DEFORMATION QUANTIZATION AND INTRINSIC NONCOMMUTATIVE DIFFERENTIAL GEOMETRY

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ABSTRACT. We provide an intrinsic formulation of the noncommutative differential geometry developed earlier by Chaichian, Tureanu, R. B. Zhang and the second author. This yields geometric definitions of covariant derivatives of noncommutative metrics and curvatures, as well as the noncommutative version of the first and the second Bianchi identities. Moreover, if a noncommutative metric and chiral coefficients satisfy certain conditions which hold automatically for quantum fluctuations given by isometric embedding, we prove that the two noncommutative Ricci curvatures are essentially equivalent. Finally, we show that the quantum fluctuations and their curvatures have close forms if (pseudo-) Riemannian metrics are given by certain type of spherically symmetric isometric embeddings. Hence the quantization of gravity is renormalizable in this case.

1. INTRODUCTION

Gravity is essentially a theory of spacetime geometry. In the concept of quantum effects of gravity, the Heisenberg uncertainty relations would result in noncommutativity of spacetime variables for sufficiently small distances. In 1947, Snyder, C.N. Yang made the first attempts to quantize spacetimes [14, 18], which are referred as Snyder's quantum space-times and Yang's quantum phase spaces [9, 10]. In their approach, spacetime variables were represented by Hermitian operators with discrete eigenvalues. This idea to encoding geometry of a space by its algebras of functions was realized prominently by Connes to establish noncommutative geometry using spectral triples [6]. And the main ingredients are the noncommutative analog of the Dirac operator acting on a representation space of the algebra, the spectrum of this generalized Dirac operator and the cyclic (co)homology. They are used to encode the information of noncommutative manifold structure, noncommutative metric and noncommutative curvature respectively. The overview of its applications to physics can be found in [5].

However, the metric and curvature information in an infinitesimal neighborhood of manifold is still lack as it is not known what means to take derivatives when coordinate variables are operators. Alternatively, deformation quantization deforms the commutative algebras of functions based on pointwise commutative multiplication to noncommutative algebras of functions based on certain noncommutative products such as the Moyal products, but still keeps spacetime variables usual functions, c.f. [3]. In recent years, there have been intensive research activities on noncommutative gravity in frame of deformation quantization, c.f. [13, 1, 2] and references therein, where general relativity is adopted to the noncommutative setting in an intuitive way, as pointed out in [15].

In [4, 16, 17], a mathematically rigorous and complete theory of noncommutative differential geometry was developed on a coordinate chart U of a (pseudo-) Riemannian manifold. The idea is to embed U isometrically into a flat (pseudo-) Euclidean space and use the isometric embedding to construct the noncommutative analogues of metric, connection and curvature. They yield the noncommutative Einstein field equations. It was found that the deformation quantization of the Schwarzschild metric does not depend on time and yields an unevaporated quantum black hole [16], and the quantum fluctuation of the plane-fronted gravitational wave is the exact solution of the noncommutative vacuum Einstein field equations [17]. We refer to [11, 12] for the review on general existence of isometric embedding and applications in physics.

The paper is organized as follows. In Section 2, we state the main theorem. In Section 3, we study the intrinsic formulation of covariant derivatives of noncommutative metrics and curvatures from the geometric point of view. This yields noncommutative version of the first and the second Bianchi identities. In Section 4, we show the two noncommutative Ricci curvatures are essentially equivalent if a noncommutative metric and chiral coefficients satisfy certain conditions. These conditions hold automatically for quantum fluctuations given by isometric embedding. In Section 5, we show that the quantum fluctuations and their curvatures have close forms coming from Moyal products of trigonometric functions if (pseudo-) Riemannian metrics are given by certain type of spherically symmetric isometric embeddings.

2. Main Theorem

In this section, we provide some basic knowledge on the noncommutative differential geometry and state the main theorem proved in the paper. Recall the intrinsic setting of noncommutative differential geometry proposed by the second author [19], without using the isometric embedding. Let M be an n-dimensional differentiable manifold and $U \subset M$ be a coordinate chart equipped with natural coordinates $(x^1, \cdots, x^n)$. Let $\hbar$ be the Planck constant viewed as an indeterminate. Denote $\mathbb{R}[[\hbar]]$ the ring of formal power series in $\hbar$ with real coefficients, and $\mathcal{A}_U$ the set of formal power series in $\hbar$ with coefficients being real smooth functions on U

$$\mathcal{A}_U = C^\infty(U)[[\hbar]] = \Big\{ \sum_{k=0}^{\infty} f_k \hbar^k \Big| f_k \in C^\infty(U) \Big\}.$$

$\mathcal{A}_U$ is an $\mathbb{R}[[\hbar]]$-module.

Throughout the paper, all the indices i, j, k, l, $\cdots$, range from 1 to n, $q \in \mathbb{N}_0$. We also use the Einstein summation convention. Given two smooth functions u, v on U, we denote uv their usual pointwise product. For any skew-symmetric $n\times n$ real constant matrix (θ^{ij}) on U, the Moyal product of u and v with respect to (θ^{ij}) is defined as

$$(u*v)(x) = \Big[\exp(\hbar\theta^{ij}\partial_i\partial'_j)u(x)v(x')\Big]_{x=x'}, \tag{2.1}$$

where x and x' denote the same coordinate system and $\partial_i = \frac{\partial}{\partial x^i}$, $\partial'_i = \frac{\partial}{\partial (x')^i}$. It is clearly that

$$u*v\in\mathcal{A}_U.$$

Extending by $\mathbb{R}[[\hbar]]$-bilinearity, the Moyal product provides an associative $\mathbb{R}[[\hbar]]$-bilinear product on $\mathcal{A}_U$, c.f. [8]. The Moyal algebra is $\mathcal{A}_U$ equipped with the Moyal product, which is a formal deformation of the algebra of real smooth functions on U.

Extend ∂_i to $\mathcal{A}_U$ by $\mathbb{R}[[\hbar]]$-linearity, the Moyal product satisfies

(i) Noncommutativity: $[x^i, x^j] = x^i * x^j - x^j * x^i = 2\hbar\theta^{ij}$;

(ii) Leibniz rule: $\partial_i(u*v) = (\partial_i u)*v + u*(\partial_i v)$, for $u, v\in\mathcal{A}_U$.

Denote $E_i = \tilde{E}_i = \partial_i$, $1\le i\le n$. The noncommutative left (resp. right) tangent bundle $\mathcal{T}_U$ (resp. $\tilde{\mathcal{T}}_U$) on U is the free left (resp. right) $\mathcal{A}_U$-module with basis $\{E_1,\cdots,E_n\}$ (resp. $\{\tilde{E}_1,\cdots,\tilde{E}_n\}$), i.e.,

$$\mathcal{T}_U = \Big\{a^i * E_i\,\Big|\, a^i\in\mathcal{A}_U,\ a^i * E_i = 0 \Longleftrightarrow a^i = 0.\Big\},$$

$$\tilde{\mathcal{T}}_U = \Big\{\tilde{E}_i * a^i\,\Big|\, a^i\in\mathcal{A}_U,\ \tilde{E}_i * a^i = 0 \Longleftrightarrow a^i = 0.\Big\}.$$

An element of $\mathcal{T}_U$ (resp. $\tilde{\mathcal{T}}_U$) is called a left (resp. right) vector field.

A noncommutative metric g on U is a homomorphism of two-sided $\mathcal{A}_U$-modules

$$g:\mathcal{T}_U\otimes_{\mathbb{R}[[\hbar]]}\tilde{\mathcal{T}}_U\longrightarrow\mathcal{A}_U,$$

such that the matrix

$$(g_{ij})\in\mathcal{A}_U^{n\times n},\quad g_{ij} = g(E_i,\tilde{E}_j)$$

is invertible, i.e., there exists a unique matrix $(g^{ij})\in\mathcal{A}_U^{n\times n}$ such that

$$g_{ik}*g^{kj} = g^{jk}*g_{ki} = \delta_i^j.$$

Let (g_l^{ij}) be the left inverse of (g_{ij}) and (g_r^{ij}) be the right inverse of (g_{ij}). Since the Moyal product is associative,

$$g_l^{ij} = g_l^{ip}*\delta_p^j = g_l^{ip}*g_{pk}*g_r^{kj} = \delta_k^i * g_r^{kj} = g_r^{ij}.$$

Therefore the left inverse and the right inverse coincide.

A noncommutative left connection ∇ is a set of operators $\{\nabla_i := \nabla_{E_i}\}$ for $1\le i\le n$, where each

$$\nabla_i:\mathcal{T}_U\longrightarrow\mathcal{T}_U$$

is called a noncommutative left covariant derivative and satisfies

(i) $\mathbb{R}[[\hbar]]$-linearity: For a, $b \in \mathbb{R}[[\hbar]]$ and V, $W \in \mathcal{T}_U$,
$$\nabla_i(aV + bW) = a\nabla_i V + b\nabla_i W;$$
(ii) Leibniz rule: For $f \in \mathcal{A}_U$ and $V \in \mathcal{T}_U$,
$$\nabla_i(f * V) = (\partial_i f) * V + f * \nabla_i V.$$

The noncommutative right connection $\tilde{\nabla} = \{\tilde{\nabla}_i := \tilde{\nabla}_{\tilde{E}_i}\}$ and the noncommutative right covariant derivatives $\tilde{\nabla}_i$ can be defined in the same way. The left and right connections are uniquely determined by connection coefficients Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$, which are elements of $\mathcal{A}_U$
$$\nabla_i E_j = \Gamma_{ij}^k * E_k, \quad \tilde{\nabla}_i \tilde{E}_j = \tilde{E}_k * \tilde{\Gamma}_{ij}^k.$$
Denote
$$\Gamma_{ijk} = \Gamma_{ij}^l * g_{lk}, \quad \tilde{\Gamma}_{ijk} = g_{kl} * \tilde{\Gamma}_{ij}^l.$$

Inspired by the Levi-Civita connection of a (pseudo-) Riemannian metric, the second author introduced the canonical connection [19]. Given a noncommutative metric g and a set of elements Υ_{ijk} of $\mathcal{A}_U$ with
$$\Upsilon_{ijk} = \Upsilon_{jik},$$
which are referred as the chiral coefficients. A noncommutative connection, which consists of a noncommutative left connection ∇ and a noncommutative right connection $\tilde{\nabla}$, is canonical with respect to g and Υ_{ijk} if it satisfies

(i) Compatibility: $\partial_k g_{ij} = g(\nabla_k E_i, \tilde{E}_j) + g(E_i, \tilde{\nabla}_k \tilde{E}_j) = \Gamma_{kij} + \tilde{\Gamma}_{kji}$;
(ii) Torsion free: $\nabla_i E_j = \nabla_j E_i$, $\tilde{\nabla}_i \tilde{E}_j = \tilde{\nabla}_j \tilde{E}_i$;
(iii) Chirality: $\Gamma_{ijk} - \tilde{\Gamma}_{ijk} = \Upsilon_{ijk}$.

