Emergence of General Relativity from Loop Quantum Gravity

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Abstract

I show that general relativity emerges from loop quantum gravity, in a relational description of gravitation field in terms of coordinates defined by matter. Local Dirac observables and coherent states are constructed for an explicit evaluation of the dynamics. The dynamics of large scales conforms with general relativity, up to the corrections near singularities.

1 Introduction

Loop quantum gravity provides a microscopic description of spacetime. However, the large scale classical limit of the theory has been a major challenge, mainly for two reasons. First, diffeomorphism symmetry forbids physical external coordinates, making it difficult to extract local Dirac observables from the quantum theory. Second, the coherent states in the theory have been poorly understood.

In this letter, I propose a model that faithfully captures the full theory and overcomes both obstacles. The local Dirac observables are defined with respect to internal coordinates provided by the matter fields. The states minimizing the uncertainty of the local Dirac observables are then taken to be the coherent states. Moreover, the evolution of the expectation values of the observables for the coherent states are explicitly calculated. Lastly, the model uses the matter coordinates to compare the evolution to the dynamics in general relativity, and shows that they are consistent up to explicit corrections near singular regions of spacetime. Details will appear in [12].

2 Loop Quantum Gravity

Loop quantum gravity [1][2] is based on general relativity in Hamiltonian formalism, using Ashtekar's conjugate variables $\{E^a_i(X), A^j_b(X)\}(a, b = x, y, z;\ i, j = 1, 2, 3;\ X \equiv (x, y, z))$. The E fields specify the spatial orthonormal frames, while the connection fields A specify the intrinsic and extrinsic curvatures of a spatial slice. Four-dimensional diffeomorphism symmetry is split into spatial and temporal symmetries. For a specific time slicing, the action is invariant under spatial diffeomorphisms and local spatial rotations of the frames; the action is also invariant under changes of time slicing. Consequently, the theory is governed by momentum, Gauss, and Hamiltonian constraints.

Following the loop quantization procedure, quantum states called knot states are obtained. Each knot state is a functional of the A fields and is specified by the following information (fig.1): 1) an oriented differential topological graph with the edges connected to the nodes; 2) an $SU(2)$ representation j_n assigned to each edge; 3) a generalized Clebsh-Gordan coefficient (intertwiner) i_n assigned to each node, to contract invariantly with the

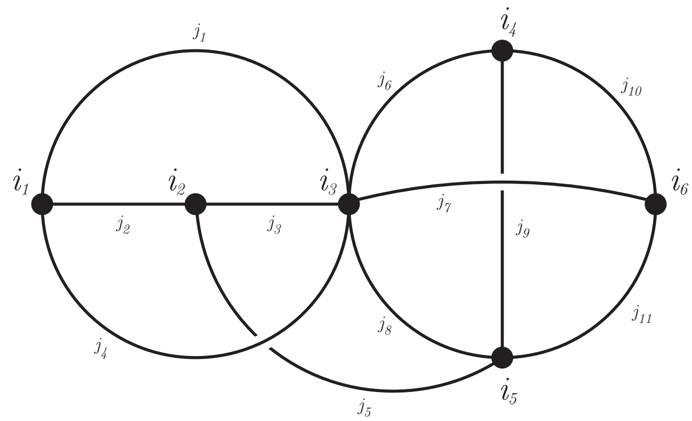

Figure 1: An example of knot states.

$SU(2)$ representations from the adjacent edges. The knot states solve the momentum and Gauss constraints and form a basis for knot space.

The operators in knot space can be constructed using the flux and holonomy operators [3]:

$$\left\{ \int_S \hat{E}_i^a ds_a \equiv \hat{F}_i(S) \right\}, \left\{ \mathcal{P} \hat{\exp}[\int_e A_b^j (\tau_j)_{\bar{l}}^{\bar{k}} de^b] \equiv \hat{h}(e)_{\bar{l}}^{\bar{k}} \right\} \tag{2.1}$$

Here, $\{S\}$ and $\{e\}$ are topological surfaces and curves defined relatively to the graph of every state in knot space. Acting on a knot state, the flux operators insert the $SU(2)$ generators into the edges cut through by the surfaces, while the holonomy operators either add new edges to the state or change the representations of the existing edges.

