The Einstein-Hilbert Action for Entropically Dominant Causal Sets

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Abstract

In the path integral formulation of causal set quantum gravity, the quantum partition function is a phase-weighted sum over locally finite partially ordered sets, which are viewed as discrete quantum spacetimes. It is known, however, that the number of "layered" sets—a class of causal sets that look nothing like spacetime manifolds—grows superexponentially with the cardinality n, giving an entropic contribution that can potentially dominate that of the action. We show here that in any dimension, the discrete Einstein-Hilbert action for a typical K-layered causal set reduces to the simple link action to leading order in n. Combined with earlier work, this completes the proof that the layered sets, although entropically dominant, are very strongly suppressed in the path sum of causal set quantum gravity whenever the discreteness scale is greater than or equal to a (mildly dimension-dependent) order one multiple of the Planck scale.

1 Introduction

A basic requirement for any viable theory of quantum gravity is that classical spacetime must emerge in the small $\hbar$ regime. In the language of path integrals, this means that the saddle point contributions to the partition function come from classical spacetimes. In the causal set approach to quantum gravity [1], the path integral over continuum spacetimes is replaced by a path sum over causal sets, that is, locally finite partially ordered sets. For a fixed cardinality n, the partition function is the path sum

$$Z(\Omega_n) = \sum_{C\in\Omega_n} \exp\frac{i}{\hbar}S(C), \tag{1}$$

where Ω_n is the space of all n-element causal sets and $S(C)$ is a causal set action. In this discrete setting, continuum spacetimes are replaced by causal sets obtained from the continuum via a random discretisation. The question is whether such "continuumlike" causal sets dominate the causal set partition function in the large n limit.

A potential challenge to the dynamical emergence of the continuum comes from a class of "layered" causal sets whose entropic contribution grows superexponentially with cardinality n [2, 3, 4]. These causal sets do not have a continuum approximation, by which we mean they cannot be obtained as random discretisations of any continuum spacetime. It is therefore important to know whether any reasonable choice of action $S(C)$ is capable of suppressing this entropic contribution in the small $\hbar$ limit.

Nonclassicality in the context of the path integral is not per se surprising. Even for a free particle, the space of paths is dominated by those that are highly nonclassical and only C^0 [5]. As we know from textbook arguments, though, these paths interfere destructively, allowing the classical paths to dominates in the $\hbar \to 0$ limit. Demonstrating such behavior in quantum gravity is of course more challenging, but progress has been made in recent years in causal set theory, where it has been shown that the path integral contributions from two- and three-layered causal sets are suppressed at leading order [6, 7, 8]. Our present work adds a finishing touch, showing that the contribution from all K-layered causal sets with $K \ll n$ is suppressed whenever the discreteness scale is greater than or equal to a (mildly dimension-dependent) order one multiple of the Planck scale.

The combinatorial results on the asymptotic growth in the number of layered partially ordered sets predate causal set theory. As shown in [2, 3, 4], the leading contribution to Ω_n comes from the three-layered Kleitman-Rothschild (KR) sets, with[1] $|\Omega_{KR}| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$ [2]. The next largest contribution comes from the symmetric bilayer sets, with $|\Omega_{\text{bilayer}}| \sim 2^{\frac{n^2}{4}+O(n)}$. This is is the start of a hierarchy of subdominant contributions, the next in line being the four-layer sets, then the five-layer sets, and so on [3, 4]. In general, for $K \ll n$,

$$|\Omega_K| \sim 2^{c(K)n^2+o(n^2)}, \tag{2}$$

where $c(K) \leq \frac{1}{4}$ is a weakly monotonically decreasing function for $K \geq 4$ [4].

It is, of course, possible to concoct an action that would suppress this overwhelming entropic contribution to $Z(\Omega_n)$, but such an action should have a reasonable physical motivation. A natural choice is the Benincasa-Dowker-Glaser (BDG) action [9, 10, 11], the discrete version of the Einstein-Hilbert action in d spacetime dimensions. For a causal set C containing n elements, this action is

$$\frac{1}{\hbar}S^{(d)}_{BDG}(C) = -\alpha_d\left(\frac{\ell}{\ell_p}\right)^{d-2}\left(n + \frac{\beta_d}{\alpha_d}\sum_{J=0}^{n_d} C^{(d)}_J N_J\right), \tag{3}$$

[1] The notation $|\Omega|$ means the cardinality of the set Ω, that is, the number of elements in Ω.

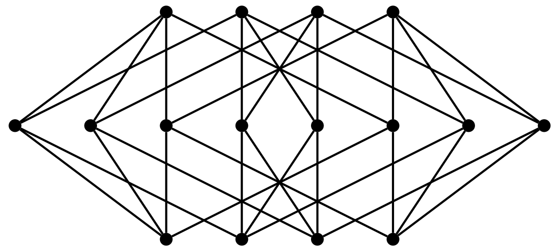
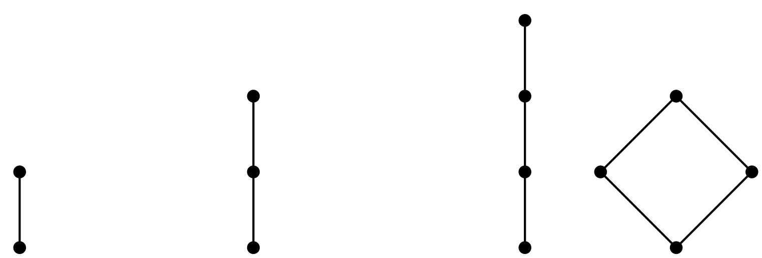


Figure 1: The Hasse diagram of a KR order. The diagram only shows links, that is, nearest neighbors; the remaining relations follow from transitive closure.

where ℓ_p is the Planck length, ℓ is a discreteness scale, and α_d, β_d, and $C_J^{(d)}$ are known dimension-dependent constants.[2] The dependence on the causal set C comes through the interval abundances N_J, which count the number of intervals in C with exactly J elements—that is, the number of pairs $e, e' \in C$ for which the cardinality $|\{e''|e \prec e'' \prec e'\}| = J$. (We adopt the irreflexive convention $e \not\prec e$.) Thus, N_0 counts the number of 0-element intervals or *links* (nearest neighbors), N_1 the number of 1-element intervals (next-to-nearest neighbors), and so on. Fig. 2 shows the first few N_J. For causal sets that are approximated by continuum spacetimes, it has been shown that this discrete action goes over to the Einstein-Hilbert action (up to boundary terms) in the continuum limit [9, 10, 11]. Here, note that



(a) N_0 (b) N_1 (c) N_2

Figure 2: Examples of intervals of size $0, 1$ and 2: N_0 counts the number of type (a) intervals or links in C, N_1 the number of type (b) intervals, and N_2 the number of any one of the two type (c) intervals. In diagram (c) the totally ordered "chain" has exactly one 4-element path and the "diamond" causal set to its right has exactly two 3-element paths.

the spacetime dimension d is not intrinsic to the causal set, but imposed by hand. We will see in what follows that this dependence on d is only marginally relevant to the question at hand.

The first concrete step towards answering the entropy versus action question in causal set theory was taken in [6], where the contribution to $Z(\Omega_n)$ of the class of bilayer causal sets

[2] See [11] for closed form expressions for α_d, β_d and $C_J^{(d)}$, where their i is our $J+1$.

Ω_{bilayer} was analyzed. These are the next subdominant class of partially ordered sets in Ω_n after Ω_{KR}. For bilayer sets, S_{BDG} simplifies drastically, since the only intervals are links, which contribute only to N_0. Thus, $N_J = 0$ for all $J > 0$, and S_{BDG} reduces to the "link action"

$$\frac{1}{\hbar} S^{(d)}_{\text{link}}(C) = -\alpha_d \left(\frac{\ell}{\ell_p}\right)^{d-2} \left(n + \frac{\beta_d}{\alpha_d} N_0\right), \tag{4}$$

where we have used the fact that in Eqn. (3), the coefficient $C_0^{(d)} = 1$ for all d [11]. The maximum number of links in a bilayer set is $\frac{n^2}{4}$ (from $\frac{n}{2}$ elements in layer 1 each linked to $\frac{n}{2}$ elements in layer 2). For a general bilayer set, we can thus write $N_0 = \frac{1}{4} p n^2$, with a "linking fraction" $p \in [0, 1]$. In [6], a bounding argument was used to count the number of causal sets for a given p, and the path integral was approximated as an integral over p and evaluated by the method of steepest descent. The result was that the bilayer contribution to the partition function was superexponentially suppressed whenever[3]

$$\tan\left[\frac{1}{2}\beta_d \left(\frac{\ell}{\ell_p}\right)^{d-2}\right] > \left(\frac{27}{4} e^{-1/2} - 1\right)^{1/2}. \tag{5}$$

This will be satisfied whenever $\ell > \ell_0$, where the lower limit ℓ_0 is mildly dimension dependent but always of order ℓ_p. For $d = 4$, $\ell_0 \approx 1.136\ell_p$, and for general d, $1.13 < (\ell_0/\ell_p) < 2.33$, giving a very conservative constraint on the discreteness scale.

For a $(K > 2)$-layer partially ordered set, however, the N_J need not be zero for $J > 0$. For a similar strategy to work one would have to find the number of causal sets that contribute to each value of the BDG action, a considerably more complicated calculation. As a warm-up, [7] considered the path sum with the BDG action replaced by the simpler link action (4). The link action is determined by the linking fraction p, and it is again possible, to leading order in n, to find the number of causal sets as a function of p. As in the bilayer case, the link action was shown to suppress all K-layer partially ordered sets with $K \ll n$, subject to the same condition (5) on the discreteness scale.

More recently we showed that to leading order, the BDG action for KR orders *is* the link action [8]. Since the arguments of [6, 7] only need the leading order contribution, this suffices to show that KR orders are in fact suppressed by the full BDG action. In this work we extend this analysis to all K-layer causal sets with $K \ll n$. This completes the proof that the entropic contribution from the K-layer causal sets is strongly suppressed by the BDG action in the causal set path sum. Because the link term appears in the BDG action in all dimensions, this suppression is dimension independent.

[3] This makes a small correction to an estimate from previous work [6] where the β_d dependence was mistakenly dropped.

In section 2 we start with some definitions and give a brief review of the $K = 3$ case of [8]. We use a slightly different perspective, which is easier to generalise to $K > 3$. Section 3 contains our main results. We begin by outlining our strategy in section 3.1. In section 3.2 we implement this explicitly for the $K = 4$ case for intervals of size $J = 1, 2, 3$, since this gives us some important general insights into the leading order contributions to N_J. In section 3.3 we prove our main result, namely that for a typical K-level partially ordered set with $K \ll n$, the N_J for $J > 0$ contribute negligibly to action for large n. In section 4 we discuss the implications of our results and some of the remaining open questions.

2 Preliminaries

We start with the basic definition. A causal set C is a locally finite partially ordered set, on elements $\{e_1, e_2 \ldots\}$, where the order relation $\prec$ is taken to be irreflexive, i.e., $e_i \not\prec e_i$. As in any partially ordered set, the order relation is *transitive*—that is, $e_i \prec e_j, e_j \prec e_k \Rightarrow e_i \prec e_k$. The irreflexive condition thus implies that there are no elements $e_i, e_j \in C$ for which both $e_i \prec e_j$ and $e_j \prec e_i$. *Local finiteness* is the statement that $\forall e_i, e_j \in C$ there are at most a finite number of $e_k \in C$ for which $e_i \prec e_k \prec e_j$; this is a discreteness condition.

An *r-element chain* in C is a totally ordered subset $e_{i_1} \prec e_{i_2} \ldots \prec e_{i_r}$ of r elements, and will be said to be of *length* r. Thus, a relation $e_i \prec e_j$ is a 2-element chain and is of length 2. The *height* $h(C)$ of the causal set is the length of the longest chain in C. An *antichain* is a set of unrelated elements, that is, a set A such that $e_i \not\prec e_j$ for any $e_i, e_j \in A$.

