Discrete Gravity with Local Lorentz Invariance

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A novel structure-preserving algorithm for general relativity in vacuum is derived from a lattice gauge theoretic discretization of the tetradic Palatini action. The resulting method is demonstrated to preserve local Lorentz invariance and symplectic structure.

I. INTRODUCTION

Since at least the 1990s, structure-preserving algorithms [1] have flourished in computational physics, having found wide adoption in subfields as diverse as orbital mechanics [2–5], geophysics [6, 7] and plasma physics [8–16]. Such algorithms are generally derived from a Lagrangian or Hamiltonian formalism and use discretizations that preserve the symplectic structure, topology, gauge symmetry, and conservation laws of their underlying physical systems. This preservation of mathematical structure can substantially improve the accuracy and fidelity of numerical simulations.

Structure-preserving discretizations of general relativity (GR) arguably have an even longer history. The most widely explored such approach was introduced in 1961: Regge calculus [17] is a discrete variational approximation of GR that encodes spacetime data on a simplicial mesh. In four spacetime dimensions, Regge calculus elegantly approximates the Einstein-Hilbert action by a sum over areas A_h and *deficit angles* δ_h, such that

$$S_{\mathrm{Regge}} = \sum_h A_h \delta_h \xrightarrow{A_h \to 0} S_{\mathrm{EH}} = \frac{1}{2}\int \mathrm{d}^4x \sqrt{-g}R \quad (1)$$

in the continuum limit. Here, h labels each 2-simplex (i.e. triangle) of the simplicial complex, and δ_h describes the failure of the 4-simplices adjoining h—i.e. $\{\sigma^4 \mid \sigma^4 \supset h\}$—to tesselate their embedding in flat $\mathbb{R}^4$ spacetime [18].

Since the 1970s, Regge calculus has not only been actively employed as the basis of many studies in quantum gravity (e.g. [19–24]) but also as an algorithmic approach to classical numerical relativity (e.g. [25–33]). Despite its many successes as a numerical tool, however, most studies in numerical relativity continue to depend on standard finite difference methods. Two reasons cited for this include the need to develop (i) a description of matter in Regge calculus, as well as (ii) a better understanding of its relationship to standard methods in numerical relativity [34, 35].

In particular, because the degrees of freedom of Regge calculus are quite distinct from those of continuum GR, it can be challenging to initialize a Regge calculus simulation with known GR initial conditions, or to test whether a particular simulation using Regge calculus recovers a known GR solution. Although various physical solutions have indeed been thoroughly and successfully benchmarked with Regge calculus [36, 37], it would seem that any given simulation generally requires a bespoke understanding of the map between discrete and continuum degrees of freedom.

It is also worth noting that, despite Regge calculus being a variational method, it nonetheless forfeits—in its complete, nonperturbative formulation—the local gauge symmetry of GR [22]. While local gauge symmetry is maintained in a Regge calculus description of flat spacetime—and even in a linearized Regge calculus of curved spacetimes [38]—this structural feature of GR is at best only partially preserved overall.

In this paper, an alternative variational approach to simulating general relativity is developed that ameliorates some of these limitations. Our effort employs familiar tools of lattice gauge theory [39] to construct a structure-preserving discretization of the tetradic Palatini action [40]. Using a Poincaré group valued connection, we describe a variational algorithm that exactly preserves Lorentz gauge symmetry, and we characterize its symplectic structure.

The approach we take is very closely related to Poincaré gauge theoretic studies of lattice quantum gravity by Menotti, et al. [41, 42]. To our knowledge, however, the classical physics of these methods, including their equations of motion, for example, have not previously been explored, nor have they been extended to define an algorithm for numerical relativity. Morover, our construction is general to simplicial and hypercubic discretizations of spacetime, and we develop a somewhat streamlined construction of the aforementioned Poincaré connection.

The remainder of this paper is organized as follows: Section II briefly reviews the tetradic Palatini action; Section III discretizes this action in a manner that preserves local Lorentz gauge invariance; Section IV derives the discrete, classical equations of motion that comprise the algorithm; and Section V describes its symplectic structure. Finally, Section VI summarizes and concludes.

