A spin foam model for general Lorentzian 4–geometries

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We derive simplicity constraints for the quantization of general Lorentzian 4–geometries. Our method is based on the correspondence between coherent states and classical bivectors and the minimization of associated uncertainties. For spacelike geometries, this scheme agrees with the master constraint method of the model by Engle, Pereira, Rovelli and Livine (EPRL). When it is applied to general Lorentzian geometries, we obtain new constraints that include the EPRL constraints as a special case. They imply a discrete area spectrum for both spacelike and timelike surfaces. We use these constraints to define a spin foam model for general Lorentzian 4–geometries.

I. INTRODUCTION

What happens if one describes geometry as a degree of freedom of quantum theory? Does geometry remain continuous or does it come in quanta? Are the singularities of classical general relativity resolved? Loop quantum gravity originates from the attempt to answer such questions. In the Hamiltonian framework, this led to canonical loop quantum gravity, and, in the path integral picture, it brought forth the notion of spin foam models[1]. A central result of both approaches is the discreteness of the area spectrum: it suggests that there is a minimal unit of area and that quantum geometry is indeed discrete.

The basic idea behind spin foam models is to divide spacetime into 4–simplices and to study how the geometry of these simplices can be quantized. More precisely, bivectors B are used to describe triangles and constraints are imposed, so that four bivectors are equivalent to a tetrahedron. The bivectors and constraints are then translated in a suitable way to the quantum theory. The main elements of this procedure were introduced by Barrett and Crane [4]. In the language of field theory, this corresponds to the transition from topological BF theory to gravity: the B in BF is constrained to be simple, so that it becomes the wedge product of two tetrads. For this reason, the constraints are called simplicity constraints.

In recent years, considerable progress was made in improving and clarifying this quantization process. The key to this progress were two new developments: firstly, Engle, Pereira, Rovelli and Livine (EPRL) defined a new model that resolves certain longstanding problems

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[1] See [1], [2] and [3] for a review.

with the Barrett–Crane model and establishes a link with canonical loop quantum gravity [5]. The quantization is based on a so–called master constraint, which is the sum of the squares of all simplicity constraints. The second important innovation was the coherent state technique introduced by Livine and Speziale [6]. It provides a better geometric understanding of quantum states, and led to the construction of the Freidel–Krasnov (FK) model [7, 8]. In this model, simplicity is imposed on expectation values of coherent states.

Both of these developments spurred further results: The FK model was reexpressed as a path integral with a simple action [9]. The semiclassical limit of the new models was analyzed [10–12], likewise the graviton propagator [13]. It was found that intertwiner states can be understood in geometric terms [14]. Recently, coherent states were constructed for entire 3–geometries [15, 16].

The present paper is motivated by two questions. The first question concerns the relation between the EPRL and coherent state approach. What is the connection between these two lines of thought? Is there a way to understand the master constraint in terms of coherent states? In the Riemannian case, we know from explicit comparison that the EPRL and FK model are closely related [9]. To this extent, the coherent state ideas apply also to the EPRL model. In the Lorentzian case, however, it is not clear how a derivation from coherent states should look like[2]. It could be very useful to have one, since coherent states provide a particularly transparent quantization of simplicity constraints.

The second motivation for this paper comes from the fact that the EPRL model is only defined for spacelike area bivectors. That is, it can only describe geometries in which all surfaces are spacelike. The obvious question is therefore: how can one specify a model that covers realistic Lorentzian geometries, where bivectors can be both spacelike and timelike?

In this article, we obtain an answer to our first question, and it turns out that we can also resolve the second question with this knowledge. What we find is a coherent state method that reproduces the EPRL constraints and applies at the same time to the cases which were not yet covered in this model. On the one hand, we recover the EPRL constraints for tetrahedra with timelike normals. Thus, the method gives the desired coherent state derivation of the EPRL spin foams. The same scheme, however, works also for tetrahedra with spacelike normals. In this case, it results in two new sets of constraints that pertain to spacelike and timelike triangles within such tetrahedra.

Our method is inspired by the Riemannian FK model and extends its logic by an additional condition. We demand that there are quantum states for which

1. The expectation value of the bivector operator is simple.

2. The uncertainty in the bivector is minimal[3].

With this, we enforce the existence of coherent states that correspond to classical simple bivectors. Such states can only exist in certain irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ and its subgroups. This determines constraints on irreps and we interpret them as the quantum version of the simplicity constraints.

Based on this, we define a new spin foam model that gives a quantization of tetrahedra with spacelike and timelike normals, and hence a quantization of general Lorentzian geometries. In addition, a coherent state vertex amplitude is specified. We should remark that

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[2] A proposal was made in ref. [7].

[3] The precise meaning of "minimal" is stated in sec. II and sec. III.

coherent states are not necessary, and, in fact, not used, when defining the spin foam sum. In our treatment, the coherent states are only essential in the derivation of the simplicity constraints.

The paper is organized as follows: in sec. II we introduce our coherent state method. It is applied to tetrahedra with a timelike normal and the constraints of the EPRL model are reproduced. In sec. III, we treat tetrahedra with a spacelike normal and obtain two new constraints that refer to spacelike and timelike triangles respectively. Section IV summarizes the constraints for the different cases. In sec. V, we use these constraints to define a spin foam model for general Lorentzian 4–geometries.

Conventions

At the outset, we make a few remarks on notation to avoid confusion due to differing conventions in the literature.

Our sign convention for the spacetime metric is $(+,-,-,-)$, and $(+,-,-)$ for 3d Minkowski spacetime. Moreover, $\epsilon_{0123} = +1$. The Immirzi parameter is γ and we assume that $\gamma > 0$. Unit normal vectors of tetrahedra are denoted by U. The letter N stands for unit normal vectors of triangles. We have the Hodge dual operator $\star$, and $\star B$ is the area bivector of triangles.

Unitary irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ are labelled by pairs (ρ,n), where $\rho \in \mathbb{R}$ and $n \in \mathbb{Z}_+$. With regard to irreducible representations of subgroups, we follow the notation in [17]: $\mathcal{D}_j$ stands for SU(2) irreps. The discrete and continuous series of SU(1,1) are designated by $\mathcal{D}_j^\pm$ and $\mathcal{C}_s^\epsilon$ respectively. Representation matrices are symbolized by $D^{(\rho,n)}(g)$ in the $\mathrm{SL}(2,\mathbb{C})$ case, and by $D^j(g)$ for subgroups.

II. SIMPLICITY CONSTRAINTS FOR THE SU(2) REDUCTION

In this section, we introduce our procedure for deriving the simplicity constraints on representations of the Lorentz group. We do so by applying it to a situation that has been already treated in the EPRL model [5].

As explained below, the simplicity constraints split into two main categories, depending on whether normal vectors U of tetrahedra are timelike or spacelike. The case considered in [5] is the one for timelike U. At the quantum level, this choice of a timelike normal is reflected by the fact that unitary irreps of $\mathrm{SL}(2,\mathbb{C})$ are analyzed in terms of irreps of the subgroup SU(2). For this case, we will find that our method produces the same simplicity constraints as the master constraint employed in the EPRL paper.

Encouraged by this agreement, we will then proceed to section III, where we apply our technique to the case when U is spacelike. Then, irreps of $\mathrm{SL}(2,\mathbb{C})$ will be decomposed into irreps of SU(1,1) and we will obtain a new set of simplicity constraints.

A. Classcial variables

Our starting point is the SO(1,3) bivector

$$J = B + \frac{1}{\gamma} \star B\,, \tag{1}$$

which is used when defining gravity as a constrained BF theory with Immirzi parameter γ. The idea is to constrain B in such a way that B becomes $B = \star(E \wedge E)$, where E is a co–tetrad. Under this constraint, the action

$$S = \int J \wedge F = \int \left(B \wedge F + \frac{1}{\gamma} \star B \wedge F \right) \tag{2}$$

reduces to the Hilbert–Palatini action with an Immirzi term.