For any $e_i \prec e_j$, the *interval* $[e_i, e_j]$ is the set $\{e_k \in C | e_i \prec e_k \prec e_j\}$, examples of which are shown in Fig. 2. An interval is the discrete, causal set analog of a causal diamond, or Alexandrov interval $I(p,q) \equiv I^+(p) \cap I^-(q)$. As in the continuum, $[e_i, e_j]$ is causally convex, i.e., for any pair $e_k, e_r \in e_i \cup e_j \cup [e_i, e_j]$, $[e_k, e_r] \subseteq [e_i, e_j]$. Hence for every $e_i \prec e_j$ we can, in a consistent way, define the *height* $h(e_i, e_j)$ as the length of the longest chain from e_i to e_j, which is also the height of the subcausal set $e_i \cup e_j \cup [e_i, e_j]$. The *size* of an interval $[e_i, e_j]$ is its cardinality $|[e_i, e_j]|$ (our convention being *not* to include the endpoints e_i, e_j in the counting). The quantities N_J associated with a causal set C, which appear in the BDG action Eqn. (3), count the number of J-size or J-element intervals in C.

A pair of elements $e_i, e_j \in C$ is said to be *linked* if $e_i \prec e_j$ and there exists no e_k such that $e_i \prec e_k \prec e_j$. Thus e_i and e_j are "nearest neighbors." We denote this relationship by $e_i \prec\!\ast\, e_j$. Whenever $e_i \prec\!\ast\, e_j$, $|[e_i, e_j]| = 0$, and hence links contribute to the 0-element intervals N_0 in the BDG action Eqn. (3).

All the relations $(e \prec e')$ in C are uniquely determined from the links($\prec\!\ast$) by implementing transitive closure. C can therefore be uniquely represented by its Hasse diagram, as in Fig. 1, or equivalently by the $n \times n$ *link matrix* $\mathcal{L}$ where

$$\mathcal{L}_{ij} = \begin{cases} 1 & e_i \prec\!* \, e_j \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Thus, the total number of links in C is

$$N_0 = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{L}_{ij}. \tag{7}$$

One can extract more information from higher powers of $\mathcal{L}$, as we shall soon see.

Consider an r-element chain where every relation in the chain is a link. We will refer to this as an *r-element path* of length r, i.e., the totally ordered subset $\{e_{i_1}, e_{i_2}, \ldots e_{i_r}\}$ with $e_{i_1} \prec\!* \, e_{i_2} \ldots \prec\!* \, e_{i_r}$. Thus, a linked pair $e_i \prec\!* \, e_j$ is a 2-element path of length 2. Since the $\prec\!*$ relation is a nearest neighbor relation, unlike chains, there can be at most one 2-element path between two elements. This is reflected by the fact that the matrix elements of $\mathcal{L}$ are either 0 or 1.

However, there can be multiple r-element paths from e_i to e_j when $e_i \prec e_j$, for $r > 2$. Consider the squared matrix $\mathcal{L}^2$

$$(\mathcal{L}^2)_{ij} = \sum_{k=1}^{n} \mathcal{L}_{ik} \mathcal{L}_{kj}. \tag{8}$$

The sum picks up a 1 for each e_k for which $e_i \prec\!* \, e_k \prec\!* \, e_j$, and a 0 otherwise. Hence $(\mathcal{L}^2)_{ij}$ is equal to the number of 3-element paths from e_i to e_j. In general, for $r \geq 2$, $(\mathcal{L}^{r-1})_{ij}$ is the number of r-element paths from e_i to e_j. It is important to note that $\mathcal{L}^{r-1}$ does not directly give information about N_J, for $J > 0$. This is illustrated in Fig. 4. However, one can use powers of $\mathcal{L}$ to put bounds on the N_J.

We are now in a position to characterise the class of causal sets of interest to us in this paper.

A causal set $C \in \Omega_n$ is said to be *K-layered* if it can be partitioned into K disjoint antichains $\mathbb{L}_i$, $C = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \ldots \mathbb{L}_K$, such that if $e \prec e'$ with $e \in \mathbb{L}_k$ and $e' \in \mathbb{L}_{k'}$, then $k < k'$. We denote the set of all K-layered partially ordered sets in Ω_n by $\Omega_{n,K} \subset \Omega_n$.

Our goal in this work is to count the number of causal sets in $\Omega_{n,K}$ with the property that $N_J = R$, for some J and some R. We will refer to the probability that a causal set has some property P as determined by the counting measure

$$Pr(C \in \Omega_n \text{ has property } P) = \frac{\text{number of sets in } \Omega_n \text{ with property } P}{\text{total number of sets in } \Omega_n}. \tag{9}$$

We consider a further subdivision of $\Omega_{n,K}$. For $C \in \Omega_{n,K}$, let $\gamma_k n \equiv |\mathbb{L}_k|$, the number of elements in layer k, where, since every element is in exactly one layer, $\sum_{k=1}^{K} \gamma_k = 1$. Define $\Omega_{n,K,\Gamma} \subset \Omega_{n,K}$ to be the set of n-element K-layer causal sets with fixed *filling fraction* $\Gamma_K = (\gamma_1, \gamma_2, \ldots \gamma_K)$. It is convenient to use the ordering of the layers $k = \{1, \ldots K\}$ to label the elements of $C \in \Omega_{n,K,\Gamma}$. We denote the elements of $\mathbb{L}_k$ by $e_i^{(k)}$, where the upper bracketed index labels the layer and the lower index $i \in [1, \gamma_k n]$ distinguishes elements within that layer. The ordering of the layers allows the link matrix to be written in block diagonal form

$$\mathcal{L} = \begin{pmatrix} 0 & \mathcal{L}^{(12)} & \mathcal{L}^{(13)} & \ldots & \mathcal{L}^{(1K)} \\ 0 & 0 & \mathcal{L}^{(23)} & \ldots & \mathcal{L}^{(2K)} \\ 0 & 0 & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & \mathcal{L}^{((K-1)K)} \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{10}$$

where each $\mathcal{L}^{(k_1 k_2)}$ is itself the $\gamma_{k_1} n \times \gamma_{k_2} n$ link matrix

$$\mathcal{L}_{ij}^{(k_1 k_2)} = \begin{cases} 1 & e_i^{(k_1)} \prec\!\ast\, e_j^{(k_2)} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

for the subcausal set $\mathbb{L}_{k_1} \sqcup \mathbb{L}_{k_2}$. Henceforth, we will consider only those labelled causal sets that are consistent with this "layer-induced" labeling.

Let $p^{(kk')}$ denote the probability for an element in $\mathbb{L}_k$ to be linked to one in $\mathbb{L}_{k'}$, $k < k'$. While the $p^{(k(k+1))}$ can all be chosen independently, this may not be possible for $k' > k+1$ because of constraints from transitivity. For example, if $p^{(k(k+1))} = 1$ for all k (i.e., $\mathbb{L}_k$ and $\mathbb{L}_{k+1}$ are maximally linked), then $p^{(kk')} = 0$ for all $k' > k+1$. Conversely, the probability for there to be a link from $e_i^{(k)} \in \mathbb{L}_k$ to $e_j^{(k')} \in \mathbb{L}_{k'}$ with $k' > k+1$, is proportional to the the probability for there to be *no* r-element paths from $e_i^{(k)}$ to $e_j^{(k')}$ for $2 < r \leq k' - (k+1) + 2$. This will be a crucial feature of our construction in what follows.

Since the specific choice of $p^{(kk')}$ will not critically affect our arguments, let us define $\vec{p} = \{p^{(kk')}\}$ as a *consistent* set of probabilities (in the above sense), and let $\Omega_{n,K,\Gamma,\vec{p}}$ correspond to causal sets generated from $\vec{p}$.

The analysis of [2, 3, 4] shows that the leading order contribution to $\Omega_{n,K}$, determined by $c(K)$ in Eqn. (2)), comes from a subclass of *dominant configurations* $\mathcal{Q}_K \subset \Omega_{n,K}$. The causal sets in this subclass have a fixed filling fraction Γ_K as well as a fixed *linking fraction* p between consecutive layers, along with the further constraint that for all $k' > k+1$, $e_i^{(k)} \prec e_j^{(k')}$ for all $e_i^{(k)} \in \mathbb{L}_k, e_j^{(k')} \in \mathbb{L}_{k'}$. The Γ_K and p for these dominant configurations themselves are functions of the number of total relations $\mathbf{R}$. Thus, for every K, the coefficient $c(K)$ in Eqn.

(2) depends not only on K but also on $\mathbf{R}$. Our results apply more generally to causal sets in $\Omega_{n,K}$, and hence also include this dominant subclass of configurations.

We stated in the introduction that causal sets in $\Omega_{n,K}$ are not *continuumlike*. What we mean by this qualification is the following. A causal set C is said to be continuumlike if it can be obtained from a continuum d-dimensional spacetime (M, g) by selecting events from M randomly via a Poisson distribution, at density ρ with respect to the volume element $\sqrt{|g|}$, with causal relations inherited from (M, g). Thus, the probability for there to be n causal set elements in a spacetime volume V is given by

$$P_V(n) = \frac{1}{n!}(\rho V)^n e^{-\rho V}. \tag{12}$$

Randomness combined with local finiteness makes it possible to recover, on average, the continuum spacetime volume of a region from the (finite) number of causal set elements in that region, since $\langle N \rangle = \rho V$. The density ρ determines a characteristic discreteness scale $\ell = \rho^{-\frac{1}{d}}$, where d is the dimension of M. Below ℓ, the causal set contains little information about M. But on scales large compared to ℓ, the dimension and much of the topology and geometry of (M, g) can be reconstructed from the causal set (see [12] for a review.) For any $C \in \Omega_{n,K}$, as long as $n \gg K$, the typical $C \in \Omega_{n,K}$ is *not* continuumlike.

We end with a brief discussion on the distinction between labelled and unlabelled causal sets. A causal set is *labelled* if each element carries a distinguishing label, essentially a discrete coordinate. Labels can simplify calculations, but ultimately they, like coordinates, must be "gauged away." The permutation group S_n on n elements acts on the set of n-element labelled causal sets as a gauge group, with each orbit corresponding to a single unlabelled causal set. A typical orbit has $n!$ causal sets, but there are exceptions. If a labelled causal set has two elements e_i, e_j with identical pasts and futures, then permuting the labels of e_i and e_j $(i \leftrightarrow j)$ leaves the causal set unchanged, so the orbit on which it lies has fewer than $n!$ labelled causal sets. An exact factorization of the gauge group thus becomes complicated.

The worst possible overcounting, however, is by a factor of $n!$, which asymptotically goes as $\sim 2^{n \log_2 n}$. Such a factor is subdominant with respect to the leading order contributions $\sim 2^{c(K)n^2}$ from the K-layer orders. We will therefore ignore the distinction between labelled and unlabelled causal sets in what follows.

2.1 Review of the three-layer ($K = 3$) case

In [8] it was shown for $K = 3$ that while $N_0 \sim n^2$, the $N_{J>0}$ grow at most as n, and thus contribute negligibly to the BDG action at large n. We review this construction briefly here to set the stage for the calculation for $K > 3$.

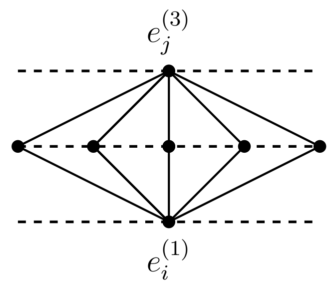


Figure 3: The number of 1-element paths for $K = 3$ from $e_i^{(1)} \in \mathbb{L}_1$ to $e_j^{(3)} \in \mathbb{L}_3$ is also the size of the interval $[e_i^{(1)}, e_j^{(3)}]$.

The method used in [8] depended on properties of the link matrix. For $K = 3$, $\mathcal{L}^{r-1} = 0$ for $r > 3$, i.e., the maximum length of a path is $r = 3$. Moreover, for $e_i^{(1)} \in \mathbb{L}_1$ and $e_j^{(3)} \in \mathbb{L}_3$, $\mathcal{L}^2_{ij}$ not only counts the number of 3-element paths from $e_i^{(1)}$ to $e_j^{(3)}$, it *also* counts the size of the interval $[e_i^{(1)}, e_j^{(3)}]$. This simplifies the $K = 3$ calculation considerably.

While the link matrix can still be used for $K > 3$, the calculation becomes more involved when there are more than three layers. For this reason, we will give a slightly different, although ultimately equivalent, proof for $K = 3$ which we can then generalise to arbitrary K.