II. THE TETRADIC PALATINI ACTION IN CONTINUOUS SPACETIME

We begin our discussion in continuous spacetime using the following index conventions:

- internal Lorentz indices are denoted $\{A, B, \dots\}$, raised and lowered by the Minkowski metric η_{AB};
- spacetime indices are denoted $\{\mu, \nu, \dots\}$, raised and lowered by the metric $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$; and
- any other indices $\{a, b, \dots\}$ will be specified as needed.

On a four-dimensional spacetime M, the tetrad (or vierbein) may be defined as a vector valued 1-form with components $e^A \in \Gamma(\wedge^1 T^*M)$ such that $e^A{}_\mu(x)$ defines an isomorphism between the tangent space $T_x M$ and the internal Lorentz space at $x \;\forall\; x \in M$. The vierbein in turn determines a metric on M according to the relation

$$g_{\mu\nu} = e^A{}_\mu \eta_{AB} e^B{}_\nu. \tag{2}$$

We further introduce the $\mathfrak{so}(3,1)$-valued 1-form with components $\omega^A{}_B \in \Gamma(\wedge^1 T^*M)$, also called the spin connection. Its curvature $F^A{}_B \in \Gamma(\wedge^2 T^*M)$ is defined by

$$F^A{}_B = \mathrm{D}\omega^A{}_B = \mathrm{d}\omega^A{}_B + \frac{1}{2}[\omega, \omega]^A{}_B \tag{3}$$

where $[\cdot,\cdot]$ extends the conventional Lie bracket to Lie algebra-valued differential forms via coefficient multiplication [43].

The Lagrangian $\mathcal{L} \in \Gamma(\wedge^4 T^*M)$ of the tetradic Palatini action $\mathcal{S} = \int_M \mathcal{L}$ is now defined in terms of the vierbein and spin connection by [40]

$$\mathcal{L} = \epsilon_{ABCD}\left(e^A \wedge e^B \wedge F^{CD}\right). \tag{4}$$

The Lorentz invariance of $\mathcal{L}$ immediately follows from the $SO(3,1)$ invariance of the Levi-Civita symbol ϵ_{ABCD}.

Unlike the metric and connection (g, Γ) of the Einstein-Hilbert action, the fields (e, ω) of Eq. (4) are taken to be independent and varied accordingly. Variation of each field yields the respective equations of motion

$$\begin{aligned} (\delta e): &\qquad 0 = \epsilon_{ABCD} e^B \wedge F^{CD} \\ (\delta\omega): &\qquad 0 = \epsilon_{ABCD}(De^A \wedge e^B - e^A \wedge De^B), \end{aligned} \tag{5}$$

where

$$De^A = \mathrm{d}e^A + \omega^A{}_B \wedge e^B. \tag{6}$$

The former relation of Eq. (5) yields Einstein's vacuum field equations while the latter is equivalent to a zero torsion condition. Despite having fewer initial assumptions (such as zero torsion), the tetradic Palatini action nevertheless recovers the equations of motion of GR in vacuum.

The following observation will also be useful in the next section that discretizes $\mathcal{L}$. Evaluated on a 4-tuple of vector fields—$\mathbf{X} = (X_1, X_2, X_3, X_4),\; X_a \in \Gamma(TM)$—the 4-form $\mathcal{L}$ of Eq. (4) gives

$$\begin{aligned} \mathcal{L}(\mathbf{X}) &= \frac{1}{2}\epsilon_{ABCD}(e^A_\mu e^B_\nu F^{CD}_{\sigma\tau})\epsilon^{abcd} X^\mu_a X^\nu_b X^\sigma_c X^\tau_d \\ &= \frac{1}{2}\epsilon_{ABCD}(e^A_\mu e^B_\nu F^{CD}_{\sigma\tau})\epsilon^{\mu\nu\sigma\tau} \det \mathbf{X} \end{aligned} \tag{7}$$

where the function $\det \mathbf{X}$ is the matrix determinant of the 4-tuple, evaluated pointwise over M.