The torsion free condition implies
$$\Gamma_{ij}^k = \Gamma_{ji}^k, \quad \tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ji}^k.$$
It is straightforward that
$$\begin{aligned} 2\Gamma_{ijk} &= \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} + \Upsilon_{ikj} + \Upsilon_{jik} - \Upsilon_{kji} \\ &= \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ji} + \Upsilon_{ijk} \\ &= \partial_i \Big(\frac{g_{jk} + g_{kj}}{2} \Big) + \partial_j \Big(\frac{g_{ki} + g_{ik}}{2} \Big) - \partial_k \Big(\frac{g_{ij} + g_{ji}}{2} \Big) + \Upsilon_{ijk}, \end{aligned} \tag{2.2}$$
and
$$\begin{aligned} 2\tilde{\Gamma}_{ijk} &= \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} + \Upsilon_{ikj} - \Upsilon_{jik} - \Upsilon_{kji} \\ &= \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ji} - \Upsilon_{ijk} \\ &= \partial_i \Big(\frac{g_{jk} + g_{kj}}{2} \Big) + \partial_j \Big(\frac{g_{ki} + g_{ik}}{2} \Big) - \partial_k \Big(\frac{g_{ij} + g_{ji}}{2} \Big) - \Upsilon_{ijk}. \end{aligned} \tag{2.3}$$

In classical Riemannian geometry, the chiral coefficients vanish and Γ_{ijk} reduce to the Christoffel symbols.

For any $f \in \mathcal{A}_U$, it is easy to verify

$$[E_i, E_j]f = [\tilde{E}_i, \tilde{E}_j]f = \partial_i \partial_j f - \partial_j \partial_i f = 0.$$

Thus the left curvature operators $\mathcal{R}_{E_i E_j}$ and the right curvature operators $\tilde{\mathcal{R}}_{\tilde{E}_i \tilde{E}_j}$ can be defined as the following $\mathcal{A}_U$-linear operators

$$\begin{aligned} \mathcal{R}_{E_i E_j} &= [\nabla_i, \nabla_j] : \ \mathcal{T}_U \longrightarrow \mathcal{T}_U, \\ \tilde{\mathcal{R}}_{\tilde{E}_i \tilde{E}_j} &= [\tilde{\nabla}_i, \tilde{\nabla}_j] : \ \tilde{\mathcal{T}}_U \longrightarrow \tilde{\mathcal{T}}_U. \end{aligned}$$

For the canonical connection, the left Riemannian curvatures R_{lkij} and right Riemannian curvatures $\tilde{R}_{lkij}$ are defined as

$$R_{lkij} = g(\mathcal{R}_{E_i E_j} E_k, \tilde{E}_l), \quad \tilde{R}_{lkij} = -g(E_k, \tilde{R}_{\tilde{E}_i \tilde{E}_j} \tilde{E}_l).$$

They satisfy

$$R_{lkij} = -R_{lkji} = \tilde{R}_{lkij}, \quad R_{lkij} \not\equiv -R_{klij}.$$

Therefore the left curvatures are sufficient for the purpose. There are two Ricci curvatures R_{kj} and Θ_{il} obtained by contracting l, i and k, j in R_{lkij} respectively

$$\begin{aligned} R_{kj} =& g(\mathcal{R}_{E_i E_j} E_k, \tilde{E}_l) * g^{li} = R_{lkij} * g^{li}, \\ \Theta_{il} =& g^{jk} * g(\mathcal{R}_{E_i E_j} E_k, \tilde{E}_l) = g^{jk} * R_{lkij}. \end{aligned}$$

Raising the index at k and l respectively, we have Ricci curvatures

$$\begin{aligned} R_j^p =& g^{pk} * g(\mathcal{R}_{E_i E_j} E_k, \tilde{E}_l) * g^{li} = g^{pk} * R_{lkij} * g^{li}, \\ \Theta_i^p =& g^{jk} * g(\mathcal{R}_{E_i E_j} E_k, \tilde{E}_l) * g^{lp} = g^{jk} * R_{lkij} * g^{lp}. \end{aligned}$$

The two Ricci curvatures R_i^p and Θ_i^p are not equal to each other in the noncommutative case. But their traces coincide and yield the same scalar curvature

$$R = R_j^j = \Theta_i^i.$$

As elements of $\mathcal{A}_U$, there are the following power series expansions

$$g_{ij} = \sum_{q=0}^{\infty} g_{ij}[q]\hbar^q, \quad g_{ij}[q] \in C^\infty(U), \tag{2.4}$$

$$\Upsilon_{ijk} = \sum_{q=0}^{\infty} \Upsilon_{ijk}[q]\hbar^q, \quad \Upsilon_{ijk}[q] \in C^\infty(U), \tag{2.5}$$

$$R_{lkij} = \sum_{q=0}^{\infty} R_{lkij}[q]\hbar^q, \quad R_{lkij}[q] \in C^\infty(U), \tag{2.6}$$

$$R_{ij} = \sum_{q=0}^{\infty} R_{ij}[q]\hbar^q, \quad R_{ij}[q] \in C^\infty(U), \tag{2.7}$$

$$\Theta_{ij} = \sum_{q=0}^{\infty} \Theta_{ij}[q]\hbar^q, \quad \Theta_{ij}[q] \in C^\infty(U), \tag{2.8}$$

$$R_j^i = \sum_{q=0}^{\infty} R_j^i[q]\hbar^q, \quad R_j^i[q] \in C^\infty(U), \tag{2.9}$$

$$\Theta_j^i = \sum_{q=0}^{\infty} \Theta_j^i[q]\hbar^q, \quad \Theta_j^i[q] \in C^\infty(U). \tag{2.10}$$

In this paper, we prove the following theorem.

Theorem 2.1. *Let M be an n-dimensional smooth manifold and $U \subset M$ a coordinate chart. Let ∇, $\tilde{\nabla}$ be the canonical connection with respect to noncommutative metric g and chiral coefficients Υ_{ijk} on U. If g_{ij} satisfy*

$$g_{ij}[2q] = g_{ji}[2q], \quad g_{ij}[2q+1] = -g_{ji}[2q+1], \tag{2.11}$$

and Υ_{ijk} satisfy

$$\Upsilon_{ijk}[2q] = 0, \tag{2.12}$$

then two Ricci curvatures are equivalent in the sense that

$$R_{ij}[2q] = \Theta_{ji}[2q], \quad R_{ij}[2q+1] = -\Theta_{ji}[2q+1] \tag{2.13}$$

and

$$R_j^i[2q] = \Theta_j^i[2q], \quad R_j^i[2q+1] = -\Theta_j^i[2q+1]. \tag{2.14}$$

In particular, if noncommutative metric and chiral coefficients are given by an isometric embedding, then (2.11), (2.12) hold and the theorem follows.

Finally, we would like to point out that, in Poisson geometry, the Moyal product is a deformation quantization of the constant Poisson structure

$$\pi = \frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j$$

for constant skew-symmetric matrix (θ^{ij}). If θ^{ij} are smooth functions, π still gives a Poisson structure if its Schouten-Nijenhuis bracket vanishes,

$$[\pi, \pi]_S = 0.$$

However, the corresponding Moyal product is not associative. In the pioneer work, Kontsevich proved that there always exists an associative noncommutative star product which provides the deformation quantization for any Poisson structure [7]. Unfortunately, this star product does not satisfy the Leibniz rule. It indicates our theory of noncommutative differential geometry depends on the choice of coordinate systems in U. As coordinate systems correspond to observers, this fits Bohr's opinion that evidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as complementary in the sense that only the totality of the phenomena exhausts the possible information about the objects.

3. Curvature operators and Bianchi identities

In this section, we study the covariant derivatives of noncommutative metrics and curvatures from the geometric point of view. This yields noncommutative version of the first and the second Bianchi identities.

Proposition 3.1. *Let M be an n-dimensional differentiable manifold and $U \subset M$ be a coordinate chart equipped with natural coordinates $(x^1, \cdots, x^n)$. Let g be a homomorphism of two-sided $\mathcal{A}_U$-modules given by (2.4) where $(g_{ij}[0])$ is not necessarily symmetric. If $(g_{ij}[0])$ is invertible on U with the inverse matrix $(g^{ij}[0])$, then (2.4) gives a noncommutative metric g on U.*

Proof: For any smooth functions $u(x)$, $v(x)$ over U, denote

$$\mu_q(u, v)(x) = \frac{1}{q!}\Big[(\theta^{ij}\partial_i\partial'_j)^q u(x)v(x')\Big]_{x=x'}. \tag{3.1}$$

Let g^{ij} have the power series expansions

$$g^{ij} = \sum_{q=0}^{\infty} g^{ij}[q]\hbar^q \in \mathcal{A}_U, \quad g^{ij}[q] \in C^\infty(U). \tag{3.2}$$

By viewing g^{ij} as the right inverse, we obtain the recursive formula for $q \in \mathbb{N}$

$$\begin{aligned} g^{ij}[q] = &- \sum_{r=1}^{q} g^{ik}[0]g_{kl}[r]g^{lj}[q-r] \\ &- \sum_{r=1}^{q}\sum_{s=0}^{q-r} g^{ik}[0]\mu_r\Big(g_{kl}[s], g^{lj}[q-r-s]\Big). \end{aligned} \tag{3.3}$$

On the other hand, by viewing g^{ij} as the left inverse, we obtain

$$\begin{aligned} g^{ij}[q] = &- \sum_{r=1}^{q} g^{ik}[q-r] g_{kl}[r] g^{lj}[0] \\ &- \sum_{r=1}^{q} \sum_{s=0}^{q-r} \mu_r \Big(g^{ik}[q-r-s], g_{kl}[s] \Big) g^{lj}[0]. \end{aligned} \tag{3.4}$$

Thus the matrix (g_{ij}) is invertible in $\mathcal{A}_U^{n\times n}$ if and only if the matrix $(g_{ij}[0](x))$ is invertible in $\mathbb{R}^{n\times n}$ for any $x \in U$. Therefore the proof of the proposition is complete. Q.E.D.

