then, we need to study the representation theory of the abstract quantum phase algebra $\hat{\mathcal{A}}_{QG}$. Once we have all the representations

$$(R_{QG} [\hat{\mathcal{A}}_{QG}], \mathcal{H}_{R_{QG}})$$

(with $R_{QG} [\hat{\mathcal{A}}_{QG}] \subset \mathcal{B}(\mathcal{H}_{R_{QG}})$) of interest, the representations that will allow us to “relationally reconstruct” the quantum space are simply the *restrictions* or *subrepresentations* of the previous representations to the subalgebra $\hat{\mathcal{A}}_{SP} \subset \hat{\mathcal{A}}_{QG}$, since the phase space product is now the same as the space product.

Definition 2.3: Assuming one has the family of possible representations $(R_{QG} [\hat{\mathcal{A}}_{QG}], \mathcal{H}_{R_{QG}})$ of $\hat{\mathcal{A}}_{QG}$, we define as

$$\hat{\mathcal{A}}_{sp.} \doteq R_{QG} [\hat{\mathcal{A}}_{SP}]$$

the *non-commutative* algebra of quantum physical 3-space that “relationally arises” from the quantum gravitational field (this algebra will be the quantum analogue of $\tilde{R}_h[\mathcal{A}_{\Sigma}]$, and R_{QG} that of $\tilde{R}_h$, in the left hand side of the diagram of Proposition 1.2). $\blacksquare$

Since we have the quantum space algebra, we now need its “Gelfand representation” to get a *manifold-like* space picture out of it. Of course, since the algebra is non-commutative, we cannot apply the standard Gelfand transform to get an ordinary differentiable manifold space. The Topos approach provides a suitable replacement, so we now recall some notions from this theory [3].

Definition 2.4: In a Hilbert space $\mathcal{H}$,

1. the category $\nu(\mathcal{H})$ is defined as the one that has as objects all commutative von Neumann sub-algebras on $\mathcal{H}$, and with sub-algebra inclusion as arrows among them;
2. the *spectral presheaf*, $\underline{M}$, is the presheaf¹² which assigns to an algebra $V \in \nu(\mathcal{H})$ the set $\underline{M}(V)$ given by its Gelfand spectrum, and to the inclusion

¹²Given a category C , a presheaf is a functor $F : C^{op} \longrightarrow \mathbf{Set}$, the latter being the category of ordinary sets and maps between them.

$V' \subseteq V$ the map which goes from $\underline{M}(V) \ni \omega : V \longrightarrow \mathbb{C}$ to $\underline{M}(V') \ni \omega|_{V'} : V' \longrightarrow \mathbb{C}$ consisting in reducing the domain of ω from V to V' . The category made by all presheaves (for a same Hilbert space), with natural transformations between the functors/presheaves as arrows, is a Topos (see the references for the definition of a Topos; informally, it's a category which "looks like" **Set** in the sense that it possesses analogues of *some* of its most relevant features (such as e.g. a sub-object classifier). ■

The key aspect [3] of this construction is that it *bijectively maps the self-adjoint operators on $\mathcal{H}$ to arrows from $\underline{M}$ to the quantity-value object $\mathbb{R}$* (which is the Topos analogue of the real numbers). In the category of classical manifolds, the analogue of these arrows are, of course, the functions from a manifold M to the real numbers $\mathbb{R}$. Thus, the general *non-commutative* algebra of bounded self-adjoint operators A on $\mathcal{H}$ is *mapped 1 – 1* to arrows $\check{A}(A) : \underline{M} \longrightarrow \mathbb{R}$, in analogy to how the Gelfand isomorphism maps the self-adjoint elements of a commutative C^* -algebra (with unit) to real valued functions $C(M) \ni f : M \longrightarrow \mathbb{R}$ on its Gelfand spectrum M (a compact topological space). This suggest the following definition:

Definition 2.5: The spectral presheaf $\underline{\Sigma}_{R_{QG}}$ based on $\nu(\mathcal{H}_{R_{QG}})$ is the space picture of quantum physical 3–space. ■

Note that, in order to physically interpret this construction as the analogue of constructing space via the Gelfand transform in the commutative case, we must first interpret the algebra product in $\hat{\mathcal{A}}_{sp.} \doteq R_{QG} [\hat{\mathcal{A}}_{SP.}]$ as the quantization of the space product (an interpretation in favor of which we argued in Section 1). Otherwise, even if one has an interpretation of the elements of the algebra as space ones, the product will still be the quantized phase space product, and therefore $\underline{\Sigma}$ will be a topos phase space (which is the usual physical interpretation made in the "topos approach" literature) rather than physical space.

Remark 2.1: Unlike an ordinary space, a spectral presheaf has no "points" or global elements [3], in concordance with the Kochen-Specker theorem. In this way, the pure states ω (on $\hat{\mathcal{A}}_{sp.} \doteq R_{QG} [\hat{\mathcal{A}}_{SP.}]$, not on $\hat{\mathcal{A}}_{SP.}$), the space's "points", are mapped to clopen sub-objects of $\underline{\Sigma}_{R_{QG}}$, i.e. our "quantum points" will be much like regions, in that they will have nonempty interiors. This will go hand in hand with the results in the next sections, which deal with the metric aspect of this. Furthermore, there's an obvious quantum nature in the points, since now we can have incompatible points (that is, points that correspond to non-commuting projectors in the Hilbert space; this is only possible because the space algebra is non-commutative). ■

Definition 2.6: The analogue of a given classical metric h_{ab} will be given by a 3–(spectral) dimensional *real first order spectral triple*

$$\mathcal{T}_{D_{QG}}^{R_{QG}} \doteq \left(R_{QG} [\hat{\mathcal{A}}_{SP.}], \mathcal{H}_{R_{QG}}, D_{QG} \right). \blacksquare$$

(Of course, the triple does contain differential geometric information, but it's encoded algebraically and functional analytically rather than in a space picture.)

Remark 2.2: In particular, one could try to classify all the *Dirac-like operators* D_{QG} there compatible with the algebra representation and *inner product* $(\cdot, \cdot)_{\mathcal{H}_{R_{QG}}}$ of its Hilbert space; *each of these operators would be an analogue of a given h_{ab}* (since $\hat{\mathcal{A}}_{sp.} \doteq R_{QG} [\hat{\mathcal{A}}_{SP.}]$), and they could be used to calculate *non-commutative* space intervals (i.e. distances) and *non-commutative* volume integrals, which, given the interpretations we made, *are the genuinely quantum distances and volumes*, since they are based on the *quantized space algebra*. ■

Definition 2.7: *The complete quantum analogue of the classical space picture (Σ, h_{ab}) will be the pair*

$$(\underline{\Sigma}_{R_{QG}}, \mathcal{T}_{D_{QG}}^{R_{QG}}). \blacksquare$$

We call all the ideas in sections 1 and 2.1 “*NCR space(-time)*” (i.e. Non-Commutative Relational space(-time)). These ideas are very general and *independent* of their concrete implementation, that is, of the selection of a particular, concrete, background independent Poisson sub-algebra $\mathcal{A}_{GR}$ of phase space functionals as required in Definition 2.1. What follows next are proposals for their *concrete* implementation. This step is quite non-trivial and currently we only have a partial answer in the sense that we're only able to build the 3-dimensional spatial part of spacetime without the temporal one (this is a consequence of the fact that the formalism of LQG is based on a $3+1$ Hamiltonian formalism instead of the so-called “covariant” one; but the $3+1$ formalism is, to date, the only one of these which has been quantized in a rigorous mathematical way, and this is one of the great results of LQG).