Like in [5], we denote the total bivector by the letter J. This choice is convenient, since J will be closely related to the generators of the Lorentz group. After quantization, the classical bivector J^{IJ} will arise from expectation values of SO(1,3) generators J^{IJ}.

At the classical level, the derivation of the simplicity constraints proceeds as follows: first we state the simplicity constraints for B, which ensure that $B = \star(E \wedge E)$. These constraints involve a vector U which is normal to the dual bivector $\star B$. Since U can be either timelike or spacelike, one obtains two classes of simplicity constraints. In the present section, we treat the case where U is timelike, as in the EPRL model. These constraints are then expressed as constraints on J—ready to be translated to the quantum theory.

In full generality, the simplicity constraint reads

$$U \cdot \star B = 0 \,, \tag{3}$$

where U is a Lorentz vector of unit norm, i.e. $U^2 = \pm 1$. This constraint implies that the dual $\star B$ is simple and of the form[4]

$$\star\, B = E_1 \wedge E_2 \,, \tag{4}$$

where E_1 and E_2 are two 4–vectors orthogonal to U (for a derivation see e.g. [7]). Moreover, B is simple and given by

$$B = A\, U \wedge N \,, \tag{5}$$

where N is a unit norm 4–vector such that $U \cdot N = 0$ and $N \cdot E_1 = N \cdot E_2 = 0$. The coefficient A in (5) is equal to the area

$$A = \sqrt{|E_1^2 E_2^2 - (E_1 \cdot E_2)^2|} \tag{6}$$

of the parallelogram spanned by E_1 and E_2.

Equations (4) and (5) elucidate the geometric meaning of the 4–vectors U and N. E_1 and E_2 correspond to the tetrad and in a discrete setting they can be regarded as the two edges of a triangle. The dual $\star B$ is the so–called area bivector of this triangle, and it is orthogonal to both U and N. To obtain a tetrahedron, one starts from four bivectors B_a, $a = 1, \ldots, 4$, and imposes the simplicity constraint (3) on each of them:

$$U \cdot \star B_a = 0 \,, \quad a = 1, \ldots, 4 \,. \tag{7}$$

This means that all four bivectors are simple and that they span a 3d subspace orthogonal to U. When the closure constraint

$$\sum_{a=1}^{4} B_a = 0 \,, \tag{8}$$

[4] $V \wedge W$ stands for the bivector $(V \wedge W)^{IJ} = V^I W^J - W^I V^J$.

is supplemented, it follows that the bivectors B_a are equivalent to a tetrahedron. The vector U is the normal of this tetrahedron, the four vectors N_a (associated to the bivectors B_a) are the normals to the four triangles, and the E's are the edges of the triangles. In this way, geometry results from constraints on bivectors. In spin foam models the analog of (7) is imposed on representations of the gauge group, while the closure constraint arises dynamically.

Let us assume now that U is timelike and gauge–fixed to $U = (1, 0, 0, 0)$. Then, the simplicity constraint becomes $(\star B)^{0i} = 0$ and $\star B$ has to be spacelike. Next we express this constraint in terms of the bivector J. Solving for J in (1) gives

$$B = \frac{\gamma^2}{\gamma^2 + 1}\left(J - \frac{1}{\gamma} \star J\right) . \tag{9}$$

It follows that the gauge–fixed simplicity constraint is equivalent to

$$J^i + \frac{1}{\gamma} K^i = 0 \tag{10}$$

where we use the usual definitions

$$J^i = \frac{1}{2}\, \epsilon^{0i}{}_{jk} J^{jk} \qquad \text{and} \qquad K^i = J^{0i} . \tag{11}$$

Eq. (10) will be the central equation for the derivation of the simplicity constraints in the quantum theory.

It is also useful to express the normal vector N in terms of J_i and K_i. On the one hand, we have that

$$B^{ij} = 0 , \qquad B^{0i} = A\, N^i . \tag{12}$$

On the other hand,

$$\begin{aligned} B^{0i} &= \frac{\gamma^2}{\gamma^2 + 1}\left(J^{0i} - \frac{1}{2\gamma}\epsilon^{0i}{}_{jk} J^{jk}\right) && (13) \\ &= \frac{\gamma^2}{\gamma^2 + 1}\left(K^i - \frac{1}{\gamma} J^i\right) && (14) \end{aligned}$$

Using the simplicity constraint (10) this yields

$$A\, N^i = -\gamma J^i \tag{15}$$

Therefore, $N = (0, \vec{N})$, where $A\vec{N} = -\gamma \vec{J}$. Observe also that the vector $\vec{N}$ is a point in the 2–sphere

$$S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1) . \tag{16}$$

and hence a point in the coadjoint orbit (or phase space) of spin $j = 1$.

B. Quantum states

Next we will describe our way of translating the simplicity constraint to quantum states. As in the EPRL model, we will not do this in a manifestly covariant way. We will start from eq. (10), which is the simplicity constraint after gauge–fixing U to $(1, 0, 0, 0)$.

The quantum analog of the bivectors are states in unitary irreducible representations of $\mathrm{SL}(2,\mathbb{C})$. The latter are labelled by pairs (ρ, n), where $\rho \in \mathbb{R}$ and $n \in \mathbb{Z}_+$. Since $U = (1,0,0,0)$ singles out SU(2) as a little group of $\mathrm{SL}(2,\mathbb{C})$, it is convenient to express everything in an "SU(2) friendly" way. This can be done by using the decomposition of $\mathrm{SL}(2,\mathbb{C})$ irreps into SU(2) irreps: namely,

$$\mathcal{H}_{(\rho,n)} \simeq \bigoplus_{j=n/2}^{\infty} \mathcal{D}_j \tag{17}$$

where $\mathcal{H}_{(\rho,n)}$ denotes the Hilbert space of the irrep (ρ, n) and $\mathcal{D}_j$ stands for the spin j irrep of SU(2) (see e.g. [22]). The corresponding completeness relation reads

$$\mathbb{1}_{(\rho,n)} = \sum_{j=n/2}^{\infty} \sum_{m=-j}^{j} |\Psi_{j\,m}\rangle \langle \Psi_{j\,m}| \,. \tag{18}$$

The states $|\Psi_{j\,m}\rangle$, $m = -j, \ldots, j$, span a subspace of $\mathcal{H}_{(\rho,n)}$ that is isomorphic to $\mathcal{D}_j$, so we identify them with states $|j\,m\rangle$ of $\mathcal{D}_j$.

Our recipe for quantizing the simplicity constraints is formulated as follows. We require the existence of quantum states for which the expectation value of the bivector J^{IJ} satisfies the simplicity constraints. Moreover, the quantum uncertainty in this bivector should be small. Here, we work with a gauge–fixing, so the components of J^{IJ} are organized in terms of $\vec{J}$ and $\vec{K}$. Let us define associated lengths $|\vec{J}| \equiv |\langle \vec{J} \rangle|$ and $|\vec{K}| \equiv |\langle \vec{K} \rangle|$, where $\langle \ \rangle$ stands for the expectation value w.r.t. the quantum state. We then demand that

$$\frac{\Delta J}{|\vec{J}|} = O\left(\frac{1}{\sqrt{|\vec{J}|}}\right) , \tag{19}$$

$$\langle \vec{J} \rangle + \frac{1}{\gamma} \langle \vec{K} \rangle = O(1) \,, \tag{20}$$

$$\frac{\Delta K}{|\vec{K}|} = O\left(\frac{1}{\sqrt{|\vec{K}|}}\right) . \tag{21}$$

Through these three conditions we establish a correspondence between classical variables and semiclassical states: the first requirement says that the states should be peaked around classical values of $\vec{J}$. The second condition states that their expectation values fulfill the simplicity constraint. The last point adds that the states should not only be peaked in $\vec{J}$, but also in the remaining components $\vec{K}$.

