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Entanglement Entropy in Causal Set Theory

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Abstract: Entanglement entropy is now widely accepted as having deep connections with quantum gravity. It is therefore desirable to understand it in the context of causal sets, especially since they provide in a natural manner the UV cutoff needed to render entanglement entropy finite. Defining entropy in a causal set is not straightforward because the type of canonical hypersurface-data on which definitions of entanglement typically rely is not available in a causal set. Instead, we will appeal to a more global expression given in [1] which, for a gaussian scalar field, expresses the entropy of a spacetime region in terms of the field's correlation function within that region. Carrying this formula over to the causal set, one obtains an entanglement entropy which is both finite and of a Lorentz invariant nature. Herein we evaluate this entropy for causal sets of 1+1 dimensions, and specifically for order-intervals ("causal diamonds") within the causal set, finding in the first instance an entropy that obeys a (spacetime) volume law instead of the expected (spatial) area law. We find, however, that one can obtain an area law by truncating the eigenvalues of a certain "Pauli-Jordan" operator that enters into the entropy formula. In connection with these results, we also study the "entropy of coarse-graining" generated by thinning out the causal set, and we compare it with what one obtains by similarly thinning out a chain of harmonic oscillators, finding the same, "universal" behaviour in both cases.

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1 Introduction

Entanglement entropy is widely believed to be an important clue to a better understanding of quantum gravity. Beginning with the original proposal that black hole entropy may be entanglement entropy in whole or in part [2], and continuing through the current surge of interest excited by Van Raamsdonk's ideas on deriving the spacetime metric from quantum entanglement [3], evidence has been accumulating that entanglement entropy has the potential to unveil some of the mysteries surrounding the interplay between the Lorentzian kinematics of general relativity and the interference-laden dynamics of quantum theory.

Despite this history, it is only recently that a workable definition of entanglement entropy has been formulated for causal sets [1]. While it turns out that a naive application of this definition leads to a counter-intuitive spacetime-volume scaling (as opposed to an entropy which scales as the spatial area in the limit of small discreteness length), we will show below how to obtain the anticipated area law by means of a suitable truncation scheme. We will also put forward an intuitive explanation of how the pre-truncation volume-scaling arises, and of why it should be regarded as spurious from the point of view of the continuum.

Ordinarily, entropy is defined by the formula

$$S = \mathrm{Tr}\rho \ln \rho^{-1}, \tag{1.1}$$

where ρ is a density matrix evaluated on a Cauchy hypersurface Σ. If Σ is divided into two complementary subregions A and B, then the reduced density matrix for subregion A is

$$\rho_A = \mathrm{Tr}_B \rho. \tag{1.2}$$

Substituting (1.2) back into (1.1), we get the entropy associated to region A as

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A \ , \tag{1.3}$$

which can be designated as the entanglement entropy between regions A and B if the original density matrix ρ was pure. (We would of course get exactly the same answer if we instead traced over the degrees of freedom of A and computed S_B.)

This definition of entanglement entropy does not work for a causal set, because we lack in that setting a notion of data on a hypersurface. (Even if this difficulty could be ignored, a hypersurface-based definition of entanglement entropy would be questionable. Essential to getting a finite entanglement entropy is a UV cutoff, and a cutoff referred to a spacelike surface has no reason to be covariant. Two partial Cauchy surfaces sharing the same boundary would then have no reason to carry equal entanglement entropies even if their domains of dependence were the same.) Fortunately, however, there exists a more covariant definition of entanglement entropy which is formally equivalent to (1.3) in a globally hyperbolic spacetime, and which does make sense for a causal set. [1] So far, this definition has been developed for the theory of a gaussian scalar field (also called a free scalar field in a quasi-free state).[1] We review this definition next.

2 The Entropy of a Gaussian Field

Let us review the definition of entanglement entropy in [1]. For a more detailed review, we refer the reader to [4], and for the full derivation to [1]. The goal is to express S directly in terms of the field correlators. Consider first a single degree of freedom. We introduce the Wightman and Pauli-Jordan matrices,

$$W = \begin{pmatrix} \langle qq \rangle & \langle qp \rangle \\ \langle pq \rangle & \langle pp \rangle \end{pmatrix} \tag{2.1}$$

[1]In a Gaussian theory, Wick's rule obtains, and the two-point function determines the theory in full.

and

$$i\Delta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ . \tag{2.2}$$

The matrix W corresponds in the field theory to $W(x, x') = \langle 0|\phi(x)\phi(x')|0\rangle$, while Δ gives the imaginary part of W and corresponds to the commutator function defined by $i\Delta(x, x') = [\phi(x), \phi(x')]$. Once we have these, we can express the entropy as a sum over the solutions λ of the generalized eigenvalue problem:

$$W\, v = i\lambda\, \Delta\, v \ , \qquad\qquad \Delta\, v \neq 0 \ , \tag{2.3}$$

where the arguments of W and Δ are restricted to the region in question. The final expression for the entropy of the region is

$$S = \sum_{\lambda} \lambda \, \ln |\lambda| \ . \tag{2.4}$$

In certain cases where we restrict W and Δ to subregions within a larger region or within an entire spacetime or causal set, (2.4) can be interpreted as an entanglement entropy. In [4], (2.4) was applied to some examples in flat two-dimensional spacetimes, and the conventional results for the scaling of the entropy with the UV cutoff were found. In the next section we will apply (2.4) to the causal set counterparts of the examples treated in [4]. In Section 4, we will apply it to obtain an entropy of coarse-graining for the specific examples of a causal set and a chain of harmonic oscillators.

3 Causal Set Entanglement Entropy

Causal set theory [5] is an approach to quantum gravity where the deep structure of spacetime is discrete. A causal set is a locally finite partially ordered set. Its elements are the "spacetime atoms", and its defining order-relation is to be interpreted as the relation of causal or temporal precedence between elements. In the continuum the causal order and the spacetime volume are enough to recover geometry.[2] An important feature of the theory is that in contrast to regular lattices, causal sets are effectively Lorentz invariant. For introductions to causal set theory, we refer the reader to [6, 7].

Consider now a free gaussian scalar field living on a causal set which is well approximated by a so-called causal diamond in a 2d Minkowski spacetime. Using the spacetime definition of entropy which was reviewed in the previous section, let us compute the entropy associated to a smaller causal-set causal diamond nested within the larger one. Our setup is shown in Figure 1, and the entropy we will compute can

[2] In the discrete case, the volume of a region simply counts the number of elements in that region.