Corollary 3.1. *For any (pseudo-) Riemannian metric $g_{ij}[0]$ on U, (2.4) provides a noncommutative metric g on U, which is referred as a quantum fluctuation of $g_{ij}[0]$.*

In classical differential geometry, the cotangent bundle is the dual of the tangent bundle. Inspired by this, we can define the noncommutative cotangent bundles as the dual modules of the noncommutative tangent bundles. As the dual of a left (resp. right) $\mathcal{A}_U$-module is a right (resp. left) $\mathcal{A}_U$-module and the dual of a free module is also free, we may use the noncommutative metric g to induce bases of the cotangent bundles dual to E_i and $\tilde{E}_j$ respectively, i.e., let E^i, $\tilde{E}^j$ be dual bases of $\tilde{E}_j$, E_i respectively, we have

$$g(E^i, \tilde{E}_j) = g(E_j, \tilde{E}^i) = \delta^i_j.$$

Definition 3.1. *The noncommutative left (resp. right) cotangent bundle $\mathcal{T}_U^*$ (resp. $\tilde{\mathcal{T}}_U^*$) on U with respect to the noncommutative metric g is the free left (resp. right) $\mathcal{A}_U$-module with basis $\{E^1, \cdots, E^n\}$ (resp. $\{\tilde{E}^1, \cdots, \tilde{E}^n\}$)*

$$\mathcal{T}_U^* = \Big\{ a_i * E^i \,\Big|\, a_i \in \mathcal{A}_U,\, a_i * E^i = 0 \Longleftrightarrow a_i = 0. \Big\},$$

and

$$\tilde{\mathcal{T}}_U^* = \Big\{ \tilde{E}^i * a_i \,\Big|\, a_i \in \mathcal{A}_U,\, \tilde{E}^i * a_i = 0 \Longleftrightarrow a_i = 0. \Big\}.$$

The left (resp. right) cotangent bundle is the dual of the right (resp. left) tangent bundle. Analogous to the classical situation, the noncommutative metric g acts as an element of $\tilde{\mathcal{T}}_U^* \otimes_{\mathcal{A}_U} \mathcal{T}_U^*$,

$$\tilde{E}^i \otimes g_{ij} * E^j = \tilde{E}^i * g_{ij} \otimes E^j. \tag{3.5}$$

The inverse matrix (g^{ij}) can be viewed as a homomorphism of two-sided modules

$$g^{-1} : \mathcal{T}_U^* \otimes_{\mathbb{R}[[\hbar]]} \tilde{\mathcal{T}}_U^* \longrightarrow \mathcal{A}_U$$

such that

$$g^{-1}(E^i, \tilde{E}^j) = g^{ij}.$$

Similarly, g^{-1} acts as an element of $\tilde{\mathcal{T}}_U \otimes_{\mathcal{A}_U} \mathcal{T}_U$,

$$\tilde{E}_i \otimes g^{ij} * E_j = \tilde{E}_i * g^{ij} \otimes E_j.$$

Similar to the classical differential geometry, a noncommutative left connection ∇ on the left tangent bundle induces a unique noncommutative right connection $\tilde{\nabla}$ on the right cotangent bundle $\tilde{\mathcal{T}}_U^*$ in terms of noncommutative metric g

$$\partial_i g(V, \tilde{W}) = g(\nabla_i V, \tilde{W}) + g(V, \tilde{\nabla}_i \tilde{W})$$

where $V \in \mathcal{T}_U$, $\tilde{W} \in \tilde{\mathcal{T}}_U^*$. It yields

$$\tilde{\nabla}_i \tilde{E}^j = -\tilde{E}^k * \Gamma_{ik}^j.$$

Moreover, a noncommutative right connection $\tilde{\nabla}$ on the right tangent bundle also induces a noncommutative left connection ∇ on the left cotangent bundle which yields

$$\nabla_i E^j = -\tilde{\Gamma}_{ik}^j * E^k.$$

The noncommutative metric g and its inverse g^{-1} can be written as,

$$\begin{aligned} g =& \tilde{E}^i \otimes g_{ij} * E^j = \tilde{E}^i * g_{ij} \otimes E^j, \\ g^{-1} =& \tilde{E}_i \otimes g^{ij} * E_j = \tilde{E}_i * g^{ij} \otimes E_j. \end{aligned}$$

This allows us to define covariant derivatives of g and g^{-1} by

$$\begin{aligned} \nabla_k g =& \tilde{E}^i \otimes \nabla_k g_{ij} * E^j = \tilde{E}^i * \nabla_k g_{ij} \otimes E^j, \\ \nabla_k g^{-1} =& \tilde{E}_i \otimes \nabla_k g^{ij} * E_j = \tilde{E}_i * \nabla_k g^{ij} \otimes E_j. \end{aligned}$$

Proposition 3.2. *Let ∇ be the canonical connection with respect to the noncommutative metric g. Then*

$$\nabla_k g = \nabla_k g^{-1} = 0 \Longleftrightarrow \nabla_k g_{ij} = \nabla_k g^{ij} = 0.$$

Proof: It is straightforward that

$$\begin{aligned} \nabla_k g &= \nabla_k (\tilde{E}^i \otimes g_{ij} * E^j) \\ &= \tilde{\nabla}_k \tilde{E}^i \otimes g_{ij} * E^j + \tilde{E}^i \otimes \partial_k g_{ij} * E^j + \tilde{E}^i \otimes g_{ij} * \nabla_k E^j \\ &= -\tilde{E}^l * \Gamma_{kl}^i \otimes g_{ij} * E^j + \tilde{E}^i \otimes \partial_k g_{ij} * E^j - \tilde{E}^i \otimes g_{ij} * \tilde{\Gamma}_{kl}^j * E^l \\ &= \tilde{E}^i \otimes (\partial_k g_{ij} - \Gamma_{kij} - \tilde{\Gamma}_{kji}) * E^j = 0. \end{aligned}$$

On the other hand, a direct computation yields

$$\begin{aligned} 0 &= \Big[\partial_k (g^{il} * g_{lr})\Big] * g^{rj} \\ &= \Big[\partial_k g^{il} * g_{lr} + g^{il} * \partial_k g_{lr}\Big] * g^{rj} \\ &= \partial_k g^{il} * g_{lr} * g^{rj} + g^{il} * (\Gamma_{klr} + \tilde{\Gamma}_{krl}) * g^{rj} \\ &= \partial_k g^{ij} + g^{il} * \Gamma_{kl}^s * g_{sr} * g^{rj} + g^{il} * g_{ls} * \tilde{\Gamma}_{kr}^s * g^{rj} \\ &= \partial_k g^{ij} + g^{il} * \Gamma_{kl}^j + \tilde{\Gamma}_{kl}^i * g^{lj}. \end{aligned}$$

Therefore,

$$\begin{aligned}\nabla_k g^{-1} =&\nabla_k(\tilde{E}_i \otimes g^{ij} * E_j)\\ =&\tilde{\nabla}_k \tilde{E}_i \otimes g^{ij} * E_j + \tilde{E}_i \otimes \partial_k g^{ij} * E_j + \tilde{E}_i \otimes g^{ij} * \nabla_k E_j\\ =&\tilde{E}_i \otimes \left(\partial_k g^{ij} + g^{il} * \Gamma^j_{kl} + \tilde{\Gamma}^i_{kl} * g^{lj}\right) * E_j = 0.\end{aligned}$$

Q.E.D.

The noncommutative left (resp. right) covariant derivative along a left (resp. right) vector field $V = a^i * E_i$ (resp. $W = \tilde{E}^i * a^i$) with $a^i \in \mathcal{A}_U$ is defined as the $\mathbb{R}[[\hbar]]$-linear map

$$\nabla_V : \mathcal{T}_U \to \mathcal{T}_U \quad (\text{resp.} \quad \tilde{\nabla}_W : \tilde{\mathcal{T}}_U \to \tilde{\mathcal{T}}_U)$$

given by

$$\nabla_V X = a^i * (\nabla_i X) \quad (\text{resp.} \quad \tilde{\nabla}_W Y = (\tilde{\nabla}_i Y) * a^i)$$

for $X \in \mathcal{T}_U$ (resp. $Y \in \tilde{\mathcal{T}}_U$).

Remark 3.1. *Noncommutative covariant derivatives along general vector fields are not compatible with the Leibniz rule. Otherwise, for any $f \in \mathcal{A}_U$,*

$$\begin{aligned}\nabla_V(f * E_j) =&(Vf) * E_j + f * \nabla_V E_j\\ =&a^i * (\partial_i f) * E_j + f * a^i * \nabla_i E_j.\end{aligned}$$

On the other hand, by the definition,

$$\begin{aligned}\nabla_V(f * E_j) =&a^i * \nabla_i(f * E_j)\\ =&a^i * (\partial_i f) * E_j + a^i * f * \nabla_i E_j.\end{aligned}$$

They are not equal to each other unless

$$a^i * f = f * a^i.$$

This is generally impossible. As a consequence, it indicates that the noncommutative covariant derivatives are not well-defined with respect to orthonormal basis.

For left and right tangent vectors

$$V = v^i * E_i, \quad W = w^j * E_j, \quad \tilde{V} = \tilde{E}_i * \tilde{v}^i, \quad \tilde{W} = \tilde{E}_j * \tilde{w}^j,$$

where $v^i, w^j, \tilde{v}^i, \tilde{w}^j \in \mathcal{A}_U$, the noncommutative Lie brackets are defined as

$$\begin{aligned}[V, W]f =&v^i * E_i\big(w^j * E_j(f)\big) - w^j * E_j\big(v^i * E_i(f)\big)\\ =&\big(v^i * E_i(w^j) - w^i * E_i(v^j)\big) * E_j(f) + [v^i, w^j] * E_i E_j(f)\\ =&-[W, V]f,\\ [\tilde{V}, \tilde{W}]f =&\tilde{E}_i * \tilde{v}^i\big(\tilde{E}_j(f) * \tilde{w}^j\big) - \tilde{E}_j * \tilde{w}^j\big(\tilde{E}_i(f) * \tilde{v}^i\big)\\ =&\tilde{E}_j(f) * \big(\tilde{E}_i(\tilde{w}^j) * \tilde{v}^i - \tilde{E}_i(\tilde{v}^j) * \tilde{w}^i\big) - \tilde{E}_i \tilde{E}_j(f) * [\tilde{v}^i, \tilde{w}^j]\\ =&-[\tilde{W}, \tilde{V}]f\end{aligned}$$

for any $f \in \mathcal{A}_U$. Analogous to the classical (pseudo-) Riemannian geometry, the noncommutative left and right curvature operators for left and right tangent vectors can be formally defined as

$$\begin{aligned}\mathcal{R}_{VW} =&[\nabla_V, \nabla_W] - \nabla_{[V,W]},\\ \tilde{\mathcal{R}}_{\tilde{V}\tilde{W}} =&[\tilde{\nabla}_{\tilde{V}}, \tilde{\nabla}_{\tilde{W}}] - \tilde{\nabla}_{[\tilde{V},\tilde{W}]}.\end{aligned}$$

It is shown that $\mathcal{R}_{E_iE_j}$, $\tilde{\mathcal{R}}_{\tilde{E}_i\tilde{E}_j}$ are left and right $\mathcal{A}_U$-module endomorphisms over left and right tangent bundles respectively [4]. But $\mathcal{R}_{VW}$, $\tilde{\mathcal{R}}_{\tilde{V}\tilde{W}}$ do not make sense unless

$$[v^i, w^j] = [\tilde{v}^i, \tilde{w}^j] = 0.$$

Thus, if $V = E_i$ (resp. $\tilde{V} = \tilde{E}_i$) or $W = E_j$ (resp. $\tilde{W} = \tilde{E}_j$), then $\mathcal{R}_{VW}E_k \in \mathcal{T}_U$ (resp. $\tilde{\mathcal{R}}_{\tilde{V}\tilde{W}}\tilde{E}_k \in \tilde{\mathcal{T}}_U$) is well-defined. This suggests to define the covariant derivatives of noncommutative curvatures by adopting the idea of classical (pseudo-) Riemannian geometry. We only consider the case of left curvatures.