The first condition is easily met by using SU(2) coherent states [18]: these states have the form

$$|j\,g\rangle \equiv D^j(g)|j\,j\rangle \,. \tag{22}$$

and arise from SU(2) rotations of the "reference" coherent state $|j\,j\rangle$. From such states we get

$$\frac{\Delta J}{|\vec{J}|} = \frac{\sqrt{j}}{j} = \frac{1}{\sqrt{j}} = O\left(\frac{1}{\sqrt{|\vec{J}|}}\right) . \tag{23}$$

Equation (20) is more subtle, as it involves $\mathrm{SL}(2,\mathbb{C})$ generators outside of SU(2). Since both $\vec{J}$ and $\vec{K}$ transform as vectors under SU(2), it is sufficient to impose (20) on the reference state $|j\,j\rangle$. If it is satisfied for $|j\,j\rangle$, then it will be true for all coherent states $|j\,g\rangle$. Therefore, we require[5]

$$\langle j\,j|\vec{J}|j\,j\rangle = -\frac{1}{\gamma}\langle j\,j|\vec{K}|j\,j\rangle\,. \tag{24}$$

It is clear from commutation relations that J^1, J^2, K^1 and K^2 change the eigenvalue of J^3, so their expectation values will be zero. We therefore only need to consider J^3 and K^3. The action of K^3 is given by[6]

$$\begin{aligned} K^3|jm\rangle \;=\; & -\sqrt{(j+m+1)(j-m+1)}\,C_{j+1}\,|j+1\,m\rangle \\ & - mA_j\,|jm\rangle \\ & + \sqrt{(j-m)(j+m)}\,C_j\,|j-1\,m\rangle\,, \end{aligned} \tag{25}$$

where

$$A_j = \frac{\rho\, n}{4j(j+1)}\,, \tag{26}$$

and

$$C_j = \frac{\mathrm{i}}{j}\sqrt{\frac{(j^2-\frac{n^2}{4})(j^2+\frac{\rho^2}{4})}{4j^2-1}}\,. \tag{27}$$

Hence eq. (24) leads to

$$j = -\frac{1}{\gamma}(-jA_j) \qquad \text{or} \qquad \gamma = A_j = \frac{\rho\, n}{4j(j+1)}\,. \tag{28}$$

In order to deal with the variance in $\vec{K}$, we recall the Casimirs of $\mathrm{SL}(2,\mathbb{C})$:

$$C_1 \;=\; 2\left(J^2-K^2\right) = \frac{1}{2}(n^2-\rho^2-4) \tag{29}$$

$$C_2 \;=\; -4J\cdot K = n\rho \tag{30}$$

As a result, one gets

$$(\Delta K)^2 \;=\; \langle K^2\rangle - \langle K\rangle^2 \tag{31}$$

$$\;=\; \langle J^2 - \frac{1}{4}(n^2-\rho^2-4)\rangle - \langle K\rangle^2 \tag{32}$$

$$\;=\; j(j+1) - \frac{1}{4}(n^2-\rho^2-4) - j^2A_j^2\,. \tag{33}$$

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[5] For the next couple of lines we omit the order symbol $O(1)$.
[6] See, for instance, [19].

By inserting the simplicity constraint $A_j = \gamma$, we obtain furthermore

$$\begin{aligned}
(\Delta K)^2 &= \frac{1}{4\gamma}\rho\, n - \frac{1}{4}(n^2 - \rho^2 - 4) - j^2\gamma^2 && (34)\\
&= \frac{1}{4\gamma}\rho\, n - \frac{1}{4}(n^2 - \rho^2) - \gamma^2 j(j+1) + \gamma^2 j + 1 && (35)\\
&= \frac{1}{4\gamma}\rho\, n - \frac{1}{4}(n^2 - \rho^2) - \frac{\gamma}{4}\rho\, n + \gamma^2 j + 1 && (36)\\
&= \frac{1}{4}\left(\frac{1}{\gamma}\rho\, n - n^2 + \rho^2 - \gamma\rho\, n\right) + j\gamma^2 + 1 && (37)\\
&= \frac{1}{4}\Big(\rho - \gamma n\Big)\Big(\rho + \frac{n}{\gamma}\Big) + \gamma^2 j + 1\,. && (38)
\end{aligned}$$

If one sets

$$\rho = \gamma n \tag{39}$$

or

$$\rho = -\frac{n}{\gamma}\,, \tag{40}$$

the first term vanishes, and

$$\frac{\Delta K}{|\vec{K}|} = \frac{\sqrt{\gamma^2 j + 1}}{\gamma j} = O\left(\frac{1}{\sqrt{|\vec{K}|}}\right)\,. \tag{41}$$

However, when plugging back (39) into the simplicity constraint (28), we obtain

$$n^2 = 4j(j+1)\,. \tag{42}$$

Since this cannot be solved for generic values of n and j, we proceed like in [5] and adopt the approximate solution

$$j = n/2\,. \tag{43}$$

One can check that (39) and (43) fulfill conditions (20) and (21). When (40) is inserted into (28), on the other hand, we get

$$n^2 = -4\gamma^2 j(j+1)\,. \tag{44}$$

This has no solution. Thus, our final result are the constraints $\rho = \gamma n$ and $j = n/2$, which are the same constraints as in the EPRL model!

The area spectrum can be derived by squaring the classical equation (15) and setting the right–hand side equal to the expectation value of the coherent state. This gives us the quantum area

$$A = \gamma\sqrt{\langle J^2\rangle} = \gamma\sqrt{j(j+1)}\,. \tag{45}$$

Up to this point, we have just used a new perspective to obtain something that was already known, i.e. the constraints of the EPRL model. This shows that the EPRL master constraint is equivalent to a set of semiclassical constraints, namely to the requirement that there exist quantum states that are peaked in $\vec{J}$ and $\vec{K}$, and that their expectation values satisfy the classical simplicity constraint.

In the next section, we will apply our procedure to derive something new: we will provide a prescription for spacelike U, and hence for bivectors $\star B$ that can be both spacelike and timelike.

III. SIMPLICITY CONSTRAINTS FOR THE SU(1,1) REDUCTION

A. Classical variables

In this section, the normal vector U is assumed to be spacelike, and we gauge–fix it to $U = (0,0,0,1)$. Then, the classical simplicity constraint is $(\star B)^{3i} = 0$, where $i = 0,1,2$. A short calculation shows that this is equivalent to

$$K^1 - \frac{1}{\gamma} J^1 = 0\,, \tag{46}$$

$$K^2 - \frac{1}{\gamma} J^2 = 0\,, \tag{47}$$

$$J^3 + \frac{1}{\gamma} K^3 = 0\,. \tag{48}$$

By defining the quantities

$$\begin{aligned} F^0 &= J^3\,, \quad F^1 = K^1\,, \quad F^2 = K^2 \\ G^0 &= K^3\,, \; G^1 = -J^1\,, \; G^2 = -J^2 \end{aligned}$$

one can write these constraints in the more symmetric form

$$F^i + \frac{1}{\gamma} G^i = 0\,, \qquad i = 0,1,2\,. \tag{49}$$

Here and below we use the indices i,j to denote vectors in 3–dimensional Minkowski spacetime, in the same way that indices i,j were used for vectors of 3–dimensional Euclidean space in the previous section. Inspection of the commutation relations reveals, in fact, that F and G transform like 3d Minkowski vectors under SU(1,1) [20]:

$$[F^i, F^j] = \mathrm{i}\, C^{ij}{}_k F^k\,, \tag{50}$$

$$[F^i, G^j] = \mathrm{i}\, C^{ij}{}_k G^k\,, \tag{51}$$

where

$$C^{01}{}_2 = -C^{10}{}_2 = C^{20}{}_1 = -C^{02}{}_1 = C^{21}{}_0 = -C^{12}{}_0 = 1\,. \tag{52}$$

Thus, eq. (49) has a similar structure as eq. (10): it relates two vectors that have the same transformation property under the little group, and one of the vectors is the generator of the little group.