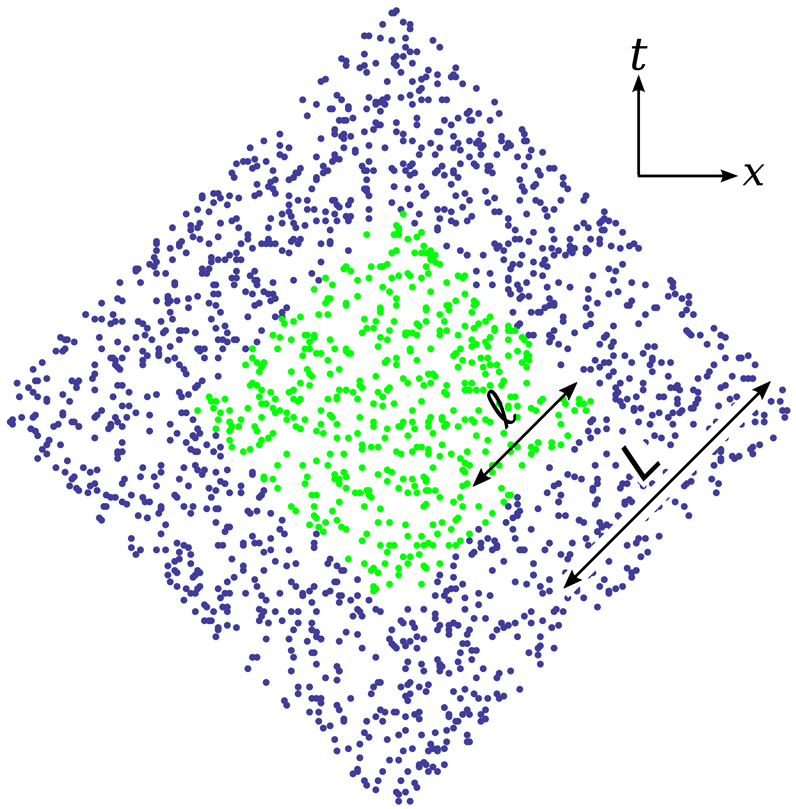

Figure 1. Causal sets of two causal diamonds.

be interpreted as that of the entanglement between the small region and its "causal complement". In less global terms, it is the entanglement entropy between the "equator" of the smaller region, and its complement within the Cauchy surface produced by extending this equator to the larger region.

In the larger diamond, we use the W of the Sorkin-Johnston vacuum [8–10], W_{SJ}, which is the positive part of the operator $i\Delta$, where

$$\Delta(x, y) = G_R(x, y) - G_R(y, x), \tag{3.1}$$

$G_R(x, y)$ being the retarded Green function. For a (free) massless scalar field, G_R is simply related to the causal matrix: $G_R = \frac{1}{2}C$, where C is the causal matrix,

$$C_{xy} := \begin{cases} 1, & \text{if } x \prec y \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

In solving (2.4), we restrict W and Δ to elements within the smaller diamond in Figure 1, keeping only the submatrices W_{xy} and Δ_{xy} such that x and y are in the smaller diamond. In order to assess how the entropy scales with the UV cutoff, we hold the ratio of the sizes of the diamonds fixed and vary the number of elements sprinkled into them. Then the UV cutoff (given by the discreteness length-scale, which is in this case square root of the density of elements) is proportional to $\sqrt{N}$ where N is the number of the causet elements. The UV cutoff is of course proportional to the square root of the number of elements in both the larger and the smaller diamond; we will use the number of elements in the smaller diamond, N_ℓ, to express it.

We find, via numerical simulations, that the entanglement entropy grows linearly with the number of elements in the smaller diamond, thus obeying a spacetime-volume law[3]! The expectation, of course, was that (in $1+1$D) the entropy would scale logarithmically with the UV cutoff (which would mean logarithmic scaling with $\sqrt{N}$ and therefore with N itself), as in the continuum theory [4, 11]. Furthermore, we find that the entropy in the causet is larger in magnitude (values of order 100) in comparison with the results in the continuum (order 1 values). Two examples of this linear scaling are shown in Figures 2 and 3, for $\ell/L = 1/4$ and $\ell/L = 1/2$, respectively. The results fit $S = aN + b$ with $a = 0.46$ and $b = -3.20$ for $\ell/L = 1/4$, and $a = 0.32$ and $b = -6.64$ for $\ell/L = 1/2$.

We also find that this spacetime-volume law persists for the massive theory, in $3+1$ dimensions, and when working with nonlocal Green functions such as that obtained from the G_R resulting from inverting the d'Alembertian defined in [12]. This suggests that it is a generic feature of the direct application of (2.4) to causal sets.

Whence comes this "extra" entropy? The spectrum of $-i\Delta^{-1}W$ on a causal set necessarily has the same form as in the continuum, in that its eigenvalues come in pairs of λ and $1-\lambda$. However, many more of these pairs contribute to the entropy than in the analogous continuum calculation. A closer look at the spectrum of $i\Delta$ reveals how this happens. In the definition (2.4) it is crucial that we exclude functions in the kernel of $i\Delta$, for which λ would not even be defined. (Doing this also ensures that we have enough constraints to enforce the equations of motion, so that only linearly independent degrees of freedom remain.) While excluding the kernel is a simple task for the continuum $i\Delta$, its meaning is not so straightforward for the causal set $i\Delta$. In the continuum, the number of "zero-modes" of $i\Delta$ is huge, but in the causet it is much smaller. Instead of strict zeroes one finds many small but finite eigenvalues that have no counterpart in the spectrum of the continuum $i\Delta$. Even though these eigenvalues are very small, they can contribute a large amount of entropy due to their being so numerous and to the inversion of Δ in $-i\Delta^{-1}W$.

[3]Notice that not only is this not an area law, but it is not even a spatial volume law. A spatial volume law would mean linear growth with $\sqrt{N}$, whereas the scaling that we obtain is linear in N.