2.2. AQG Relational Space

First, we need the quantized phase space algebra $\hat{\mathcal{A}}_{QG}$. The immediate choice would be the kinematic quantum algebra $\hat{\mathcal{A}}_{LQG}$ from LQG. Unfortunately, this algebra makes a heavy use of a continuum background manifold Σ ; for example, the “electric-flux” elements $\hat{E}(S, \mathbf{k}) \in \hat{\mathcal{A}}_{LQG}$, where $\mathbf{k} \in C^\infty(S, \mathfrak{su}(2))$ and $S \subset \Sigma$ is a *surface*, are *parameterized* by the surfaces S , which are submanifolds of Σ . Thus, if we use that algebra, we would be falling into the physical contradictions which we were seeking to avoid with all this approach.

Now, the principle of canonical quantization suggests that *all*¹³ the properties of the quantum theory descend to the classical one, since one can reconstruct the quantum theory only by knowing the structure of the phase space of the

¹³It's a common myth that properties such as the quantum spin are “purely” quantum and don't have a classical analogue: this has proven to be false by results from Geometric Quantization and other more modern approaches [10].

classical theory. On the other hand, one suspects that the continuum manifold only appears in the classical limit, and, therefore, any properties in the classical theory that depend on it *cannot* be considered genuine properties that descended from the fundamental quantum theory, but rather, *spurious* properties introduced by the classical limit and which only exist at that level. Thus, we adopt the following *hypothesis*:

Hypothesis 2.2: *If one takes the set of classical properties in classical phase space that allows to relationally reconstruct the classical manifold, and strips out all what makes reference to the continuum manifold, what remains are the true fundamental properties that will allow to relationally reconstruct the quantum space in the quantum theory once these properties have been canonically quantized. ■*

(This should clarify the process “Dis.” of Definition 2.2, which stands for a kind of *fundamental* discretization, i.e. one without a cutoff parameter, since what’s being taken out is considered simply to be unnecessary to begin with, rather than something we just take out for the sake of approximation.)

A way to implement this is perhaps provided by a close cousin of LQG called Algebraic Quantum Gravity (AQG) [8], to which we now turn our attention.

Definition 2.8: [2] In LQG, phase space is coordinatized by the canonical pairs $[\mathbf{A}_a, \mathbf{E}^a]$, where $\mathbf{A}_a$ is a $SU(2)$ Yang-Mills connection on Σ and $\mathbf{E}^a$ its associated “Yang-Mills electric field” (both fields are $\mathfrak{su}(2)$ –valued, as represented by the bold font). Then, in LQG, the classical $\mathcal{A}_{LQG}$ is generated by the following phase space properties¹⁴:

$$\mathbf{H}(e)([\mathbf{A}_a, \mathbf{E}^a]) \doteq \mathcal{P} \exp \int_e \mathbf{A}_a \quad \text{and} \quad \mathbf{E}(S, \mathbf{k})([\mathbf{A}_a, \mathbf{E}^a]) \doteq \int_S * \mathbf{E}_{ab} \cdot \mathbf{k}. \blacksquare$$

Proposition 2.1: The variables in Definition 2.8 *can* be used to obtain $C^\infty(S)$ via R_h .

Proof: pick any $\mathbf{k}_0 \in C^\infty(S, \mathfrak{su}(2))$ such that $\mathbf{k}_0 \neq 0$ in *all* of S and write

$$\mathbf{k} \doteq f \mathbf{k}_0, \quad f \in C^\infty(S).$$

Since $* \mathbf{E}_{ab} \cdot \mathbf{k}$ is a 2–form on S , we have

$$\begin{aligned} * \mathbf{E}_{ab} \cdot \mathbf{k} &= f * \mathbf{E}_{ab} \cdot \mathbf{k}_0 \\ &= f f_{\mathbf{E}^a, \mathbf{k}_0} \epsilon_{ab}(q), \end{aligned}$$

where $f_{\mathbf{E}^a, \mathbf{k}_0} \in C^\infty(S)$ depends on $\mathbf{E}^a$ and $\mathbf{k}_0$ (and q comes from the h in $[h_{ab}, \pi^{ab}]$ determined by the point $[\mathbf{A}_a, \mathbf{E}^a]$; see [2] for how these variables transform into each other). For $f_1, f_2 \in C^\infty(S)$, consider the product $\cdot_S$ defined by:

¹⁴Where $*E_{ab}^{(i)} \doteq e_{abc} E_{(i)}^c$, $i = 1, 2, 3$, e is a path, $S \subset \Sigma$ a surface, and $\mathbf{k} \in C^\infty(S, \mathfrak{su}(2))$.

$$(E(S, f_1 \mathbf{k}_0) \cdot_S E(S, f_2 \mathbf{k}_0)) ([\mathbf{A}_a, \mathbf{E}^a]) \doteq E(S, f_1 f_2 \mathbf{k}_0), \forall [\mathbf{A}_a, \mathbf{E}^a] \in X$$

$$= \int_S f_1 f_2 f_{\mathbf{E}^a, \mathbf{k}_0} \epsilon(q).$$

With this product and for *each* $\mathbf{k}_0$, define the algebras:

$$\mathcal{W}_{\mathbf{k}_0} \doteq \left(\{E(S, f \mathbf{k}_0)\}_{f \in C^\infty(S)}, \cdot_S \right).$$

Now, clearly, since $\mathbf{k}_0 \neq 0$ in *all* of S , the algebras for different $\mathbf{k}_0$ are *all isomorphic and equivalent to* $C^\infty(S)$ (since $f \mathbf{k}_0 = f' \mathbf{k}_0$ implies $f = f'$). In this way, for any solution characterized by $[\mathbf{A}_a, \mathbf{E}^a]$ such that $\mathbf{E}^a \neq 0$ in *all* of S , there's at least one $\mathbf{k}_0$ such that¹⁵

$$f_{\mathbf{E}^a, \mathbf{k}_0} = 1 \text{ in all of } S,$$

and, in this way, the algebra for that $\mathbf{k}_0$ gets *bijectively* mapped to $C^\infty(S)$ by the relational representation R_h . Thus, for any point $[\mathbf{A}_a, \mathbf{E}^a]$ such that $\mathbf{E}^a \neq 0$ in *all* of S , there's always one of those algebras that gets bijectively mapped to $C^\infty(S)$ by the relational representation R_h ; but, since all the algebras are isomorphic, this means that the relational representation induces the algebra $C^\infty(S)$ of the surface S from a *single* algebra of phase space functions for *any* of those points $[\mathbf{A}_a, \mathbf{E}^a]$. $\square$

Definition 2.9: [8] In AQG, one considers an *abstract algebraic (countably infinite) graph* α (of distinguishable edges e) and an *embedding* $\gamma \doteq \psi(\alpha)$ on Σ that makes it *dual* to a *triangulation* γ^* . Thus, for each $e \doteq \psi(e)$ there's a *unique face* S_e in γ^* which intersects e , and it does so only at an interior point p_e of both S_e and e . For each $x \in S_e$, choose a path $\rho_e(x)$ which starts at ∂e , runs along e until p_e , and then within S_e until x . Then, we define the following phase space functions:

$$\mathbf{H}(e) ([\mathbf{A}_a, \mathbf{E}^a]) \doteq \mathcal{P} \exp \int_{\psi(e)} \mathbf{A}_a$$

and

$$E_{(j)}(e) ([\mathbf{A}_a, \mathbf{E}^a]) \doteq \int_{S_{\psi(e)}} e_{abc}(x) \left[\mathbf{H}_{\rho_{\psi(e)}(x)}(\mathbf{A}_a) \mathbf{E}^c(x) \mathbf{H}_{\rho_{\psi(e)}(x)}^{-1}(\mathbf{A}_a) \right] \cdot \tau_j,$$

where $\mathbf{H}_{\rho_e(x)}(\mathbf{A}_a)$ is the holonomy and $\{\tau_j\}_{j=1,2,3}$ is a basis of $\mathfrak{su}(2)$ (of course, $\tau_i \cdot \tau_j = \delta_{ij}$ and $\text{tr}(\tau_i \tau_j) = 2\delta_{ij}$).