As before, we determine the relation between the vector N and the generators. On the one hand,

$$B^{ij} = 0\,, \qquad B^{3i} = A\, N^i\,, \qquad \text{where } i,j = 0,1,2\,. \tag{53}$$

On the other hand, we also have

$$B^{3i} = \frac{\gamma^2}{\gamma^2+1}\left(J^{3i} - \frac{1}{2\gamma}\epsilon^{3i}{}_{jk}J^{jk}\right) \tag{54}$$

For $i = 0$, this gives

$$B^{30} = \frac{\gamma^2}{\gamma^2+1}\left(J^{30} - \frac{1}{\gamma}\epsilon^{30}{}_{12}J^{12}\right) \tag{55}$$

$$= \frac{\gamma^2}{\gamma^2+1}\left(-K^3 + \frac{1}{\gamma}J^3\right) \tag{56}$$

On account of the simplicity constraint this reduces to

$$AN^0 = \gamma J^3\,. \tag{57}$$

Similarly, we obtain

$$AN^1 = -\gamma K^2 \qquad \text{and} \qquad AN^2 = \gamma K^1 \tag{58}$$

for the cases $i = 1$ and $i = 2$. Altogether we get

$$A\begin{pmatrix} N^0 \\ N^1 \\ N^2 \end{pmatrix} = \gamma \begin{pmatrix} J^3 \\ -K^2 \\ K^1 \end{pmatrix} = \gamma \begin{pmatrix} F^0 \\ -F^2 \\ F^1 \end{pmatrix}\,. \tag{59}$$

Like in the SU(2) case, the vector $\vec{N} = (N^0, N^1, N^2)$ is related to quotient spaces of SU(1,1): timelike vectors $\vec{N}$ coordinatize the two–sheeted hyperboloid

$$\mathbb{H}_+ \cup \mathbb{H}_-\,, \qquad \mathbb{H}_\pm = \{\,\vec{N} \mid N^2 = 1\,, N^0 \gtrless 0\,\}\,, \tag{60}$$

and each sheet is isomorphic to the quotient $\mathrm{SU}(1,1)/\mathrm{U}(1)$. Spacelike $\vec{N}$ parametrize the single–sheeted hyperboloid $\mathcal{H}_s = \{\vec{N}\,|\,N^2 = -1\}$. This hyperboloid is isomorphic to the quotient $\mathrm{SU}(1,1)/(\mathrm{G}_1 \otimes \mathbb{Z}_2)$, where G_1 is the one–parameter subgroup of SU(1,1) generated by K_1 [21]. Appendix A describes the details of these isomorphisms.

B. Quantum states

Since we have set U equal to $(0,0,0,1)$, the little group is SU(1,1). Accordingly, we should work with unitary irreducible representations of SU(1,1). These come in a discrete and a continuous series. In both cases, the irreps can be built from eigenstates $|j\,m\rangle$ of J_3, satisfying

$$\langle j\,m|j\,m'\rangle \;=\; \delta_{mm'}\,, \tag{61}$$

$$J_3\,|j\,m\rangle \;=\; m|j\,m\rangle\,. \tag{62}$$

Combinations of K_1 and K_2 act as raising and lowering operators. The Casimir is given by $Q = (J^3)^2 - (K^1)^2 - (K^2)^2$.

In irreps of the discrete series, one has

$$Q\,|j\,m\rangle = j(j-1)|j\,m\rangle\,, \qquad \text{where } j = \tfrac{1}{2},\,1,\,\tfrac{3}{2},\,\ldots \tag{63}$$

The eigenvalue m can take the values

$$m = j,\ j+1,\ j+2,\ \ldots \qquad \text{or} \qquad m = -j,\ -j-1,\ -j-2,\ \ldots \tag{64}$$

We use the same notation as in [17] and denote the irrep consisting of states $|j\,m\rangle$ with $m \gtrless 0$ by $\mathcal{D}_j^{\pm}$. The fact that $Q = j(j-1)$ is positive for $j \geq 3/2$ suggests that the discrete series contains coherent states corresponding to timelike 3–vectors.

For the continuous series,

$$Q\,|j\,m\rangle = j(j+1)|j\,m\rangle\,, \qquad \text{where } j = -\tfrac{1}{2} + \mathrm{i}s, \quad 0 < s < \infty, \tag{65}$$

and

$$m = 0,\,\pm 1,\,\pm 2,\,\ldots \qquad \text{or} \qquad m = \pm\frac{1}{2},\,\pm\frac{3}{2},\,\ldots \tag{66}$$

Irreps of this series are denoted by $\mathcal{C}_j^{\epsilon}$. The label $\epsilon = 0, \frac{1}{2}$ designates the irreps with integer m and half–integer m respectively. In this case, the Casimir $Q = j(j+1) = -s^2 - \frac{1}{4}$ is always negative, so we expect coherent states to be associated to spacelike 3–vectors.

Similarly as for SU(2), the $\mathrm{SL}(2,\mathbb{C})$ irrep (ρ, n) can be expanded in an SU(1,1)–adapted basis (see [20, 22] and also [23]). The resulting completeness relation involves states $\left|\Psi^{\pm}_{j\,m}\right\rangle$ and $|\Psi^{(\alpha)}_{s\,m}\rangle$, $\alpha = 1, 2$, that correspond to states $|j\,m\rangle$ in the discrete and continuous series respectively:

$$\begin{aligned}
\mathbb{1}_{(\rho,n)} \;=\; & \sum_{j>0}^{n/2} \sum_{m=j}^{\infty} \left|\Psi^{+}_{j\,m}\right\rangle \left\langle \Psi^{+}_{j\,m}\right| \\
& + \int_0^\infty \mathrm{d}s\; \mu_\epsilon(s) \sum_{\pm m=\epsilon}^{\infty} \left|\Psi^{(1)}_{s\,m}\right\rangle \left\langle \Psi^{(1)}_{s\,m}\right| \\
& + \sum_{j>0}^{n/2} \sum_{-m=j}^{\infty} \left|\Psi^{-}_{j\,m}\right\rangle \left\langle \Psi^{-}_{j\,m}\right| \\
& + \int_0^\infty \mathrm{d}s\; \mu_\epsilon(s) \sum_{\pm m=\epsilon}^{\infty} \left|\Psi^{(2)}_{s\,m}\right\rangle \left\langle \Psi^{(2)}_{s\,m}\right|
\end{aligned} \tag{67}$$

The sum over j extends over values such that $j - n/2$ is integral. Moreover, ϵ has a value such that $\epsilon - n/2$ is an integer. The measure factors are given by

$$\mu_\epsilon(s) = \begin{cases} 2s\tanh(\pi s)\,, & \epsilon = 0\,, \\[2ex] 2s\coth(\pi s)\,, & \epsilon = 1/2\,. \end{cases} \tag{68}$$

When SL(2, $\mathbb{C}$) is restricted to SU(1,1), the states $\left|\Psi^{\pm}_{j\,m}\right\rangle$ furnish irreducible representations that are isomorphic to those of the discrete series:

$$\left\langle\Psi^{\pm}_{j\,m'}\middle|\Psi^{\pm}_{j\,m}\right\rangle = \delta_{m'm} \tag{69}$$

$$\left\langle\Psi^{\pm}_{j\,m'}\middle|D^{(\rho,n)}(g)\middle|\Psi^{\pm}_{j\,m}\right\rangle = \langle j\,m'|D^{j}(g)|j\,m\rangle \quad \text{for } g \in \mathrm{SU}(1,1). \tag{70}$$

We therefore identify $\left|\Psi^{\pm}_{j\,m}\right\rangle$ with $|j\,m\rangle$ in $\mathcal{D}^{\mp}_{j}$.