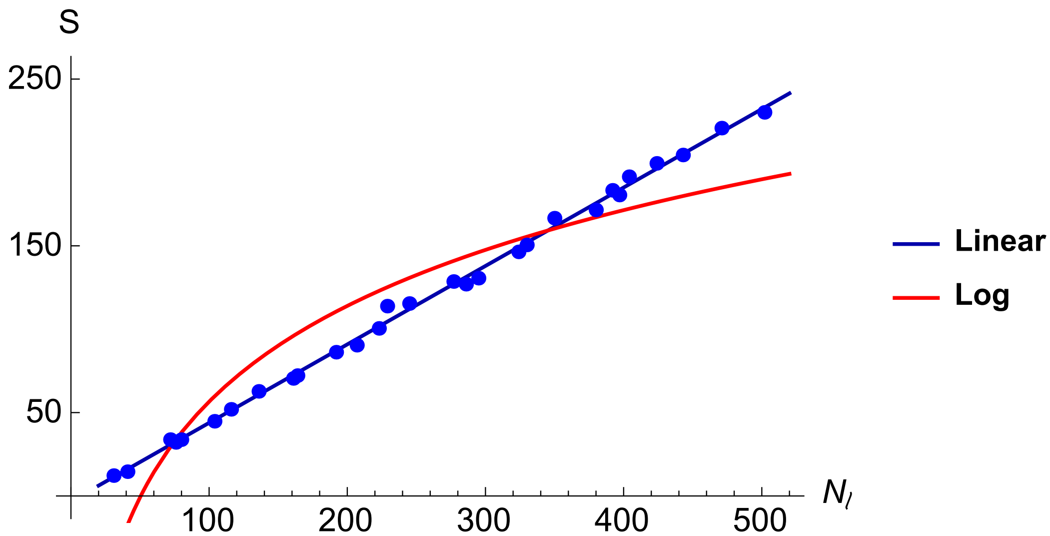

Figure 2. S vs N_ℓ when $\ell/L = 1/4$, along with best fits for linear and logarithmic functions. N_ℓ is the number of causet elements in the smaller diamond.

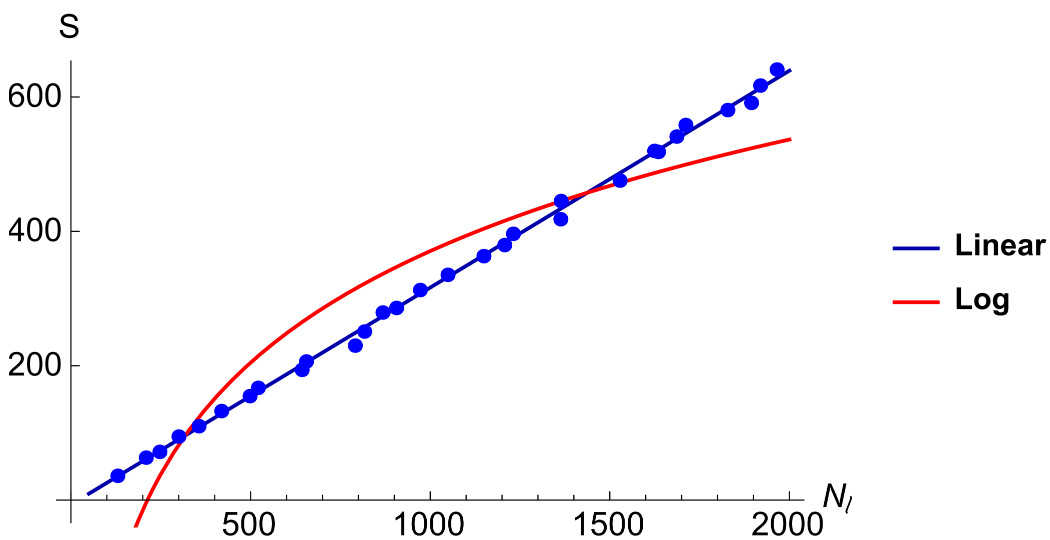

Figure 3. S vs N_ℓ when $\ell/L = 1/2$, along with best fits for linear and logarithmic functions. N_ℓ is the number of causet elements in the smaller diamond.

This observation leads to the idea that (as suggested to us by Siavash Aslanbeigi) these "almost zero-modes" of $i\Delta$ might be the source of the discrepancy[4] between causet

[4]We say "discrepancy" and not "error" since we don't wish to take a position on which, if either,

and continuum, and that they should be excluded from the entropy calculation if one aims at agreement with the continuum. If we start removing the smallest eigenvalues $\tilde{\lambda}$ of $i\Delta$, the scaling of the entropy with the cutoff indeed becomes logarithmic.[5] If the magnitude of the smallest eigenvalue we keep is approximately $\tilde{\lambda}_{min} \sim \sqrt{N}/4\pi$, then we get not only the expected scaling-law but also the expected coefficient 1/3 [13].

An example of the logarithmic shape of the data points after the truncation of $i\Delta$ is shown in Figure 4 for $\ell/L = 1/2$. In Figure 4, the spectrum of $i\Delta$ has been truncated such that $\tilde{\lambda}_{min} \sim \sqrt{N_L}/4\pi$ in the larger diamond and $\tilde{\lambda}_{min} \sim \sqrt{N_\ell}/4\pi$ when the restriction is made to the smaller diamond. (We first truncate both Δ and W in the larger diamond (Δ being the antisymmetric part of W). We then restrict both matrices to the smaller diamond. Call these restricted matrices W^R and Δ^R. We then do a second truncation on them, based on the spectrum of Δ^R.) The best fit to $S = a\ln(x) + b$, with x being $\sqrt{N_\ell}/4\pi$ in the smaller diamond yielded $a = 0.334$ and $b = 1.901$. It is worth emphasizing that the truncation has to be done both in the larger diamond and in the smaller diamond.

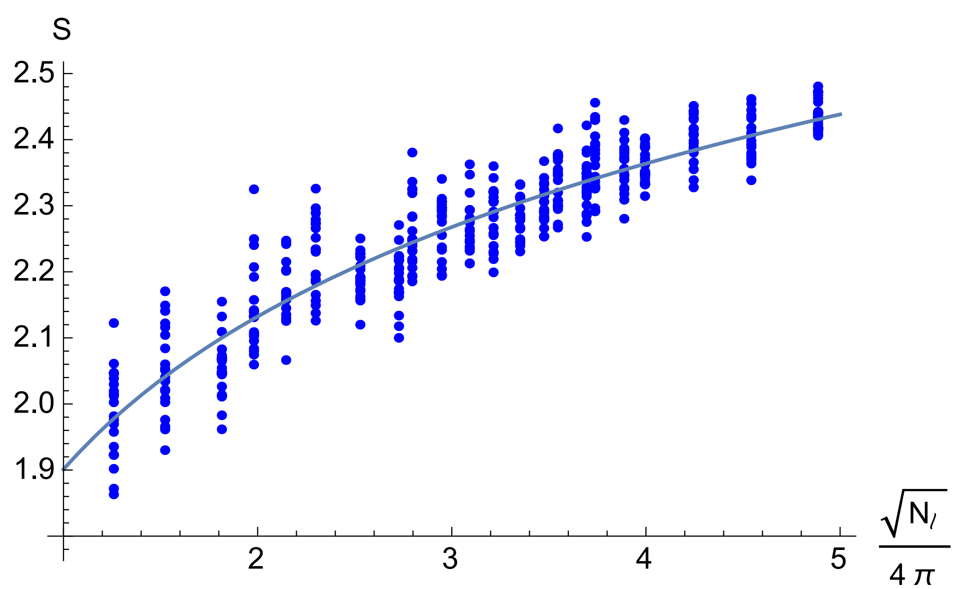

Figure 4. S vs. $\sqrt{N_\ell}/4\pi$, after the spectrum of $i\Delta$ has been truncated such that $\tilde{\lambda}_{min} \sim \sqrt{N_L}/4\pi$ in the larger diamond and $\tilde{\lambda}_{min} \sim \sqrt{N_\ell}/4\pi$ in the smaller diamond.