We denote by $\mathcal{A}_{AQG}$ the algebra *linearly generated* (with *real* coefficients) by the above phase space functions. $\blacksquare$

Lemma 2.1: [8] The Poisson brackets for the variables in Definition 2.9 are given by:

¹⁵Indeed, just pick $\mathbf{E}^a$ (normalized) as one of the elements of a triad basis of both the tangent spaces and $\mathfrak{su}(2)$, and then, in that basis, $(k_0)_{(\Delta)} = \frac{\sqrt{q}}{E_{(\Delta)}^*}$, since $E_{(\Delta)}^* \neq 0$ in S .

$$\{\mathbf{H}(\mathbf{e}), \mathbf{H}(\mathbf{e}')\} = 0,$$

$$\{E_{(j)}(\mathbf{e}), \mathbf{H}(\mathbf{e}')\} = -\delta_{\mathbf{e}\mathbf{e}'} \tau_j \mathbf{H}(\mathbf{e}),$$

$$\{E_{(j)}(\mathbf{e}), E_{(k)}(\mathbf{e}')\} = \delta_{\mathbf{e}\mathbf{e}'} \sum_{l=1}^3 \varepsilon_{jkl} E_{(l)}(\mathbf{e}). \square$$

Lemma 2.2: ($G = SU(2)$, in our case.) The Poisson bracket relations of Lemma 2.1 for $\mathbf{e} = \mathbf{e}'$ are equivalent to those of the phase space defined by the cotangent bundle $T^*G \cong \mathfrak{g}^* \times G$ with the Poisson brackets given by the so-called semidirect product Poisson structure (see [10] for some details), where the $E_{(j)}(\mathbf{e})$ are simply the generators of $\mathfrak{su}(2)$. $\square$

Remark 2.3: As one can see, *both the variables and their Poisson brackets only depend on the abstract algebraic graph and all reference to the continuum manifold Σ has completely vanished.* ■

Lemma 2.3: The variables of the proof of Propostion 2.1, $E(S, f\mathbf{k}_0)$ (considering the set of *all* the elements from *all* the algebras $\mathcal{W}_{\mathbf{k}_0} \doteq (\{E(S, f\mathbf{k}_0)\}_{f \in C^\infty(S)}, \cdot, S)$ that were defined), subjected to the process of Definition 2.2/Hypothesis 2.2, result in the variables of Definition 2.9.

Proof: the process in Definition 2.2/Hypothesis 2.2 amounts to only retaining the generic constant functions c instead of the general f and a constant generic $\mathfrak{su}(2)$ matrix $\mathbf{C} = \sum_{j=1}^3 C_j \tau_j$ instead of the general function $\mathbf{k}_0$. Thus,

$$\begin{aligned} E(S_e, c\mathbf{C}) ([\mathbf{A}_a, \mathbf{E}^a]) &\doteq \sum_{j=1}^3 cC_j \int_{S_e} *E_{ab}^{(j)} \\ &= \sum_{j=1}^3 cC_j E(S_e, \tau_j) ([\mathbf{A}_a, \mathbf{E}^a]), \end{aligned}$$

which indicates that the functions to quantize are of the form (absorbing c into $\mathbf{C}$)

$$E(S_e, \mathbf{C}) = \sum_{j=1}^3 C_j E_{(j)}(S_e),$$

which gives the previous linearly generated algebra $\mathcal{A}_{AQG}$ for the E part. $\square$

Now, in AQG, the abstract quantized commutation relations of $\widehat{\mathcal{A}}_{AQG}$ are:

$$[\widehat{\mathbf{H}}(\mathbf{e}), \widehat{\mathbf{H}}(\mathbf{e}')] = 0,$$

$$[\widehat{E}_{(j)}(\mathbf{e}), \widehat{\mathbf{H}}(\mathbf{e}')] = -i\delta_{\mathbf{e}\mathbf{e}'} \tau_j \widehat{\mathbf{H}}(\mathbf{e}),$$

$$\left[\widehat{\mathbf{E}}_{(j)}(\mathbf{e}), \widehat{\mathbf{E}}_{(k)}(\mathbf{e}')\right] = i\delta_{\mathbf{e}\mathbf{e}'} \sum_{l=1}^3 \varepsilon_{jkl} \widehat{\mathbf{E}}_{(l)}(\mathbf{e}).$$

Definition 2.10: We define: $\mathcal{F}_{Dis} \equiv \mathcal{A}_{SP} \equiv \mathcal{A}_{S_e} \doteq \mathfrak{su}(2) \subset \mathcal{A}_{AQG}$, generated by the $\mathbf{E}_{(j)}(\mathbf{e})$. Now, we use the $\widehat{\mathbf{E}}_{(j)}(S_e)$ variables from AQG to define, following Definition 2.2, the quantum algebra

$$\widehat{\mathcal{A}}_{SP} \equiv \widehat{\mathcal{A}}_{\widehat{S_e}} \doteq \widehat{\mathfrak{su}(2)} \subset U(\mathfrak{su}(2))$$

(where the generators of $\widehat{\mathfrak{su}(2)}$ in $U(\mathfrak{su}(2))$ are now $-i\widehat{\mathbf{E}}_{(j)}(S_e)$) which corresponds to the quantization $\widehat{S_e}$ of the 1-punctured surface S_e . ■

Lemma 2.4: [8] A concrete self-adjoint representation R_{AQG} of $\widehat{\mathcal{A}}_{AQG}$ is implemented on the infinite tensor product Hilbert space

$$\mathcal{H}^{\otimes \infty} \doteq \otimes_e \mathcal{H}^e, \quad \text{where} \quad \mathcal{H}^e \doteq L^2(SU(2), \nu)$$

(which is the closure of the finite linear span of vectors of the form $\otimes_f \doteq \otimes_e f_e$, where $f_e \in \mathcal{H}^e$; ν is the Haar measure of $SU(2)$) by the operators

$$\widehat{\mathbf{H}}(\mathbf{e}) \otimes_f \doteq \left[\widehat{\mathbf{H}}(\mathbf{e})f_e\right] \otimes [\otimes_{e' \neq e} f_{e'}],$$

$$\widehat{\mathbf{E}}_{(j)}(\mathbf{e}) \otimes_f \doteq \left[\widehat{\mathbf{E}}_{(j)}(\mathbf{e})f_e\right] \otimes [\otimes_{e' \neq e} f_{e'}],$$

where

$$\left[\widehat{\mathbf{H}}(\mathbf{e})f_e\right](g) \doteq gf_e(g),$$

$$\left[\widehat{\mathbf{E}}_{(j)}(\mathbf{e})f_e\right](g) \doteq i \left[\frac{d}{dt} f_e(g e^{-t\tau_j})\right] \Big|_{t=0} . \square$$

Since we are studying the quantization $\widehat{S_e}$ of the punctured surface S_e , we now turn our attention to the algebra $\widehat{\mathcal{A}}_{\widehat{S_e}}$ for the $\mathbf{e}$ part and its representation space $\mathcal{H}^e \doteq L^2(SU(2), \nu)$ (we will omit the $\mathbf{e}$ from now on).