III. THE DISCRETE ACTION

We now reformulate the Lagrangian of the previous section in a discrete setting. Our formalism is general to orientable simplicial and cubic discretizations. We denote the set of all k-cells $\Sigma^k = \{\sigma^k\}$, such that any oriented k-cell will be denoted σ^k or will otherwise be specified by an ordered label of its vertices. $\sigma_{ij} \in \Sigma^1$, for example, denotes an edge oriented from vertex $\sigma_i \in \Sigma^0$ to vertex $\sigma_j \in \Sigma^0$. We define $N_i(\sigma^k) = \{j \neq i \mid \sigma_{ij} \subset \sigma^k\}$ as the set of labels of neighboring vertices in the cell σ^k that share an edge with basepoint σ_i. In both simplicial and cubic discretizations in four dimensions, for example, $\#N_i(\sigma^4) = 4$ if $\sigma_i \subset \sigma^4$ and 0 otherwise. We denote the permutation set of these neighboring vertex labels as $\Pi_i(\sigma^4) = S[N_i(\sigma^4)]$.

We now define a Poincaré group-valued connection $U(\sigma_{ij}) = U_{ij}$ along each edge $\sigma^1 \in \Sigma^1$. Its representation $SO(3,1) \ltimes \mathbb{R}^4 \to GL_5(\mathbb{R})$ is defined such that the Poincaré group element (Λ, ℓ) characterizing the Lorentz transformation Λ and the translation ℓ yields the following connection:

$$U_{ij} = \begin{bmatrix} \Lambda_{ij} & \ell_{ij} \\ \mathbf{0} & 1 \end{bmatrix} \qquad \text{s.t.} \qquad U_{ji} = U_{ij}^{-1} = \begin{bmatrix} \Lambda_{ij}^{-1} & -\Lambda_{ij}^{-1}\ell_{ij} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{8}$$

The edge data $\Lambda_{ij} \in SO(3,1)$ and $\ell_{ij} \in \mathbb{R}^4$ will comprise the degrees of freedom of our discrete action. We note that $\Lambda_{ji} = (\Lambda_{ij})^{-1}$ and $\ell_{ji} = -\Lambda_{ji}\ell_{ij}$ above.

For notational ease, we define the connection along each edge $U(\sigma^1)$ to act from the right. Therefore, given an arbitrary Lorentz gauge transformation defined at each vertex, say

$$\{g_i = g(\sigma_i) \in SO(3,1) \;\forall\; \sigma_i \in \Sigma^0\}, \tag{9}$$

the connection U_{ij} transforms as

$$\begin{aligned} U'_{ij} &= \begin{bmatrix} g_i & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}^{-1} U_{ij} \begin{bmatrix} g_j & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \left[\begin{array}{c|c} g_i^{-1}\Lambda_{ij} g_j & g_i^{-1}\ell_{ij} \\ \hline \mathbf{0} & 1 \end{array}\right]. \end{aligned} \tag{10}$$

We denote the $(A,B)^{\text{th}}$ component of the Lorentz connection along edge σ_{ij} by Λ^A_{ijB}, and the A^{th} component of the corresponding translation connection by ℓ_{ij}^A. We also adopt a notation for a Lorentz holonomy with an arbitrary number of edges. In particular, for the holonomy comprised of $(n-1)$ connections between the vertices $\sigma_{i_1}, \ldots, \sigma_{i_n}$, we write

$$\begin{aligned}\Lambda^A_{i_1\cdots i_n B} &= \left(\Lambda_{i_1 i_2}\Lambda_{i_2 i_3}\cdots\Lambda_{i_{n-1}i_n}\right)^A{}_B \\ &= \Lambda^A_{i_1 i_2 C}\Lambda^C_{i_2 i_3 D}\cdots\Lambda^E_{i_{n-1}i_n B}\end{aligned} \tag{11}$$

where intermediate Lorentz indices $\{C, D, \ldots, E\}$ are all contracted. When a holonomy is comprised of the connections along the edges of a single face $\sigma^2 \in \Sigma^2$ (i.e. when it is a "minimal" nontrivial loop), we will use the symbol Ω for its Lorentz holonomy rather than Λ, and we will suppress some of its indices. In particular, given a loop $\sigma_i\sigma_j\cdots\sigma_k\sigma_i$ around a single face, we denote its holonomy Ω^A_{ijkB}. This notation is general to the simplicial and cubic setting, such that, for example,

$$\begin{aligned}\text{Simplicial:}\quad \Omega^{AB}_{ijk} &= \left(\Lambda_{ij}\Lambda_{jk}\Lambda_{ki}\right)^A{}_C\eta^{CB} \\ \text{Cubic:}\quad \Omega^{AB}_{ijk} &= \left(\Lambda_{ij}\Lambda_{ji'}\Lambda_{i'k}\Lambda_{ki}\right)^A{}_C\eta^{CB}.\end{aligned} \tag{12}$$