With hindsight we can understand why the magnitude of the smallest eigenvalue

of the two entropies is the "correct" one.

[5]We use $\tilde{\lambda}$ to refer to the spectrum of $i\Delta$, to avoid confusion with λ which are the eigenvalues of $-i\Delta^{-1}W$ that go into (2.4).

has to be $\sim \sqrt{N}/4\pi$ for consistency with the continuum results. The spectrum of $i\Delta$ in the continuum has dimensions of area, while its spectrum in the causal set is dimensionless. This dimensional observation, together with a comparison of the largest eigenvalues of $i\Delta$ between continuum and causal set, shows that the two spectra can be related by a density factor: $\tilde{\lambda}^{cs} = \rho\tilde{\lambda}^{cont}$, where $\rho = N_\ell/4\ell^2$. Converting our $\tilde{\lambda}^{cs}_{min}$ to a $\tilde{\lambda}^{cont}_{min}$, we find

$$\begin{aligned} \tilde{\lambda}^{cont}_{min} &= \tilde{\lambda}^{cs}_{min}/\rho \\ &= \sqrt{N}/4\pi\rho \\ &\sim \frac{\ell^2}{\pi\sqrt{N}}, \end{aligned} \tag{3.3}$$

This is precisely the minimum eigenvalue which we retained in the continuum, after imposing our cutoff on the wavelength of the eigenmodes of $i\Delta$. This is reviewed in Appendix A. Eigenvalues smaller than $\tilde{\lambda}^{cs}_{min}$ thus correspond to solutions beyond the cutoff, and are the ones we wish to exclude.

Another way to think of where the $\sqrt{N}/4\pi$ comes from is the following. On one hand, the causet provides a fundamental length given (in 2d) by $\rho^{-1/2}$, and in this sense it serves as a "low pass filter" in relation to the continuum. On the other hand, in the continuum we know exactly the relation between wavelength and eigenvalue for eigenfunctions of Δ in a causal diamond. If by means of this relation, we convert a cutoff at wavelength $\rho^{-1/2}$ into a cutoff on the spectrum of Δ, we obtain the truncation rule stated above. To the extent, then, that the asymptotic form of the wavelength-eigenvalue relation is asymptotically universal (as one might expect it to be), one would expect to use the same eigenvalue-cutoff, not just for a causal diamond (order-interval), but for a spacetime region of any shape.

Truncating the spectrum of $i\Delta$ in the causal set by requiring its smallest eigenvalue to be $\tilde{\lambda}_{min} \sim \sqrt{N}/4\pi$ reduces the size of the spectrum from $\sim N$ to $\sim \sqrt{N}$. Thus, a large number of these approximate kernel-modes need to be eliminated if one wishes to to recover an area law.

Figure 5 compares the positive spectrum of $i\Delta$ in the causal set with that in the continuum, using a log-log plot. The causal set comprises 200 elements sprinkled with a density of 50. The red dots are the continuum eigenvalues, the blue dots those of the causet appropriately rescaled for the comparison, and the green dashed line is at $\tilde{\lambda}^{cs} = \sqrt{N}/4\pi$, where we would expect the causet spectrum to end if it were to agree with the continuum. As one sees, the eigenvalues above this line are in good agreement between causet and continuum, but in very poor agreement below it. In particular, there is a clear "break" in the causet spectrum just where the truncation has to be done. Evidently, this spectral feature could also be used as a guide for where to apply the truncation.

In general, then, one can expect to recover continuum-cum-cutoff behaviour from the causal set by modifying the condition $i\Delta\, v \neq 0$ in (2.3) to $i\Delta\, v \neq \tilde{\lambda}_0 v$ when $\tilde{\lambda}_0 < \sqrt{N}/4\pi$.

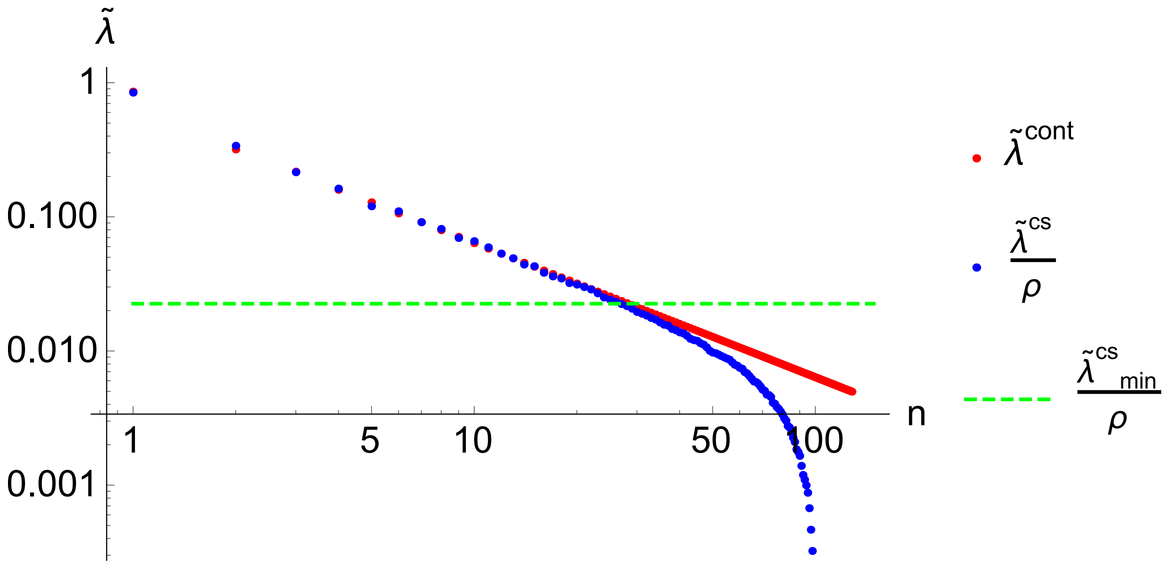

Figure 5. Comparison of the spectrum of $i\Delta$ in the continuum and causal set. The causal set has 200 elements and a density of 50. The green dashed line is where $\tilde{\lambda}^{cs} = \sqrt{N}/4\pi$.