Before continuing, we're going to need the following examples:

Example 2.1.1: (cf. [13] for details and proofs.) Consider a (compact) Riemannian symmetric space (M, ν) ($\dim M = m$) with isotropy (Lie) group G . As is well known, if K is the isotropy (or stabilizer) group of a fixed point $p \in M$, then M can be identified with the homogeneous coset space, that is, $M \cong \frac{G}{K}$. The Lie algebra $\mathfrak{g}$ of G can then be split into $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (where $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$). Now, to fix a homogeneous spin structure on M is to have a homomorphism $\widetilde{\text{Ad}} : K \longrightarrow Spin(\mathfrak{m})$ such that the following diagram commutes (where λ is the usual covering map):

$$\begin{array}{ccc}
& & Spin(\mathfrak{m}) \\
& \nearrow \widetilde{\text{Ad}} & \downarrow \lambda \\
K & \xrightarrow{\text{Ad}} & SO(\mathfrak{m})
\end{array}$$

Let $\kappa : Spin(\mathfrak{m}) \rightarrow GL(\Delta)$ be the spin representation (where Δ is the vector space of spinors). A spinor field $\check{\psi}$ on M is identified with a function $\check{\psi} : G \rightarrow \Delta$ that satisfies the invariance condition $\check{\psi}(gk) = \kappa \widetilde{\text{Ad}}(k^{-1})\check{\psi}(g)$. For $X \in \mathfrak{g}$, the left invariant vector field X on G is defined in the standard way, $X(\check{\psi})(g) \doteq \frac{d}{dt} [\check{\psi}(ge^{-tX})] |_{t=0}$. With this, we can (locally) define the Dirac operator as

$$\check{D}\check{\psi} \doteq -i \sum_{j=1}^m \gamma^j X_j(\check{\psi}).$$

Then, if $\Omega_G = -\sum_{j=1}^{\dim G} X_j^2$ is the Casimir operator of G , the following important result holds: $\check{D}^2 = \Omega_G + \frac{1}{8}\mathfrak{s}$. The power of this formula lies in that it allows us to calculate the eigenvalues of $\check{D}^2$ and $|\check{D}|$ purely by means of the representation theory of G . Furthermore, the canonical, *commutative* spectral triple of Example 1.1, $(C^\infty(M), L^2(\mathcal{S}, \epsilon(g_{ab})), \check{D}_g)$, can now be written as $(C^\infty(\frac{G}{K}), L^2(\check{\mathcal{S}}, \nu), \check{D})$. ■

Example 2.1.2: In the particular case of $G = SU(2)$, we have two examples of interest:

1. The first one is $K = \{e\}$, which means that $M \cong \frac{G}{K} = G = SU(2) \cong \mathbb{S}^3$, i.e. the 3-sphere. The Dirac operator is

$$\check{D}_{\mathbb{S}^3}\check{\psi} \doteq -i \sum_{j=1}^3 \sigma^j X_j(\check{\psi}),$$

where σ^j are the standard Pauli matrices¹⁶. Note that, in this case, since $\Delta = \mathbb{C}^{2^1}$ ($3 = n = 2m + 1$), we have $L^2(\check{\mathcal{S}}_{\mathbb{S}^3}, \nu) \cong \mathbb{C}^2 \otimes L^2(SU(2), \nu)$.

2. The second example is $K = U(1)$, which means $M \cong \frac{SU(2)}{U(1)} \cong \mathbb{S}^2$, i.e. the 2-sphere. The Dirac operator is

$$\check{D}_{\mathbb{S}^2}\check{\psi} \doteq -i \sum_{j=1}^2 \sigma^j X_j(\check{\psi}).$$

In this case, $L^2(\check{\mathcal{S}}_{\mathbb{S}^2}, \nu) \cong \mathbb{C}^{2^1} \otimes L^2(\mathbb{S}^2, \nu)$ ($2 = n = 2m$).

¹⁶We use the convention: $\sigma^1 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Lemma 2.5: (cf. [14] for some details.) Up to normalization constants, the spinors¹⁷

$$Y_{s'\mu'}^+ \doteq \begin{pmatrix} \Pi_{-\frac{1}{2}, \mu'}^{(s')} |_{(-\varphi, \theta, \varphi)} \\ i\Pi_{\frac{1}{2}, \mu'}^{(s')} |_{(-\varphi, \theta, \varphi)} \end{pmatrix}, \quad Y_{s'\mu'}^- \doteq \begin{pmatrix} -\Pi_{-\frac{1}{2}, \mu'}^{(s')} |_{(-\varphi, \theta, \varphi)} \\ i\Pi_{\frac{1}{2}, \mu'}^{(s')} |_{(-\varphi, \theta, \varphi)} \end{pmatrix},$$

$\forall s' \in \mathbb{N}_0 + \frac{1}{2}, \forall \mu' \in \underbrace{\{-s', -s' + 1, \dots, s' - 1, s'\}}_{(2s'+1)}$, where the $\Pi_{\pm\frac{1}{2}, \mu'}^{(s')} |_{(-\varphi, \theta, \varphi)}$

$e^{\mp i(\psi+\varphi)/2} = \Pi_{\pm\frac{1}{2}, \mu'}^{(s')} |_{(\psi, \theta, \varphi)}$ form an *eigenbasis* of $\check{\mathcal{D}}_{\mathbb{S}^2}$ in $L^2(\check{\mathbb{S}}_{\mathbb{S}^2}, \nu)$, such that

$$\check{\mathcal{D}}_{\mathbb{S}^2} Y_{s'\mu'}^+ = \left(s' + \frac{1}{2}\right) Y_{s'\mu'}^+, \quad \check{\mathcal{D}}_{\mathbb{S}^2} Y_{s'\mu'}^- = -\left(s' + \frac{1}{2}\right) Y_{s'\mu'}^-. \square$$

In this way, the eigenvalues of $\check{\mathcal{D}}_{\mathbb{S}^2}^2$ are such that $(s' + \frac{1}{2})^2 = s'(s' + 1) + \frac{1}{4}$, so we can then recognize $s'(s' + 1)$ as the eigenvalues of the Casimir element Ω_G and the value for the scalar curvature $\mathfrak{s}_{\mathbb{S}^2} = 2$ of $(\mathbb{S}^2, \nu)$. Noting that $\mathfrak{s}_{\mathbb{S}^2} = \frac{2}{\rho_{\mathbb{S}^2}^2}$, where $\rho_{\mathbb{S}^2}$ is the radius of $(\mathbb{S}^2, \nu)$, we will write the eigenvalues $d_{s'}$ of $\check{\mathcal{D}}_{\mathbb{S}^2}$ as

$$d_{s'} = \sqrt{s'(s' + 1) + \left(\frac{1}{4\rho_{\mathbb{S}^2}^2}\right)}. \blacksquare$$

Thus, these examples show that we have at least two Dirac operators that we can define on the representation space of AQG. But the one we will use is inspired in $\check{\mathcal{D}}_{\mathbb{S}^2}$, i.e. that one corresponding to the 2-sphere, since we are quantizing a surface.

Definition 2.11: For each $s \in \frac{1}{2}\mathbb{N}_0$, we define $\mathcal{H}_\mu^{(s)} \subset \underbrace{\mathcal{H}_{(s)} \oplus \dots \oplus \mathcal{H}_{(s)}}_{(2s+1)}$

to be the space spanned by $\left\{ \Pi_{m\mu}^{(s)} \right\}_{1 \leq m \leq (2s+1)}$. Then, we reduce to $\mathcal{H}_\mu^{(s)}$ the

¹⁷By the Peter-Weyl theorem [12], we know that the functions $SU(2) \ni g \mapsto \Pi^{(s)}(g)_{ij}, i, j = 1, \dots, (2s+1), \forall s \in \frac{1}{2}\mathbb{N}_0$, given by the matrix elements of the irreducible representations of $SU(2)$ on the (necessarily, *finite dimensional*) Hilbert spaces $\mathcal{H}_{(s)}$ (where $\dim \mathcal{H}_{(s)} = 2s+1$), comprise a basis of the *separable* $L^2(SU(2), \nu)$. Also, the same theorem states that $L^2(SU(2), \nu)$ decomposes into a direct sum of spaces on which the left regular representation $[\mathfrak{L}_h(f_e)](g) \doteq f_e(gh^{-1}), \forall h, g \in SU(2), \forall f_e \in L^2(SU(2), \nu)$ of $SU(2)$ on $L^2(SU(2), \nu)$ [note that *this representation is the one used to define $\widehat{\mathcal{E}}_{(j)}(e)$ above*] is *irreducible*. All irreducible representations of $SU(2)$ appear in this decomposition and the one for spin s (which appears $(2s+1)$ times in the decomposition) is realized by the matrices $\Pi^{(s)}(g)$ acting on the sub-space of $L^2(SU(2), \nu)$ spanned by the previous $\Pi_{ij}^{(s)}, i, j = 1, \dots, (2s+1)$ (or, in a different convention, $\Pi_{m\mu}^{(s)}, m, \mu = -s, -s+1, \dots, s-1, s$), and isomorphic to $\mathcal{H}_{(s)}$, i.e.