With regard to the continuous series, the situation is more subtle. Firstly, the continuous series states $\left|\Psi^{(\alpha)}_{s\,m}\right\rangle$ appear twice, which is indicated by the index $\alpha = 1, 2$. Moreover, these states are not normalizable:

$$\left\langle\Psi^{(\alpha')}_{s'\,m'}\middle|\Psi^{(\alpha)}_{s\,m}\right\rangle = \frac{\delta(s'-s)}{\mu_{\epsilon}(s)}\,\delta_{\alpha'\alpha}\,\delta_{m'm}\,, \tag{71}$$

As a result, the analog of eq. (70) requires an integration over s:

$$\int\limits_{0}^{\infty} \mathrm{d}s'\;\mu_{\epsilon}(s')\left\langle\Psi^{(\alpha)}_{s'\,m'}\middle|D^{(\rho,n)}(g)\middle|\Psi^{(\alpha)}_{s\,m}\right\rangle = \langle j\,m'|D^{j}(g)|j\,m\rangle \quad \text{for } g \in \mathrm{SU}(1,1). \tag{72}$$

With this qualification in mind, we can say that

$$\mathcal{H}_{(\rho,n)} \quad \simeq \quad \left(\bigoplus_{j>0}^{n/2}\mathcal{D}^{+}_{j} \oplus \int\limits_{0}^{\infty}\mathrm{d}s\;\mathcal{C}^{\epsilon}_{s}\right) \oplus \left(\bigoplus_{j>0}^{n/2}\mathcal{D}^{-}_{j} \oplus \int\limits_{0}^{\infty}\mathrm{d}s\;\mathcal{C}^{\epsilon}_{s}\right)\,. \tag{73}$$

The proof of this decomposition proceeds similarly as for SU(2) [22]. By homogeneity, the representation of SL(2, $\mathbb{C}$) on functions of $\mathbb{C}^2$ reduces to a representation on pairs of functions (φ_1, φ_2) of SU(1,1). Such functions can be expanded into matrix elements of SU(1,1) [23], in analogy to the Peter–Weyl theorem for compact groups. Covariance properties require that certain irreps do not appear in this decomposition—in the same way that spins $j < n/2$ do not appear in the decomposition into SU(2) irreps. Thus, one obtains the first two lines in (67) from the first component of the pair, and the last two lines from the second component. Explicitly, the states $\left|\Psi^{\pm}_{j\,m}\right\rangle$ and $\left|\Psi^{(\alpha)}_{s\,m}\right\rangle$ are given by

$$\Psi^{+}_{j\,m}(g) = \sqrt{2j-1}\begin{pmatrix} D^{j}_{n/2,m}(g) \\ 0 \end{pmatrix}, \qquad \Psi^{(1)}_{s\,m}(g) = \begin{pmatrix} D^{-1/2+\mathrm{i}s}_{n/2,m}(g) \\ 0 \end{pmatrix},$$

$$\Psi^{-}_{j\,m}(g) = \sqrt{2j-1}\begin{pmatrix} 0 \\ D^{j}_{-n/2,m}(g) \end{pmatrix}, \qquad \Psi^{(2)}_{s\,m}(g) = \begin{pmatrix} 0 \\ D^{-1/2+\mathrm{i}s}_{-n/2,m}(g) \end{pmatrix},$$

where $g \in \mathrm{SU}(1,1)$.

C. Constraints for the discrete series

Let us begin by deriving the simplicity constraints of the discrete series. The main difference to the SU(2) case is that we are now dealing with Minkowksi 3–vectors. How can one generalize the notions of minimal uncertainty and coherent states to a relativistic setting?

In a relativistic theory physical quantities are Lorentz invariant. Thus, it seems natural to define the uncertainty in the Minkowski vector F by

$$(\Delta F)^2 = \left\langle (F - \langle F\rangle)^i (F - \langle F\rangle)_i \right\rangle \tag{74}$$

$$= \langle F^i F_i\rangle - \langle F^i\rangle\langle F_i\rangle \tag{75}$$

$$= j(j-1) - m^2 = j^2 - j - m^2 \,. \tag{76}$$

Since $j > 0$, and $|m| \geq j$, we see that $(\Delta F)^2$ is always negative.

Our semiclassical conditions can be easily adapted to this new situation: to accommodate for minus signs, we define $\Delta F \equiv \sqrt{|\,(\Delta F)^2\,|}$ and $|\vec{F}| \equiv \sqrt{|\langle\vec{F}\rangle^2|}$, and demand that

$$\frac{\Delta F}{|\vec{F}|} = O\left(\frac{1}{\sqrt{|\vec{F}|}}\right) , \tag{77}$$

$$\langle\vec{F}\rangle + \frac{1}{\gamma}\langle\vec{G}\rangle = O(1)\,, \tag{78}$$

$$\frac{\Delta G}{|\vec{G}|} = O\left(\frac{1}{\sqrt{|\vec{F}|}}\right) . \tag{79}$$

The first equation can be solved by choosing states with $m = \pm j$. More generally, we can use any state of the form

$$|j\,g\rangle_+ \equiv D^j(g)|j\,j\rangle\,, \tag{80}$$

$$|j\,g\rangle_- \equiv D^j(g)|j\,{-j}\rangle\,, \tag{81}$$

where $g \in \mathrm{SU}(1,1)$. The states $|j\,g\rangle_+$ are the coherent states defined by Perelomov for SU(1,1) [18]. Observe that the expectation value of F w.r.t. $|j\,\mathbb{1}\rangle_+$ gives the vector $\vec{N} = (1,0,0)$, while $|j\,\mathbb{1}\rangle_-$ produces $\vec{N} = (-1,0,0)$.

These states exhibit a nice relation with the hyperboloids $\mathbb{H}_+$ and $\mathbb{H}_-$ mentioned above: every element $g \in \mathrm{SU}(1,1)$ can be written as

$$g = g_0 h\,, \tag{82}$$

where $h \in \mathrm{U}(1)$. We see therefore that a representative g_0 of the coset $\mathbb{H}_\pm \simeq \mathrm{SU}(1,1)/\mathrm{U}(1)$ is sufficient to determine the coherent states $|j\,g\rangle_\pm$ up to a phase. Moreover, in a completeness relation the coherent states appear both as a bra and a ket, so that the phase cancels. Thus, it is sufficient to consider states

$$|j\,\vec{N}\rangle \equiv |j\,g(\vec{N})\rangle_\pm\,, \tag{83}$$

where $\vec{N} \in \mathbb{H}_\pm$ and $g(\vec{N})$ is a representative in the coset defined by $\vec{N}$ (see appendix A).