4 Entropy of Coarse-Graining

In this section we study the entropy of coarse-graining by decimation and blocking. We study decimation in both causal sets and a chain of harmonic oscillators, and blocking in a chain of harmonic oscillators. There is no known way to coarse-grain by blocking in a causal set. We use (2.4) for the causal set calculation, and the formalism of [14] for the oscillator calculations.

The Lagrangian for the chain of oscillators we consider is

$$\mathcal{L} = \frac{1}{2}\left(\sum_{N=1}^{N_{max}} \dot{\hat{q}}_N^2 - \sum_{N,M=1}^{N_{max}} V_{MN}\, \hat{q}_N\, \hat{q}_M\right) = \frac{1}{2}\sum_{N=1}^{N_{max}} [\dot{\hat{q}}_N^2 - m^2\hat{q}_N^2 - k(\hat{q}_{N+1} - \hat{q}_N)^2], \quad (4.1)$$

where k is the coupling strength between the oscillators, and in terms of the spatial UV cutoff a, $k = 1/a^2$ [15]. We set $k = 10^6$. We consider the massless theory with periodic boundary conditions and mass regulator[6] $m^2 = 10^{-6}$.

In coarse-graining by decimation, we iteratively remove 10% of the causet elements and oscillators. In the causet we remove each element with probability 0.1, and in the chain of oscillators we remove each tenth oscillator. A simple relation is obtained in both cases. The entropy depends quadratically on the number of degrees of freedom (DoF's) remaining after coarse-graining. Initially, when all DoF's are present, the entropy is 0. It rises and reaches a maximum when about half of the DoF's remain, after which it drops, symmetrically, until it reaches near 0 again around when there are no more DoF's left.

The causal set result without truncating $i\Delta$ is shown in Figure 6, where the entropy is plotted versus the number of elements remaining in the causal diamond. Initially the diamond contained 4048 sprinkled elements. The results fit $S = aN^2 + bN + c$ with $a = -0.00014$, $b = 0.56$, and $c = 41.06$.

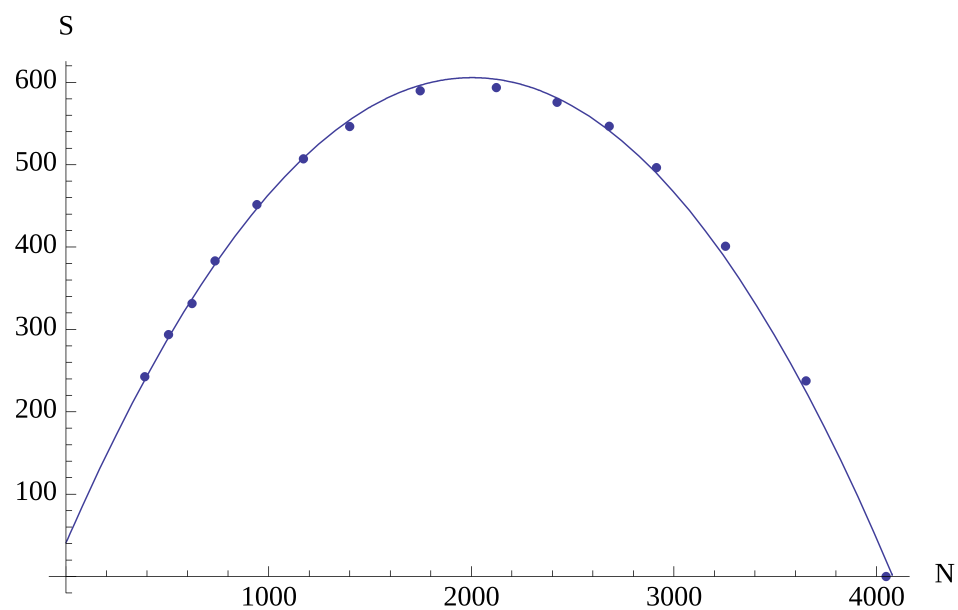

Figure 6. S vs. N in a causet under coarse-graining (without truncating $i\Delta$) by decimation: we remove elements with probability 0.1.

The causal set result with truncated $i\Delta$ is shown in Figure 7, where the entropy is plotted versus the square root of the number of elements remaining in the causal

[6] A mass regulator is introduced since the $m = 0$ theory is IR divergent [16].

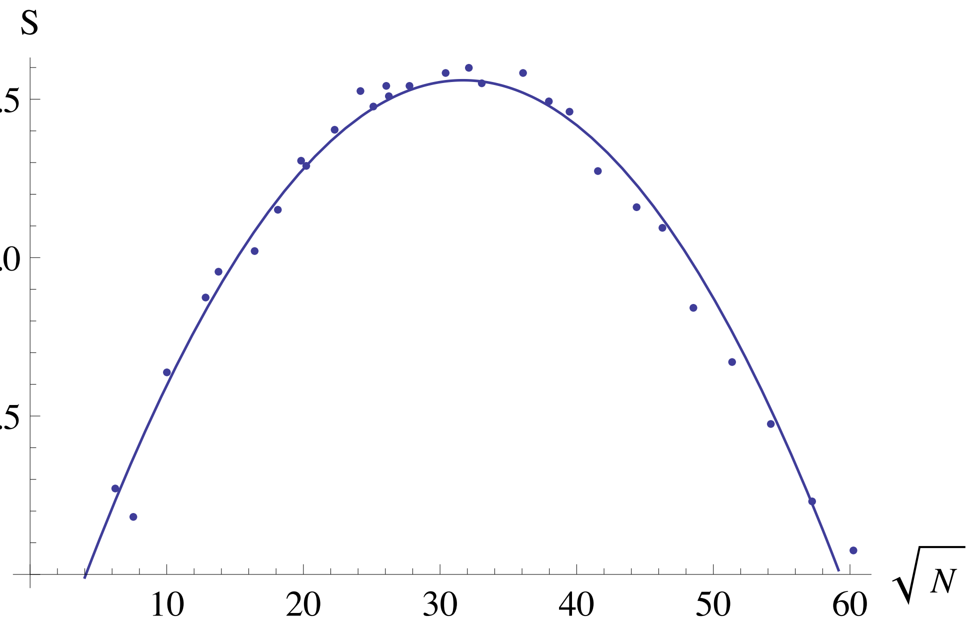

Figure 7. S vs. $\sqrt{N}$ in a causet under coarse-graining (with truncated $i\Delta$) by decimation: we remove elements with probability 0.1.

diamond. Initially the diamond contained 4048 sprinkled elements. The results fit $S = aN + b\sqrt{N} + c$ with $a = -0.0020$, $b = 0.13$, and $c = -0.50$.