Let C be a three-layer causal set, and fix elements $e_i^{(1)} \in \mathbb{L}_1$ and $e_j^{(3)} \in \mathbb{L}_3$. As shown in Fig. 3, the interior of the interval $[e_i^{(1)}, e_j^{(3)}]$ is a set of causally unrelated elements—an antichain—all lying in layer 2. Let us first ask, for a fixed $e_i^{(1)}$ and $e_j^{(3)}$, for the probability $q(J)$ that this antichain has exactly J elements, with $0 \leq J \leq \gamma_2 n$. For a set of J *fixed* elements in layer 2, this probability is clearly

$$(p^{(12)}p^{(23)})^J(1 - p^{(12)}p^{(23)})^{\gamma_2 n - J}$$

where the first factor is the probability that the chosen J elements are included and the second factor is the probability that the remaining $\gamma_2 n - J$ elements are excluded. Since there are $\binom{\gamma_2 n}{J}$ ways of selecting the J elements in layer 2, the total probability that $[e_i^{(1)}, e_j^{(3)}]$ has exactly J elements is

$$q(J) = \binom{\gamma_2 n}{J}(p^{(12)}p^{(23)})^J(1 - p^{(12)}p^{(23)})^{\gamma_2 n - J}. \tag{13}$$

For n large and $J \ll n$,

$$\binom{\gamma_2 n}{J} \sim \frac{(\gamma_2 n)^J}{J!}. \tag{14}$$

Defining $\alpha = 1 - p^{(12)}p^{(23)} < 1$, we have

$$q(J) \sim \frac{(\gamma_2 p^{(12)}p^{(23)})^J}{J!} n^J \alpha^{\gamma_2 n} = A_J n^J 2^{-\gamma_2 n|\log_2 \alpha|}, \tag{15}$$

which isolates the leading order n-dependence. Note that for n large, $q(J)$ is exponentially small; this will be the key cause of suppression of J-element intervals for three-layer sets.

We can now ask for the probability that the number of J-element intervals in the causal set C is exactly $N_J = R_J$. There are $\gamma_1\gamma_3 n^2$ possible pairs $e_i^{(1)} \in \mathbb{L}_1, e_j^{(3)} \in \mathbb{L}_3$, hence by the same reasoning that led to (13),

$$\mathcal{P}(N_J = R_J) = \binom{\gamma_1\gamma_3 n^2}{R_J} q(J)^{R_J}(1-q(J))^{\gamma_1\gamma_3 n^2 - R_J}. \tag{16}$$

Since the asymptotic behavior of $q(J)$ is dominated by the term $\alpha^{\gamma_2 n}$, the factor involving $(1-q(J))$ goes to one at large n. Indeed, $-q-q^2 \leq \ln(1-q) \leq -q$ for $0 < q < \frac{1}{2}$, so for q of the form (15),

$$\ln(1-q)^{bn^2} \sim n^{J+2}\alpha^{\gamma_2 n} \tag{17}$$

which goes to zero for large n. Using the fact that $\binom{\gamma_1\gamma_3 n^2}{R_J} \lesssim 2^{\gamma_1\gamma_3 n^2}$ for any choice of R_J, we thus have

$$\mathcal{P}(N_J = R_J) \sim 2^{\beta n^2 - \gamma_2 n R_J |\log_2 \alpha|} \tag{18}$$

for large n, where $\beta = \gamma_1\gamma_3$ is independent of n. Since there are $\sim 2^{\frac{n^2}{4}+o(n^2)}$ three-layer causal sets, the number of such sets with $N_J = R_J$ is thus

$$\mathcal{N}(N_J = R_J) \sim 2^{\beta' n^2 - \gamma_2 n R_J |\log_2 \alpha|}, \tag{19}$$

where $\beta' = \frac{1}{4} + \gamma_1\gamma_3$.

Since $N_0 \propto n^2$ for the dominant $K = 3$ layer posets, for the terms $N_{J>0}$ to matter in the BDG action (3) in the large n limit, they, too, would have to be at least of order n^2. But from (19), the number of three-layer sets with $N_J \sim \kappa n^2$ is suppressed by a factor of $2^{-\kappa\gamma_2|\log_2\alpha| n^3}$, becoming negligible at large n. As shown in [8], one can take this argument further and show that N_J grows as n, with a calculable proportionality factor, but we will not need that here.

It is useful to develop a more intuitive understanding of this suppression. When we ask for the number of J-element intervals in a causal set, we are counting the intervals that have J elements *and no more*. For an interval $[e, e']$ in a large layered causal set, there are a very large number of paths that could potentially connect e and e', and we must exclude most of them. The factor $\alpha^{\gamma_2 n}$ in (15) comes from this exclusion, and leads in turn to the exponential suppression in the number (19). We will see below that for causal sets with more layers, the same kind of suppression occurs.

3 Generalisation

For general K, the calculation becomes more complicated. In particular, the information contained in $(\mathcal{L}^r)_{ij}$ is not sufficient to infer the size of the interval $[e_i^{(k)}, e_j^{(k')}]$. Fig. 4 shows an example for $K = 4$, in which the paths shown are all 2-element paths obtained from $\mathcal{L}^3$. It is evident from the figure that $(\mathcal{L}^3)_{ab} = (\mathcal{L}^3)_{cd} = 2$—that is, there are exactly two 4-element paths starting from e_a and ending at e_b, and also exactly two 4-element paths starting from e_c and ending at e_d —but the interval sizes differ, $|[e_a, e_b]| = 4$ and $|[e_c, e_d]| = 3$. Thus, one has to be careful in extracting N_J from the $\mathcal{L}^r$ for $K > 3$.

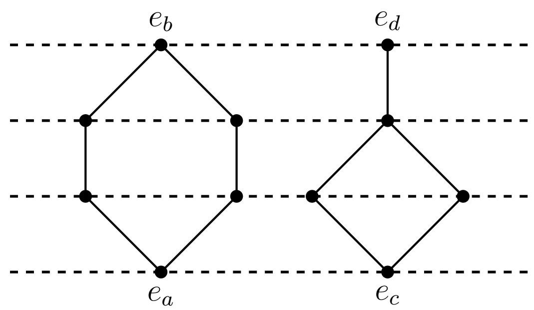


Figure 4: Two causal set intervals $[e_a, e_b]$ and $[e_c, e_d]$ with $(\mathcal{L}^3)_{ab} = (\mathcal{L}^3)_{cd} = 2$, but $|[e_a, e_b]| = 4$ while $|[e_c, e_d]| = 3$.

3.1 Strategy

The BDG action $S_{\mathrm{BDG}}(C)$ of (3) is a function of the N_J, the number of intervals in C of size J. The basic question we wish to ask is whether, for a K-layer causal set, the $N_{J>0}$ contribute significantly to the action or whether, as in the three-layer case of section 2.1, they become negligible for large n. As in the three-layer case, we will first calculate the probability $\mathcal{P}(N_J = R_J)$ of having exactly R_J J-element intervals. We will again find that the number of K-layer causal sets for which $R_J \sim \epsilon n^2$—that is, the number that contribute significantly to the BDG action—is again very highly suppressed for large n. It is worth emphasizing that this result is particular to *layered* sets; for sets obtained from a Poisson sprinkling of d-dimensional Minkowski spacetime, the numbers N_J all grow as $n^{2-\frac{2}{d}}$ for large n [13].

To demonstrate our claimed suppression, consider a J-element interval in a causal set C. The interior of each such interval is itself a J-element causal set, $c \in \Omega_J$. In the $K = 3$ case above, the J-element antichain is the only possible interior, but for $K > 3$ many other possibilities exist. As in Fig. 2(c), different $c \in \Omega_J$ can contribute to N_J, and this “fine structure" provides an important characterisation of C. An example of such fine structure is given in Fig. 5.

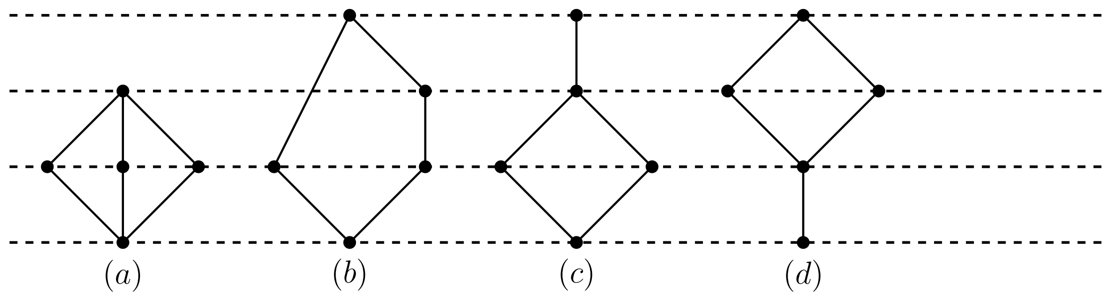


Figure 5: Examples of different types of 3-element intervals in a $K = 4$ level causal set. While there are five possible unlabelled 3-element causal sets, only four can be realised as the interiors of intervals for $K = 4$. The 3-element chain cannot be realised since it needs $K \geq 5$.

In what follows, we will show that for large n, the contribution to N_J for a typical K-level causal set is dominated by a particular subset of Ω_J, the J-antichains, whose elements occupy a single level. While our main interest is in small values of J—for instance, $J = 1, 2, 3$ for the BDG action in $d = 4$—we will show a much more general result as long as J and K remain fixed. Our basic strategy is to split the calculation of the probability $\mathcal{P}(N_J = R)$ into three stages, starting with the types of causal sets that can occur as interiors of intervals and successively refining their description. Each stage, though, will be based on the same mathematical structure, which we now summarize.

Suppose we have a set of objects that can come in M "colors" indexed by $\alpha = 1, \ldots, M$.[4] In order to pick some number R of objects, we must have $R^{(1)}$ objects of color 1, $R^{(2)}$ of color 2, etc., with

$$\sum_{\alpha=1}^{M} R^{(\alpha)} = R, \qquad R^{(\alpha)} \geq 0. \tag{20}$$

A sum of this form is called a *weak composition* of R into M parts, and is an example of what is sometimes known in combinatorics as a walls and balls problem ("divide a row of R balls into segments using $M - 1$ walls"). Note that some of the $R^{(\alpha)}$ can be zero. Denote the set of such weak compositions of R as $\Pi(R)$, where the number of parts is implicit in the notation.

Now suppose instead that the numbers of objects of color α is a random variable $N_J^{(\alpha)}$, for

[4]For example, take the set of objects to be the set of all J-element intervals, each "colored" by which J-element causal set in Ω_J it is.

which the value $R^{(\alpha)}$ occurs with probability $\mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)})$. It is then clear that

$$\mathcal{P}(N = R) = \sum_{\{R^{(\alpha)}\}\in\Pi(R)} \left(\prod_{\alpha=1}^{M} \mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) \right), \tag{21}$$

where $\{R^{(\alpha)}\}$ denotes a particular weak composition in the set of weak compositions $\Pi(R)$ of R. This will be the fundamental equation we use for each stage.

In general, this expression for $\mathcal{P}(N = R)$ is quite hard to calculate. For the setting we are interested in, however, it is possible to find a useful bound. Note first that the number of weak compositions $\Pi(R)$ is $\binom{R+M-1}{R}$, which reaches its maximum at $R = M - 1$, with

$$\binom{R+M-1}{R} \lesssim \frac{2^{2R}}{\sqrt{\pi R}}. \tag{22}$$

Hence

$$\mathcal{P}(N = R) \lesssim \frac{2^{2R}}{\sqrt{\pi R}} \max_{\{R^{(\alpha)}\}\in\Pi(R)} \left(\prod_{\alpha=1}^{M} \mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) \right). \tag{23}$$

We will see below that the probabilities relevant to counting intervals in a causal set take the general form

$$\mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) \sim 2^{-\beta_\alpha n^{r_\alpha} R^{(\alpha)}}, \tag{24}$$

where β_α and r_α are nonnegative constants and n is the cardinality of the set, which we will take to be very large. Without loss of generality, we can choose the indexing to order the exponents $r_1 > r_2 > r_3 > \dots$ (the case of two or more exponents being equal is a trivial generalization). Then for weak compositions for which $R^{(1)} \neq 0$, the $\alpha = 1$ term will dominate the product in (21), with

$$\prod_{\alpha=1}^{M} \mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) \sim 2^{-\beta_1 n^{r_1} R^{(1)}}$$

For weak compositions in which $R^{(1)} = 0$, on the other hand, the $\alpha = 2$ term will dominate, with

$$\prod_{\alpha=2}^{M} \mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) \sim 2^{-\beta_2 n^{r_2} R^{(2)}} > 2^{-\beta_1 n^{r_1} R^{(1)}}$$

Continuing this process, we see that the maximum in (23) will be attained when $R^{(1)} = R^{(2)} = \cdots = R^{(M-1)} = 0$ and $R^{(M)} = R$, with

$$\max_{\{R^{(\alpha)}\}\in\Pi(R)} \left(\prod_{\alpha=1}^{M} \mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) \right) \sim 2^{-\beta_M n^{r_M} R} \tag{25}$$

where r_M is the smallest of the exponents in the probabilities (24). The problem thus reduces to a computation of these probabilities.