Two ingredients are needed for the dynamics in the classical limit – local degrees of freedom, and the corresponding coherent states that solve Hamiltonian constraints. These would be extremely difficult if dealing with a pure gravitational system. Instead, my model localizes the flux and holonomy operators using physical coordinates and frames defined by matter. The localized flux and holonomy operators serve as the local degrees of freedom, whose coherent dynamics will be compared to general relativity.

3 The model

Consider a system of gravitational fields and a set of N uncharged fermion fields. Loop quantization of the system [4] leads to the new knot states, whose intertwiners are now functions of the fermion fields at the nodes. The new knot states span the new knot space K', and the operators in K' are now composites of the flux, the holonomy, and also the fermion field operators acting on the nodes $\{n\}$.

Suppose that a set of operators containing the scalar $\{\phi^0(n), \phi^1(n), \phi^2(n), \phi^3(n)\}$, current $\{\hat{J}_I^i(n), \hat{U}_I^{\bar{i}}(n)\}$ and conjugate current $\{\hat{\bar{J}}_i^I(n), \hat{\bar{U}}_{\bar{i}}^{\bar{I}}(n)\}$ operators can be constructed from the fermion fields operators ($I = 1, 2, 3$; $\bar{I} = 1, 2$). Under such a setting, we consider the subspace

K of K' satisfying the conditions: (1) K contains only the knot states with an identical graph of six edges to each node, with a locally cubic lattice structure; (2) every state $|s\rangle$ in K satisfies $(\hat{\phi}^1, \hat{\phi}^2, \hat{\phi}^3)(n)|s\rangle = (x, y, z)(n)|s\rangle$, $(\hat{J}^i_I \hat{\bar{J}}^I_j)(n)|s\rangle = \delta^i_j|s\rangle$ and $(\hat{U}^{\bar{i}}_{\bar{I}} \hat{\bar{U}}^{\bar{I}}_{\bar{j}})(n)|s\rangle = \delta^{\bar{i}}_{\bar{j}}|s\rangle$ for any node n. Here the quantity $(x, y, z)(n)$ assigned to n is the eigenvalue of $(\hat{\phi}^1, \hat{\phi}^2, \hat{\phi}^3)(n)$ acting on $|s\rangle$, and adjacent eigenvalues are assigned to adjacent nodes. Given such conditions, we can view the scalar and current operators as providing physical coordinates and frames to K through their eigenvalues[1].

In K, the spatially localized flux and holonomy operators can be written out simply as [12]:

$$\left\{\widehat{F}(S_{X,\Delta X})_I \equiv \widehat{J}(X)^i_I \widehat{F}(S_{X,\Delta X})_i\right\}, \left\{\widehat{h}(e_{X,\Delta X})^{\bar{I}}_{\bar{J}} \equiv \widehat{\bar{U}}(X+\Delta X)^{\bar{I}}_{\bar{i}} \widehat{h}(e_{X,\Delta X})^{\bar{i}}_{\bar{j}} \widehat{U}(X)^{\bar{j}}_{\bar{J}}\right\}. \quad (3.1)$$

Here the nodes are identified with their spatial coordinates X, and ΔX refers to the coordinate gap across a specific edge. The surface $S_{X,\Delta X}$ cuts through the edge of the graph that connects X to $X + \Delta X$, at the end point X, and is oriented in the ΔX direction. The curve $e_{X,\Delta X}$ is identical to the edge going from X to $X + \Delta X$. As a notation, set $e_{X,\Delta X,\Delta Y}$ to be the minimal square loop on the graph with the outgoing and incoming edges assigned by ΔX and ΔY. Notice that all of the invariant components are expressed using the current frames.