Definition 3.2. *The covariant derivatives of noncommutative curvature operators are defined as follows.*

$$\begin{aligned}(\nabla_k\mathcal{R})_{E_iE_j}E_p =&\nabla_k(\mathcal{R}_{E_iE_j}E_p) - \mathcal{R}_{(\nabla_{E_k}E_i)E_j}E_p\\ &- \mathcal{R}_{E_i(\nabla_{E_k}E_j)}E_p - \mathcal{R}_{E_iE_j}(\nabla_{E_k}E_p).\end{aligned}$$

Definition 3.3. *The covariant derivatives of noncommutative curvature tensors, noncommutative Ricci curvatures and noncommutative scalar curvature are defined as follows.*

$$\begin{aligned}\nabla_s R_{lkij} =&g\big((\nabla_s\mathcal{R})_{E_iE_j}E_k, \tilde{E}_l\big),\\ \nabla_s R_j^p =&g^{pk} * g\big((\nabla_s\mathcal{R})_{E_iE_j}E_k, \tilde{E}_l\big) * g^{li} = g^{pk} * \nabla_s R_{lkij} * g^{li},\\ \nabla_s \Theta_i^p =&g^{jk} * g\big((\nabla_s\mathcal{R})_{E_iE_j}E_k, \tilde{E}_l\big) * g^{lp} = g^{jk} * \nabla_s R_{lkij} * g^{lp},\\ \nabla_s R =&g^{jk} * g\big((\nabla_s\mathcal{R})_{E_iE_j}E_k, \tilde{E}_l\big) * g^{li}.\end{aligned}$$

Remark 3.2. *If $V = E_i$ or $W = E_j$, then the operator $\mathcal{R}_{VW} : \mathcal{T}_U \to \mathcal{T}_U$ is well-defined but generally not $\mathcal{A}_U$-linear. In fact assume $V = E_i$ and*

$W = a^j * E_j$, *then*

$$\begin{aligned}\mathcal{R}_{VW}(f * E_k) =& \nabla_i\Big(a^j * \nabla_j(f * E_k)\Big) - a^j * \nabla_j\Big(\nabla_i(f * E_k)\Big)\\ &- \nabla_{[E_i, a^j * E_j]}(f * E_k)\\ =&(\partial_i a^j) * \nabla_j(f * E_k) + a^j * \nabla_i\Big(\nabla_j(f * E_k)\Big)\\ &- a^j * \nabla_j\Big(\nabla_i(f * E_k)\Big) - \nabla_{(\partial_i a^j) * E_j}(f * E_k)\\ =&a^j * \mathcal{R}_{E_i E_j}(f * E_k) + (\partial_i a^j) * \nabla_j(f * E_k)\\ &- (\partial_i a^j) * \nabla_j(f * E_k)\\ =&a^j * f * \mathcal{R}_{E_i E_j} E_k.\end{aligned}$$

The same computation yields that

$$\mathcal{R}_{VW} E_k = a^j * \mathcal{R}_{E_i E_j} E_k.$$

Hence $\mathcal{R}_{VW}(f * E_k)$ *and* $f * \mathcal{R}_{VW} E_k$ *are not equal unless* $a^j * f = f * a^j$. *The above computation also yields*

$$\mathcal{R}_{E_i(a^j * E_j)} = a^j * \mathcal{R}_{E_i E_j}.$$

Similarly, we have

$$\mathcal{R}_{(a^i * E_i)E_j} = a^i * \mathcal{R}_{E_i E_j}.$$

As an operator, $(\nabla_k \mathcal{R})_{E_i E_j}$ *dose not give rise a left* $\mathcal{A}_U$*-module endomorphism over left tangent bundle. This is because*

$$\begin{aligned}(\nabla_k \mathcal{R})_{E_i E_j}(f * E_p) =&\nabla_k\Big(\mathcal{R}_{E_i E_j}(f * E_p)\Big) - \mathcal{R}_{(\nabla_k E_i)E_j}(f * E_p)\\ &- \mathcal{R}_{E_i(\nabla_k E_j)}(f * E_p) - \mathcal{R}_{E_i E_j}\Big(\nabla_k(f * E_p)\Big)\\ =&\nabla_k(f * \mathcal{R}_{E_i E_j} E_p) - \mathcal{R}_{(\nabla_k E_i)E_j}(f * E_p)\\ &- \mathcal{R}_{E_i(\nabla_k E_j)}(f * E_p)\\ &- \mathcal{R}_{E_i E_j}\Big((\partial_k f) * E_p + f * \nabla_k E_p\Big)\\ =&(\partial_k f) * \mathcal{R}_{E_i E_j} E_p + f * \nabla_k(\mathcal{R}_{E_i E_j} E_p)\\ &- \mathcal{R}_{(\nabla_k E_i)E_j}(f * E_p) - \mathcal{R}_{E_i(\nabla_k E_j)}(f * E_p)\\ &- (\partial_k f) * \mathcal{R}_{E_i E_j} E_p - f * \mathcal{R}_{E_i E_j}(\nabla_k E_p)\\ =&f * \nabla_k(\mathcal{R}_{E_i E_j} E_p) - \mathcal{R}_{(\nabla_k E_i)E_j}(f * E_p)\\ &- \mathcal{R}_{E_i(\nabla_k E_j)}(f * E_p) - f * \mathcal{R}_{E_i E_j}(\nabla_k E_p)\\ \not\equiv& f * (\nabla_k \mathcal{R})_{E_i E_j} E_p\end{aligned}$$

as $\mathcal{R}_{(\nabla_k E_i)E_j}$ *and* $\mathcal{R}_{E_i(\nabla_k E_j)}$ *are not* $\mathcal{A}_U$*-module endomorphisms in general.*

Theorem 3.1. *The first (algebraic) Bianchi identity*

$$\mathcal{R}_{E_i E_j} E_k + \mathcal{R}_{E_j E_k} E_i + \mathcal{R}_{E_k E_i} E_j = 0$$

and the second (differential) Bianchi identity

$$(\nabla_i\mathcal{R})_{E_jE_k}E_p+(\nabla_j\mathcal{R})_{E_kE_i}E_p+(\nabla_k\mathcal{R})_{E_iE_j}E_p=0$$

hold for $1\leq i,j,k,p\leq n$.

Proof: Since the connection is torsion free, we have

$$\begin{aligned}&\mathcal{R}_{E_iE_j}E_k+\mathcal{R}_{E_jE_k}E_i+\mathcal{R}_{E_kE_i}E_j\\ &\quad=\nabla_i\nabla_jE_k-\nabla_j\nabla_iE_k+\nabla_j\nabla_kE_i-\nabla_k\nabla_jE_i\\ &\qquad+\nabla_k\nabla_iE_j-\nabla_i\nabla_kE_j\\ &\quad=\nabla_i(\nabla_jE_k-\nabla_kE_j)+\nabla_j(\nabla_kE_i-\nabla_iE_k)\\ &\qquad+\nabla_k(\nabla_iE_j-\nabla_jE_i)\\ &\quad=0.\end{aligned}$$

Thus the first Bianchi identity holds. As

$$\begin{aligned}(\nabla_i\mathcal{R})_{E_jE_k}E_p=&\nabla_i\nabla_j\nabla_kE_p-\nabla_i\nabla_k\nabla_jE_p-\mathcal{R}_{(\nabla_iE_j)E_k}E_p\\ &-\mathcal{R}_{E_j(\nabla_iE_k)}E_p-\nabla_j\nabla_k\nabla_iE_p+\nabla_k\nabla_j\nabla_iE_p,\\ (\nabla_j\mathcal{R})_{E_kE_i}E_p=&\nabla_j\nabla_k\nabla_iE_p-\nabla_j\nabla_i\nabla_kE_p-\mathcal{R}_{(\nabla_jE_k)E_i}E_p\\ &-\mathcal{R}_{E_k(\nabla_jE_i)}E_p-\nabla_k\nabla_i\nabla_jE_p+\nabla_i\nabla_k\nabla_jE_p,\\ (\nabla_k\mathcal{R})_{E_iE_j}E_p=&\nabla_k\nabla_i\nabla_jE_p-\nabla_k\nabla_j\nabla_iE_p-\mathcal{R}_{(\nabla_kE_i)E_j}E_p\\ &-\mathcal{R}_{E_i(\nabla_kE_j)}E_p-\nabla_i\nabla_j\nabla_kE_p+\nabla_j\nabla_i\nabla_kE_p,\end{aligned}$$

we obtain

$$\begin{aligned}&(\nabla_i\mathcal{R})_{E_jE_k}E_p+(\nabla_j\mathcal{R})_{E_kE_i}E_p+(\nabla_k\mathcal{R})_{E_iE_j}E_p\\ &=-\mathcal{R}_{(\nabla_iE_j)E_k}E_p-\mathcal{R}_{E_j(\nabla_iE_k)}E_p-\mathcal{R}_{(\nabla_jE_k)E_i}E_p\\ &\quad-\mathcal{R}_{E_k(\nabla_jE_i)}E_p-\mathcal{R}_{(\nabla_kE_i)E_j}E_p-\mathcal{R}_{E_i(\nabla_kE_j)}E_p.\end{aligned}$$

The torsion free condition implies

$$\begin{aligned}\mathcal{R}_{(\nabla_iE_j)E_k}E_p+\mathcal{R}_{E_k(\nabla_jE_i)}E_p=0,\\ \mathcal{R}_{E_j(\nabla_iE_k)}E_p+\mathcal{R}_{(\nabla_kE_i)E_j}E_p=0,\\ \mathcal{R}_{(\nabla_jE_k)E_i}E_p+\mathcal{R}_{E_i(\nabla_kE_j)}E_p=0.\end{aligned}$$

Therefore

$$(\nabla_i\mathcal{R})_{E_jE_k}E_p+(\nabla_j\mathcal{R})_{E_kE_i}E_p+(\nabla_k\mathcal{R})_{E_iE_j}E_p=0.$$

Thus the second Bianchi identity holds. Q.E.D.