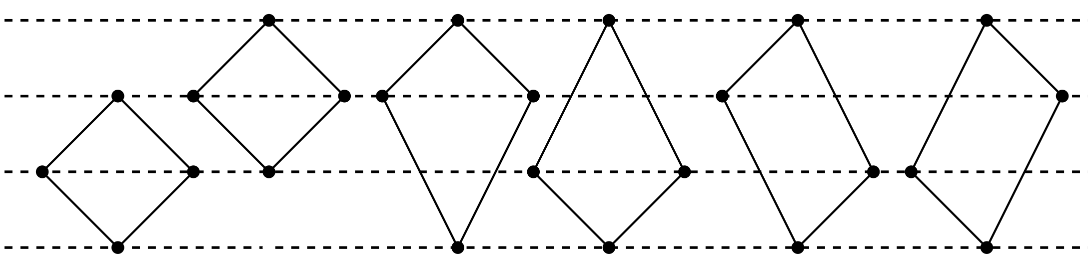
We can now describe the three stages of our calculation:

Stage 1: Let C be a randomly chosen n-element K-layer causal set. We are interested in the number of J-element intervals in C, or more precisely in the probability $\mathcal{P}(N_J = R)$ that C contains precisely R such intervals. As noted above, the interior of each such interval is itself a J-element causal set. List the members $c^{(\alpha)} \in \Omega_J$ that are "allowed" as interiors of J-element intervals in some arbitrary but fixed order, $(c^{(1)}, c^{(2)}, \ldots, c^{(M)})$, where $M \leq |\Omega_J|$. For example, for $K = 3$, the only possible interior of a J-element interval is an antichain $\mathbf{a}_J$, so for this case $M = 1$ and $c^{(1)} = \mathbf{a}_J$. We therefore need to consider all weak compositions of R into M parts. Then by the argument above, the probability that C contains exactly R J-element intervals is

$$\mathcal{P}(N_J = R) = \sum_{\{R^{(\alpha)}\}\in\Pi(R)} \left(\prod_{\alpha=1}^{M} \mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) \right) \tag{26}$$

To compute the probability (26),we need the set Ω_J of J-element causal sets, the set of weak compositions of R, and the individual probabilities $\mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)})$. For the first, causal sets are completely classified up to $J = 16$ [14]; above that their number grows very rapidly, but we will see below that the counting is dominated by a small subset of all causal sets. For the second, the weak compositions of an integer are well understood [15]. We therefore focus on the third problem, the probabilities $\mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)})$.

Stage 2: As the next stage, consider a particular allowed J-element causal set $c^{(\alpha)} \in \Omega_J$. This set can occur as the interior of an interval in many different ways; Fig. 6, for instance, shows six different embeddings of the two-element antichain as the interior of an interval in a four-layer causal set.



(a) (b) (c) (d) (e) (f)

Figure 6: Six embeddings of a two-element antichain as the interior of an interval in a four-layer causal set. Examples (e) and (f) are isomorphic, but generically additional connections of the "left" and "right" elements will distinguish them.

As a first refinement, note that an interval with interior $c^{(\alpha)}$ may begin at any level k and end at any later level k', as long as $k' - k \geq h(c^{\alpha}) + 2$, where $h(c^{\alpha})$ is the height of $c^{(\alpha)}$

(as defined in section 2). In other words, $R^{(\alpha)}$ receives contributions from pairs of layers (k,k') such that $[e_i^{(k)}, e_j^{(k')}] = c^{(\alpha)}$, $e_i^{(k)} \in \mathbb{L}_k$, $e_j^{(k')} \in \mathbb{L}_{k'}$. If we denote the number of such contributions as $R^{(\alpha,k,k')}$, we again have a weak composition:

$$\sum_{(k,k')\in\kappa} R^{(\alpha,k,k')} = R^{(\alpha)}, \qquad R^{(\alpha,k,k')} \geq 0 \tag{27}$$

where κ is the set of pairs (k,k') with $k \in \{1,\ldots,K-2\}$ and $k' \in \{k+h_\alpha+2,\ldots,K\}$. Then, just as in stage 1,

$$\mathcal{P}(N_J^{(\alpha)} = R^{(\alpha)}) = \sum_{\{R^{(\alpha,k,k')}\}\in\Pi(R^{(\alpha)})} \left(\prod_{(k,k')\in\kappa} \mathcal{P}\left(N_J^{(\alpha,k,k')} = R^{(\alpha,k,k')}\right)\right) \tag{28}$$

Stage 3: Even after we have sorted the sets $c^{(\alpha)}$ by their initial and final layers, there is further fine structure. In Fig. 6, for instance, the intervals (c), (d), (e), and (f) all start at layer 1 and end at layer 4, contributing to $R^{(1,4)}$. For a further refinement, we now fix k and k' and consider embeddings $c^{(\alpha,k,k')} \hookrightarrow C$. Each embedding will determine a set of numbers $\vec{n} = (n_{k_1}, \ldots n_{k_r})$, where n_{k_a} is the the number of elements of $c^{(\alpha,k,k')}$ lying in the layer k_a, with the ordering $k < k_1 < \ldots < k_r < k'$. The intervals with each $\vec{n}$, in turn, will contribute a number $R^{(\alpha,k,k',\vec{n})}$ to $R^{(\alpha,k,k')}$. We thus have another weak composition, with

$$\sum_{\vec{n}_c\in\mathcal{E}_\alpha^{(k,k')}} R^{(\alpha,k,k',\vec{n})} = R^{(\alpha)}, \qquad R^{(\alpha,k,k',\vec{n})} \geq 0 \tag{29}$$

where $\mathcal{E}_\alpha^{(k,k')}$ is the set of possible embeddings $\vec{n}$ of $c^{(\alpha)}$ in the intervening $k'-k$ layers, and

$$\mathcal{P}(N_J^{(\alpha,k,k')} = R^{(\alpha,k,k')}) = \sum_{\{R^{(\alpha,k,k',\vec{n})}\}\in\Pi(R^{(\alpha,k,k')})} \left(\prod_{\vec{n}_c\in\mathcal{E}_\alpha^{(k,k')}} \mathcal{P}\left(N_J^{(\alpha,k,k',\vec{n})} = R^{(\alpha,k,k',\vec{n})}\right)\right) \tag{30}$$

Now let $q(\vec{n}_\alpha)$ denote the probability of a particular set $c^{(\alpha)}$ occurring as the interior of an interval between layers k and k' with an internal pattern given by $\vec{c}$—that is, the probability of an interval $[e_i^{(k)}, e_j^{(k')}] = c^{(\alpha)}, e_i^{(k)} \in \mathbb{L}_k, e_j^{(k')} \in \mathbb{L}_{k'}$ with embedding $\vec{n}_\alpha$. Then as we saw in the $K = 3$ case,

$$\mathcal{P}(N_\alpha^{(k,k',\vec{n}_c)} = R) = \binom{\gamma_k\gamma_{k'}n^2}{R} q(\vec{n}_c)^R (1-q(\vec{n}_c))^{\gamma_k\gamma_{k'}n^2-R}. \tag{31}$$

To evaluate this expression, we face the daunting task of calculating $q(\vec{n}_\alpha)$, for each $\vec{n}_\alpha$ associated with each pair of layers (k,k') and each $c^{(\alpha)} \in \Omega_J$. To do so explicitly for general J and K is highly nontrivial: the number of possible interiors in stage 1 grows as $2^{J^2/4}$, for example.

Note, however, that if a particular set of probabilities dominates in the large n limit, we can ignore the others to leading order in sums over weak compositions. In particular, for probabilities of the form (24), the sum over compositions is bounded by the much simpler expression (25).

A further simplification comes from the structure of the pairs (k, k') in Stage 2. Suppose we have an interval $[e_i^{(k)}, e_j^{(k')}]$ whose initial element lies in a layer $k > 1$. By definition, the interval depends only on elements to the future of e, and thus to the future of layer k. This means that in calculating probabilities, we can ignore all the layers that come before $\mathbb{L}_k$. Similarly, we can ignore the layers that come after $\mathbb{L}_{k'}$. This reduces the calculation to one for a causal set with fewer layers.

3.2 Four layers ($K = 4$)

We now explicitly work out the four-layer case for $J = 1, 2, 3$. This provides the necessary intuition for generalisation to arbitrary K and J. The reader may at any point skip to the general case, section 3.3.

3.2.1 $J = 1$

Fig. 7 illustrates the various possible types of $J = 1$ intervals when $K = 4$. $\Omega_{J=1}$ consists of the single element e, hence for Stage 1 the $|\Omega_1|$ partition of R is trivial (Eqn. (26)). Next, for Stage 2, $h(c) = 1$, hence we require $k' \geq k + 2$. This gives us the possible pairs $\kappa = \{(1, 3), (2, 4), (1, 4)\}$. For Stage 3, the pairs $(1, 3)$ and $(2, 4)$ each have only one possible embedding each: $\vec{n} = (n_2 = 1)$ and $\vec{n} = (n_3 = 1)$, respectively. The pair $(1, 4)$, on the other hand, has two possible embeddings, $\vec{n} = (n_2 = 1)$ and $\vec{n} = (n_3 = 1)$.

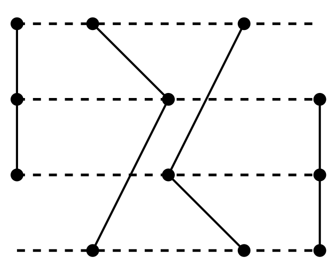


Figure 7: For $K = 4, J = 1$, there are different choices of pairs $\{(k, k')\}$, and therefore also different choices of embeddings. For the pair $(1, 3)$, there is only one embedding, in which e lies in layer 2. For the pair $(2, 4)$ there is again only one choice, in which e lies in layer 3. For the pair $(1, 4)$, on the other hand, e can lie in either layer 2 or layer 3. All four possibilities must be considered when calculating $\mathcal{P}(N_1 = R)$.

For $(k,k') = (1,3)$ or $(2,4)$, the analysis is the same as the $K=3$ case of section 2.1. As in Eqn. (13), specialized to $J=1$, we have

$$q_{13} = \gamma_2 n \pi_{13}(1-\pi_{13})^{\gamma_2 n-1}, \quad q_{24} = \gamma_3 n \pi_{24}(1-\pi_{24})^{\gamma_3 n-1} \tag{32}$$

where $\pi_{13} = p^{(12)}p^{(23)}$ and $\pi_{24} = p^{(23)}p^{(34)}$. In the large n limit these behave asymptotically as

$$q_{13} \sim 2^{-\gamma_2 n|\log_2 \alpha_{13}|}, \quad q_{24} \sim 2^{-\gamma_3 n|\log_2 \alpha_{24}|} \tag{33}$$

where, as in section 2.1, $\alpha_{ij} = 1 - \pi_{ij}$.