The flux operators can be further localized. Note that any node X is shared by eight cubes, each with eight nodes and twelve edges. The operator $\hat{F}(S_{X,\Delta X})_I$ picks up the four cubes among them that contain the edge $e_{X,\Delta X}$. Utilizing the coordinates, $\hat{F}(S^{(n)}_{X,\Delta X})_I$ is defined to pick up the nth cube among the four ($n = 1, 2, 3, 4$). These operators satisfy [12]:

$$\sum_{n=1}^{4} \widehat{F}(S^{(n)}_{X,\Delta X})_I = \widehat{F}(S_{X,\Delta X})_I$$

$$\left[\hat{h}(e_{Z,\Delta Z,\Delta Y})^{\bar{J}}_{\bar{K}}, \hat{F}(S^{(n)}_{X,\Delta X})_I\right] = \frac{1}{2}\left[\hat{h}(e_{Z,\Delta Z,\Delta Y})^{\bar{J}}_{\bar{K}}, \hat{F}(S_{X,\Delta X})_I\right] \quad (3.2)$$

if $e_{Z,\Delta Z,\Delta Y}$ lies in the nth cube, otherwise the commutator vanishes. The sets $\{\hat{F}(S^{(n)}_{X,\Delta X})_I\}$ and $\{\hat{h}(e_{X,\Delta X})^{\bar{I}}_{\bar{J}}\}$ constitute the spatially local operators in the model.

The Hamiltonian constraint operator in the model, with which the localized Dirac observables must commute, has the gravitational term [3]:

$$\hat{H}_g(N) = C \sum_X N(X) \sum_{i,j,k=1}^{6} \mathrm{sgn}\left(\Delta X_i, \Delta X_j, \Delta X_k\right) \left(\hat{h}(e_{X,\Delta X_i,\Delta X_j}) - \hat{h}^{-1}(e_{X,\Delta X_i,\Delta X_j})\right)^{\bar{I}}_{\bar{J}}$$
$$\left(\hat{h}^{-1}(e_{X,\Delta X_k})\right)^{\bar{L}}_{\bar{I}} \cdot \left[\left(\hat{h}(e_{X,\Delta X_k})\right)^{\bar{J}}_{\bar{L}}, \hat{V}(X)\right] \quad (3.3)$$

where Immirzi parameter is set to be i, and

$$\hat{V}(X) = \left[\sum_{l,m,n=1}^{6} \mathrm{sgn}\left(\Delta X_l, \Delta X_m, \Delta X_n\right) \hat{\epsilon}^{PQR} \widehat{F}(S_{X,\Delta X_l})_P \widehat{F}(S_{X,\Delta X_m})_Q \widehat{F}(S_{X,\Delta X_n})_R\right]^{\frac{1}{2}} \quad (3.4)$$

[1]This is the quantum analog of using massless scalar fields to define harmonic coordinates; see [5] [6] for a related use of fields to determine coordinates.

In standard loop quantum gravity [3], the holonomy operators in the Hamiltonian constraint are taken over infinitesimal triangle loops. In the model, the holonomy operators in $\hat{H}_g$ are taken over minimal square loops existing in the graph. As a result, the standard loop quantum gravity is graph changing while the model is graph preserving. That is the only difference between the two.

The full Hamiltonian operator for the model is $\hat{H}(N) = \hat{H}_g(N) + l_p^2 \hat{H}_t(N)$, with $\hat{H}_t(N) = \sum_X N(X)\hat{P}_{\phi^0}(X)^2$, where P_{ϕ^0} is the effective conjugate momenta to ϕ^0. I will approximate the system by ignoring the matter back reaction, and will consider only the pure gravitational dynamics. Moreover, ϕ^0 is translated by $\hat{H}_t(N)$, while ϕ^1, ϕ^2 and ϕ^3 commute with the constraint; this condition is equivalent to a choice of the physical coordinates.