$$L^2(SU(2), \nu) \cong \mathbb{C} \oplus \mathcal{H}_{(\frac{1}{2})} \oplus \mathcal{H}_{(\frac{1}{2})} \oplus \dots \oplus \underbrace{\mathcal{H}_{(s)} \oplus \dots \oplus \mathcal{H}_{(s)}}_{(2s+1)} \oplus \dots$$

(the $j_0 - \text{th}, 1 \leq j_0 \leq (2s+1)$, space $\mathcal{H}_{(s)}$ in $\underbrace{\mathcal{H}_{(s)} \oplus \dots \oplus \mathcal{H}_{(s)}}_{(2s+1)}$ is spanned by

$$\left\{ \Pi_{ij_0}^{(s)} \right\}_{1 \leq i \leq (2s+1)}.$$

action of the algebra representation $R_{AQG} \left[\widehat{\mathfrak{su}(2)} \right]$. We denote the result as $R_{AQG} \left[\widehat{\mathfrak{su}(2)} \right]_{\mu}^{(s)}$. ■

Definition 2.12: We define

$$D_{\mu}^{(s)} \doteq -i \sum_{j=1}^2 \sigma^j X_j$$

on $\check{\mathcal{H}}_{\mu}^{(s)} \doteq \mathbb{C}^2 \otimes \mathcal{H}_{\mu}^{(s)}$, $\forall s \in \frac{1}{2}\mathbb{N}_0$, where $-\widehat{iE}_{(j)}(S_e) = X_j$ (which are therefore *skew-adjoint* rather than self-adjoint, like the $\widehat{E}_{(j)}$). ■

Proposition 2.2: On the spaces $\check{\mathcal{H}}_{\mu}^{(s)}$, we have

$$\left(D_{\mu}^{(s)} \right)^2 u_{m\mu}^{\pm} = d_{s,m}^2 u_{m\mu}^{\pm},$$

$$u_{m\mu}^{\pm} = \begin{pmatrix} \pm \Pi_{-m\mu}^{(s)} \\ i \Pi_{m\mu}^{(s)} \end{pmatrix}, \quad d_{s,m}^2 = s(s+1) - m^2 - m, \quad \forall s \in \frac{1}{2}\mathbb{N}_0, \quad \forall m \in \underbrace{\{-s, -s+1, \dots, s-1, s\}}_{(2s+1)}.$$

Proof: we have:

$$\begin{aligned} D^2 &= -(\sigma^1 X_1 + \sigma^2 X_2)^2 \\ &= -I_{2 \times 2} X_1^2 - I_{2 \times 2} X_2^2 - \sigma^1 \sigma^2 X_1 X_2 - \sigma^2 \sigma^1 X_2 X_1 \\ &= -I_{2 \times 2} X_1^2 - I_{2 \times 2} X_2^2 - \sigma^1 \sigma^2 [X_1, X_2] \\ &= I_{2 \times 2} \Omega_G - I_{2 \times 2} (-X_3^2) - \sigma^3 (iX_3); \end{aligned}$$

since $(iX_3) \Pi_{m\mu}^{(s)} = -m \Pi_{m\mu}^{(s)}$, $\forall \mu$, the $d_{s,m}^2$ are eigenvalues of D^2 for $u_{m\mu}^{\pm}$. □

Remark 2.4: In this way, by using the relation $\Pi_{\pm \frac{1}{2}, \mu}^{(s')} |_{(-\varphi, \theta, \varphi)} \mapsto \Pi_{\pm \frac{1}{2}, \mu}^{(s')} |_{(-\varphi, \theta, \varphi)}$
 $e^{\mp i(\psi + \varphi)/2} = \Pi_{\pm \frac{1}{2}, \mu}^{(s')} |_{(\psi, \theta, \varphi)} \in \mathcal{H}_{\mu}^{(s')}$, $\forall s' \in \{\mathbb{N}_0 + \frac{1}{2}\}$, we can see that $\left[\left(D_{\mu}^{(s')} \right)^2 u_{\frac{1}{2}\mu}^{\pm} \right] |_{(-\varphi, \theta, \varphi)} =$
 $\left(\check{D}_{\mathbb{S}^2} \right)^2 Y_{s', \mu}^{\pm} = d_{s', \mu}^2 Y_{s', \mu}^{\pm}$. Thus, $\left(D_{\mu}^{(s')} \right)^2$ mimics the spectral properties of
 $\left(\check{D}_{\mathbb{S}^2} \right)^2$ for eigenvectors based on the vectors in $\mathcal{H}_{\mu}^{(s')}$ with biggest negative and
smallest positive $m' \in \underbrace{\{-s', -s'+1, \dots, s'-1, s'\}}_{(2s'+1)}$, respectively, $\Pi_{-\frac{1}{2}, \mu}^{(s')} |_{(\psi, \theta, \varphi)}$
and $\Pi_{\frac{1}{2}, \mu}^{(s')} |_{(\psi, \theta, \varphi)}$, as just seen in Proposition 2.3. Thus, the factor $\frac{1}{4}$ in $d_{s, -\frac{1}{2}}^2$
for our $\left(D_{\mu}^{(s')} \right)^2$ can be thought as coming from the sphere. ■

Remark 2.5: Certainly, we could define the Dirac operator directly as $D_\mu^{(s)} \doteq -i \sum_{j=1}^2 \sigma^j X_j$, without any mention to the sphere. But we preferred to do it the other way around in order to highlight the similitudes and differences between our spectral triple and that of the sphere (in particular, the factor $\frac{1}{4}$ in $d_{s, -\frac{1}{2}}^2$). We also wanted to show with this why the Casimir element appears in the area, something which is not obvious at all, but is generic to Dirac operators (which are called for by NCG to calculate metric properties) on homogeneous spaces, as seen in Examples 2.1.1-2. Furthermore, the actual concrete reason here is that, by the properties of the algebra, $[D, a]_{1,1}$ and $[D, a]_{1,2}$ belong again to the algebra, and this makes D^2 appear in the steps for calculating the distance (see Theorem 2.1). Now, the difference is that in NCG one has a whole geometric machinery to justify why such a Dirac operator must be used to calculate metric properties (in particular, the reconstruction theorem for the commutative case, which shows that the information of the classical metric can be recovered from the Dirac operator), while in LQG one directly canonically quantizes the classical phase space area functional, a process which gives us a spectrum that resembles the one of the Casimir element, but which suffers from the unavoidable obscurity and black-box-like character of standard canonical quantization. We believe our approach is more transparent, and that it also gives further insight (see Remark 2.7-8 later). ■

Of course, we only saw how Dirac operators of this form just form the usual spectral triple for the sphere with its corresponding commutative algebra, $C^\infty(\mathbb{S}^2)$, while what we need here is a spectral triple with respect to the *non-commutative* algebra $R_{AQG}[\widehat{\mathfrak{su}(2)}]$. The obvious option would be to try to combine D^e on $\mathbb{C}^2 \otimes \mathcal{H}^e$ with $R_{AQG}[\widehat{\mathfrak{su}(2)}]$ in some way to obtain a spectral triple. But this won't work because the $\widehat{E}_{(j)}(e)$ are *unbounded operators* on $\mathcal{H}^e$. Thus, we will reduce the domain of the representation to the *finite dimensional* subspaces $\tilde{\mathcal{H}}_\mu^{(s)}$ where it's *irreducible*.