For the pair $(1,4)$, we can see from Fig. 7 that there are two embeddings, which we can label $\vec{n}_2 = (n_2 = 1, n_3 = 0)$ and $\vec{n}_3 = (n_2 = 0, n_3 = 1)$. Let us first consider $\vec{n}_2$. To obtain a probability for this configuration, for a fixed initial element $e_i^{(1)} \in \mathbb{L}_1$ and a fixed final element $e_j^{(4)} \in \mathbb{L}_4$, we must require that

- There is exactly one 3-element path from $e_i^{(1)}$ to $e_j^{(4)}$ via an element in $\mathbb{L}_2$. This contributes a factor $\gamma_2 n \pi_{124}(1-\pi_{124})^{\gamma_2 n-1}$, where $\pi_{124} = p^{(12)}p^{(24)}$.
- There is *no* path from $e_i^{(1)}$ to $e_j^{(4)}$, via an element in $\mathbb{L}_3$, since $[e_i^{(1)}, e_j^{(4)}]$ would then contain more than one element. This contributes a factor $(1-\pi_{134})^{\gamma_3 n}$, where $\pi_{134} = p^{(13)}p^{(34)}$.
- There is *no* 4-element path from $e_i^{(1)}$ to $e_j^{(4)}$ via an element in $\mathbb{L}_2$ and an element in $\mathbb{L}_3$, since again $[e_i^{(1)}, e_j^{(4)}]$ would contain more than one element. Note that this condition is equivalent to a restriction on the link matrix, $(\mathcal{L}^3)_{ij} = 0$. Since there are $\gamma_2\gamma_3 n^2$ pairs of elements $(e_{i'}^{(2)} \in \mathbb{L}_2, e_{j'}^{(3)} \in \mathbb{L}_3)$, this contributes a factor of $(1-\pi_{1234})^{\gamma_2\gamma_3 n^2}$, where $\pi_{1234} = p^{(12)}p^{(23)}p^{(34)}$.

The calculation for the second embedding, $\vec{n}_3$, is identical, except that layers 2 and 3 are switched. Thus

$$\begin{aligned} q_{124} &= \gamma_2 n \pi_{124}(1-\pi_{124})^{\gamma_2 n-1}(1-\pi_{134})^{\gamma_3 n}(1-\pi_{1234})^{\gamma_2\gamma_3 n^2} \\ q_{134} &= \gamma_3 n \pi_{134}(1-\pi_{134})^{\gamma_3 n-1}(1-\pi_{124})^{\gamma_2 n}(1-\pi_{1234})^{\gamma_2\gamma_3 n^2}, \end{aligned} \tag{34}$$

The last term in each expression clearly dominates in the large n limit, giving identical leading asymptotic behavior,

$$q_{124} \sim q_{134} \sim 2^{-\gamma_2\gamma_3 n^2|\log_2 \alpha_{1234}|} \tag{35}$$

with $\alpha_{1234} = 1 - \pi_{1234} < 1$.

This example illustrates an important feature: the suppression is enhanced when the number of intervening layers increases. More layers allow more paths between the endpoints of an interval, which must be excluded to keep the interval size fixed. In fact, as we have seen, more layers allow *many* more paths, since one can independently specify elements in each layer.

Applying Eqn. (31) to the two partitions $\vec{n}_2$ and $\vec{n}_3$ of $(1,4)$, we have

$$\begin{aligned}\mathcal{P}(N_1^{(1,4,\vec{n}_2)} = R) &= \tbinom{\gamma_1\gamma_4 n^2}{R} q_{124}^R (1-q_{124})^{\gamma_1\gamma_4 n^2 - R} \sim 2^{\beta_{(1,4)} n^2 - \gamma_2\gamma_3 n^2 R |\log_2 \alpha_{1234}|} \\ \mathcal{P}(N_1^{(1,4,\vec{n}_3)} = R) &= \tbinom{\gamma_1\gamma_4 n^2}{R} q_{134}^R (1-q_{134})^{\gamma_1\gamma_4 n^2 - R} \sim 2^{\beta_{(1,4)} n^2 - \gamma_2\gamma_3 n^2 R |\log_2 \alpha_{1234}|}\end{aligned} \tag{36}$$

where $\beta_{(1,4)} = \gamma_1\gamma_4$. Here we have used the facts that $\binom{\gamma_1\gamma_4 n^2}{R} \lesssim 2^{\gamma_1\gamma_4 n^2}$ and that both $(1-q_{124})^{\gamma_1\gamma_4 n^2}$ and $(1-q_{134})^{\gamma_1\gamma_4 n^2}$ go to 1 for large n, as in Eqn. (17). Inserting into Eqn. (30),

$$\begin{aligned}\mathcal{P}(N_1^{(1,4)} = R) &= \sum_{R^{(1,4,\vec{n}_2)} + R^{(1,4,\vec{n}_3)} = R} \mathcal{P}(N_1^{(1,4,\vec{n}_2)} = R^{(1,4,\vec{n}_2)}) \mathcal{P}(N_1^{(1,4,\vec{n}_3)} = R^{(1,4,\vec{n}_3)}) \\ &\sim 2^{\beta_{(14)} n^2 - \gamma_2\gamma_3 n^2 R |\log_2 \alpha_{1234}|}.\end{aligned} \tag{37}$$

Since both $(1,3)$ and $(2,4)$ admit just a single partition, from Eqn. (31)

$$\begin{aligned}\mathcal{P}(N_1^{(1,3)} = R) &= \tbinom{\gamma_1\gamma_3 n^2}{R} q_{13}^R (1-q_{13})^{\gamma_1\gamma_3 n^2 - R} \sim 2^{\beta_{(13)} n^2 - \gamma_2 n R |\log_2 \alpha_{13}|} \\ \mathcal{P}(N_1^{(2,4)} = R) &= \tbinom{\gamma_2\gamma_4 n^2}{R} q_{24}^R (1-q_{24})^{\gamma_2\gamma_4 n^2 - R} \sim 2^{\beta_{(24)} n^2 - \gamma_3 n R |\log_2 \alpha_{24}|},\end{aligned} \tag{38}$$

where $\beta_{(1,3)} = \gamma_1\gamma_3$ and $\beta_{(2,4)} = \gamma_2\gamma_4$. Comparing Eqns. (37) and (38), we note that the leading negative term in the exponent in the latter is of lower order in n than the former and hence when $R \propto n^2$ (to leading order) it dominates for any finite composition. This is a crucial feature of our calculation: when there are more layers in between k and k', i.e., $k'-k > h(c)+1$, the leading negative term in the exponent is of higher order in n, and hence is suppressed relative to the "tightest fit" $k'-k = h(c)+1$. Hence, of the $|\kappa|$ compositions in Eqn. (27), the dominant ones are those for which $R_{14} \sim 0$. If $\tilde{\gamma}|\log_2 \tilde{\alpha}|$ is the smaller of $\gamma_2|\log_2 \alpha_{13}|$ and $\gamma_3|\log_2 \alpha_{24}|$, then

$$\mathcal{P}(N_1 = R) \sim 2^{\tilde{\beta} n^2 - \tilde{\gamma} n R |\log_2 \tilde{\alpha}|}. \tag{39}$$

This is identical in form to Eqn. (18): the analysis has thus reduced to that for $K=3$.

3.2.2 $J = 2$

$\Omega_{J=2}$ consists of two causal sets: the two-element antichain $\mathbf{a}_2$ (• •) and the two-element chain $\mathbf{c}_2$ ( ). Since $h(\mathbf{c}_2) = 2$, for $K = 4$, there is only one choice of initial and final layer, $(k,k') = (1,4)$. For $\mathbf{a}_2$, on the other hand, we have the three possibilities: $\kappa = \{(1,3),(2,4),(1,4)\}$. The first two of these are similar to the $K=3$ case; each admits a single

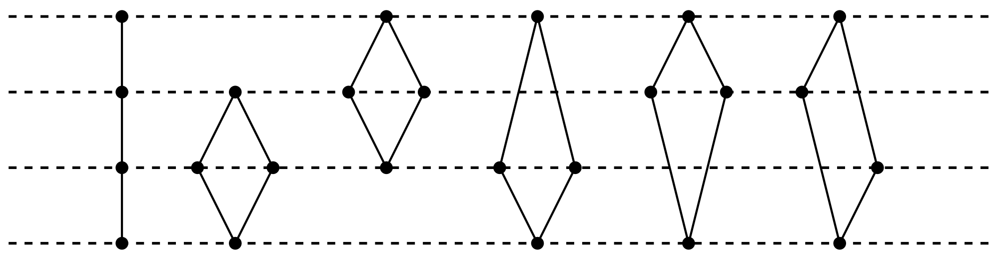


Figure 8: The six different ways in which the 2-element interval can be realised in a $K = 4$ layer poset. To the extreme left is the chain $\mathbf{c}_2$, which has only one realisation, while the antichain $\mathbf{a}_2$ can be realised in five different ways.

embedding, $\vec{n}_2 = (n_2 = 2)$ and $\vec{n}_3 = (n_3 = 2)$. For the choice $(1, 4)$, on the other hand, there are three possible embeddings, shown in Fig. 8: $\mathcal{E} = \{(n_2 = 2), (n_3 = 2), (n_2 = 1, n_3 = 1)\}$. We calculate $q(\vec{n}_c)$ for each of these possibilities:

We first consider the two-element chain $\mathbf{c}_2$. For this set, there is only one possible embedding: $(k, k') = (1, 4)$ with $\vec{n} = (n_2 = 1, n_3 = 1)$. To obtain a probability for this configuration, for a fixed initial element $e_i^{(1)} \in \mathbb{L}_1$ and a fixed final element $e_j^{(4)} \in \mathbb{L}_4$, we must require that

- There is exactly one 4-element path from $e_i^{(1)}$ to $e_j^{(4)}$, which goes from $e_i^{(1)}$ to an element in $\mathbb{L}_2$ to an element in $\mathbb{L}_3$ and then to $e_j^{(4)}$. This contributes a factor $\gamma_2\gamma_3 n^2 \pi_{1234}(1 - \pi_{1234})^{\gamma_2\gamma_3 n^2 - 1}$, where $\pi_{1234} = p^{(12)}p^{(23)}p^{(34)}$.
- There is no 3-element path from $e_i^{(1)}$ to $e_j^{(4)}$. This contributes a factor $(1 - \pi_{124})^{\gamma_2 n}$ (to exclude paths with an element in $\mathbb{L}_2$) and a factor $(1 - \pi_{134})^{\gamma_3 n}$ (to exclude paths with an element in $\mathbb{L}_3$), where $\pi_{124} = p^{(12)}p^{(24)}$ and $\pi_{134} = p^{(13)}p^{(34)}$.

Hence

$$q_{1234} = (\gamma_2\gamma_3 n^2)\, \pi_{1234}(1 - \pi_{1234})^{\gamma_2\gamma_3 n^2 - 1}(1 - \pi_{124})^{\gamma_2 n}(1 - \pi_{134})^{\gamma_3 n} \sim 2^{-\gamma_2\gamma_3 n^2 |\log_2 \alpha_{1234}|}, \quad (40)$$

where again $\alpha_{1234} = 1 - \pi_{1234}$. The insight we gain from this example is that for $[e_i^{(k)}, e_j^{(k')}] = c$, the leading order behavior goes as $2^{-\beta n^r}$, where $r = k' - k - 1$. Thus, the smaller r is, the more dominant is the term.

For the two-element antichain $\mathbf{a}_2$, the possible embeddings are shown in Figs. 6 and 8. There are three possible (k, k') values: $\kappa = \{(1, 3), (2, 4), (1, 4)\}$. The first two reduce to the the $K = 3$ case

$$\begin{aligned} q_{13} &= \binom{\gamma_2 n}{2} \pi_{123}^2 (1 - \pi_{123})^{\gamma_2 n - 2} \sim 2^{-\gamma_2 n |\log_2 \alpha_{123}|} \\ q_{24} &= \binom{\gamma_3 n}{2} \pi_{234}^2 (1 - \pi_{234})^{\gamma_3 n - 2} \sim 2^{-\gamma_3 n |\log_2 \alpha_{234}|} \end{aligned} \quad (41)$$

where again $\pi_{123} = p^{(12)}p^{(23)}$, $\pi_{234} = p^{(23)}p^{(34)}$, and $\alpha_{ijk} = 1 - \pi_{ijk}$.