Here, i' labels the vertex diagonal to i on the appropriate face of the cubic lattice; in a more typical notation, $(i, j, i', k) = (\mathbf{n}, \mathbf{n}+\hat{a}, \mathbf{n}+\hat{a}+\hat{b}, \mathbf{n}+\hat{b})$. Ω thereby characterizes the Lorentz curvature of a face σ^2.

With the preceding notations in mind, we now define the discrete action and Lagrangian on hypercells $\{\sigma^4\} = \Sigma^4$ as follows:

$$\begin{aligned}S &= \sum_{\sigma^4\in\Sigma^4} L(\sigma^4) \\ L(\sigma^4) &= \sum_{\substack{\sigma_i\subset\sigma^4\\ \pi\in\Pi_i(\sigma^4)}} \frac{(-1)^{|\pi|}}{2\rho_f n_v}\epsilon_{ABCD}\left(\ell^A_{i\pi(1)}\ell^B_{i\pi(2)}\Omega^{CD}_{i\pi(3)\pi(4)}\right).\end{aligned} \tag{13}$$

The summand of Eq. (13) can be roughly understood as evaluating the Lagrangian of Eq. (7) on a frame determined by edges adjoining σ_i. π denotes a permutation of these edges whose orientation of parity $|\pi|$ is inherited from the mesh itself. n_v denotes the number of vertices per hypercell, so that $L(\sigma^4)$ averages over frames constructed at each vertex of σ^4. ρ_f denotes a shape factor that accounts for the hypercell volume in a setting with edges of unit length. The determinant appearing in Eq. (7) is now unnecessary; implicit in ℓ^A_{ij} and Ω^{AB}_{ijk} are the lengths and areas, respectively, of the discrete connection, providing a volume factor that need only be normalized by ρ_f. Specifically, $(\rho_f, n_v) = (12, 5)$ and $(1, 16)$ in the simplicial and cubic settings, respectively.

Finally, to demonstrate how the group-valued fields of Eq. (13) approximate Eq. (7), we note that

$$\begin{bmatrix}\Lambda & \ell \\ \mathbf{0} & 1\end{bmatrix} = \exp\begin{bmatrix}\omega & e \\ \mathbf{0} & 0\end{bmatrix} \tag{14}$$

for some $\omega \in \mathfrak{so}(3,1)$ and $e \in \mathbb{R}^4$. Thus, $\Lambda^A{}_B \sim \delta^A_B + \omega^A{}_B$ and $\ell^A \sim e^A$ to least nontrivial order. Similarly, $\Omega^{AB} \sim \eta^{AB} + F^{AB}$ for some $F \in \mathfrak{so}(3,1)$, and η^{AB} is annihilated by the Levi-Civita symbol in Eq. (13). To leading order, therefore, fields of the discrete Lagrangian recover the Lie algebraic fields of Eq. (7).

We further note the local Lorentz invariance of $L(\sigma^4)$. In particular, under an arbitrary gauge transformation $\{g_i \in SO(3,1)\}_{\sigma_i\in\Sigma^0}$ using Eq. (10) we find

$$\begin{aligned}&\left(\epsilon_{ABCD}\ell^A_{i\pi(1)}\ell^B_{i\pi(2)}\Omega^{CD}_{i\pi(3)\pi(4)}\right)' \\ &\quad= \epsilon_{ABCD}\left(g_i^{-1}\ell_{i\pi(1)}\right)^A\left(g_i^{-1}\ell_{i\pi(2)}\right)^B\left(g_i^{-1}\Omega_{i\pi(3)\pi(4)}g_i\right)^{CD} \\ &\quad= \epsilon_{ABCD}\ell^A_{i\pi(1)}\ell^B_{i\pi(2)}\Omega^{CD}_{i\pi(3)\pi(4)}.\end{aligned} \tag{15}$$

The last equality above follows from the Lorentz group relation $(g_i)^E{}_F\eta^{FD} = (g_i^{-1})^D{}_F\eta^{FE}$ and the $SO(3,1)$-invariance of the Levi-Civita symbol.