It should be noted that the DoF's in terms of which we get a parabolic relation for the entropy of course-graining are different for the truncated and full $i\Delta$. For the full $i\Delta$ the DoF's are counted by the number of elements remaining in the diamond, N, and for the truncated $i\Delta$ they are counted by $\sqrt{N}$.

The result for the chain of oscillators is shown in Figure 8, where the entropy is plotted versus the number of oscillators remaining in the chain. Initially the chain contained 500 oscillators. The results fit $S = aN^2 + bN + c$ with $a = -8.8 \times 10^{-4}, b = 0.45, c = 10.82$.

In coarse-graining by blocking, we rewrite the q_i's in terms of $Q_i^{\pm}$'s defined as $Q_1^{\pm} \equiv (q_1 \pm q_2)/2$, $Q_2^{\pm} \equiv (q_3 \pm q_4)/2$, ... We then discard all Q^{-}'s, thus reducing the DoF's by half. In the next iteration we work in terms of $(Q_1^{+} \pm Q_2^{+})/2$, $(Q_3^{+} \pm Q_4^{+})/2$... and repeat. The result for the entropy of coarse-graining by blocking in a chain of oscillators is shown in Figure 9. The entropy is shown versus the number of oscillators remaining in the chain. Initially the chain contained 2^{12} oscillators. The results fit $S = aN^2 + bN + c$ with $a = -2.8 \times 10^{-5}$, $b = 0.14$, and $c = 1.38$.

Thus entropy of coarse-graining by both decimation and blocking have led to a

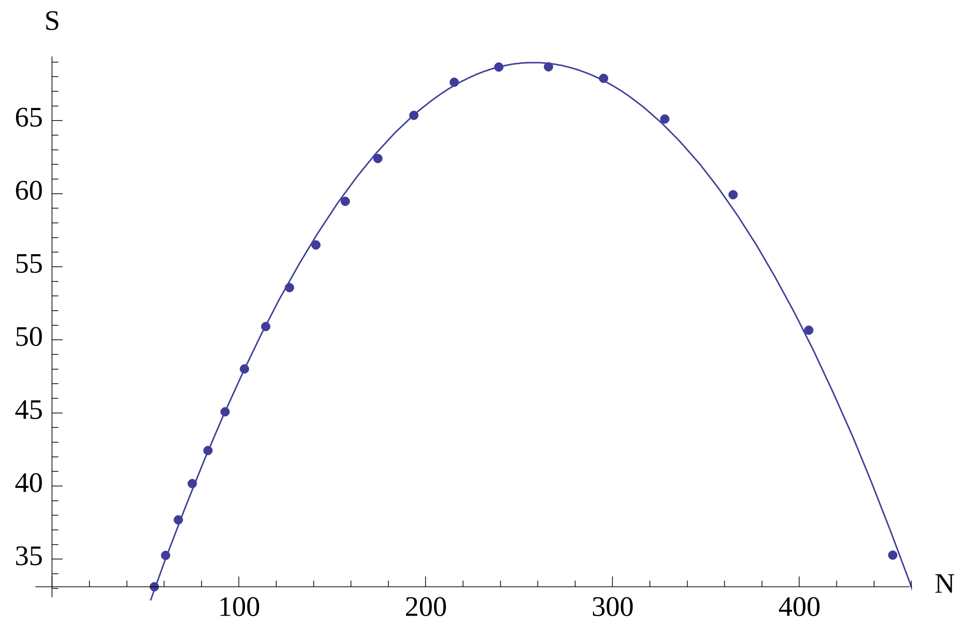

Figure 8. S vs. N in a chain of oscillators under coarse-graining by decimation: we remove elements with probability 0.1.

parabolic dependence on the number of remaining DoF's, in our examples. Our results suggest that this entropy of coarse-graining might have universal properties that would be interesting to investigate further. We frequently deal with coarse-grained versions of certain systems, and there seems to be an entropy associated to this coarse-graining which has universal properties that would be useful to understand.

5 Conclusions

In the present paper, we have studied (primarily by computer simulations) the entanglement entropy of a free scalar field in causal sets well approximated by regions of $1+1$D flat spacetime. Initially we found unexpectedly that instead of the conventional spatial area law (logarithmic scaling of entropy with the UV cutoff), a spacetime-volume scaling was obtained. We attributed this difference between the causet and the continuum, to a difference in the near-zero part of the spectrum of $i\Delta$. With this in mind, we identified, in the causet case, a minimum eigenvalue of $i\Delta$ which answers to the fundamental discreteness scale embodied in the causet itself. And we found that when the spectrum of $i\Delta$ was truncated there, the continuum area law was recovered.

With these findings, we are beginning to understand entanglement entropy in causal set theory. This is important for causal sets, of course, but it also demonstrates an

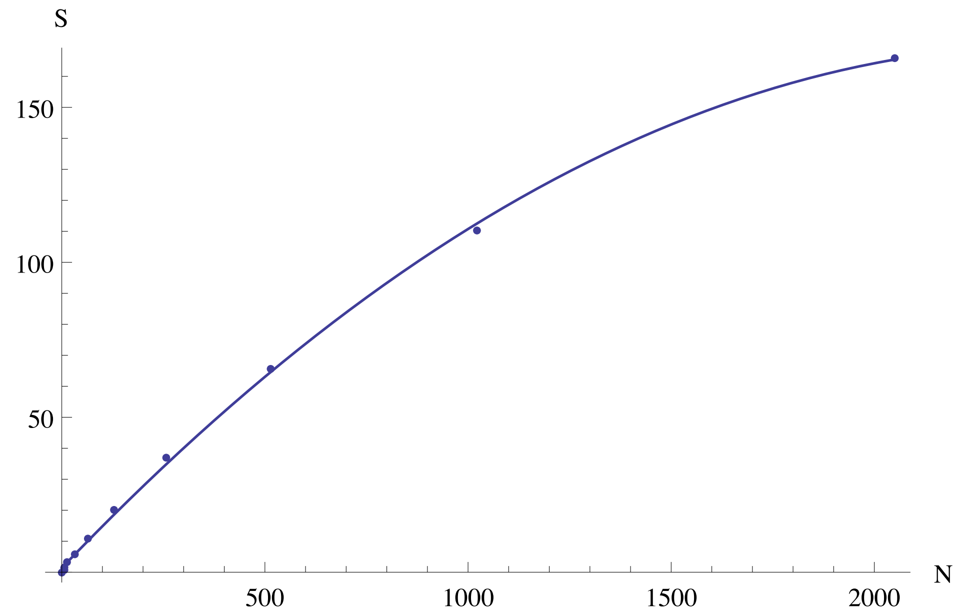

Figure 9. S vs. N in a chain of oscillators under coarse-graining by blocking.

important point of principle, namely that the UV cutoff needed to render entanglement entropy finite can be introduced without doing violence to Lorentz symmetry. The way now seems open to begin to address questions which hinge on understanding the entropy of entanglement associated with black hole horizons, ultimately the question whether most or all of the horizon entropy can be traced to entanglement of one sort or another.