For $(k, k') = (1, 4)$, the possible embeddings $\mathcal{E}$ are $\vec{n}_1 \equiv (n_2 = 2, n_3 = 0)$, $\vec{n}_2 \equiv (n_2 = 0, n_3 = 2)$, and $\vec{n}_3 \equiv (n_2 = 1, n_3 = 1)$. Denote the associated probabilities by $q_{14}^{\vec{n}_1}$, $q_{14}^{\vec{n}_2}$, and $q_{14}^{\vec{n}_3}$. Consider first $q_{14}^{\vec{n}_1}$ with an initial element $e_i^{(1)} \in \mathbb{L}_1$ and a final element $e_j^{(4)} \in \mathbb{L}_4$. As in the previous cases, we must require that

- There are exactly two 3-element paths from $e_i^{(1)}$ to $e_j^{(4)}$ via two distinct elements in $\mathbb{L}_2$. This contributes a factor $\binom{\gamma_2}{2}(\pi_{124})^2(1 - \pi_{124})^{\gamma_2 n - 2}$.
- There is no 3-element path from $e_i^{(1)}$ to $e_j^{(4)}$ that passes through an element in $\mathbb{L}_3$. This contributes a factor $(1 - \pi_{134})^{\gamma_3 n}$ (to exclude all paths with an element in $\mathbb{L}_3$).
- There is no 4-element path from $e_i^{(1)}$ to $e_j^{(4)}$ via elements in $\mathbb{L}_2$ and $\mathbb{L}_3$. This is again equivalent to the requirement that $(\mathcal{L}^3)_{ij} = 0$, and it contributes a factor of $(1 - \pi_{1234})^{\gamma_2\gamma_3 n^2}$ to the probability.

For $q_{14}^{\vec{n}_2}$, and $q_{14}^{\vec{n}_3}$ the arguments are almost identical. We thus obtain

$$\begin{aligned} q_{14}^{\vec{n}_1} &= \binom{\gamma_2 n}{2}\pi_{124}^2(1-\pi_{124})^{\gamma_2 n-2}(1-\pi_{134})^{\gamma_3 n}(1-\pi_{1234})^{\gamma_2\gamma_3 n^2} \sim 2^{-\gamma_2\gamma_3 n^2|\log_2\alpha_{1234}|} \\ q_{14}^{\vec{n}_2} &= \binom{\gamma_3 n}{2}\pi_{134}^2(1-\pi_{134})^{\gamma_3 n-2}(1-\pi_{124})^{\gamma_2 n}(1-\pi_{1234})^{\gamma_2\gamma_3 n^2} \sim 2^{-\gamma_2\gamma_3 n^2|\log_2\alpha_{1234}|} \\ q_{14}^{\vec{n}_3} &= \gamma_2\gamma_3 n^2\pi_{124}\pi_{134}(1-\pi_{124})^{\gamma_2 n-1}(1-\pi_{134})^{\gamma_3 n-1}(1-\pi_{1234})^{\gamma_2\gamma_3 n^2} \sim 2^{-\gamma_2\gamma_3 n^2|\log_2\alpha_{1234}|}. \end{aligned} \tag{42}$$

We see that the leading n dependence is completely dictated by the factor $(1 - \pi_{1234})^{\gamma_2\gamma_3 n^2}$, the term that excludes "extra" 4-element paths. Comparing Eqns. (41) and (42) we can see that, as in the $J = 1$ case, the "tight fit" $k' = k + 2$, configurations dominate over the "loose fit" configurations $k' > k + 2$.

Putting these together, we have for the $\mathbf{c}_2$ contribution

$$\mathcal{P}(N_2^{(\mathbf{c}_2)} = R) = \binom{\gamma_1\gamma_4 n^2}{R} q_{1234}^R (1 - q_{1234})^{\gamma_1\gamma_4 n^2 - R} \lesssim 2^{\beta_{14} n^2 - \gamma_2\gamma_3 R n^2 |\log_2(\alpha_{1234})|}, \tag{43}$$

where $\beta_{14} = \gamma_1\gamma_4$ and we have again used the bound $\binom{M}{N} \lesssim \binom{M}{M/2} \lesssim 2^M$.

The $\mathbf{a}_2$ contributions are

$$\begin{aligned} \mathcal{P}(N_2^{(\mathbf{a}_2,1,3)} = R) &= \binom{\gamma_1\gamma_3 n^2}{R} q_{13}^R (1 - q_{13})^{\gamma_1\gamma_3 n^2 - R} \lesssim 2^{\beta_{13} n^2 - \gamma_2 n R|\log_2\alpha_{123}|} \\ \mathcal{P}(N_2^{(\mathbf{a}_2,2,4)} = R) &= \binom{\gamma_2\gamma_4 n^2}{R} q_{24}^R (1 - q_{24})^{\gamma_2\gamma_4 n^2 - R} \lesssim 2^{\beta_{24} n^2 - \gamma_3 n R|\log_2\alpha_{234}|}, \end{aligned} \tag{44}$$

where $\beta_{13} = \gamma_1\gamma_3, \beta_{24} = \gamma_2\gamma_4$, and

$$\begin{aligned}
\mathcal{P}(N_2^{(\mathbf{a}_2,1,4,\vec{n}_1)} = R) &= \binom{\gamma_1\gamma_4 n^2}{R}(q_{14}^{\vec{n}_1})^R(1-q_{14}^{\vec{n}_1})^{\gamma_1\gamma_4 n^2 - R} \lesssim 2^{\beta_{14}n^2 - \gamma_2\gamma_3 n^2 R|\log_2 \alpha_{1234}|} \\
\mathcal{P}(N_2^{(\mathbf{a}_2,1,4,\vec{n}_2)} = R) &= \binom{\gamma_1\gamma_4 n^2}{R}(q_{14}^{\vec{n}_2})^R(1-q_{14}^{\vec{n}_2})^{\gamma_1\gamma_4 n^2 - R} \lesssim 2^{\beta_{14}n^2 - \gamma_2\gamma_3 n^2 R|\log_2 \alpha_{1234}|} \\
\mathcal{P}(N_2^{(\mathbf{a}_2,1,4,\vec{n}_3)} = R) &= \binom{\gamma_1\gamma_4 n^2}{R}(q_{14}^{\vec{n}_3})^R(1-q_{14}^{\vec{n}_3})^{\gamma_1\gamma_4 n^2 - R} \lesssim 2^{\beta_{14}n^2 - \gamma_2\gamma_3 n^2 R|\log_2 \alpha_{1234}|}.
\end{aligned} \tag{45}$$

Comparing the contributions, it is clear that $\mathcal{P}(N_2 = R)$ is dominated by the partition with R coming from the tight fit contributions of $\mathbf{a}_2$, Eqn. (44). Thus

$$\mathcal{P}(N_2 = R) \lesssim 2^{\tilde{\beta}n^2 - \tilde{\gamma}nR|\log_2 \tilde{\alpha}|} \tag{46}$$

where $\tilde{\gamma}|\log_2 \tilde{\alpha}|$ is the smaller of $\gamma_2|\log_2 \alpha_{123}|$ and $\gamma_3|\log_2 \alpha_{234}|$. We again see that the dominant contribution comes from the three-layer-like tight embeddings of antichains, and the probabilities have the same structure as those in the three-layer case.

3.2.3 $J = 3$

Here we extend the analysis to the significantly more involved case $J = 3$. This is a special case of the more general analysis of section 3.3, but the details may help with intuition.

For $J = 3$, one must consider all three-element intervals $[e_i^{(k)}, e_j^{(k')}]$ whose interiors are the sets in Fig. 9.

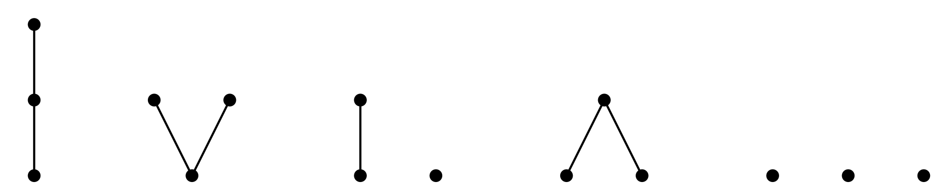


Figure 9: The set of unlabelled 3-element causal sets

Of these, the chain (a) is forbidden, since it won't fit in the interior of a four-layer set. The others can all be embedded in layers $\mathbb{L}_2$ and $\mathbb{L}_3$, and hence are allowed. Sets (b) and (d) each have one embedding: $\vec{n} = (n_2 = 1, n_3 = 2)$ for (b) and $\vec{n} = (n_2 = 2, n_3 = 1)$ for (d). Set (c) has two possible embeddings, $\vec{n} = (n_2 = 1, n_3 = 2)$ and $\vec{n} = (n_2 = 2, n_3 = 1)$. Set (e), the antichain $\mathbf{a}_3$, has four possible embeddings, $\vec{n} = (n_2 = 2, n_3 = 1)$, $\vec{n} = (n_2 = 1, n_3 = 2)$, $\vec{n} = (n_2 = 3, n_3 = 0)$, and $\vec{n} = (n_2 = 0, n_3 = 3)$. In all but the last two cases, the initial element $e_i^{(k)}$ must lie in $\mathbb{L}_1$ and the final element $e_j^{(k')}$ must lie in $\mathbb{L}_4$. For the antichain embedding $\vec{n} = (n_2 = 3, n_3 = 0)$, though, $e_j^{(k')}$ can lie in either $\mathbb{L}_3$ or $\mathbb{L}_4$, and for $\vec{n} = (n_2 = 0, n_3 = 3)$, $e_i^{(k)}$ can lie in either $\mathbb{L}_1$ or $\mathbb{L}_2$. We will see that the "tight" intervals dominate.

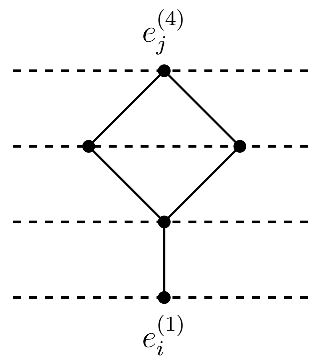


Figure 10: For (b) there are exactly two 4-element paths from $e_i^{(1)}$ to $e_j^{(4)}$, which coincide in the first leg from $e_i^{(1)}$ to an element in $\mathbb{L}_2$ and then separate thereafter to rejoin at $e_j^{(4)}$.

Since sets (b) and (d) are time reversals of each other and have only one realisation each, we start with them. For (b) (see Fig. 10), we must require that

- There are two 4-element paths from $e_i^{(1)} \in \mathbb{L}_1$ to $e_j^{(4)} \in \mathbb{L}_4$ such that there is only one link from $e_i^{(1)}$ to an element $e_{i'}^{(2)} \in \mathbb{L}_2$, and subsequently exactly two 3-element paths from $e_{i'}^{(2)}$ to $e_j^{(4)}$ via two distinct elements in $\mathbb{L}_3$. Such a pattern occurs with probability $p^{(12)}(p^{(23)}p^{(24)})^2$, and comes with combinatorial factors $\gamma_2 n$ (the choice of an element in $\mathbb{L}_2$) and $\binom{\gamma_3 n}{2}$ (the choice of two elements in $\mathbb{L}_3$).
- There are no additional 3-element paths from $e_i^{(1)}$ to $e_j^{(4)}$ either via an element in $\mathbb{L}_2$ or an element in $\mathbb{L}_3$. These give a factor $(1-p^{(12)}p^{(24)})^{\gamma_2 n-1}(1-p^{(13)}p^{(34)})^{\gamma_3 n-1}$.
- There are no additional 4-element paths from $e_i^{(1)}$ to $e_j^{(4)}$. This contributes a factor $(1-p^{(12)}p^{(23)}p^{(34)})^{\gamma_2\gamma_3 n^2-2}$.

The leading behavior comes from the last term, which has an asymptotic form

$$q^{(b)} \sim 2^{-\gamma_2\gamma_3 n^2 |\log_2 \alpha_{1234}|} \tag{47}$$

where, as usual, $\alpha_{1234} = 1 - p^{(12)}p^{(23)}p^{(34)}$. Set (d) is the time reversal of set (b), whose probability can be obtained by simply interchanging layers $\mathbb{L}_2$ and $\mathbb{L}_3$. The leading large n behavior is thus identical to (47).