IV. THE DISCRETE EQUATIONS OF MOTION

We now compute equations of motion (EOM) by varying the discrete action with respect to the connection. To compactify notation, when an element of the permutation π appears in an index, it will hereafter be denoted by a corresponding underlined number, for example, $\underline{1} = \pi(1)$. As usual in a first-order formalism, we assume Λ_{ij} and ℓ_{ij} to be independent. Varying the action with respect to ℓ^A_{ij}, and applying the expression for U_{ji} from Eq. (8) where appropriate, we find

$$\begin{aligned}0 = \frac{\partial S}{\ell^A_{ij}} = \sum_{\sigma^4\supset\sigma_{ij}}\Bigg[&\sum_{\substack{\pi\in\Pi_i(\sigma^4)\\ \pi(1)=j}}\frac{(-1)^{|\pi|}}{\rho_f n_v}\epsilon_{ABCD}\ell^B_{i\underline{2}}\Omega^{CD}_{i\underline{34}} \\ &- \sum_{\substack{\pi\in\Pi_j(\sigma^4)\\ \pi(1)=i}}\frac{(-1)^{|\pi|}}{\rho_f n_v}\epsilon_{EBCD}\Lambda^E_{jiA}\ell^B_{j\underline{2}}\Omega^{CD}_{j\underline{34}}\Bigg].\end{aligned} \tag{16}$$

The first sum of Eq. (16) arises from terms with basepoint i and the second from terms with basepoint j. We note that although frames are permuted at distinct basepoints in these two lines, their parities are understood to be induced by a global orientation and are therefore mutually consistent. Eq. (16) is counterpart to (δe) of Eq. (5), and constitutes a discrete reformulation of Einstein's vacuum equations.

We now derive the the Lorentz connection EOM, exercising caution to ensure that the variation of Λ_{ij} is constrained to the $SO(3,1)$ manifold. In particular, $\Lambda^T\eta\Lambda = \eta$ implies $(\Lambda^{-1}\delta\Lambda)^T\eta + \eta(\Lambda^{-1}\delta\Lambda) = 0$, so that $\Lambda^{-1}\delta\Lambda \in \mathfrak{so}(3,1)$ for a variation $\delta\Lambda$. We can impose this constraint by taking a variation that satisfies $(\Lambda^{-1}\delta\Lambda)^{AB} = (\Lambda^{-1}\delta\Lambda)^{[AB]}$, but is otherwise arbitrary.

To that end, we consider as an example the variation of $\Omega^{CD}_{i\underline{34}} = \left(\Lambda_{i\underline{3}}\Lambda_{\underline{34}}\Lambda_{\underline{4}i}\right)^{CD}$ in the simplicial setting with respect to the Lorentz connection on the edge $\sigma_{i\underline{3}}$:

$$\begin{aligned}\delta\Omega^{CD}_{i\underline{34}} &= \left(\Lambda^{C}_{i\underline{3}E}\Lambda^{E}_{\underline{3}iF}\right)\delta\Lambda^{F}_{i\underline{3}G}\Lambda^{G}_{\underline{34}H}\Lambda^{HD}_{\underline{4}i} \\ &= \Lambda^{C}_{i\underline{3}[E|}\Lambda_{\underline{34}i|G]}{}^{D}\left(\Lambda^{E}_{\underline{3}iF}\delta\Lambda^{FG}_{i\underline{3}}\right)\end{aligned}$$