The methods we have used in our simulations could also prove valuable in a continuum context, as they illustrate how simulating entanglement entropy via sprinkled causal sets can expedite calculations which would otherwise be more tedious.

6 Acknowledgements

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A Entanglement Entropy in Continuum Diamonds

A.1 Entanglement Entropy

In this Appendix, we review the main results of [10].

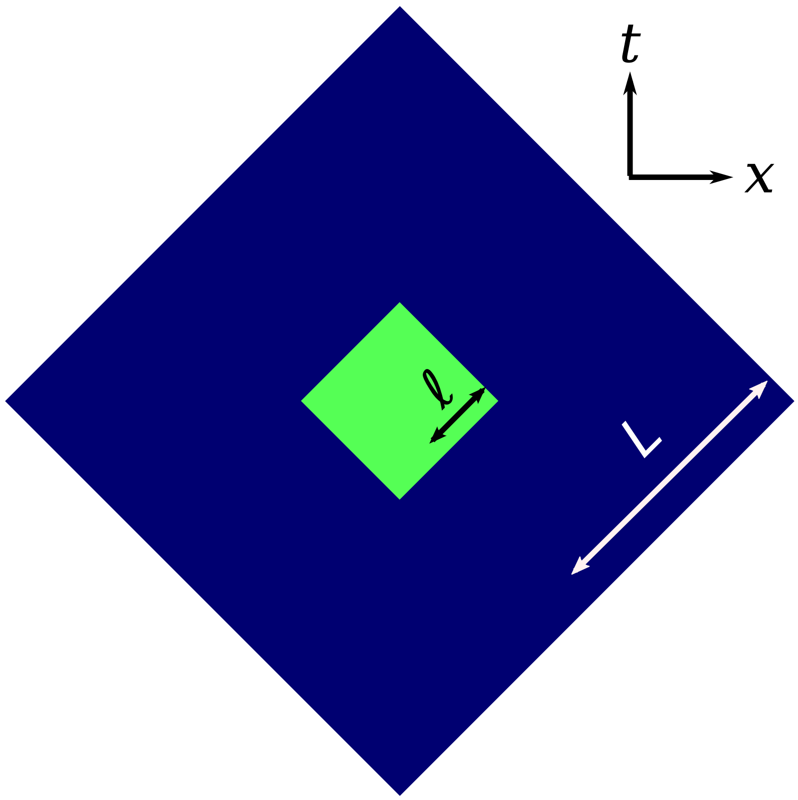

Figure 10. Two causal diamonds.

We wish to use (2.4) to compute the entanglement entropy of a scalar field, resulting from restricting it to a smaller causal diamond within a larger one in $1+1$D Minkowski spacetime. The setup is the continuum analogue of Figure 1, shown in Figure 10. In Minkowski lightcone coordinates $u = \frac{t+x}{\sqrt{2}}$ and $v = \frac{t-x}{\sqrt{2}}$,

$$\Delta(u, v; u', v') = \frac{-1}{2}[\theta(u - u') + \theta(v - v') - 1], \tag{A.1}$$

and

$$W = -\frac{1}{4\pi}\ln|\Delta u \Delta v| - \frac{i}{4}\text{sgn}(\Delta u + \Delta v)\theta(\Delta u \Delta v) - \frac{1}{2\pi}\ln\frac{\pi}{4L} + \epsilon + \mathcal{O}\left(\frac{\delta}{L}\right), \tag{A.2}$$

where $\epsilon \approx -0.063$ when $\ell \ll L$, and δ collectively denotes the coordinate differences $u-u', v-v', u-v', v-u'$. We set $\frac{\ell}{L} = .01$.

We completely understand the properties of Δ and W in this spacetime [10].

We represent W and Δ as matrices in the eigenbasis of $i\Delta$ which consists of two sets of eigenfunctions:

$$\begin{aligned} f_k(u,v) &:= e^{-iku} - e^{-ikv}, && \text{with } k = \frac{n\pi}{\ell},\ n = \pm 1, \pm 2, \dots \\ g_k(u,v) &:= e^{-iku} + e^{-ikv} - 2\cos(k\ell), && \text{with } k \in \mathcal{K}, \end{aligned} \tag{A.3}$$

where $\mathcal{K} = \{k \in \mathbb{R} \,|\, \tan(k\ell) = 2k\ell \text{ and } k \neq 0\}$.

The eigenvalues are $\lambda_k = \ell/k$, and the L^2-norms are $||f_k||^2 = 8\ell^2$ and $||g_k||^2 = 8\ell^2 - 16\ell^2\cos^2(k\ell)$.

For the representation of W, we computed $\langle f_k|W|f_{k'}\rangle$ and $\langle g_k|W|g_{k'}\rangle$. The terms $\langle f_k|W|g_{k'}\rangle$ vanish, making W block diagonal in this basis.

The matrices representing W and Δ are truncated to retain only a finite number of eigenfunctions f_k and g_k up to a maximum value $k = k_{max}$. Initial conditions described by functions with wavelengths greater than $\lambda_{min} \sim 1/k_{max}$, can be expanded in terms of these modes. A natural choice for the cutoff is therefore $1/k_{max}$. In the calculations, we keep ℓ/L fixed and vary k_{max}. The eigenvalues are all of order one with absolute value less than 3. All but a handful of the eigenvalue-pairs have values close to 1 and 0.

The obtained values of S, shown in Figure 11, are fit almost perfectly by the curve

$$S = b \ln\left[\frac{\ell}{a}\right] + c \tag{A.4}$$

with $b = 0.33277$ and $c = 0.70782$.