The model uses group average method [7] [8] [12] to solve the Hamiltonian constraint. Set $\hat{\mathbb{P}}$ to be the group average operator using $(i\hbar l_p^2)^{-1}\hat{H}(N)$ as the generators and summing over all $N(X)$. For every state $|s\rangle$ in K there is a state $\hat{\mathbb{P}}|s\rangle$ in K_{distr}, where K_{distr} is the distributional extension of K. With a proper inner product, $\hat{\mathbb{P}}K \equiv \mathbb{K}$ is the physical Hilbert space of quantum spacetimes which solves the Hamiltonian constraint. On the other hand, for each spatially localized operator $\hat{O}(X)$ on K there is an operator $\hat{\mathbb{P}}\hat{O}(X)\hat{\Pi}_T \equiv \hat{O}(X,T)$ on $\mathbb{K}$, where $\hat{\Pi}_T$ is the projection operator onto the eigenstates of $\hat{\phi}^0$ with eigenvalues in a small range around T. Finally, we identify the local Dirac observables in $\mathbb{K}$ to be [12] $\{\widehat{F}(S^{(n)}_{X,\Delta X}, T)_I\}$ and $\{\widehat{h}(e_{X,\Delta X}, T)^{\bar{I}}_{\bar{J}}\}$.

4 Dynamics

The next step is to identify the coherent states, on which the dynamics of the observables can be calculated. With the local Dirac observables at hand, we pick a state $|\mathcal{S}\rangle \in \mathbb{K}$ coherent in $\{\widehat{F}(S^{(n)}_{X,\Delta X}, T_0)_I\}$, $\{\widehat{h}(e_{X,\Delta X}, T_0)^{\bar{I}}_{\bar{J}}\}$, and also $\{\hat{P}_{\phi^0}(X, T_0)\}$. Notice that the coherency is established around clock time T_0, the moment when the quantum spacetime $|\mathcal{S}\rangle$ is expected to be classical . Through the coherent state $|\mathcal{S}\rangle$, the clock time derivatives of the local Dirac observables can be calculated at T_0 [12]:

$$\frac{d}{dT}\Big|_{T_0}\langle\hat{O}(X,T)\rangle = \langle\frac{1}{i\hbar l_p^2}\hat{\mathbb{P}}\left[\hat{O}(X), \hat{H}\left(\frac{1}{\langle\hat{P}_{\phi^0}(T_0)\rangle}\right)\right]\hat{\Pi}_{T_0}\rangle + \epsilon_T\left(\sim \Delta\phi^0\Delta P_{\phi^0}\right) \tag{4.1}$$

where ϵ_T is the correction due to quantum fluctuations of the clock field. Setting $\hat{O}$ to be the flux and holonomy observables, a long calculation leads to a result of the form [12]:

$$\begin{aligned}\frac{d}{dT}\Big|_{T_0}\langle\widehat{F}(S^{(n)}_{X,\Delta X}, T)_I\rangle &= \Phi_F\left(\left\{\langle\widehat{F}(S^{(n)}_{Z,\Delta Z}, T_0)_K\rangle\right\}, \left\{\langle\widehat{h}(e_{Z,\Delta Z}, T_0)^{\bar{K}}_{\bar{L}}\rangle\right\}\right)\\ &\quad +\epsilon_F\left(\sim \Delta F\Delta h, \langle[F,F]\rangle\right) + \epsilon_T\\ \frac{d}{dT}\Big|_{T_0}\langle\widehat{h}(e_{X,\Delta X}, T)^{\bar{I}}_{\bar{J}}\rangle &= \Phi_h\left(\left\{\langle\widehat{F}(S^{(n)}_{Z,\Delta Z}, T_0)_K\rangle\right\}, \left\{\langle\widehat{h}(e_{Z,\Delta Z}, T_0)^{\bar{K}}_{\bar{L}}\rangle\right\}\right)\\ &\quad +\epsilon_h\left(\sim \Delta F\Delta h\right) + \epsilon_T\end{aligned} \tag{4.2}$$