The term in parentheses on the first line is a conveniently chosen form of the Kronecker delta δ^{C}_{F}, and the second line follows from the notation of Eq. (11) and from asserting the antisymmetry of the variation $(\Lambda^{-1}\delta\Lambda)^{[EG]}$. Continuing in this way, noting that $\Lambda^{A}_{ijB} = \eta^{AD}\Lambda^{C}_{jiD}\eta_{CB}$, and denoting $(-1)^{|\pi|}/\rho_f n_v = a^{|\pi|}$ for brevity, we vary S with respect to $(\Lambda_{ji}\delta\Lambda_{ij})^{[MN]}$ to find, in the simplicial case:

$$\begin{aligned}0 &= \frac{\partial S_{\text{simplicial}}}{(\Lambda_{ji}\delta\Lambda_{ij})^{[MN]}} \\ &= \sum_{\sigma^4\supset\sigma_{ij}}\Bigg[\sum_{\substack{\pi\in\Pi_i(\sigma^4)\\ \pi(3)=j}} a^{|\pi|}\epsilon_{ABCD}\ell^{A}_{i\underline{1}}\ell^{B}_{i\underline{2}}\left(\Lambda^{D}_{i\underline{4}j[M|}\Lambda^{C}_{ij|N]}\right) \\ &+ \sum_{\substack{k\in\sigma^4\\ k\neq i,j}}\sum_{\substack{\pi\in\Pi_k(\sigma^4)\\ \pi(3)=i\\ \pi(4)=j}} a^{|\pi|}\epsilon_{ABCD}\ell^{A}_{k\underline{1}}\ell^{B}_{k\underline{2}}\left(\Lambda^{D}_{kj[M|}\Lambda^{C}_{kij|N]}\right) \\ &+ \sum_{\substack{\pi\in\Pi_j(\sigma^4)\\ \pi(4)=i}} a^{|\pi|}\epsilon_{ABC[M|}\ell^{A}_{j\underline{1}}\ell^{B}_{j\underline{2}}\left(\Omega^{C}_{j\underline{3}i|N]}\right)\Bigg].\end{aligned} \tag{17}$$

The first line of Eq. (17) arises from terms with basepoint i, the middle line from terms with basepoint $k \neq i, j$ in σ^4 and the last from terms with basepoint j. Eq. (17) enforces a discrete zero-torsion condition analogous to $(\delta\omega)$ of Eq. (5).

The Lorentz EOM for a cubic discretization follows similarly:

$$\begin{aligned}0 &= \frac{\partial S_{\text{cubic}}}{(\Lambda_{ji}\delta\Lambda_{ij})^{[MN]}} \\ &= \sum_{\sigma^4\supset\sigma_{ij}}\Bigg[\sum_{\substack{\pi\in\Pi_i(\sigma^4)\\ \pi(3)=j}} a^{|\pi|}\epsilon_{ABCD}\ell^{A}_{i\underline{1}}\ell^{B}_{i\underline{2}}\left(\Lambda^{D}_{i\underline{4}i'j[M|}\Lambda^{C}_{ij|N]}\right) \\ &+ \sum_{\substack{k\in\sigma^4\\ k\neq i,j}}\sum_{\substack{\pi\in\Pi_k(\sigma^4)\\ \pi(3)=i\\ k'=j}} a^{|\pi|}\epsilon_{ABCD}\ell^{A}_{k\underline{1}}\ell^{B}_{k\underline{2}}\left(\Lambda^{D}_{k\underline{4}j[M|}\Lambda^{C}_{kij|N]}\right) \\ &+ \sum_{\substack{k\in\sigma^4\\ k\neq i,j}}\sum_{\substack{\pi\in\Pi_k(\sigma^4)\\ k'=i\\ \pi(4)=j}} a^{|\pi|}\epsilon_{ABCD}\ell^{A}_{k\underline{1}}\ell^{B}_{k\underline{2}}\left(\Lambda^{D}_{kj[M|}\Lambda^{C}_{k\underline{3}ij|N]}\right) \\ &+ \sum_{\substack{\pi\in\Pi_j(\sigma^4)\\ \pi(4)=i}} a^{|\pi|}\epsilon_{ABC[M|}\ell^{A}_{j\underline{1}}\ell^{B}_{j\underline{2}}\left(\Omega^{C}_{j\underline{3}i|N]}\right)\Bigg].\end{aligned} \tag{18}$$

Eqs. (16-18) define the desired algorithm for vacuum numerical relativity. However, while these equations suffice to compute simulation steps in the *bulk*, the evolution of *boundary connections*—including connections along both spacelike and timelike boundaries—still requires some explanation. In particular, even if initial and boundary connections are known a priori, Eqs. (16-18) involve data from holonomies that generally extend outside of the boundary wall, and are therefore underspecified on the boundary.