Definition 2.13: The terms in $(\widehat{\mathcal{A}}_\mu^{(s)}, \tilde{\mathcal{H}}_\mu^{(s)}, D_\mu^{(s)}; \Gamma, J)$ correspond to $\widehat{\mathcal{A}}_\mu^{(s)} \doteq I_{2 \times 2} R_{AQG}[U(\mathfrak{su}(2))]_\mu^{(s)}$, and Γ, J are the corresponding operators on the 2-sphere transferred to our spaces, i.e. $\Gamma \doteq \sigma^3 I_{\mathcal{H}_\mu^{(s)}}$ and, for $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \tilde{\mathcal{H}}_\mu^{(s)}$, $J \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \doteq -i\sigma^2 \overline{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}} = \begin{pmatrix} -\bar{v}_2 \\ \bar{v}_1 \end{pmatrix}$. ■

Proposition 2.3: The triple $(\widehat{\mathcal{A}}_\mu^{(s)}, \tilde{\mathcal{H}}_\mu^{(s)}, D_\mu^{(s)}; \Gamma, J)$ is a spectral triple of spectral dimension 0, on which, for elements in $R_{AQG}[\widehat{\mathfrak{su}(2)}]_\mu^{(s)} \doteq \widehat{\mathcal{A}}_{sp, \mu}^{(s)} \subset \widehat{\mathcal{A}}_\mu^{(s)}$ (as a Lie algebra only¹⁸), the conditions for the following properties are verified:

1. real structure,
2. first order,

¹⁸For what we do here, only the Lie brackets commutation relations will be needed, along with the fact that they're realized via the matrix commutator of the representation spaces (see Remark 2.9 at the end). Thus, despite having some similitudes, this triple is *not* the so-called “fuzzy sphere”, since in the latter the algebra is given by the usual associative matrix algebra $\mathcal{M}(n, \mathbb{C})$.

3. and KO -dimension 2.

(This doesn't form a sub-triple in the strict sense, though; on the other hand, since *only the Lie bracket is needed to prove them*, we could call it a “Lie sub-triple”, since $\widehat{\mathcal{A}}_{sp,\mu}^{(s)}$ is indeed a true Lie sub-algebra of $\widehat{\mathcal{A}}_\mu^{(s)}$).

Proof: That $\left[D_\mu^{(s)}, a\right]$ doesn't vanish or is ill-defined for $a \in \widehat{\mathcal{A}}_{sp,\mu}^{(s)}$ will be seen below in point 2; both the commutator and a are bounded because $\check{\mathcal{H}}_\mu^{(s)}$ is finite dimensional (and is also of spectral dimension 0 for this very reason); furthermore, $D_\mu^{(s)}$ is obviously self-adjoint.

1) The only property we need to check is $[a, Jb^*J^{-1}] = 0, \forall a, b \in \mathcal{A}$, since the others don't depend on the algebra and therefore are identical to the ones of the 2-sphere. First, note that the condition is asking $Jb^*J^{-1} = b^0$ to be a representation of the *opposite algebra* $\mathcal{A}^0$ (that is, b^0 is such that $b \in \mathcal{A}$, but with product $a^0b^0 = (ba)^0$, or, in Lie bracket terms, $[a^0, b^0] = ([b, a])^0$ that commutes with $\mathcal{A}$. Thus, given that $J^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \doteq \begin{pmatrix} \bar{v}_2 \\ -\bar{v}_1 \end{pmatrix}$, $b^* = -b$ (by skew-adjointness), and that

$$\begin{aligned} b^{1,2} f^{1,2} &= X_b^L(f^{1,2})(g) \doteq \frac{d}{dt} [f^{1,2}(ge^{-tb})] \big|_{t=0} = \frac{d}{dt} [f^{1,2}(\varphi - tb_\varphi, \theta - tb_\theta, \psi - tb_\psi)] \big|_{t=0} \\ &= \sum_{j=1}^3 \frac{\partial f^{1,2}}{\partial \varphi_j} (-b_j) \end{aligned}$$

(since $\nabla_{\varphi_j} f^{1,2} = \frac{\partial f^{1,2}}{\partial \varphi_j}$ for scalar functions), we get (the b_j are *real*):

$$\begin{aligned} b^0 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= Jb^*J^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = -Jb \begin{pmatrix} \bar{f}_2 \\ -\bar{f}_1 \end{pmatrix} = -J \begin{pmatrix} \sum_{j=1}^3 -b_j \frac{\partial \bar{f}_2}{\partial \varphi_j} \\ -\sum_{j=1}^3 -b_j \frac{\partial \bar{f}_1}{\partial \varphi_j} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=1}^3 b_j \frac{\partial f_1}{\partial \varphi_j} \\ \sum_{j=1}^3 b_j \frac{\partial f_2}{\partial \varphi_j} \end{pmatrix} = \frac{d}{dt} \left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (ge^{tb}) \right] \big|_{t=0} = X_b^R \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) (g), \end{aligned}$$

which is the *right* representation, and then $[a, b^0] = 0$, since a is defined in terms of the *left* representation, and these two representations *always commute* with each other:

$$\begin{aligned} [a, b^0] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (g) &= \left[a \frac{d}{dt} \left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (ge^{tb}) \right] \big|_{t=0} - b^0 \frac{d}{dt'} \left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (e^{t'a}g) \right] \big|_{t'=0} \right] \\ &= \left[\frac{d}{dt'} \frac{d}{dt} \left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (e^{t'a}ge^{tb}) \right] \big|_{t,t'=0} - \frac{d}{dt} \frac{d}{dt'} \left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (e^{t'a}ge^{tb}) \right] \big|_{t',t=0} \right] = 0, \end{aligned}$$

by commutation of partial derivatives and since the b_j are constants.

2) $D = -i(\sigma^1 X_1 + \sigma^2 X_2)$ and $a = I_{2 \times 2} \sum_{j=1}^3 a_j X_j$. Then:

$$\begin{aligned} [D, a] &= -i\sigma^1 \sum_{j=1}^3 a_j [X_1, X_j] - i\sigma^2 \sum_{j=1}^3 a_j [X_2, X_j] \\ &= -i\sigma^1 (a_2 X_3 - a_3 X_2) - i\sigma^2 (-a_1 X_3 + a_3 X_1), \end{aligned}$$

and, in this way, since $\sigma_{11}^1(a_2 X_3 - a_3 X_2)$ and $i\sigma_{12}^2(-a_1 X_3 + a_3 X_1)$ are in $\mathcal{A}_{11;12}$, they must commute with $\mathcal{A}_{11;12}^0$ by point 2, i.e. $[[D, a], b^0] = 0, \forall a, b \in \mathcal{A}$.

3) Verbatim as in the case of the 2-sphere since the operators involved for the calculation are simply the same. $\square$

Before continuing, we will need the following examples:

Example 2.2: The following are exemplary of the general form taken by the hermitian matrices iX_j on the irreducible spaces $\mathcal{H}_\mu^{(s)}$:

- $s = 1$:

$$iX_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad iX_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad iX_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

- $s = \frac{3}{2}$:

$$iX_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad iX_2 = \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix},$$

$$iX_3 = \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix};$$

- $s \in \frac{1}{2}\mathbb{N}_0$.¹⁹

$$iX_1^{hk} = \frac{1}{2}(\delta_{h,k+1} + \delta_{h+1,k})\sqrt{(s+1)(h+k-1) - hk},$$

$$iX_2^{hk} = \frac{1}{2i}(\delta_{h,k+1} - \delta_{h+1,k})\sqrt{(s+1)(h+k-1) - hk},$$

$$iX_3^{hk} = (s+1-k)\delta_{hk}, \quad 1 \leq h, k \leq (2s+1). \blacksquare$$

Theorem 2.1: The *noncommutative distance* for the previous triple for elements in $\widehat{\mathcal{A}}_{sp,\mu}^{(s)}$, between “the origin” Ω_- and the pure state Ω_+ at the maximum possible distance from it, is

$$d(\Omega_+, \Omega_-) = d_s^{-1},$$

¹⁹We will use the letters h, k for this convention of values of the indexes, and m, μ for the one of Proposition 2.2.

where $d_s = d_{s, -\frac{1}{2}}$ for $2s + 1$ even, and $d_s = d_{s, 0}$ for $2s + 1$ odd.