We next look at set (c) (Fig. 11), with embeddings $\vec{n}_1 = (n_2 = 2, n_3 = 1)$ and $\vec{n}_2 = (n_2 = 1, n_3 = 2)$. We start with the embedding $\vec{n}_1 = (n_2 = 2, n_3 = 1)$. We must now require that

- There is one 4-element path from $e_i^{(1)} \in \mathbb{L}_1$ to $e_j^{(4)} \in \mathbb{L}_4$ via some $e_{i_1}^{(2)} \in \mathbb{L}_2$ and $e_{j_1}^{(3)} \in \mathbb{L}_3$. Such a pattern occurs with probability $p^{(12)}p^{(23)}p^{(34)}$, and comes with combinatorial factors $\gamma_2 n$ and $\gamma_3 n$.

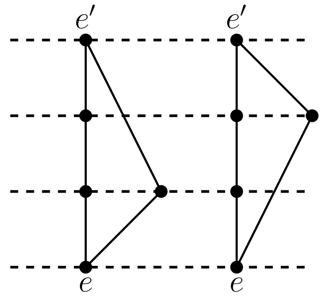


Figure 11: For (c) there are two possible embeddings corresponding to $\vec{n}_1 = (n_2 = 2, n_3 = 1)$ and $\vec{n}_2 = (n_2 = 1, n_3 = 2)$. For $\vec{n}_1$, there is exactly one 4-element path from $e_i^{(1)}$ to $e_j^{(4)}$ and exactly one 3-element path from $e_i^{(1)}$ to $e_j^{(4)}$ via an element in $\mathbb{L}_2$. For $\vec{n}_2$, there is again exactly one 4-element path from $e_i^{(1)}$ to $e_j^{(4)}$ and also exactly one 3-element path from $e_i^{(1)}$ to $e_j^{(4)}$, but now via an element in $\mathbb{L}_3$.

- There is also one 3-element path from $e_i^{(1)}$ to $e_j^{(4)}$ via an element $e_{i_2}^{(2)} \in \mathbb{L}_2$ that does not intersect the first path at $\mathbb{L}_2$. Such a pattern occurs with probability $p^{(12)}p^{(24)}$, and comes with a combinatorial factor $\gamma_2 n - 1$ (since $e_{i_2}^{(2)}$ must be distinct from the element $e_{i_1}^{(2)}$ in the 4-element path).
- There are no additional 3-element paths from $e_i^{(1)}$ to $e_j^{(4)}$. Such paths can occur either via elements in $\mathbb{L}_2$ or $\mathbb{L}_3$ that are not already in either the 4-element path or the 3-element path in Fig. 11. This contributes a probability $(1 - p^{(12)}p^{(24)})^{\gamma_2 n - 2}(1 - p^{(13)}p^{(34)})^{\gamma_3 n - 1}$.
- There are no additional 4-element paths from $e_i^{(1)}$ to $e_j^{(4)}$. Such an extra path could include any of the elements $e_{i_1}^{(2)}$, $e_{j_1}^{(3)}$, or $e_{i_2}^{(2)}$, but not both $e_{i_1}^{(2)}$ and $e_{j_1}^{(3)}$, since that would be a duplicate. The resulting factor is thus $(1 - p^{(12)}p^{(23)}p^{(34)})^{\gamma_2\gamma_3 n^2 - 1}$.

We see again that the leading behavior comes from the last term, and that the large n asymptotics are again given by (47). Since the other embedding of set (c), $\vec{n}_2 = (n_2 = 2, n_3 = 1)$, is again a time reversal, its probability can be obtained by interchanging layers $\mathbb{L}_2$ and $\mathbb{L}_3$, and the asymptotic behavior is the same.

Finally, we turn to the antichain, set (e). Begin with the embedding $\vec{n}_1 = (n_2 = 2, n_3 = 1)$ (Fig. 12). We must now require that

- There are exactly two 3-element paths (and no more) from $e_i^{(1)} \in \mathbb{L}_1$ to $e_j^{(4)} \in \mathbb{L}_4$ via two elements $e_{i_1}^{(2)}, e_{i_2}^{(2)} \in \mathbb{L}_2$. These give a contribution $(p^{(12)}p^{(24)})^2(1 - p^{(12)}p^{(24)})^{\gamma_2 n - 2}$ to the probability, with a combinatorial factor $\binom{\gamma_2 n}{2}$.
- There is exactly one path 3-element path from $e_i^{(1)}$ to an element $e_{j_1}^{(3)} \in \mathbb{L}_3$ to $e_j^{(4)}$. This gives a contribution $p^{(13)}p^{(34)}(1 - p^{(13)}p^{(34)})^{\gamma_3 n - 1}$, with a combinatorial factor $\gamma_3 n$.

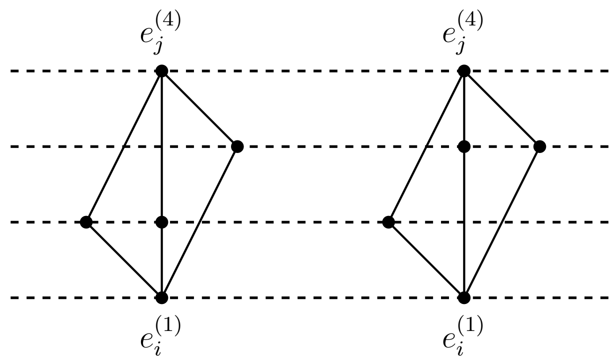


Figure 12: For (e) with $e_i^{(1)} \in \mathbb{L}_1$, and $e_j^{(4)} \in \mathbb{L}_4$ there are two possible embeddings, corresponding to $\vec{n}_1 = (n_2 = 2, n_3 = 1)$ and $\vec{n}_2 = (n_2 = 1, n_3 = 2)$. For each there are exactly three 3-element paths from $e_i^{(1)}$, to $e_j^{(4)}$. In the former, two paths pass through $\mathbb{L}_2$ and the third through $\mathbb{L}_3$, while in the latter, one path passes through $\mathbb{L}_2$ and the other two through $\mathbb{L}_3$.

- There is no 4-element path from $e_i^{(1)}$ to $e_j^{(4)}$. Such an extra path cannot include the elements $e_{i_1}^{(2)}$ or $e_{i_2}^{(2)}$ in $\mathbb{L}_2$, since by assumption these are links—that is, nearest neighbors—to $e_i^{(1)}$. Similarly, it cannot contain $e_{j_1}^{(3)}$, which is linked to $e_j^{(4)}$. The resulting factor is thus $(1 - p^{(12)}p^{(23)}p^{(34)})^{(\gamma_2 n - 2)(\gamma_3 n - 1)}$.

As this example makes clear, the fine details of the combinatorics can be complicated. But the leading asymptotic behavior is not affected; it is again given by (47). The embedding $\vec{n}_2 = (n_2 = 1, n_3 = 2)$ is again related by time reversal, and has the same asymptotic behavior.

Next, we consider the antichain embedding $\vec{n}_3 = (n_2 = 3, n_3 = 0)$ (Fig. 13). There are now two possible configurations: the final element $e_j^{(k)}$ can lie in layer three (i.e., $(k, k') = (1, 3)$) or in layer four (i.e., $(k, k') = (1, 4)$).

We start with $(k, k') = (1, 4)$. We must require that

- There are exactly three 3-element paths from $e_i^{(1)} \in \mathbb{L}_1$ to $e_j^{(4)} \in \mathbb{L}_4$ via elements $e_{i_1}^{(2)}, e_{i_2}^{(2)}, e_{i_3}^{(2)} \in \mathbb{L}_2$. This gives a contribution $(p^{(12)}p^{(24)})^3(1 - p^{(12)}p^{(24)})^{\gamma_2 n - 3}$ to the probability, with a combinatorial factor $\binom{\gamma_2 n}{3}$.
- There is no 3-element path from $e_i^{(1)}$ to $e_j^{(4)}$ via elements in $\mathbb{L}_3$. This gives a contribution $(1 - p^{(13)}p^{(34)})^{\gamma_3 n}$, with a combinatorial factor $\gamma_3 n$.
- There is no additional 4-element path from $e_i^{(1)}$ to $e_j^{(4)}$. As in the previous example, such an extra path cannot include elements $e_{i_1}^{(2)}, e_{i_2}^{(2)}$ or $e_{i_3}^{(2)}$. The resulting factor is thus $(1 - p^{(12)}p^{(23)}p^{(34)})^{(\gamma_2 n - 3)\gamma_3 n}$.

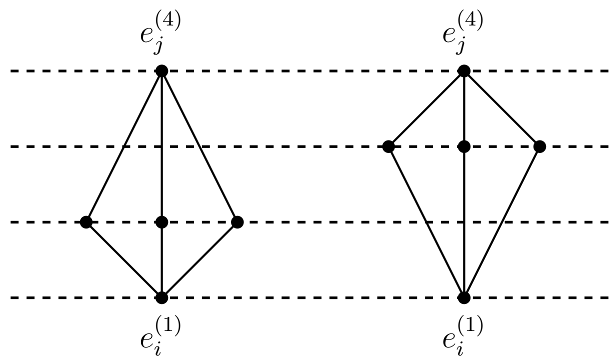


Figure 13: For the interior (e) with $e_i^{(1)} \in \mathbb{L}_1$, and $e_j^{(4)} \in \mathbb{L}_4$ there are two other possible embeddings, corresponding to $\vec{n}_3 = (n_2 = 3, n_3 = 0)$ and $\vec{n}_4 = (n_2 = 0, n_3 = 3)$. For each there are exactly three 3-element paths from $e_i^{(1)}$ to $e_j^{(4)}$; in the former, all pass through $\mathbb{L}_2$, while in the latter, all pass through $\mathbb{L}_3$.

Once again, the last term dominates the asymptotic behavior, which is again given by (47). As before, the embedding $\vec{n}_4 = (n_2 = 0, n_3 = 3)$ with $(k, k') = (1, 4)$ is the time reversal, and has the same asymptotic behavior.

We now have two remaining possibilities, the "tight fit" cases with $k' = k + 2$. There are two possibilities here: $(k, k') = (1, 3)$ with $\vec{n} = (n_2 = 3, n_3 = 0)$ and $(k, k') = (2, 4)$ with $\vec{n} = (n_2 = 0, n_3 = 3)$. But these are already familiar from the $K = 3$-layer case. For $(k, k') = (1, 3)$, layer $\mathbb{L}_4$ is irrelevant; all that matters is the three-layer set $\mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3$, for which the computation was already done in section 2.1, with a probability given by (15). Similarly, for $(k, k') = (2, 4)$, layer $\mathbb{L}_1$ is irrelevant, and the result is again given by (15).

Comparing (15) and 47), we see that at large n, the probabilities for tight fit antichains fall off exponentially more slowly than any of the other interval interiors. By the argument of section 3.1, these also dominate the probabilities $\mathcal{P}(N_3 = R)$, reducing the problem to the three-layer case.

3.3 General K, J

The experience with four-layer sets with $J = 1, 2, 3$ suggests that the dominant contribution to $\mathcal{P}(N_J = R)$ comes from tight fit antichains $\mathbf{a}_J$. As a first step, we therefore calculate $\mathcal{P}(N_J^{\mathbf{a}_J})$ and compare the contributions from the tight fit and loose fit antichains for general K, J. We find that, as expected, the former dominate.

3.3.1 Antichain Contributions

For $J > 0$, the allowed (k, k') pairs are $k = 1, \ldots, K-2$ and $k' = k+2, \ldots K$. We begin with the by now familiar calculation of the contribution from tight fit antichains, that is, antichains with $k' = k+2$. As in the three-layer case, these admit a single embedding, $\vec{n} = (\ldots, n_k = 0, n_{k+1} = J, n_{k+2} = 0, \ldots)$. The probability for the interior of the interval $[e_i^{(k)}, e_j^{(k+2)}]$ to be the antichain $\mathbf{a}_J$ can be simply copied from the $K = 3$ case of section 2.1:

$$q_k = \binom{\gamma_{k+1} n}{J} (\pi_k)^J (1-\pi_k)^{\gamma_{k+1} n - J} \qquad \text{with } \pi_k \equiv p^{(k(k+1))} p^{((k+1)(k+2))}, \tag{48}$$

a trivial generalisation of Eqn. (13). Thus for fixed J, to leading order in n,

$$q_k \sim 2^{-\gamma_{k+1} n |\log_2(\alpha_k)|} \tag{49}$$

with $\alpha_k = 1 - \pi_k$.