The strategy we adopt [44] to derive equations of motion for boundary connections, therefore, is to extend all spacelike and timelike boundary surfaces outward from the bulk, creating a narrow 'double wall' of some fiducial thickness ϵ around the simulation domain. This double wall is then populated with cells of width ϵ, such that connections between an inner wall vertex $\sigma_{i_{\text{in}}}$ and an outer wall vertex $\sigma_{i_{\text{out}}}$ will have $\Lambda_{i_{\text{in}}i_{\text{out}}} \sim \mathbb{1} + \mathcal{O}(\epsilon)$ and $\ell_{i_{\text{in}}i_{\text{out}}} \sim \mathcal{O}(\epsilon)$. The connections lying along the outer wall itself are chosen to copy the initial or boundary conditions of the inner wall. Then, equations of motion for the inner wall connections can be derived as usual from Eqs. (16-18), as they now behave as connections in the bulk. Finally, we take $\epsilon \to 0$ in the resulting equations of motion for the (inner wall) boundary connections.

It is worth noting that not all boundary and initial conditions will satisfy the discrete equations of motion. Just as boundary constraints must be satisfied in the continuum theory, care must be taken to ensure that Eqs. (16-18) are satisfied on the initial surfaces of the discrete theory.

V. SYMPLECTIC STRUCTURE OF THE DISCRETE ACTION

Variational integrators for field theories have a natural multisymplectic structure (see e.g. Refs. [45, 46] and references therein), generalizing the ordinary symplectic structure possessed by variational integrators in particle mechanics [47]. Here we review the proof that variational integrators are naturally (multi)symplectic, thereby confirming the multisymplectic structure of Eqs. (16-18).

In a variational integrator for particle mechanics, the action evaluated on a temporal cell $[t_i, t_{i+1}]$ provides a generating function for a canonical (symplectic) transformation across the cell. Specifically, the discrete action S_i for cell i is a generating function for the canonical transformation $(q_i, p_i) \to (q_{i+1}, p_{i+1})$, where q_i, q_{i+1} are the particle coordinates at the left and right end of the cell respectively, $p_i = -\frac{\partial S_i}{\partial q_i}$, and $p_{i+1} = \frac{\partial S_i}{\partial q_{i+1}}$. The equations of motion (e.g. $\frac{\partial S_{i-1}}{\partial q_i} + \frac{\partial S_i}{\partial q_i} = 0$) guarantee that the momentum at a point is identical whether using the left or right cell to define it. In this way the symplectic transformations inside the cells are glued consistently across cells to produce a global symplectic evolution.

In field theory, the situation is slightly different. A field ϕ has a multimomentum π^μ (one for each dimension of space-time) [48], which in the case of a scalar field can be recast as a 3-form $\pi = \pi^\mu \mathrm{d}^3x_\mu = \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}\pi^\mu\,\mathrm{d}x^\alpha \wedge$

$\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$. For any spacetime region R, we then have the boundary fields $\phi(\sigma), \pi(\sigma)$ living on $\Sigma = \partial R$, where $\pi(\sigma) = \pi\big|_\Sigma$ can be regarded as a pseudo-scalar field. The action evaluated over R, $S(R)$, is a generating function for a submanifold $\left\{\left(\phi(\sigma), \pi(\sigma) = \frac{\delta S(R)}{\delta \phi}\right)\right\}$ in this "boundary phase space". One may regard $S(R)$ (imprecisely) as a generating function for a canonical transformation between any two parts of the boundary. In the case when the boundary of R consists of two disconnected pieces corresponding to two different times, $S(R)$ is the generating function for a canonical transformation between those times.