A.2 Renyi Entropies

We can extend the results of [4] to include Renyi Entropies. The spacetime definition of entropy given in [1] can be generalized for Renyi entropies of order n, $S^{(n)}$, in the following way:

$$S^{(n)} = \sum_{\lambda} \frac{-1}{1-n} \ln(\lambda^n - (\lambda-1)^n) \tag{A.5}$$

where λ and $1-\lambda$ are solutions to the generalized eigenvalue problem (2.4). The spacetime we apply this formula to is again Figure 10. The expected result [13] is that the entropies should take the form:

$$S^{(n)} = \frac{1}{6}(1+\frac{1}{n}) \ln(\frac{\ell}{a}) + c_n \tag{A.6}$$

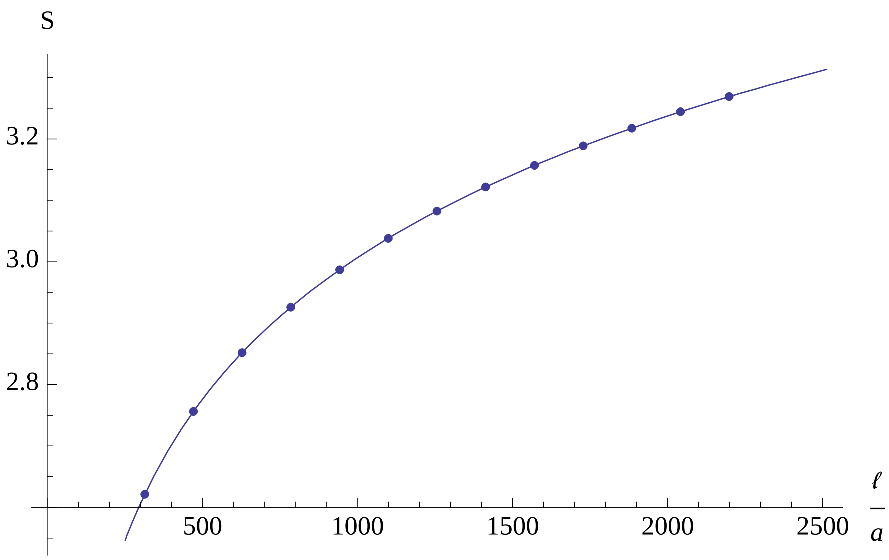

Figure 11. Data points represent calculated values of $S = \sum \lambda \ln |\lambda|$ in the continuum causal diamonds of Figure 10.

where c_n are non-universal constants.

Figures 12-13 show the results from (A.5) for $S^{(2)}$ and $S^{(3)}$. There is good agreement between them and (A.6), with more deviation present for the higher order Renyi entropies. The scaling coefficients found from (A.5) for $S^{(2)}$ to $S^{(10)}$ are:

$$\{0.24961, 0.221498, 0.206892, 0.197726, 0.191411, 0.18682, 0.183354, 0.18066, 0.178517\}, \tag{A.7}$$

to be compared with those from (A.6):

$$\{0.25, 0.222222, 0.208333, 0.2, 0.194444, 0.190476, 0.1875, 0.185185, 0.183333\}. \tag{A.8}$$

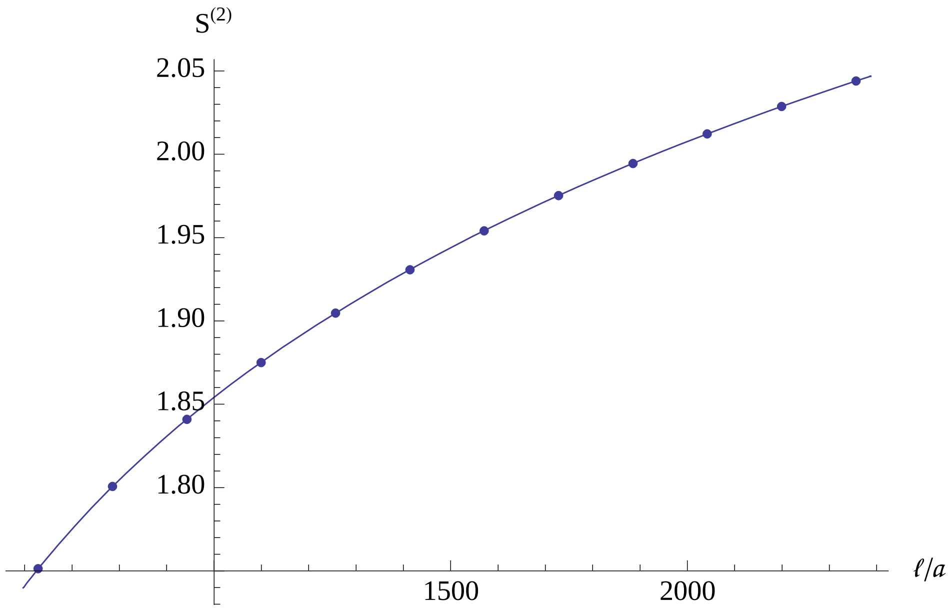

Figure 12. 2nd order Renyi entropy $S^{(2)}$ vs. ℓ/a.

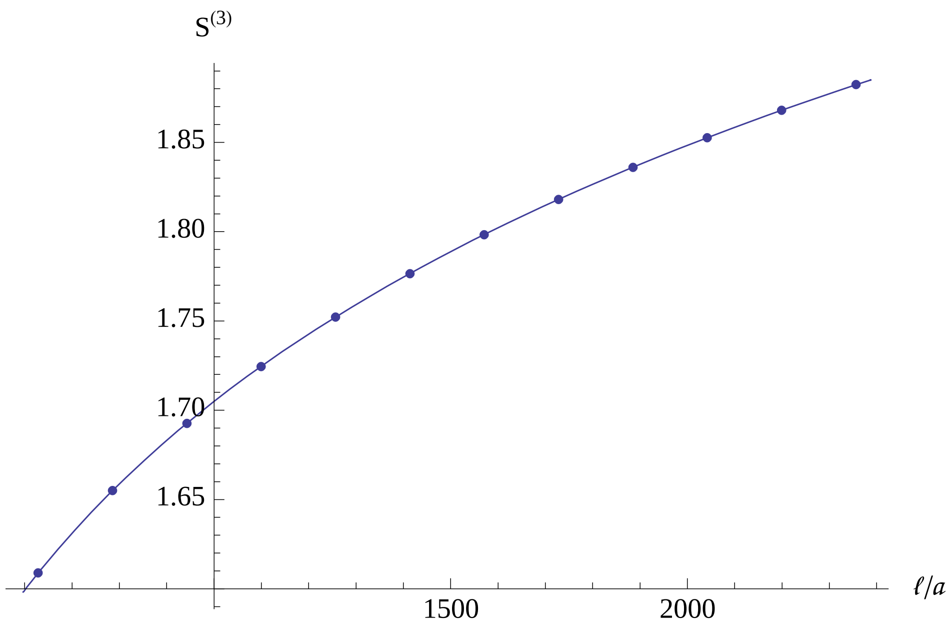

Figure 13. 3rd order Renyi entropy $S^{(3)}$ vs. ℓ/a.

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