The treatment of condition (78) and (79) is in many ways analogous to what we did in the previous section. Consider first the simplicity constraint (78): since F and G transform in the same way under SU(1,1), it suffices to compute the expectation values of the reference coherent states $|j\,-j\rangle$ and $|j\,j\rangle$. Due to the commutation relations, K_1, K_2, J_1 and J_2 change the eigenvalue of J_3, so their expectation values will be zero. It remains to evaluate the expectation values of J_3 and K_3. According to ref. [20] the action of K_3 is given by

$$K_3\,|j\,m\rangle = (\ldots)|j+1\,m\rangle - mA_j|j\,m\rangle + (\ldots)|j-1\,m\rangle\,. \tag{84}$$

$(\ldots)$ stands for factors that we do not need below, since we are only interested in expectation values. Therefore, for both $|j\,j\rangle$ and $|j\,-j\rangle$, the simplicity constraint leads to

$$j + \frac{1}{\gamma}(-jA_j) = 0 \tag{85}$$

or

$$\gamma = A_j = \frac{\rho\,n}{4j(j-1)}\,, \tag{86}$$

in analogy to the SU(2) case. Consider next the variance of G:

$$(\Delta G)^2 \equiv (\Delta G)^i(\Delta G)_i = \langle G^i G_i\rangle - \langle G^i\rangle\langle G_i\rangle \tag{87}$$

Noting that

$$\frac{1}{2}C_1 = J^2 - K^2 = Q - G^2 \tag{88}$$

we get

$$\langle G^2\rangle = j(j-1) - \frac{1}{4}\left(n^2-\rho^2-4\right) \tag{89}$$

and for a state $|j\,m\rangle$

$$(\Delta G)^2 = j(j-1) - \frac{1}{4}\left(n^2-\rho^2-4\right) - (mA_j)^2\,. \tag{90}$$

Interestingly, this satisfies, like $(\Delta F)^2$, a negativity property[7]:

$$\begin{aligned}
(\Delta G)^2 &= j(j-1) - \frac{1}{4}\left(n^2-\rho^2-4\right) - (mA_j)^2 && (91)\\
&= j^2 - j - \left(\frac{n}{2}\right)^2 + \frac{\rho^2}{4} + 1 - \left(m\frac{\rho\,n}{4j(j-1)}\right)^2 && (92)\\
&= j^2 - \left(\frac{n}{2}\right)^2 - j + 1 - \frac{\rho^2}{4}\left[\left(\frac{m}{j}\frac{n/2}{j-1}\right)^2 - 1\right] && (93)\\
&< 0 \quad \text{for } j > 1\,. && (94)
\end{aligned}$$

[7] Thanks to Laurent Freidel for pointing this out.

When applied to coherent states that are subject to the simplicity constraint, the first line gives

$$(\Delta G)^2 \;=\; j(j-1) - \frac{1}{4}\left(n^2 - \rho^2 - 4\right) - (jA_j)^2 \tag{95}$$

$$\;=\; \frac{1}{4}\Big(\rho - \gamma n\Big)\Big(\rho + \frac{n}{\gamma}\Big) - \gamma^2 j + 1 \tag{96}$$

The expressions are almost the same as in the SU(2) case. By going through analogous arguments we arrive again at the conditions $\rho = \gamma n$ and $j = n/2$. These are the same equations as before, but $j = n/2$ refers now to irreps of SU(1,1).

The quantum area can be obtained from relation (59). By squaring this equation and setting the right–hand side equal to the expectation value of the coherent state, we find that

$$A^2 = \gamma^2 \langle Q \rangle = \gamma^2 j(j-1)\,, \tag{97}$$

or

$$A = \gamma\sqrt{j(j-1)} \qquad \text{for } j \geq 1\,. \tag{98}$$

Alternatively, we could start from

$$A^2 = \frac{1}{2}(\star B)^{IJ}(\star B)_{IJ} \tag{99}$$

for timelike N, and compute

$$\frac{1}{2}(\star B)^{IJ}(\star B)_{IJ} = \frac{1}{2}(\star B)^{ij}(\star B)_{ij} = \gamma^2 Q\,, \qquad i,j = 0,1,2\,, \tag{100}$$

using the simplicity constraints. This gives again eq. (97).

Clearly, the results for the discrete series are very similar to those for the SU(2) irreps. This makes sense when we consider the corresponding classical quantities: the coherent states correspond to spacelike U and timelike N, so the area bivector $\star B$ is spacelike, like in the SU(2) case. Thus, we have described the same classical object in two different gauges: first for timelike U and then for spacelike U.

That we have these two possibilities is important: if the normals U of tetrahedra were always timelike, we could never have timelike triangles. By allowing also spacelike normals U we permit tetrahedra to contain both spacelike and timelike triangles. The discrete series deals with the spacelike triangles in such tetrahedra. In the next subsection, we will provide the formalism for the timelike triangles.

D. Constraints for the continuous series

When we come to the continuous series, the most important question is: what are the appropriate coherent states? Since now $Q < 0$, the classical vectors N should be spacelike. For the continuous series, Perelomov uses the state $|j\, m = 0\rangle$ and its SU(1,1) transformations [18]. This is not the state we are looking for, since it has zero expectation value with regard to J_3, K_1 and K_2. It produces the zero vector $N = 0$ classically.

What we need is a spacelike vector: eigenstates of J_3 are not suited for this, since they lead to vectors $(\pm 1, 0, 0)$. This suggests that we use eigenstates of K_1 or K_2 instead. Such states have been studied by Mukunda [24], Barut and Phillips [25] and Lindblad and Nagel [26]. We adopt the notation of [26] and write

$$K_1\,|j\,\lambda\,\sigma\rangle = \lambda|j\,\lambda\,\sigma\rangle\,. \tag{101}$$

The spectrum of K_1 is the real line and it is two–fold degenerate. For this reason, eigenstates carry an additional label $\sigma = \pm$ which denotes two orthogonal states with the same eigenvalue λ. Due to the non–compactness of the K_1 subgroup, the states $|j\,\lambda\,\sigma\rangle$ are not normalizable:

$$\langle j\,\lambda'\,\sigma'|j\,\lambda\,\sigma\rangle = \delta(\lambda'-\lambda)\delta_{\sigma'\sigma} \tag{102}$$

In this respect, they are similar to eigenstates of momentum and a rigorous definition can be given by using rigged Hilbert space techniques (see [26] for details).

In the following, we will construct coherent states that are based on eigenstates of K_1, with the aim of satisfying the semiclassical conditions (77), (78) and (79). When doing so, we have to take into account that the states are not normalizable. This non-normalizability appears at two independent levels: firstly, we are using eigenstates of K_1, so they are not normalizable due to eq. (102). Secondly, the states $|j\,\lambda\,\sigma\rangle$ correspond to states in $\mathcal{H}_{(\rho,n)}$—let us denote them by $\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}\right\rangle$—and these are not normalizable because of eq. (71).

We deal with this by smearing the states suitably with a Gaussian wavefunction in λ and s. The smearing comes with a parameter δ that we will send to zero in the end[8]. For the smearing we use the Gaussian

$$f_\delta(s) = \frac{1}{(2\pi)^{1/4}\delta^{1/2}}\,\mathrm{e}^{-s^2/4\delta^2}\,. \tag{103}$$

The smeared states are defined by

$$\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right\rangle \equiv \int\limits_0^\infty \mathrm{d}s'\,\sqrt{\mu_\epsilon(s')}f_\delta(s'-s)\int\limits_{-\infty}^\infty \mathrm{d}\lambda'\,f_\delta(\lambda'-\lambda)\left|\Psi^{(\alpha)}_{s'\,\lambda'\,\sigma}\right\rangle\,, \tag{104}$$

and have norm 1. Their variance in F is

$$(\Delta F)^2 = \left\langle\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right| F^iF_i\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right\rangle - \left\langle\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right| F^i\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right\rangle\left\langle\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right| F_i\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right\rangle\,. \tag{105}$$

This looks more complicated than it is, since in the limit $\delta\to 0$ the result reduces to a simple analog of eq. (76). For instance,

$$\left\langle\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right| K^1\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}(\delta)\right\rangle = \int\limits_0^\infty \mathrm{d}s'\,f^2_\delta(s'-s)\int\limits_{-\infty}^\infty \mathrm{d}\lambda'\,\lambda' f^2_\delta(\lambda'-\lambda)\ \stackrel{\delta\to 0}{\to}\ \lambda\,. \tag{106}$$

Thus, the variance becomes

$$(\Delta F)^2\ \stackrel{\delta\to 0}{=}\ -s^2-\frac{1}{4}+\lambda^2 \tag{107}$$

[8] Compare this with the use of momentum wavefunctions in scattering amplitudes.