The Φ_F and Φ_h are obtained by explicitly evaluating the commutators in (4.1), then substituting the operators in the result by their expectation values. The substitution brings in

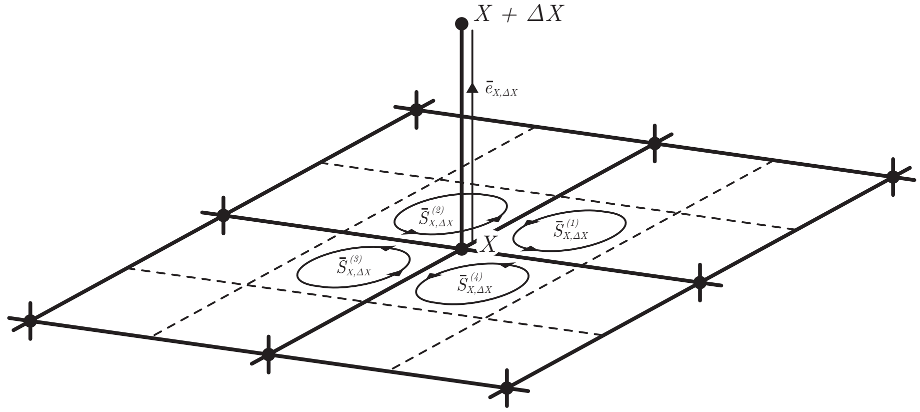

Figure 2: An illustration of $\bar{S}^{(n)}_{X,\Delta X}$ and $\bar{e}_{X,\Delta X}$ in $\mathbb{R}^3$. The solid lines represent a part of the embedded graph.

further quantum corrections ϵ_F and ϵ_h. Moreover, the Φ terms are expected to dominate in large scales with coherent states, since the quantum fluctuations are small compared with the expectation values.

The final step of the model is to relate to the classical equations of motions for the E and A fields. Using the spatial coordinates on the nodes, we may smoothly embed the graph of $|\mathcal{S}\rangle$ into $\mathbb{R}^3$. The embedding naturally induces a cell decomposition of $\mathbb{R}^3$, with embedded edges being the 1-skeletons and each face bounded by four embedded edges. Also, restrict the state $|\mathcal{S}\rangle$, the embedding and cell-decomposition such that any two adjacent cells are almost congruent parallelepipeds. After specifying the above, we pick a fitting algorithm mapping the values of $\{\langle\hat{F}(S^{(n)}_{Z,\Delta Z},T)_K\rangle, \langle\hat{h}(e_{Z,\Delta Z},T)^{\bar{K}}_{\bar{L}}\rangle\}$ to the values of $\{E^a_I(X,T), A^J_b(X,T)\}$ in $\mathbb{R}^3$, that contains the following rules:

$$\int_{\bar{S}^{(n)}_{X,\Delta X}} E^a_I(T) ds_a \equiv \langle\widehat{F}_I(S^{(n)}_{X,\Delta X},T)\rangle$$

$$\mathcal{P}\exp\left[\int_{\bar{e}_{X,\Delta X}} A^J_b(T)(\tau_J)^{\bar{K}}_{\bar{L}} de^b\right] \equiv \langle\widehat{h}(e_{X,\Delta X},T)^{\bar{K}}_{\bar{L}}\rangle \tag{4.3}$$