In the discrete setting, we take $R = \sigma^d$, a hypercell of maximum dimension in our lattice (d is the spacetime dimension). The boundary phase space no longer consists of fields, but of pairs $(\phi_i, \pi_i(\sigma^d))$ for each vertex $\sigma_i \subset \sigma^d$. The discrete action over σ^d, $L(\sigma^d)$, is a generating function for a manifold in the boundary phase space: $\left\{\left(\phi_i, \pi_i(\sigma^d) = \frac{\partial L(\sigma^d)}{\partial \phi_i}\right)\right\}$, in agreement with the continuum multisymplectic structure discussed above. Note that in this case, the momentum at a vertex is not unique, but rather depends on the hypercell σ^d used to compute it (the same holds true in the continuum: the momentum depends on both the location and the boundary used to define it). The equations of motion (e.g. $\sum_{\sigma^d \supset \sigma_i} \frac{\partial L(\sigma^d)}{\partial \phi_i} = 0$) do not guarantee a unique momentum at each vertex, but rather that the sum of momenta defined for each region/boundary containing that vertex vanishes. This guarantees the integrator will be symplectic when stepping in time. To see this, let vertex σ_i be associated with time t_0 and define $\sigma^d_+ = \{\sigma^d \supset \sigma_i \text{ with } t > t_0\}$ and $\sigma^d_- = \{\sigma^d \supset \sigma_i \text{ with } t < t_0\}$. Then if $\pi_i^+ = -\sum_{\sigma^d \in \sigma^d_+} \pi_i(\sigma^d)$ (the minus sign takes care of the orientation for convenience) and $\pi_i^- = \sum_{\sigma^d \in \sigma^d_-} \pi_i(\sigma^d)$, we get the usual gluing of symplectic transformations under time-stepping: $\pi_i^- = \pi_i^+$. Furthermore, this will hold true no matter how we choose to define our time and associated time-stepping (i.e. if there are multiple ways to perform time-stepping in our cellular complex, each of them will be guaranteed to result in symplectic evolution).

In the case of gravity in four dimensions, we are using 1-form fields rather than scalar fields, so the multimomentum is more naturally a 2-form. Additionally, our two primary fields (e and ω) are conjugate to each other (in the sense that the multimomentum of ω is a function of e, while the multimomentum of e vanishes leaving behind an $e - \omega$ phase space). All of this is captured by the discrete Eqs. (16-18). The boundary phase space of a cell σ^4 consists of pairs $(\ell^A_{ij}, \Lambda^A_{ijB})$ for each edge $\sigma_{ij} \subset \sigma^4$ (rather than for each vertex, as in the case of a scalar field). The discrete action $L(\sigma^4)$ is a generating function, which defines momenta conjugate to ℓ^A_{ij} and Λ^A_{ijB} as the bracketed summands of Eqs. (16) and Eqs. (17-18), respectively. The momentum conjugate to Λ is a function of ℓ, while the initialization of the algorithm (see the end of Sec. IV) ensures the vanishing of the momentum conjugate to ℓ in the initial (non-ϵ) layer of cells (the symplectic structure of the time-stepping will ensure the momentum vanishes for the entire complex). In this way the discrete multisympletcic structure of our gravitational integrator mirrors the continuous multisympletctic structure of GR.

VI. CONCLUSION

We have presented a new numerical scheme for general relativity, detailed in Eqs. (16-18). This scheme preserves both the (multi)sympletcic structure and local Lorentz invariance of the tetrad formulation of GR. Furthermore, its discrete variables have a clear connection with their continuum counterparts. As such, this scheme holds promise as an integrator for numerical relativity (the structure preservation leading to exact conservation laws or bounded errors for the discrete equations) and for studying the classical limits of certain quantum gravity theories (such as loop quantum gravity, spin foams, etc.). In these roles, the scheme's symplectic structure promises an improvement over non-symplectic finite-difference and spectral methods, while its natural association with continuum variables makes it a more viable alternative than other symplectic approaches to discrete gravity (most notably Regge calculus). In future work, implementations of this algorithm will be needed to demonstrate its practical utility. Furthermore, like Regge calculus, further study is required to incorporate matter into our approach (though the way forward seems clearer).

It may also be of interest to explore the potential union between the algorithm defined here and other structure-preserving discretizations suitable for numerical relativity. For example, it may be useful to explore the relationship between our holonomy-centric approach and the recently developed technique of group-equivariant interpolation in symmetric spaces [49, 50]. It may also be useful to compare our effort with finite element cochain complexes suitable for applications in numerical relativity [51]. In this way, the algorithm we introduce can be a starting point for fruitful explorations into structure-preserving discrete gravity theories.

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