Proof: the operator norm of T is defined as $\|T\|^2 \doteq \sup_{v \in \mathcal{H}} \{(v, T^*Tv)_{\mathcal{H}} / \|v\| = 1\}$.

We compute:

$$[D, a]^*[D, a] = [\mathrm{i}\sigma^1(a_2X_3 - a_3X_2) + \mathrm{i}\sigma^2(-a_1X_3 + a_3X_1)] [\mathrm{i}\sigma^1(a_2X_3 - a_3X_2) + \mathrm{i}\sigma^2(-a_1X_3 + a_3X_1)]$$

$$= -I_{2 \times 2}(a_2X_3 - a_3X_2)^2 - \sigma^2\sigma^1(-a_1X_3 + a_3X_1)(a_2X_3 - a_3X_2) - \sigma^1\sigma^2(a_2X_3 - a_3X_2)(-a_1X_3 + a_3X_1)$$

$$-I_{2 \times 2}(-a_1X_3 + a_3X_1)^2$$

$$= -I_{2 \times 2}(a_2^2X_3^2 - a_2a_3\{X_3, X_2\} + a_3^2X_2^2) + \sigma^1\sigma^2[(-a_1X_3 + a_3X_1), (a_2X_3 - a_3X_2)]$$

$$-I_{2 \times 2}(a_1^2X_3^2 - a_1a_3\{X_3, X_2\} + a_3^2X_1^2)$$

$$= a_3^2D^2 + I_{2 \times 2}a_3(a_1 + a_2)\{X_2, X_3\} + I_{2 \times 2}(a_1^2 + a_2^2)(-X_3^2) - \sigma^3a_3(a_1\mathrm{i}X_1 + a_2\mathrm{i}X_2).$$

Consider, now, any $v = \omega \otimes V \in \mathbb{C}^2 \otimes \mathcal{H}$ such that $\|\omega \otimes V\| = 1$ (recall that $(\omega \otimes V, \omega \otimes V)_{\mathbb{C}^2 \otimes \mathcal{H}} = (\omega, \omega)_{\mathbb{C}^2}(V, V)_{\mathcal{H}}$). We would, then, get something of the form:

$$(v, [D, a]^*[D, a]v)_{\mathbb{C}^2 \otimes \mathcal{H}} = a_3^2D^2v + \|\omega\|^2 \sum_{\alpha=0,3} (V, M_\alpha V)_{\mathcal{H}} + \sigma(\omega) \sum_{\alpha=1,2} (V, M_\alpha V)_{\mathcal{H}},$$

with

$$M_0 = a_3(a_1 + a_2)\{X_2, X_3\}, \quad M_1 = -a_3a_1\mathrm{i}X_1, \quad M_2 = -a_3a_2\mathrm{i}X_2, \quad M_3 = (a_1^2 + a_2^2)(-X_3^2),$$

$$(V, M_\alpha V)_{\mathcal{H}} = \sum_{h,k=1}^{2s+1} M_\alpha^{hk} V^h \overline{V}^k, \quad \sigma(\omega) = (\omega, \sigma^3\omega)_{\mathbb{C}^2} = |\omega_1|^2 - |\omega_2|^2.$$

Now, we claim that, for each a , we have

$$\|[D, a]\|^2 - a_3^2d_s^2 \doteq Q(a)^2 \geq 0.$$

Thus, we get²⁰:

$$\|[D, a]\| = \sqrt{a_3^2d_s^2 + Q(a)^2}.$$

²⁰the square power is just for emphasis regarding the positivity of the term; $Q(a)$ is, of course, *real*.

It remains to prove the claim. The problematic terms are those of M_0 , M_1 , and M_2 , since, in them, the components a_1 , a_2 , a_3 , appear *unsquared* so that their sign could make $\| [D, a] \|^2 - a_3^2 d_s^2$ *negative*, because the growth of the only remaining term, $(V, M_3 V)_{\mathcal{H}}$ —which is always strictly positive— is *bounded* from above by the norm of iX_3 (which equals $\| iX_3 \|_{\mathcal{B}(\mathcal{H})} = s$), that is: $\| iX_3 V \|_{\mathcal{H}} \leq \| iX_3 \|_{\mathcal{B}(\mathcal{H})} \| V \|_{\mathcal{H}}$, $\forall V \in \mathcal{H}$.

Consider the normalized eigenvector $V_{(3)} = (0, \dots, \underbrace{1}_{m=-\frac{1}{2}; 0}, \dots, 0)$ of $-X_3^2$ with the biggest value, d_s^2 , for $d_{s,m}^2$ (then, for even $2s+1$ we get $m_3 = -\frac{1}{2}$ and so $d_s^2 = s(s+1) + \frac{1}{4}$, and for odd $2s+1$ we get $m_3 = 0$ and so $d_s^2 = s(s+1)$) where $\omega_{(3)} = (1, 0)$. We then get $(v_{(3)} = \omega_{(3)} \otimes V_{(3)})$:

$$(v_{(3)}, [D, a]^* [D, a] v_{(3)})_{\mathbb{C}^2 \otimes \mathcal{H}} - a_3^2 d_s^2 = m_3^2 (a_1^2 + a_2^2) \geq 0,$$

since²¹ $M_{\alpha}^{-\frac{1}{2}, -\frac{1}{2}} = 0$, $M_{\alpha}^{00} = 0$, $\forall \alpha \neq 3$. Thus, given that *there's at least one* $v_{(3)} \in \mathbb{C}^2 \otimes \mathcal{H}$ *satisfying*

$$(v_{(3)}, [D, a]^* [D, a] v_{(3)})_{\mathbb{C}^2 \otimes \mathcal{H}} \geq a_3^2 d_s^2 \geq a_3^2 d_{s,m}^2,$$

and that, by definition,

$$\| [D, a] \|^2 \geq (v, [D, a]^* [D, a] v)_{\mathbb{C}^2 \otimes \mathcal{H}}, \forall v \in \mathbb{C}^2 \otimes \mathcal{H},$$

we must *necessarily* have

$$\| [D, a] \|^2 \geq (v_{(3)}, [D, a]^* [D, a] v_{(3)})_{\mathbb{C}^2 \otimes \mathcal{H}} \geq a_3^2 d_s^2 \geq a_3^2 d_{s,m}^2,$$

which means that

$$\| [D, a] \|^2 - a_3^2 d_s^2 \doteq Q(a)^2 \geq 0.$$

Consider now the density matrices ρ^+ , ρ^- (which represent *pure* states in $\widehat{\mathcal{A}}_{\mu}^{(s)}$ in this case) given by:

- $n = 2s + 1$ even: $\rho_{ij}^+ = 1$ for $i = j = \frac{n}{2}$, and $\rho_{ij}^+ = 0$ otherwise; $\rho_{ij}^- = 1$ for $i = j = \frac{n}{2} + 1$, and $\rho_{ij}^- = 0$ otherwise.
- $n = 2s + 1$ odd: $\rho_{ij}^+ = 1$ for $i = j = \frac{n-1}{2}$ and $\rho_{ij}^+ = 0$ otherwise; $\rho_{ij}^- = 1$ for $i = j = \frac{n-1}{2} + 1$ and $\rho_{ij}^- = 0$ otherwise.