Remark 3.3. *Bianchi identities also hold for noncommutative right curvature tensors.*

Proposition 3.3. *The second Bianchi identity gives that*

$$\nabla_iR^i_j+\nabla_i\Theta^i_j-\delta^i_j\nabla_iR=0.$$

Proof: By the second Bianchi identity, we have

$$\nabla_i R_{qpjk} + \nabla_j R_{qpki} + \nabla_k R_{qpij} = 0.$$

Multiplying g^{ip} from the left side and g^{qk} from the right side, we obtain

$$g^{ip} * \nabla_i R_{qpjk} * g^{qk} + g^{ip} * \nabla_j R_{qpki} * g^{qk} + g^{ip} * \nabla_k R_{qpij} * g^{qk} = 0.$$

Taking summation for i, k, p and q, we obtain

$$-\nabla_i R_j^i + \nabla_j R - \nabla_k \Theta_j^k = 0.$$

Therefore the proof of the proposition is complete. Q.E.D.

4. Equivalence of noncommutative Ricci curvatures

In this section, we show that two Ricci curvatures R_j^i and Θ_j^i are equivalent under certain conditions. In particular, they are satisfied if noncommutative metric and chiral coefficients are given by an isometric embedding.

Let noncommutative metric g_{ij}, its inverse g^{ij} and chiral coefficients Υ_{ijk} have power series expansions (2.4), (3.2) and (2.5).

Lemma 4.1. *If noncommutative metric g satisfies (2.11), then*

$$g^{ij}[2q] = g^{ji}[2q], \quad g^{ij}[2q+1] = -g^{ji}[2q+1]. \tag{4.1}$$

Proof: Since $(g_{ij}[0])$ is symmetric and invertible on U, the inverse matrix $(g^{ij}[0])$ is also symmetric, i.e.,

$$g^{ij}[0] = g^{ji}[0].$$

For $u, v \in C^\infty(U)$, (3.1) indicates that

$$\mu_{2q}(u, v) = \mu_{2q}(v, u), \quad \mu_{2q+1}(u, v) = -\mu_{2q+1}(v, u). \tag{4.2}$$

By (4.2) and the recursive formulas (3.3), (3.4), we have

$$\begin{aligned} g^{ij}[1] &= -g^{ik}[0] g_{kl}[1] g^{lj}[0] - g^{ik}[0] \mu_1\Big(g_{kl}[0], g^{lj}[0]\Big) \\ &= g^{jl}[0] g_{lk}[1] g^{ki}[0] + \mu_1\Big(g^{jl}[0], g_{lk}[0]\Big) g^{ki}[0] \\ &= -g^{ji}[1]. \end{aligned}$$

Next, let $q \in \mathbb{N}$, using the recursive formula (3.3), we have

$$\begin{aligned} g^{ij}[2q] =& -\sum_{r=1}^{q} g^{ik}[0]g_{kl}[2r]g^{lj}[2q-2r] \\ & -\sum_{r=1}^{q} g^{ik}[0]g_{kl}[2r-1]g^{lj}[2q-2r+1] \\ & -\sum_{r=1}^{q}\sum_{s=0}^{q-r} g^{ik}[0]\mu_{2r}\Big(g_{kl}[2s], g^{lj}[2q-2r-2s]\Big) \\ & -\sum_{r=1}^{q}\sum_{s=1}^{q-r} g^{ik}[0]\mu_{2r}\Big(g_{kl}[2s-1], g^{lj}[2q-2r-2s+1]\Big) \\ & -\sum_{r=1}^{q}\sum_{s=0}^{q-r} g^{ik}[0]\mu_{2r-1}\Big(g_{kl}[2s], g^{lj}[2q-2r-2s+1]\Big) \\ & -\sum_{r=1}^{q}\sum_{s=0}^{q-r} g^{ik}[0]\mu_{2r-1}\Big(g_{kl}[2s+1], g^{lj}[2q-2r-2s]\Big). \end{aligned}$$

Therefore, by induction and (4.2) and recursive formula (3.4), we obtain

$$\begin{aligned} g^{ij}[2q] =& -\sum_{r=1}^{q} g^{jl}[2q-2r]g_{lk}[2r]g^{ki}[0] \\ & -\sum_{r=1}^{q} g^{jl}[2q-2r+1]g_{lk}[2r-1]g^{ki}[0] \\ & -\sum_{r=1}^{q}\sum_{s=0}^{q-r} \mu_{2r}\Big(g^{jl}[2q-2r-2s], g_{lk}[2s]\Big)g^{ki}[0] \\ & -\sum_{r=1}^{q}\sum_{s=1}^{q-r} \mu_{2r}\Big(g^{jl}[2q-2r-2s+1], g_{lk}[2s-1]\Big)g^{ki}[0] \\ & -\sum_{r=1}^{q}\sum_{s=0}^{q-r} \mu_{2r-1}\Big(g^{jl}[2q-2r-2s+1], g_{lk}[2s]\Big)g^{ki}[0] \\ & -\sum_{r=1}^{q}\sum_{s=0}^{q-r} \mu_{2r-1}\Big(g^{jl}[2q-2r-2s], g_{lk}[2s+1]\Big)g^{ki}[0] \\ =& g^{ji}[2q]. \end{aligned}$$

Similarly, we can prove

$$g^{ij}[2q+1] = -g^{ji}[2q+1].$$

Q.E.D.

Lemma 4.2. *If noncommutative metric g satisfies (2.11), then, for any $f \in C^\infty(U)$,*

$$\begin{aligned}\big(g^{ij} * f * g^{kl}\big)[2q] &= \big(g^{lk} * f * g^{ji}\big)[2q],\\ \big(g^{ij} * f * g^{kl}\big)[2q+1] &= -\big(g^{lk} * f * g^{ji}\big)[2q+1].\end{aligned}$$

Proof: For any $u, v \in C^\infty(U)$, we have

$$\begin{aligned}\big(g^{ij} * u * v\big)[2q] =&\sum_{r=0}^{q}\sum_{s=0}^{q-r}\mu_{2r}\Big(g^{ij}[2s], \mu_{2q-2r-2s}(u,v)\Big)\\ &+\sum_{r=0}^{q}\sum_{s=1}^{q-r}\mu_{2r}\Big(g^{ij}[2s-1], \mu_{2q-2r-2s+1}(u,v)\Big)\\ &+\sum_{r=1}^{q}\sum_{s=0}^{q-r}\mu_{2r-1}\Big(g^{ij}[2s], \mu_{2q-2r-2s+1}(u,v)\Big)\\ &+\sum_{r=1}^{q}\sum_{s=0}^{q-r}\mu_{2r-1}\Big(g^{ij}[2s+1], \mu_{2q-2r-2s}(u,v)\Big).\end{aligned}$$

Using Lemma 4.1 and (4.2), we obtain

$$\begin{aligned}\big(g^{ij} * u * v\big)[2q] =&\sum_{r=0}^{q}\sum_{s=0}^{q-r}\mu_{2r}\Big(\mu_{2q-2r-2s}(v,u), g^{ji}[2s]\Big)\\ &+\sum_{r=0}^{q}\sum_{s=1}^{q-r}\mu_{2r}\Big(\mu_{2q-2r-2s+1}(v,u), g^{ji}[2s-1]\Big)\\ &+\sum_{r=1}^{q}\sum_{s=0}^{q-r}\mu_{2r-1}\Big(\mu_{2q-2r-2s+1}(v,u), g^{ji}[2s]\Big)\\ &+\sum_{r=1}^{q}\sum_{s=0}^{q-r}\mu_{2r-1}\Big(\mu_{2q-2r-2s}(v,u), g^{ji}[2s+1]\Big)\\ =&\big(v * u * g^{ji}\big)[2q].\end{aligned}$$

Similarly, we can show

$$\big(g^{ij} * u * v\big)[2q+1] = -\big(v * u * g^{ji}\big)[2q+1].$$

Using them, we obtain

$$\begin{aligned}
\big(g^{ij} * f * g^{kl}\big)[2q] =& \sum_{r=0}^{q} \Big(g^{ij} * f * (g^{kl}[2r])\Big)[2q-2r] \\
&+ \sum_{r=1}^{q} \Big(g^{ij} * f * (g^{kl}[2r-1])\Big)[2q-2r+1] \\
=& \sum_{r=0}^{q} \Big((g^{kl}[2r]) * f * g^{ji}\Big)[2q-2r] \\
&- \sum_{r=1}^{q} \Big((g^{kl}[2r-1]) * f * g^{ji}\Big)[2q-2r+1] \\
=& \sum_{r=0}^{q} \Big((g^{lk}[2r]) * f * g^{ji}\Big)[2q-2r] \\
&+ \sum_{r=0}^{q} \Big((g^{lk}[2r-1]) * f * g^{ji}\Big)[2q] \\
=& \big(g^{lk} * f * g^{ji}\big)[2q],
\end{aligned}$$

and, similarly

$$\big(g^{ij} * f * g^{kl}\big)[2q+1] = -\big(g^{lk} * f * g^{ji}\big)[2q+1].$$

Q.E.D.