Next consider loose fit antichains, that is, antichains with $k' > k+2$. Suppose there are $r > 1$ distinct layers sandwiched between k and k', that is, $k' = k + r + 1$. For a fixed (k, k') the distribution of elements is again a weak composition problem: we wish to distribute J elements among r layers. There are $\binom{r+J-1}{J-1}$ such embeddings. Consider one such embedding

$$\vec{n} = (\ldots, n_{k_1}, n_{k_2} \ldots n_{k_s}, \ldots), \text{ with } k < k_1 \ldots < k_s < k',\ 1 < s \leq r, \text{ and } \sum_{i=1}^{s} n_{k_i} = J. \tag{50}$$

Fix an initial element $e_i^{(k)} \in \mathbb{L}_k$ and a final element $e_j^{(k')} \in \mathbb{L}_{k'}$. In order for the interval $[e_i^{(k)}, e_j^{(k')}]$ to be the antichain $\mathbf{a}_J$, we must require

- For each k_i, there are exactly n_{k_i} distinct 3-element paths from $e_i^{(k)}$ to $e_j^{(k')}$ via an element in layer $\mathbb{L}_{k_i}$. Each k_i thus contributes a factor

  $$\binom{\gamma_{k_i} n}{n_{k_i}} \pi_{k_i}^{n_{k_i}} (1-\pi_{k_i})^{\gamma_{k_i} n - n_{k_i}},$$

  where $\pi_{k_i} = p^{(k k_i)} p^{(k_i k')}$.

- There are no t-element paths from $e_i^{(k)}$ to $e_j^{(k')}$ for $t > 3$. In terms of the link matrix, this is equivalent to the requirement that $(\mathcal{L}^{t-1})_{ij} = 0$ for every $3 < t \leq r+2$.

Thus

$$q_{kk'} = \left(\prod_{i=1}^{s} \binom{\gamma_{k_i} n}{n_{k_i}} \pi_{k_i}^{n_{k_i}} (1-\pi_{k_i})^{\gamma_{k_i} n - n_{k_i}} \right) \prod_{t=4}^{r+2} \mathcal{P}\left((\mathcal{L}^{t-1})_{ij} = 0 \right) \tag{51}$$

The suppression for $r > 1$ comes from the second product: multi-element paths from $e_i^{(k)}$ to $e_j^{(k')}$ are common, and intervals that exclude them are very rare. The dominant contribution comes from $t = r + 2$, for which

$$\mathcal{P}((\mathcal{L}^{r+1})_{ij} = 0) = (1 - \pi_r)^{\Gamma_r n^r + \mathcal{O}(n^{r-1})}, \tag{52}$$

where $\pi_r = p^{(k(k+1))} p^{((k+1)(k+2))} \dots p^{((k+r-1)(k+r))} p^{((k+r)(k+r+1))}$ and $\Gamma_r = \gamma_{k+1}\gamma_{k+2}\dots\gamma_{k+r}$. As in section 3.2.3, there are corrections to the exponent coming from the fact that a small set of multi-element paths are already excluded. If we know that the element $e_1 \in \mathbb{L}_{k_1}$ is a member of our candidate antichain (50), that is, $e \prec\!* \, e_1 \prec\!* \, e'$ with $e_1 \in \mathbf{a}_J$, then by the definition of a link, there can be no multi-element paths of the form $e \prec\!* \, e_1 \prec\!* \, e_2 \prec\!* \, e'$ or $e \prec\!* \, e_2 \prec\!* \, e_1 \prec\!* \, e'$. But the resulting modifications are subdominant; at most they require that we replace some factors $\gamma_{k_i} n$ in the exponent with $\gamma_{k_i}(n - n_{k_i})$, with $n_{k_i} < J \ll n$. Thus

$$q_{kk'} \sim 2^{-\Gamma_r n^r |\log_2 \alpha_r|}, \tag{53}$$

where again $\alpha_r = 1 - \pi_r$. Comparing to Eqn. (49), we see that the contribution of the loose antichains ($r > 1$) is subdominant. Hence the leading order contribution to $\mathcal{P}(N_J^{\mathbf{a}_J} = R)$ comes from the tight antichains, with

$$\mathcal{P}(N_J^{\mathbf{a}_J} = R) \sim 2^{-\tilde{\gamma}|\log_2 \tilde{\alpha}| R n}, \tag{54}$$

where $\tilde{\gamma}|\log_2 \tilde{\alpha}|$ is the smallest of the $\gamma_{k+1}|\log_2 \alpha_k|$ for $k \in (1, \dots K-2)$.

3.3.2 Non-Antichain Contributions

We will now prove that for any allowed $c \in \Omega_J$ that is not the antichain, and for any pair of layers (k, k'), the probability for that an interval $[e_i^{(k)}, e_j^{(k')}]$ has an interior c with embedding $\vec{n}$ is, to leading order

$$q(c) \sim 2^{-\eta(c) n^a} \tag{55}$$

where $\eta(c) > 0$ and $a > 1$. This implies that for large n, the tight antichains of the preceding subsection are always dominant.

First note that any causal set c that is not an antichain has a height $h(c) > 1$, as defined in section 2. Hence unless $k' - k = r + 1 \geq h(c) + 1$, such a c cannot contribute, as we saw already for the 3-element chain in the $K = 4, J = 3$ calculation of section 3.2.3. Therefore we need $r \geq h(c) > 1$, which also means that $K - 2 \geq h(c) > 1$.

Consider a causal set c with an embedding labeled by $(k, k', \vec{n} = (\dots, n_{k_1}, n_{k_2}, \dots, n_{k_s}, \dots))$, with $k < k_1 \dots < k_s < k'$ and $\sum_{i=1}^{s} n_{k_i} = J$, $1 < h(c) \leq s \leq r$. The probability $q(c; k, k', \vec{n})$ will have two contributions:

- The set c must occur, with the embedding $(k, k', \vec{n})$, in the interior of the interval $[e_i^{(k)}, e_j^{(k')}]$. Call this probability $\mathcal{P}_1$.

- There must be no other paths connecting $e_i^{(k)}$ and $e_j^{(k')}$. Call this probability $\mathcal{P}_2$.

We first place an upper bound on $\mathcal{P}_1$. This probability will depend on the topology of c, and will in general be quite complicated. Given an exact embedding $c \hookrightarrow [e_i^{(k)}, e_j^{(k')}]$, though, this probability is certainly bounded by 1.

Note, however, that a specification of $\vec{n}$ does not completely determine the embedding of c, since we are still free to choose the locations of n_{k_1} elements in layer $\mathbb{L}_{k_1}$, n_{k_2} elements in layer $\mathbb{L}_{k_2}$, etc. This gives an additional combinatorial factor of

$$\binom{\gamma_{k_1} n}{n_{k_1}}\binom{\gamma_{k_2} n}{n_{k_2}}\dots\binom{\gamma_{k_s} n}{n_{k_s}} \sim n^{n_{k_1}} n^{n_{k_2}} \dots n^{n_{k_s}} = n^J \tag{56}$$

where the asymptotic behavior comes from the fact that $\gamma_{k_i} n \gg J > n_{k_i}$. Hence $\mathcal{P}_1$ is bounded above by a constant times n^J. Note that this matches the behavior we found for $J = 1, 2, 3$ in the $K = 4$ layer case of sections 3.2.1–3.2.3.

We now turn to $\mathcal{P}_2$, the probability that there is no other path from $e_i^{(k)}$ to $e_j^{(k')}$. For any finite J element interval, or equivalently any finite c, of cardinality $J \ll n$, there are only a finite number of paths from $e_i^{(k)}$ to $e_j^{(k')}$ in c. All others in C must be excluded. Therefore the argument is virtually the same as that of section 3.3.1. As in that case, the leading suppression comes from the need to exclude any additional $(k'-k+1)$-element paths that start at $e_i^{(k)} \in \mathbb{L}_k$, hit all of the intermediate layers $\mathbb{L}_{k+1}, \mathbb{L}_{k+2}, \dots, \mathbb{L}_{k'-1}$, and end at $e_j^{(k')} \in \mathbb{L}_{k'}$.

The one new complication comes from the word "additional": there may be $(k'-k+1)$-element paths that already lie in c, and others that intersect with c in a way that already excludes them, like the paths described after Eqn. 52. These should not be separately excluded. We can again obtain an upper bound, though, by considering paths that do not intersect c at all. That is, we must at least exclude any path that starts at $e_i^{(k)} \in \mathbb{L}_k$, links to any of the $\gamma_{k+1}n - n_{k+1}$ elements in layer $\mathbb{L}_{k+1}$ that are not in c, then to any of the $\gamma_{k+2}n - n_{k+2}$ elements in layer $\mathbb{L}_{k+2}$ that are not in c, and so on, and ends at $e_j^{(k')} \in \mathbb{L}_{k'}$. For large n, the number of such paths is

$$(\gamma_{k+1}n - n_{k+1})(\gamma_{k+2}n - n_{k+2})\dots(\gamma_{k'-1}n - n_{k'}) \sim \Gamma_r n^r + \mathcal{O}(n^{r-1}), \tag{57}$$

where, again, $r = k' - k - 1$ and $\Gamma_r = \gamma_{k+1}\gamma_{k+2}\dots\gamma_{k+r}$, just as in Eqn. (52). Thus, asymptotically,

$$q(c; k, k', \vec{n}) = \mathcal{P}_1\mathcal{P}_2 \lesssim 2^{-\Gamma_r n^r |\log_2 \alpha_r|}, \tag{58}$$

where again $\alpha_r = 1 - \pi_r$.

Comparing to Eqn. (49), we see that the probabilities are dominated by those of tight fit antichains, whose probabilities take the same form as those for the three-layer sets discussed in section 2.1. One might worry that an "entropy" factor from the number of causal sets c might overwhelm the suppression (58). But the number of J-element causal sets grows as $2^{J^2/4}$, so as long as $n \gg J$, this will not happen. An additional combinatorial factor can come from the number of weak partitions, that is, the number of ways of distributing N_J over different interval interiors. As discussed in section 3.1, however, these are also not large enough to overcome the exponential suppression (58).

4 Conclusions

Let us summarize our results and place them in context. Our interest is in causal set theory as a discrete model of spacetime. The fundamental problem we are addressing is the fact that almost all causal sets are not, in fact, anything like spacetimes. The set of causal sets is dominated by "layered" sets, sets with only a small number of moments of time, which must somehow be suppressed in any physically sensible model.

An obvious candidate for such a physically sensible model is the path integral (technically, the path sum) with the BDG action, the discrete version of the Einstein-Hilbert action. In [6], it was shown that for a wide range of coupling constants, this path integral does, in fact, very strongly suppress the simplest layered causal sets, the bilayer sets. In [7] this result was extended to all layered sets, but using the link action, a truncation of the BDG action, which no longer approximates the continuum Einstein-Hilbert action.

In [8], we took the next step, by showing that for three-layer sets this truncation is harmless. Specifically, in the limit that the number n of causal set elements becomes large, the difference between the BDG action and the link action for three-layer sets becomes negligible. What we have proved in this paper is that the same is true for K-layer causal sets, for any $K \ll n$. In other words, the BDG path sum—the causal set version of the standard gravitational path integral—strongly suppresses the entropically dominant layered causal sets.

It is worth emphasizing that the suppression is *extremely* strong. For a causal set with n elements, the suppression factor takes the form 2^{-kn^2}. If the discreteness scale is the Planck scale, a spacetime region of $1\,\mathrm{ns} \times 1\,\mathrm{cm}^3$ has $n \sim 10^{133}$, giving a suppression of order $2^{-10^{266}}$.

The BDG action is one of many nonlocal actions, which are constructed using a nonlocality scale $\xi \geq \ell$ [9, 10, 11]. As ξ becomes larger compared to the discreteness scale ℓ, it brings in more contributions from larger J-element intervals to the action. Since our analysis works for all $K, J \ll n$, though, these still do not contribute significantly to the nonlocal action, which therefore again reduces to the link action.

While an important question in causal set theory has been settled by these arguments, a number of other questions remain. Of the remaining non-layered causal sets in Ω_n, many, and plausibly most, are also not continuumlike. What role do they play in the semiclassical regime? Even if we were to restrict ourselves to continuumlike causal sets, these can have any spacetime dimension. Does an action of dimension d then pick out only spacetimes of dimension d and suppress those of dimension $d' \neq d$? These are important questions going forward.

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