Clearly, the variance can now take on arbitrary positive and negative values, if we choose s and λ suitably.

Condition (77) requires us to bring this uncertainty close to zero, so we should choose $\lambda = \pm\sqrt{s^2+1/4}$ or $\lambda = \pm s$, for instance. For simplicity, we will select $\lambda = \pm s$ in the following. The difference between these two choices will not make any difference in the final simplicity constraints.

Our new coherent states are therefore given by

$$|j\, g\, \sigma\rangle_s \equiv D^j(g)|j\, s\, \sigma\rangle\,, \qquad g \in \mathrm{SU}(1,1)\,, \tag{108}$$

for states in $\mathcal{C}_s^\epsilon$, or

$$\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}\, g\right\rangle_s \equiv D^{(\rho,n)}(g)\left|\Psi^{(\alpha)}_{s\,\lambda\,\sigma}\, g\right\rangle\,, \qquad g \in \mathrm{SU}(1,1)\,, \tag{109}$$

for states in $\mathcal{H}_{(\rho,n)}$. The subscript s indicates that expectation values of these states correspond to points of the spacelike one–sheeted hyperboloid $\mathbb{H}_s$. Modulo phase and action of $\mathbb{Z}_2$, the coherent states can be parametrized by vectors $\vec{N} \in \mathbb{H}_s \simeq \mathrm{SU}(1,1)/(G_1 \otimes \mathbb{Z}_2)$, i.e.

$$|j\, \vec{N}\, \sigma\rangle \equiv |j\, g(\vec{N})\, \sigma\rangle_s\,, \tag{110}$$

where $g(\vec{N})$ is a representative in the coset defined by $\vec{N} \in \mathbb{H}_s$.

When dealing with the simplicity constraint (78), we restrict our attention to the reference vectors $|j\, s\, \sigma\rangle$, since the other states are covered by SU(1,1) covariance. K^2, J^2, J^3 and K^3 do not commute with K_1 and change the eigenvalue of λ, so their expectation values vanish. Hence we only need to analyze the F^1 and G^1 component of (78).

In another paper [27], we derive the action of $\mathrm{SL}(2,\mathbb{C})$ generators on eigenstates of K_1 and obtain

$$J_1\,|j\,\lambda\,\sigma\rangle = (\ldots)|j+1\,\lambda\,\sigma'\rangle - \lambda A_j|j\,\lambda\,\sigma\rangle + (\ldots)|j-1\,\lambda\,\sigma'\,\rangle\,. \tag{111}$$

This equation holds for each of the two continuous series in the decomposition (73). The simplicity constraint yields therefore

$$s - \frac{1}{\gamma}(-s\,A_j) = 0 \tag{112}$$

or

$$\gamma = -A_j = -\frac{\rho\, n}{4j(j+1)}\,. \tag{113}$$

The final step comes from the variance of G:

$$(\Delta G)^2 \overset{\delta\to 0}{=} j(j+1) - \frac{1}{4}\left(n^2 - \rho^2 - 4\right) + (s\,A_j)^2\,. \tag{114}$$

Due to the minus sign in the simplicity constraint, we get this time

$$(\Delta G)^2 = \frac{1}{4}\left(\rho + \gamma n\right)\left(\rho - \frac{n}{\gamma}\right) - \frac{1}{4}\gamma^2 + 1\,. \tag{115}$$

The first term vanishes when

$$\rho = -\gamma n \tag{116}$$

or

$$\rho = \frac{n}{\gamma}\,. \tag{117}$$

Plugging back (116) into (113) results in a contradiction:

$$\frac{n^2}{4j(j+1)} = -\frac{n^2}{4(s^2+1/4)} = 1\,. \tag{118}$$

When inserting (117) into (113), on the other hand, one obtains

$$-\frac{\rho^2}{4j(j+1)} = \frac{\rho^2}{4(s^2+1/4)} = 1 \qquad \text{or} \qquad \frac{\rho}{2} = \sqrt{s^2+1/4}\,. \tag{119}$$

We therefore arrive at the constraints $n = \gamma\rho$ and $\rho/2 = \sqrt{s^2+1/4}$, which is qualitatively different from the SU(2) case and the discrete series of SU(1,1)[9]. The second condition implies $\rho > 1$ and hence $n > \gamma$. The area becomes

$$A = \gamma\sqrt{-\langle Q\rangle} = \gamma\sqrt{s^2+1/4} = \gamma\frac{\rho}{2} = \frac{n}{2}\,, \tag{120}$$

leading to the conclusion that the area of timelike surfaces is quantized!

IV. SUMMARY OF CONSTRAINTS

In this section, we summarize the simplicity constraints and area spectra that we have obtained. Recall that U is, roughly speaking, the normal vector of a tetrahedron and N is the normal to a triangle in this tetrahedron. The associated bivector is given by $B = A\,U\wedge N$. In our analysis, several irreducible representations played a role: the unitary irreps $\mathcal{H}_{(\rho,n)}$ of $\mathrm{SL}(2,\mathbb{C})$, the unitary irrep $\mathcal{D}_j$ of SU(2), and irreps $\mathcal{D}_j^{\pm}$ and $\mathcal{C}_s^{\epsilon}$ of the discrete and continuous series of SU(1,1).

The following table lists our results for the different choices of U and N. The constraints of the EPRL model correspond to the first column.

[9] Note that we could have also chosen $s = \rho/2$ and nevertheless satisfied our semiclassical conditions. We have picked $\rho^2/4 = s^2+1/4$, since it leads to a particularly simple expression for the area spectrum.

classical data	$U=(1,0,0,0)$ N spacelike $\star B$ spacelike	$U=(0,0,0,1)$ N timelike $\star B$ spacelike	$U=(0,0,0,1)$ N spacelike $\star B$ timelike
little group	SU(2)	SU(1,1)	SU(1,1)
relevant irreps	$\mathcal{D}_j$	$\mathcal{D}_j^{\pm}$	$\mathcal{C}_s^{\epsilon}$
constr. on (ρ, n)	$\rho = \gamma n$	$\rho = \gamma n$	$n = \gamma \rho$
constr. on irreps	$j = n/2$	$j = n/2$	$s^2 + 1/4 = \rho^2/4$
reference coherent states	$\|j\, j\rangle \in \mathcal{D}_j$	$\|j\ \pm j\rangle \in \mathcal{D}_j^{\pm}$	$\|j\, s\, \sigma\rangle \in \mathcal{C}_s^{\epsilon}$
coadjoint orbit	S^2	$\mathbb{H}_{\pm}$	$\mathbb{H}_s$
area spectrum	$\gamma\sqrt{j(j+1)}$	$\gamma\sqrt{j(j-1)}$	$\gamma\sqrt{s^2+1/4} = n/2$
	EPRL		

V. A SPIN FOAM MODEL FOR GENERAL LORENTZIAN 4–GEOMETRIES

We now dispose of constraints for the quantization of spacelike and timelike simple bivectors. This allows us to define a new spin foam model that extends the EPRL model and describes realistic Lorentzian 4–geometries, where both spacelike and timelike surfaces appear.