Lemma 4.3. *If noncommutative metric g satisfies (2.11), then, for $u, v \in C^\infty(U)$,*

$$\begin{aligned}
\big(u * g^{ij} * v\big)[2q] =& \big(v * g^{ji} * u\big)[2q], \\
\big(u * g^{ij} * v\big)[2q+1] =& - \big(v * g^{ji} * u\big)[2q+1].
\end{aligned}$$

Proof: For any $u \in C^\infty(U)$, we have

$$\begin{aligned}
\big(u * g^{ij}\big)[2q] =& \sum_{r=0}^{q} \mu_{2r}\Big(u, g^{ij}[2q-2r]\Big) \\
&+ \sum_{r=1}^{q} \mu_{2r-1}\Big(u, g^{ij}[2q-2r+1]\Big).
\end{aligned}$$

Using Lemma 4.1 and (4.2), we obtain

$$\begin{aligned}\big(u*g^{ij}\big)[2q] =&\sum_{r=0}^{q}\mu_{2r}\Big(u,g^{ij}[2q-2r]\Big)\\&+\sum_{r=1}^{q}\mu_{2r-1}\Big(u,g^{ij}[2q-2r+1]\Big)\\=&\sum_{r=0}^{q}\mu_{2r}\Big(g^{ji}[2q-2r],u\Big)\\&+\sum_{r=1}^{q}\mu_{2r-1}\Big(g^{ji}[2q-2r+1],u\Big)\\=&\big(g^{ji}*u\big)[2q].\end{aligned}$$

Similarly we have

$$\big(u*g^{ij}\big)[2q+1] = -\big(g^{ji}*u\big)[2q+1].$$

Then using Lemma 4.1 and (4.2) again, we have

$$\begin{aligned}\big(u*g^{ij}*v\big)[2q] =&\sum_{r=0}^{q}\mu_{2r}\Big(\big(u*g^{ij}\big)[2q-2r],v\Big)\\&+\sum_{r=1}^{q}\mu_{2r-1}\Big(\big(u*g^{ij}\big)[2q-2r+1],v\Big)\\=&\sum_{r=0}^{q}\mu_{2r}\Big(v,\big(g^{ji}*u\big)[2q-2r]\Big)\\&+\sum_{r=1}^{q}\mu_{2r-1}\Big(v,\big(g^{ji}*u\big)[2q-2r+1]\Big)\\=&\big(v*g^{ji}*u\big)[2q].\end{aligned}$$

Similarly we have

$$\big(u*g^{ij}*v\big)[2q+1] = -\big(v*g^{ji}*u\big)[2q+1].$$

Q.E.D.

Lemma 4.4. *Let ∇, $\tilde{\nabla}$ be the canonical connection with respect to the non-commutative metric g and chiral coefficients Υ_{ijk} on U. If (2.11), (2.12) hold, then*

$$\Gamma_{ijk}[2q] = \tilde{\Gamma}_{ijk}[2q], \quad \Gamma_{ijk}[2q+1] = -\tilde{\Gamma}_{ijk}[2q+1]. \tag{4.3}$$

Proof: By chirality and (2.12), we have

$$\Gamma_{ijk}[2q] - \tilde{\Gamma}_{ijk}[2q] = \Upsilon_{ijk}[2q] = 0.$$

By (2.2), (2.3) and (2.11), we have

$$\begin{aligned}\Gamma_{ijk}[2q+1] =& \frac{1}{2}\Upsilon_{ijk}[2q+1] \\ =& -\tilde{\Gamma}_{ijk}[2q+1].\end{aligned}$$

Q.E.D.

Proposition 4.1. *Let M be an n-dimensional smooth manifold and $U \subset M$ a coordinate chart. Let ∇, $\tilde{\nabla}$ be the canonical connection with respect to noncommutative metric g and chiral coefficients Υ_{ijk} on U. Let Riemannian curvatures have the power series expansions (2.6). If (2.11), (2.12) hold, then*

$$R_{lkij}[2q] = -R_{klij}[2q], \quad R_{lkij}[2q+1] = R_{klij}[2q+1]. \tag{4.4}$$

Proof: In terms of connection coefficients, the Riemannian curvatures are

$$\begin{aligned}R_{lkij} =& g\big((\nabla_i\nabla_j - \nabla_j\nabla_i)E_k, \tilde{E}_l\big) \\ =& \partial_i g\big(\nabla_j E_k, \tilde{E}_l\big) - g\big(\nabla_j E_k, \tilde{\nabla}_i \tilde{E}_l\big) \\ & - \partial_j g\big(\nabla_i E_k, \tilde{E}_l\big) + g\big(\nabla_i E_k, \tilde{\nabla}_j \tilde{E}_l\big) \\ =& \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma_{ik}^r * g_{rs} * \tilde{\Gamma}_{jl}^s - \Gamma_{jk}^r * g_{rs} * \tilde{\Gamma}_{il}^s \\ =& \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma_{iks} * g^{sr} * \tilde{\Gamma}_{jlr} - \Gamma_{jks} * g^{sr} * \tilde{\Gamma}_{ilr}.\end{aligned}$$

Since

$$\begin{aligned}\partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} =& \partial_i g\big(\nabla_j E_k, \tilde{E}_l\big) - \partial_j g\big(\nabla_i E_k, \tilde{E}_l\big) \\ =& \partial_i\big(\partial_j g(E_k, \tilde{E}_l) - g(E_k, \tilde{\nabla}_j \tilde{E}_l)\big) \\ & - \partial_j\big(\partial_i g(E_k, \tilde{E}_l) - g(E_k, \tilde{\nabla}_i \tilde{E}_l)\big) \\ =& \partial_i\partial_j g_{kl} - \partial_i \tilde{\Gamma}_{jlk} - \partial_j\partial_i g_{kl} + \partial_j \tilde{\Gamma}_{ilk} \\ =& \partial_j \tilde{\Gamma}_{ilk} - \partial_i \tilde{\Gamma}_{jlk},\end{aligned}$$

we obtain

$$R_{lkij} = \partial_j \tilde{\Gamma}_{ilk} - \partial_i \tilde{\Gamma}_{jlk} + \Gamma_{iks} * g^{sr} * \tilde{\Gamma}_{jlr} - \Gamma_{jks} * g^{sr} * \tilde{\Gamma}_{ilr}.$$

Using Lemma 4.4, we have

$$\begin{aligned}\big(\partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl}\big)[2q] =& -\big(\partial_j \tilde{\Gamma}_{ikl} - \partial_i \tilde{\Gamma}_{jkl}\big)[2q] \\ \big(\partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl}\big)[2q+1] =& \big(\partial_j \tilde{\Gamma}_{ikl} - \partial_i \tilde{\Gamma}_{jkl}\big)[2q+1].\end{aligned}$$

Moreover, using Lemma 4.3 and Lemma 4.4, we obtain

$$
\begin{aligned}
&\big(\Gamma_{iks} * g^{sr} * \tilde{\Gamma}_{jlr}\big)[2q] \\
=&\sum_{\alpha=0}^{q}\sum_{\beta=0}^{q-\alpha}\big(\Gamma_{iks}[2\alpha] * g^{sr} * \tilde{\Gamma}_{jlr}[2\beta]\big)[2q-2\alpha-2\beta] \\
&+\sum_{\alpha=0}^{q}\sum_{\beta=1}^{q-\alpha}\big(\Gamma_{iks}[2\alpha] * g^{sr} * \tilde{\Gamma}_{jlr}[2\beta-1]\big)[2q-2\alpha-2\beta+1] \\
&+\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\big(\Gamma_{iks}[2\alpha-1] * g^{sr} * \tilde{\Gamma}_{jlr}[2\beta]\big)[2q-2\alpha-2\beta+1] \\
&+\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\big(\Gamma_{iks}[2\alpha-1] * g^{sr} * \tilde{\Gamma}_{jlr}[2\beta+1]\big)[2q-2\alpha-2\beta] \\
=&\sum_{\alpha=0}^{q}\sum_{\beta=0}^{q-\alpha}\big(\Gamma_{jlr}[2\beta] * g^{rs} * \tilde{\Gamma}_{iks}[2\alpha]\big)[2q-2\alpha-2\beta] \\
&+\sum_{\alpha=0}^{q}\sum_{\beta=1}^{q-\alpha}\big(\Gamma_{jlr}[2\beta-1] * g^{rs} * \tilde{\Gamma}_{iks}[2\alpha]\big)[2q-2\alpha-2\beta+1] \\
&+\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\big(\Gamma_{jlr}[2\beta] * g^{rs} * \tilde{\Gamma}_{iks}[2\alpha-1]\big)[2q-2\alpha-2\beta+1] \\
&+\sum_{\alpha=1}^{q}\sum_{\beta=0}^{q-\alpha}\big(\Gamma_{jlr}[2\beta+1] * g^{rs} * \tilde{\Gamma}_{iks}[2\alpha-1]\big)[2q-2\alpha-2\beta] \\
=&\big(\Gamma_{jls} * g^{sr} * \tilde{\Gamma}_{ikr}\big)[2q].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\big(\Gamma_{iks} * g^{sr} * \tilde{\Gamma}_{jlr}\big)[2q+1] =& -\big(\Gamma_{jls} * g^{sr} * \tilde{\Gamma}_{ikr}\big)[2q+1], \\
\big(\Gamma_{jks} * g^{sr} * \tilde{\Gamma}_{ilr}\big)[2q] =&\big(\Gamma_{ils} * g^{sr} * \tilde{\Gamma}_{jkr}\big)[2q], \\
\big(\Gamma_{jks} * g^{sr} * \tilde{\Gamma}_{ilr}\big)[2q+1] =& -\big(\Gamma_{ils} * g^{sr} * \tilde{\Gamma}_{jkr}\big)[2q+1].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
R_{lkij}[2q] =&\big(\partial_i\Gamma_{jkl} - \partial_j\Gamma_{ikl} + \Gamma_{iks} * g^{sr} * \tilde{\Gamma}_{jlr} - \Gamma_{jks} * g^{sr} * \tilde{\Gamma}_{ilr}\big)[2q] \\
=& -\big(\partial_j\tilde{\Gamma}_{ikl} - \partial_i\tilde{\Gamma}_{jkl} + \Gamma_{ils} * g^{sr} * \Gamma_{jkr} - \Gamma_{jls} * g^{sr} * \tilde{\Gamma}_{ikr}\big)[2q] \\
=& -R_{klij}[2q].
\end{aligned}
$$

Similarly, we obtain

$$
R_{lkij}[2q+1] = R_{klij}[2q+1].
$$

Q.E.D.