The embedded $\bar{S}^{(n)}_{X,\Delta X}$ and $\bar{e}_{X,\Delta X}$ are as shown in fig.2. Notice that each $\bar{S}^{(n)}_{X,\Delta X}$ is set to be a quarter segment of a face from the cell decomposition. To explore large scale limits, choose the state such that the corresponding E and A fields are almost constant within a single cell. Set $\bar{S}$ to be any (oriented) union of the faces from the cell decomposition, and $\bar{e}$ any union of embedded edges. Then, with a typical value of $|\Delta X| \equiv d$ equation (4.2) leads to [12]:

$$\begin{aligned}
\frac{d}{dT}\Big|_{T_0} \int_{\bar{S}} E_I^a(T) ds_a = &\left\{ \int_{\bar{S}} E_I^a ds_a, H\left(\frac{1}{\langle \hat{P}_{\phi^0}(T_0)\rangle}\right)\right\}(T_0) \\
&+ \Phi_F^{holo}(\bar{S})(\sim |A|d) + \Phi_F^{QG}(\bar{S})(\sim \hbar l_p^2) + \epsilon_F(\bar{S}) + \epsilon_T(\bar{S}) \\
\frac{d}{dT}\Big|_{T_0} \mathcal{P}\exp\left[\int_{\bar{e}} A_b^J(T)(\tau_J)_{\bar{L}}^{\bar{K}} de^b\right] = &\left\{ \mathcal{P}\exp\left[\int_{\bar{e}} A_b^J(\tau_J)_{\bar{L}}^{\bar{K}} de^b\right], H\left(\frac{1}{\langle \hat{P}_{\phi^0}(T_0)\rangle}\right)\right\}(T_0) \\
&+ \Phi_h^{holo}(\bar{e})(\sim |A|d) + \Phi_h^{QG}(\bar{e})(\sim \hbar l_p^2) + \epsilon_h(\bar{e}) + \epsilon_T(\bar{e})
\end{aligned} \tag{4.4}$$

where the $H(N)$ is exactly the classical Hamiltonian constraint:

$$H(N) = \int d^3X N \frac{E_B^b E_D^d}{\sqrt{\det E}} \epsilon^{BDN} \left(\partial_b A_d^N - \partial_d A_b^N + \epsilon^{NKL} A_b^K A_d^L\right) \tag{4.5}$$

The equations hold for any embedding, cell decomposition, and fitting algorithm specified above.

Each equation in (4.4) has the Poisson bracket, Φ^{holo}, and Φ^{QG} adding up to the corresponding Φ term in (4.2), and it also has the quantum correction ϵ terms. The Poisson brackets conform exacly with general relativity, with the shift functions equal to zero. The G^{holo} terms are the holonomy corrections due to the nonlinear contributions from the holonomies, which appear because of the discretized structure of space. The Φ^{QG} terms are the quantum geometry corrections due to Thiemann's regularization of the inverse volume factors, which signify the removal of the initial singularity [9][10] in the quantum theory.

General relativity will emerge if the following conditions hold for the state $|\mathcal{S}\rangle$. First, $|A|d$ everywhere must be small such that the holonomy corrections are small, with $|A|$ being the norm of the A fields. When the A fields are finite, the nodes must be embedded in $\mathbb{R}^3$ tightly, so the space would appear continuous for the observer. When the A fields become singular, this condition cannot hold and the corrections would take over. Second, $\hbar l_p^2$ must be small compared with d^2, so that the quantum geometry corrections and the quantum fluctuation corrections are small. When the space has a large volume, the measurements must be made in large scales. When the spatial volume becomes absolutely small the corrections become dominant.

5 Conclusions

I have shown that loop quantum gravity recovers the local dynamics of general relativity with the appropriate coherent state $|\mathcal{S}\rangle$. The holonomy and quantum geometry corrections are the trademarks of loop quantum gravity, and they are also explicitly formulated in the model in terms of E, A, and d. The stability of the emergence of general relativity with respect to the clock time is under investigation [11] [12].

In contrast to the fundamental symmetries, it is remarkable that emergent diffeomorphism symmetry in the model is broken by the corrections near singularities or in small scales. Therefore it would be of great interests to study the cosmological implications of the model, and would be instructive to the study of loop quantum cosmology from the full theory's perspective.

6 Acknowledgments

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