Then, for an arbitrary $a \in \widehat{\mathcal{A}}_{sp, \mu}^{(s)}$, we get:

- even case: $\Omega_{\rho^+}(a) \doteq \text{tr}(\rho^+ a) = \frac{1}{2} i a_3$ and $\Omega_{\rho^-}(a) \doteq \text{tr}(\rho^- a) = -\frac{1}{2} i a_3$;
- odd case: $\Omega_{\rho^+}(a) = \text{tr}(\rho^+ a) = i a_3$ and $\Omega_{\rho^-}(a) = \text{tr}(\rho^- a) = 0$.

Thus, the noncommutative distance is:

$$\mathcal{d}(\Omega_{\rho^+}, \Omega_{\rho^-}) = \sup \{ |a_3|, \forall a \in \mathcal{A} / \| [D, a] \| \leq 1 \}.$$

²¹ $\{X_2, X_3\}^{hk} = (\delta_{h,k+1} - \delta_{h+1,k})(2i)(2s+2-h-k)\sqrt{(s+1)(h+k-1)-hk}$.

Going back to $\| [D, a] \|$: since it's a sum of *positive* terms, $\| [D, a] \| = \sqrt{a_3^2 d_s^2 + Q(a)^2}$, and $a_3^2 d_s^2$ *only depends on* a_3 , it's therefore clear that the second term should go away if we want to maximize $|a_3|$ under the constraint $\| [D, a] \| \leq 1$ (otherwise, when varying a_1, a_2 , the other positive term will diminish the part of 1 that goes to $a_3^2 d_s^2$, so to speak), so our only option is $a_{\text{sup.}} = (0, 0, a_3)$, where we get $Q(a_{\text{sup.}})^2 = 0$. In this way, $\| [D, a_{\text{sup.}}] \| = \sqrt{a_3^2 d_s^2} = |a_3| d_s = 1$, i.e. $|a_3| = d_s^{-1}$. Therefore, $\not{d} = d_s^{-1}$. $\square$

Remark 2.6: Note that the actual *physical* distance will be $\not{d} = d_s$ because the triple we have is for the algebra, while our initial variables were the quantization of momenta on the *dual* algebra (Lemma 2.2), which means that the physical metric is the *inverse* (or dual) of the metric on the algebra. $\blacksquare$

Remark 2.7: The obtained values for the area *are precisely the only values allowed by the area operator from LQG for such a punctured surface*²², since we can make $\left(\frac{1}{4\rho_{s^2}^2}\right)$ as small as we want by just letting ρ_{s^2} get bigger. Nevertheless, *our approach makes a further prediction*: d_s are just the biggest distance values, but by choosing (recall that $n = 2s + 1$) $\rho_{ij}^+ = 1, i = j = \frac{n}{2} - h$ ($0 \leq h \leq \frac{n}{2} - 1$), $\rho_{ij}^- = 1, i = j = \frac{n}{2} + k$ ($1 \leq k \leq \frac{n}{2}$), for the even case, and $\rho_{ij}^+ = 1, i = j = \frac{n-1}{2} - h$, ($0 \leq h \leq \frac{n-1}{2} - 1$), $\rho_{ij}^- = 1, i = j = \frac{n-1}{2} + k$ ($1 \leq k \leq \frac{n+1}{2}$), for the odd case, *we get smaller values*:

$$\not{d}_{h,k} = \frac{d_s}{h+k}.$$

The physical relevance and consequences of this difference with respect to the prediction of LQG remains a topic for further research. $\blacksquare$

Remark 2.8: The triple has *spectral dimension* 0 and *KO*-dimension 2. This seems to be a clear case of *spectral dimensional drop* (from 2, in the classical case²³, to 0 here) upon quantization²⁴ for the surface. The interpretation that we make of the fact that the irreducible geometries are 0-dimensional is the following: recall, from Example 1.5-6, that the spectral triples that describe the geometry of a set with a *finite* number of points, and the Euclidean distances among them, are spectral 0-dimensional, too. There's an important difference, though, with respect to the classical points, since here what's coming as a non-commutative *distance* should actually be physically interpreted as an *area*. This means that the “points” must be some sort of *irreducible* “string-like” objects

²²All of this, of course, also applies to the surfaces dual to *each* of the edges e in the graph α , and one can also consider other graphs as well. Since the representation is carried out on the tensor product space, this means that the area values for the surfaces corresponding to *different* edges are just *added* for obtaining the total area for the “union” of those surfaces, which now will have *two punctures*; again, this is exactly the result that one gets from the area operator in standard LQG.

²³We are *not* referring to the 2-sphere of Example 2.1.2-2 (which was just a mathematical auxiliary for the construction of D), but to some arbitrary surface in spacetime, whose coordinate algebra was related to the $E(S, \mathbf{k})$ variables from GR that were quantized here.

²⁴The stripping out of the continuum played a key role on this, since it transformed the *infinite* dimensional (real) algebra of continuous functions of the classical case into a *finite* dimensional one, thus opening the possibility of irreducible representations on finite dimensional spaces.

at the physical level. *Thus, this approach offers new and more detailed insight into the nature of area in QG.* ■

Remark 2.9: The reason for working on the triple based on the “ambient” given by $R_{AQG}[U(\mathfrak{su}(2))]$ lies in the fact that the matrix product of the representation space doesn’t close on the Lie algebra $R_{AQG}[\widehat{\mathfrak{su}(2)}]$, whereas the Lie bracket in terms of that product does. On the other hand, on physical grounds, only the use of elements in $\mathfrak{su}(2)$ is justified, and that’s why only that sub-space of elements was considered when verifying the properties and in the calculation of the distance. That is, we make the hypothesis that the latter *lower bound* on the distance for the triple of Definition 2.13 (or, the *true distance* of what we called a “Lie sub-triple”) *is the actual physical distance.* ■

3. Conclusions

In NCG, what determines the structure of spacetime (in particular, if it’s a classical differentiable manifold or not) is its algebra $\mathcal{U}_{st}$. Inspired by a relationalist conception of spacetime, we made a detailed analysis of the relation between this algebra and the phase space algebra of the Gravitational Field. We proposed an approach which, using mathematical tools from non-commutative geometry (NCG) à la Connes and the Topos approach to quantum theory à la Isham, sheds new light on the obscure issue of the space(-time) picture in (canonical) quantum gravity. We then applied our scheme to the particular algebra of Algebraic Quantum Gravity, a cousin of LQG best suited to our purposes. In this context, we obtained a novel way of deducing the quantization of the possible values for the area of a surface, which came expressed in a *purely combinatorial* way (this being a long sought thing in the context of canonical LQG [15, 16]). We got the same values than those of LQG, but *also additional, smaller values*. Last but not least, *at no point of our derivation* (in the quantized theory) *did we use the classical continuum manifold*, thus avoiding the possibility of our assumptions contradicting our conclusions, as well as obtaining a physically clear picture of quantum space in return. In LQG it has often been repeated that “edges of a graph carry quanta of area”. This however is only *indirectly* and *partially* hinted in LQG by the quantized phase space functionals appearing there, but not *explicitly* since spatial surfaces and regions (which are the very things that carry areas and volumes) can only be identified relationally *once a solution has been chosen*, and not at the level of the whole phase space, where they’re devoid of physical meaning. Instead, this claim becomes fully realized and rigorously established under our approach. We directly obtain a well defined non-commutative space (in the sense of Connes’ NCG, that is, a so-called *spectral triple*) whose non-commutative metric geometry (characterized by Dirac-like operators) is quantized and related to graphs in the same way as in LQG. These spaces can be identified with the physical space that relationally arises from the canonically quantized Gravitational Field (quantum relational space being another long-sought entity in the LQG approach [9]).

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