Theorem 4.1. *Let M be an n-dimensional smooth manifold and $U \subset M$ a coordinate chart. Let ∇, $\tilde{\nabla}$ be the canonical connection with respect to noncommutative metric g and chiral coefficients Υ_{ijk} on U. Let two Ricci curvatures have the power series expansions (2.7), (2.8), (2.9), (2.10). If (2.11), (2.12) hold, then*

$$R_{ij}[2q] = \Theta_{ji}[2q], \quad R_{ij}[2q+1] = -\Theta_{ji}[2q+1], \tag{4.5}$$

$$R^i_j[2q] = \Theta^i_j[2q], \quad R^i_j[2q+1] = -\Theta^i_j[2q+1]. \tag{4.6}$$

Proof: By definition

$$R_{ij} = R_{likj} * g^{lk}, \quad \Theta_{ij} = g^{lk} * R_{jkil}.$$

Using Lemma 4.1, Proposition 4.1 and (4.2), we obtain

$$\begin{aligned}
R_{ij}[2q] =& \sum_{r=0}^{q}\sum_{s=0}^{q-r} \mu_{2r}\Big(R_{likj}[2s], g^{lk}[2q-2r-2s]\Big) \\
&+ \sum_{r=0}^{q}\sum_{s=1}^{q-r} \mu_{2r}\Big(R_{likj}[2s-1], g^{lk}[2q-2r-2s+1]\Big) \\
&+ \sum_{r=1}^{q}\sum_{s=0}^{q-r} \mu_{2r-1}\Big(R_{likj}[2s], g^{lk}[2q-2r-2s+1]\Big) \\
&+ \sum_{r=1}^{q}\sum_{s=0}^{q-r} \mu_{2r-1}\Big(R_{likj}[2s+1], g^{lk}[2q-2r-2s]\Big) \\
=& - \sum_{r=0}^{q}\sum_{s=0}^{q-r} \mu_{2r}\Big(g^{kl}[2q-2r-2s], R_{ilkj}[2s]\Big) \\
&- \sum_{r=0}^{q}\sum_{s=1}^{q-r} \mu_{2r}\Big(g^{kl}[2q-2r-2s+1], R_{ilkj}[2s-1]\Big) \\
&- \sum_{r=1}^{q}\sum_{s=0}^{q-r} \mu_{2r-1}\Big(g^{kl}[2q-2r-2s+1], R_{ilkj}[2s]\Big) \\
&- \sum_{r=1}^{q}\sum_{s=0}^{q-r} \mu_{2r-1}\Big(g^{kl}[2q-2r-2s], R_{ilkj}[2s+1]\Big) \\
=& - \big(g^{kl} * R_{ilkj}\big)[2q] \\
=& \big(g^{kl} * R_{iljk}\big)[2q] \\
=& \Theta_{ji}[2q].
\end{aligned}$$

Similarly, we have

$$R_{ij}[2q+1] = -\Theta_{ji}[2q+1].$$

Then since

$$R^i_j = g^{ik} * R_{kj}, \quad \Theta^i_j = \Theta_{jk} * g^{ki},$$

we obtain

$$
\begin{aligned}
R^i_j[2q] =& \sum_{r=0}^{q}\sum_{s=0}^{q-r}\mu_{2r}\Big(g^{ik}[2s], R_{kj}[2q-2r-2s]\Big) \\
&+ \sum_{r=0}^{q}\sum_{s=1}^{q-r}\mu_{2r}\Big(g^{ik}[2s-1], R_{kj}[2q-2r-2s+1]\Big) \\
&+ \sum_{r=1}^{q}\sum_{s=0}^{q-r}\mu_{2r-1}\Big(g^{ik}[2s], R_{kj}[2q-2r-2s+1]\Big) \\
&+ \sum_{r=1}^{q}\sum_{s=0}^{q-r}\mu_{2r-1}\Big(g^{ik}[2s+1], R_{kj}[2q-2r-2s]\Big) \\
=& \sum_{r=0}^{q}\sum_{s=0}^{q-r}\mu_{2r}\Big(\Theta_{jk}[2q-2r-2s], g^{ki}[2s]\Big) \\
&+ \sum_{r=0}^{q}\sum_{s=1}^{q-r}\mu_{2r}\Big(\Theta_{jk}[2q-2r-2s+1], g^{ki}[2s-1]\Big) \\
&+ \sum_{r=1}^{q}\sum_{s=0}^{q-r}\mu_{2r-1}\Big(\Theta_{jk}[2q-2r-2s+1], g^{ki}[2s]\Big) \\
&+ \sum_{r=1}^{q}\sum_{s=1}^{q-r}\mu_{2r-1}\Big(\Theta_{jk}[2q-2r-2s], g^{ki}[2s+1]\Big) \\
=& \big(\Theta_{jk} * g^{ki}\big)[2q] \\
=& \Theta^i_j[2q].
\end{aligned}
$$

Similarly, we have

$$R^i_j[2q+1] = -\Theta^i_j[2q+1].$$

Q.E.D.

Now we provide the quantum fluctuation of a pseudo-Riemannian metric $g[0]$ on U in terms of isometric embedding [4, 16, 17]. Recall that $(U, g[0])$ can always be isometrically embedded into a pseudo-Euclidean space, c.f. [11], i.e., there exist a differentiable map

$$X : U \longrightarrow \mathbb{R}^{p,m-p}$$

such that

$$g_{ij}[0] = \sum_{\alpha=1}^{m} \eta_{\alpha\alpha}\partial_i X^\alpha \cdot \partial_j X^\alpha,$$

where $\eta = \mathrm{diag}(-1, \cdots, -1, 1, \cdots, 1)$ is the flat metrics of $\mathbb{R}^{p,m-p}$. The quantum fluctuation of $g[0]$ is

$$g\big(E_i, \tilde{E}_j\big) = \sum_{\alpha=1}^{m} \eta_{\alpha\alpha} \partial_i X^\alpha * \partial_j X^\alpha, \tag{4.7}$$

where $E_i = \tilde{E}_i = \partial_i$. It yields a canonical connection with the connection and chiral coefficients

$$\Gamma_{ijk} = \sum_{\alpha=1}^{m} \eta_{\alpha\alpha} \partial_i \partial_j X^\alpha * \partial_k X^\alpha, \tag{4.8}$$

$$\tilde{\Gamma}_{ijk} = \sum_{\alpha=1}^{m} \eta_{\alpha\alpha} \partial_k X^\alpha * \partial_i \partial_j X^\alpha, \tag{4.9}$$

$$\Upsilon_{ijk} = \sum_{\alpha=1}^{m} \eta_{\alpha\alpha} \big(\partial_i \partial_j X^\alpha * \partial_k X^\alpha - \partial_k X^\alpha * \partial_i \partial_j X^\alpha\big). \tag{4.10}$$

The noncommutative metric (4.7) also induce a noncommutative scalar product on the high order partial derivatives of isometric embedding X. For multi-index γ, δ,

$$g\big(\partial^{|\gamma|} X, \partial^{|\delta|} X\big) = \sum_{\alpha=1}^{m} \eta_{\alpha\alpha} \partial^{|\gamma|} X^\alpha * \partial^{|\beta|} X^\alpha. \tag{4.11}$$

Corollary 4.1. *Let g be given by (4.7) for isometric embedding*

$$X = (X^1, \cdots, X^m) \in C^\infty(U, \mathbb{R}^m),$$

where $X^\alpha \in C^\infty(U)$, $1 \leq \alpha \leq m$. Let R^i_j and Θ^i_j be the two Ricci curvatures of the canonical connection induced by X. Then (4.5) and (4.6) hold.

Proof: We only need to check (4.7), (4.10) satisfy (2.11) and (2.12). Indeed, it is a direct consequence of (4.2). Q.E.D.

Remark 4.1. *The following noncommutative Einstein field equations were proposed in* [4]

$$R^i_j + \Theta^i_j - \delta^i_j R = T^i_j.$$

As it may not capture all information of noncommutative metrics, the second author gave the strong version in [19]

$$R^i_j - \frac{1}{2}\delta^i_j R = T^i_j, \quad \Theta^i_j - \frac{1}{2}\delta^i_j R = \tilde{T}^i_j.$$

Theorem 4.1 and Corollary 4.1 indicate that only the first one is sufficient and the noncommutative Einstein field equations should be

$$R^i_j - \frac{1}{2}\delta^i_j R = T^i_j$$

if (2.11), (2.12) hold, in particular, if noncommutative metrics are given by isometric embedding.

5. Spherically symmetric isometric embeddings

In this section, we show that the quantum fluctuations and their curvatures have close forms coming from Moyal products of trigonometric functions if (pseudo-) Riemannian metrics are given by certain type of spherically symmetric isometric embeddings. This indicates that the quantization of gravity is renormalizable in this case.

Theorem 5.1. *Let open set*

$$U = (0,\infty) \times (0,2\pi) \times (0,\pi) \times \cdots \times (0,\pi) \subset \mathbb{R}^n,$$

which is equipped with coordinates $(x^1, x^2, \cdots, x^n) = (\rho, \theta_1, \cdots, \theta_{n-1})$*. Let* $(U, g[0])$ *be a (pseudo-) Riemannian metric given by a spherically symmetric isometric embedding*

$$X : U \longrightarrow \mathbb{R}^{p,m-p}$$

with

$$\begin{aligned} X^1 &= f^1(\rho), \\ &\cdots\cdots \\ X^{m-n} &= f^{m-n}(\rho), \\ X^{m-n+1} &= f^{m-n+1}(\rho) \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_2 \sin\theta_1, \\ X^{m-n+2} &= f^{m-n+2}(\rho) \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_2 \cos\theta_1, \\ &\cdots\cdots \\ X^{m-2} &= f^{m-2}(\rho) \sin\theta_{n-1} \sin\theta_{n-2} \cos\theta_{n-3}, \\ X^{m-1} &= f^{m-1}(\rho) \sin\theta_{n-1} \cos\theta_{n-2}, \\ X^m &= f^m(\rho) \cos\theta_{n-1}, \end{aligned}$$

where $f^1(\rho)$, $\cdots$, $f^m(\rho)$ *are smooth functions of* ρ, $m-n+1 > p$ *and*

$$f^{m-n+1}(\rho) = f^{m-n+2}(\rho) = f(\rho).$$

Fix some $l \in [3,n]$*, define the Moyal product in terms of skew-symmetric matrix* (θ^{ij}) *with nonzero elements*

$$\theta^{2l} = -\theta^{l2} = \lambda \neq 0.$$

Then the quantum fluctuation of $g[0]$ *and their curvatures have close forms coming from absolutely convergent power series expansions on* U*.*

Proof: Note that only the term

$$\partial_i X^{m-n+1} * \partial_j X^{m-n+1} + \partial_i X^{m-n+2} * \partial_j X^{m-n+2}$$

cannot be reduced to the usual commutative product in noncommutative metric (4.7). Denote

$$g_{ij}^{(\alpha,\beta)} = \partial_i X^\alpha * \partial_j X^\beta.$$

And denote $a_0 = m - n + 1$ for short. Using the formulas in the appendix, we obtain, for $2 < i < j \leq n, 2 < k \leq n$ and $i, j, k \neq l$,

$$\begin{aligned}
g_{11}^{(a_0,a_0)} + g_{11}^{(a_0+1,a_0+1)} =& \big(f'\big)^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_l \sin^2\theta_{l-2}\cdots\sin^2\theta_2\\
& \Big(\sin^2\theta_{l-1}\cosh^2(\lambda\hbar) - \cos^2\theta_{l-1}\sinh^2(\lambda\hbar)\Big),\\
g_{12}^{(a_0,a_0)} + g_{12}^{(a_0+1,a_0+1)} =& 2ff' \sin^2\theta_{n-1}\cdots\sin^2\theta_l \sin^2\theta_{l-2}\cdots\sin^2\theta_2\\
& \sin\theta_{l-1}\cos\theta_{l-1}\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
g_{1l}^{(a_0,a_0)} + g_{1l}^{(a_0+1,a_0+1)} =& ff' \sin^2\theta_{n-1}\cdots\sin^2\theta_l \sin^2\theta_{l-2}\cdots\sin^2\theta_2\\
& \sin\theta_{l-1}\cos\theta_{l-1}\Big(1 + 2\sinh^2(\lambda\hbar)\Big),\\
g_{22}^{(a_0,a_0)} + g_{22}^{(a_0+1,a_0+1)} =& f^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_l \sin^2\theta_{l-2}\cdots\sin^2\theta_2\\
& \Big(\sin^2\theta_{l-1}\cosh^2(\lambda\hbar) - \cos^2\theta_{l-1}\sinh^2(\lambda\hbar)\Big),\\
g_{2l}^{(a_0,a_0)} + g_{2l}^{(a_0+1,a_0+1)} =& f^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_l \sin^2\theta_{l-2}\cdots\sin^2\theta_2\\
& \Big(\sin^2\theta_{l-1} - \cos^2\theta_{l-1}\Big)\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
g_{ll}^{(a_0,a_0)} + g_{ll}^{(a_0+1,a_0+1)} =& f^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_l \sin^2\theta_{l-2}\cdots\sin^2\theta_2\\
& \Big(\cos^2\theta_{l-1}\cosh^2(\lambda\hbar) - \sin^2\theta_{l-1}\sinh^2(\lambda\hbar)\Big),\\
g_{1k}^{(a_0,a_0)} + g_{1k}^{(a_0+1,a_0+1)} =& ff' \sin^2\theta_{n-1}\cdots\sin^2\theta_k \sin\theta_{k-1}\cos\theta_{k-1}\sin^2\theta_{k-2}\cdots\\
& \Big(\sin^2\theta_{l-1}\cosh^2(\lambda\hbar) - \cos^2\theta_{l-1}\sinh^2(\lambda\hbar)\Big),\\
g_{2k}^{(a_0,a_0)} + g_{2k}^{(a_0+1,a_0+1)} =& -2f^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_k \sin\theta_{k-1}\cos\theta_{k-1}\sin^2\theta_{k-2}\cdots\\
& \sin\theta_{l-1}\cos\theta_{l-1}\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
g_{lk}^{(a_0,a_0)} + g_{lk}^{(a_0+1,a_0+1)} =& f^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_k \sin\theta_{k-1}\cos\theta_{k-1}\sin^2\theta_{k-2}\cdots\\
& \sin\theta_{l-1}\cos\theta_{l-1}\Big(1 + 2\sinh^2(\lambda\hbar)\Big),\\
g_{kk}^{(a_0,a_0)} + g_{kk}^{(a_0+1,a_0+1)} =& f^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_k \cos^2\theta_{k-1}\sin^2\theta_{k-2}\cdots\\
& \Big(\sin^2\theta_{l-1}\cosh^2(\lambda\hbar) - \cos^2\theta_{l-1}\sinh^2(\lambda\hbar)\Big),\\
g_{ij}^{(a_0,a_0)} + g_{ij}^{(a_0+1,a_0+1)} =& f^2 \sin^2\theta_{n-1}\cdots\sin^2\theta_j \sin\theta_{j-1}\cos\theta_{j-1}\sin^2\theta_{j-2}\cdots\\
& \sin^2\theta_i \sin\theta_{i-1}\cos\theta_{i-1}\sin^2\theta_{i-2}\cdots\\
& \Big(\sin^2\theta_{l-1}\cosh^2(\lambda\hbar) - \cos^2\theta_{l-1}\sinh^2(\lambda\hbar)\Big).
\end{aligned}$$

These indicate that

$$g_{2k} = -g_{k2}, \quad k \neq 2$$

but other metric components are symmetric, and the quantum fluctuation $g = (g_{ij})$ of $g[0]$ have close forms which are smooth functions not depending on θ_1. Therefore the Moyal product relating to (g_{ij}) becomes usual commutative product. This means the inverse matrix (g^{ij}) coincides with the inverse matrix in the sense of usual commutative product, and its elements do not depend on θ_1 neither. By (4.8), (4.9), similar calculation yields that the connection coefficients Γ_{ijk}, $\tilde{\Gamma}_{ijk}$ also have close forms which are smooth functions not depending on θ_1.

As all quantities relating to the quantum fluctuation and the connection coefficients do not depend on θ_1, the Moyal product in deriving the curvatures becomes usual commutative product. Therefore the curvatures have close forms, which depend only on ρ, $\theta_2, \cdots, \theta_n$ and $\hbar$. Q.E.D.

Appendix A. Moyal Products of Trigonometric Functions

Let $U \subset \mathbb{R}^2$ be an open subset with coordinates $(x^1, x^2) = (\theta_1, \theta_2)$. Define the Moyal product $*$ on $\mathcal{A}_U = C^\infty(U)[[\hbar]]$ by the matrix

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix},$$

for some constant $\lambda \neq 0$. Moyal products of trigonometric functions are given as follows.

A.1.

$$\begin{aligned}
&(\sin\theta_1 \sin\theta_2) * (\sin\theta_1 \sin\theta_2) \\
&\quad = \sin^2\theta_1 \sin^2\theta_2 \cosh^2(\lambda\hbar) - \cos^2\theta_1 \cos^2\theta_2 \sinh^2(\lambda\hbar), \\
&(\sin\theta_1 \sin\theta_2) * (\sin\theta_1 \cos\theta_2) \\
&\quad = \sin\theta_2 \cos\theta_2 \big(\sin^2\theta_1 + \sinh^2(\lambda\hbar)\big) - \sin\theta_1 \cos\theta_1 \cosh(\lambda\hbar) \sinh(\lambda\hbar), \\
&(\sin\theta_1 \sin\theta_2) * (\cos\theta_1 \sin\theta_2) \\
&\quad = \sin\theta_1 \cos\theta_1 \big(\sin^2\theta_2 + \sinh^2(\lambda\hbar)\big) + \sin\theta_2 \cos\theta_2 \cosh(\lambda\hbar) \sinh(\lambda\hbar), \\
&(\sin\theta_1 \sin\theta_2) * (\cos\theta_1 \cos\theta_2) \\
&\quad = \sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 + (\sin^2\theta_1 - \sin^2\theta_2) \cosh(\lambda\hbar) \sinh(\lambda\hbar),
\end{aligned}$$

A.2.

$$
\begin{aligned}
&(\sin\theta_1\cos\theta_2) * (\sin\theta_1\sin\theta_2)\\
&\quad = \sin\theta_2\cos\theta_2\big(\sin^2\theta_1 + \sinh^2(\lambda\hbar)\big) + \sin\theta_1\cos\theta_1\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
&(\sin\theta_1\cos\theta_2) * (\sin\theta_1\cos\theta_2)\\
&\quad = \sin^2\theta_1\cos^2\theta_2\cosh^2(\lambda\hbar) - \cos^2\theta_1\sin^2\theta_2\sinh^2(\lambda\hbar),\\
&(\sin\theta_1\cos\theta_2) * (\cos\theta_1\sin\theta_2)\\
&\quad = \sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2 + (\cos^2\theta_1 - \sin^2\theta_2)\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
&(\sin\theta_1\cos\theta_2) * (\cos\theta_1\cos\theta_2)\\
&\quad = \sin\theta_1\cos\theta_1\big(\cos^2\theta_2 + \sinh^2(\lambda\hbar)\big) - \sin\theta_2\cos\theta_2\cosh(\lambda\hbar)\sinh(\lambda\hbar),
\end{aligned}
$$

A.3.

$$
\begin{aligned}
&(\cos\theta_1\sin\theta_2) * (\sin\theta_1\sin\theta_2)\\
&\quad = \sin\theta_1\cos\theta_1\big(\sin^2\theta_2 + \sinh^2(\lambda\hbar)\big) - \sin\theta_2\cos\theta_2\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
&(\cos\theta_1\sin\theta_2) * (\sin\theta_1\cos\theta_2)\\
&\quad = \sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2 + (\sin^2\theta_1 - \cos^2\theta_2)\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
&(\cos\theta_1\sin\theta_2) * (\cos\theta_1\sin\theta_2)\\
&\quad = \cos^2\theta_1\sin^2\theta_2\cosh^2(\lambda\hbar) - \sin^2\theta_1\cos^2\theta_2\sinh^2(\lambda\hbar),\\
&(\cos\theta_1\sin\theta_2) * (\cos\theta_1\cos\theta_2)\\
&\quad = \sin\theta_2\cos\theta_2\big(\cos^2\theta_1 + \sinh^2(\lambda\hbar)\big) + \sin\theta_1\cos\theta_1\cosh(\lambda\hbar)\sinh(\lambda\hbar),
\end{aligned}
$$

A.4.

$$
\begin{aligned}
&(\cos\theta_1\cos\theta_2) * (\sin\theta_1\sin\theta_2)\\
&\quad = \sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2 + (\cos^2\theta_1 - \cos^2\theta_2)\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
&(\cos\theta_1\cos\theta_2) * (\sin\theta_1\cos\theta_2)\\
&\quad = \sin\theta_1\cos\theta_1\big(\cos^2\theta_2 + \sinh^2(\lambda\hbar)\big) + \sin\theta_2\cos\theta_2\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
&(\cos\theta_1\cos\theta_2) * (\cos\theta_1\sin\theta_2)\\
&\quad = \sin\theta_2\cos\theta_2\big(\cos^2\theta_1 + \sinh^2(\lambda\hbar)\big) - \sin\theta_1\cos\theta_1\cosh(\lambda\hbar)\sinh(\lambda\hbar),\\
&(\cos\theta_1\cos\theta_2) * (\cos\theta_1\cos\theta_2)\\
&\quad = \cos^2\theta_1\cos^2\theta_2\cosh^2(\lambda\hbar) - \sin^2\theta_1\sin^2\theta_2\sinh^2(\lambda\hbar).
\end{aligned}
$$

Acknowledgement. The work is supported by Chinese NSF grants 11731001, the special foundation for Junwu and Guangxi Ba Gui Scholars.

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