A. Configurations

The spin foam model is defined on a 4–dimensional simplicial complex Δ and its dual complex Δ^*. We denote edges, triangles, tetrahedra and 4–simplices of Δ by l, t, τ and σ respectively. For dual vertices, edges and faces we use v, e and f respectively.

Let us start by explaining the way geometrical data are assigned to cells of these complexes. If we speak in terms of classical variables, we have the following assignments: there is a normal vector U for each tetrahedron τ, a normal vector N for each triangle t inside a tetrahedron τ (i.e. for each pair τt), and a bivector B or $\star B$ for each triangle t. In terms of the dual Δ^*, this means that we assign a vector U_e to each edge e, a vector N_{ef} to each pair ef, where f contains e, and a bivector B_f to each face f. Since we fix the gauge, the value of U_e amounts to a choice between $(1,0,0,0)$ and $(0,0,0,1)$.

At the quantum level, these degrees of freedom translate to the following data: we choose a little group for each edge e, either SU(2) or SU(1,1), and an irrep of this little group for each pair ef, which can be $\mathcal{D}_j$, $\mathcal{D}_j^\pm$ or $\mathcal{C}_s^\epsilon$. Furthermore, there is an SL$(2,\mathbb{C})$ irrep (ρ_f, n_f) for each face f.

These data are subject to the simplicity constraints: the allowed configurations are given by the three columns in the table of sec. IV. The first thing to note is that we cannot assign the three possibilities freely to each edge. The SL$(2,\mathbb{C})$ irrep (ρ, n) is chosen for an entire face f, so if one selects the first column and $\rho = \gamma n$ for one edge e of this face, one cannot pick the third column and $n = \gamma\rho$ for another edge e' in it. Classically, this corresponds to the fact that the bivector $\star B$ for a face f is either spacelike or timelike, and this affects all edges of the face.

An admissible configuration can be obtained by first assigning a representation $(\gamma^\pm n_f, n_f)$ to each face of the dual complex, where ± 1 corresponds to spacelike/timelike $\star B$. This is followed by an assignment of a normal vector U_e (and hence little group) to each edge e that is consistent with the given choice of constraints on (ρ_f, n_f). Finally, one selects an irrep of this little group for each pair ef, $f \supset e$, according to the following scheme:

$$\begin{array}{llll}
\mathcal{D}_{n_f/2}, & \text{if} \quad \star B \text{ spacelike, } U \text{ timelike} & \rightarrow & N \text{ spacelike,} \\[2ex]
\mathcal{D}^\pm_{n_f/2}, & \text{if} \quad \star B \text{ spacelike, } U \text{ spacelike} & \rightarrow & N \text{ timelike future/past,} \\[2ex]
\mathcal{C}^\epsilon_{\sqrt{n_f^2/\gamma^2-1}}, & \text{if} \quad \star B \text{ timelike, } \; U \text{ spacelike} & \rightarrow & N \text{ spacelike.}
\end{array}$$

A second set of variables comes from the connection. It is implemented as an assignment of SL$(2,\mathbb{C})$ elements g_e to each edge e of the dual complex Δ^*. More precisely, we split each edge e into two half–edges ve and ev', where v and v' are the endpoints of the edge. To these half–edges, we assign group elements $g_{ve} = g_{ev}^{-1}$ and $g_{ev'} = g_{v'e}^{-1}$.

B. Partition function

Next we specify the partition function of the model. We first state the definition, and then explain how it arises from BF theory by imposition of the simplicity constraints.

For the definition, we will use projectors from the SL$(2,\mathbb{C})$ irrep $\mathcal{H}_{(\rho,n)}$ to the little group irreps $\mathcal{D}_j$, $\mathcal{D}_j^\pm$ and $\mathcal{C}_s^\epsilon$:

$$P_{(\rho,n),j} = \sum_{m=-j}^{j} |\Psi_{j\,m}\rangle \langle \Psi_{j\,m}| \tag{121}$$

$$P^\pm_{(\rho,n),j} = \sum_{\pm m=j}^{\infty} \left|\Psi^\pm_{j\,m}\right\rangle \left\langle \Psi^\pm_{j\,m}\right| \tag{122}$$

$$P^\epsilon_{(\rho,n),s}(\delta) = \sum_{\alpha=1,2} \sum_{\pm m=\epsilon}^{\infty} \left|\Psi^{(\alpha)}_{s\,m}(\delta)\right\rangle \left\langle \Psi^{(\alpha)}_{s\,m}(\delta)\right| \tag{123}$$

(124)

In the case of the continuous series, we project onto the smeared states

$$\left|\Psi^{(\alpha)}_{s\,m}(\delta)\right\rangle \equiv \int\limits_0^\infty \mathrm{d}s' \,\sqrt{\mu_\epsilon(s')} f_\delta(s'-s) \left|\Psi^{(\alpha)}_{s'\,m}\right\rangle \,, \tag{125}$$

in accordance with our discussion on normalizability (see sec. III). Matrix elements are defined in the limit, where the smearing parameter δ goes to zero.

The partition function is given by the sum

$$Z = \int\limits_{\mathrm{SL}(2,\mathbb{C})} \prod_{ev} \mathrm{d}g_{ev} \sum_{n_f} \sum_{\zeta_f=\pm 1} \sum_{U_e} \prod_f (1+\gamma^{2\zeta_f}) n_f^2 \, A_f\big((\gamma^{\zeta_f} n_f, n_f), \zeta_f; U_e; g_{ev}\big) \,. \tag{126}$$

Let us explain the different elements of this formula: The $\mathrm{SL}(2,\mathbb{C})$ group elements g_{ev} are integrated over with the Haar measure. For each face, we sum over the positive integers n_f. Furthermore, there is a variable $\zeta_f = \pm 1$ that indicates whether the area bivector of a face is spacelike or timelike respectively. The normal vector U_e is summed over the two possibilities $(1,0,0,0)$ and $(0,0,0,1)$. Each face f carries an amplitude A_f which arises from a contraction of parallel transports and projectors from the edges e of the face (see Fig. 1):

$$A_f((\rho,n),\zeta; U_e; g_{ev}) = \lim_{\delta\to 0}\, \mathrm{tr}\left[\prod_{e\subset f} D^{(\rho,n)}(g_{ve}) P_{(\rho,n),\zeta,U_e}(\delta) D^{(\rho,n)}(g_{ev'})\right] \tag{127}$$

Here, v and v' denote the vertices at the beginning and end of each edge. The projector $P_{(\rho,n),\zeta,U_e}(\delta)$ depends on ζ and the U_e's and implements the simplicity constraints on irreps of the little group:

$$P_{(\rho,n),\zeta,U}(\delta) = \begin{cases} P_{(\rho,n),n/2}\,, & \text{if} \quad \zeta = 1\,, \quad U = (1,0,0,0)\,, \\[2ex] P^+_{(\rho,n),n/2} + P^-_{(\rho,n),n/2}\,, & \text{if} \quad \zeta = 1\,, \quad U = (0,0,0,1)\,, \\[2ex] \theta(n-\gamma) P^\epsilon_{(\rho,n),\sqrt{n^2/\gamma^2-1}}(\delta)\,, & \text{if} \quad \zeta = -1\,, \ U = (0,0,0,1)\,, \\[2ex] 0\,, & \text{if} \quad \zeta = -1\,, \ U = (1,0,0,0)\,. \end{cases} \tag{128}$$

Formulas (126) and (127) are the result of imposing simplicity constraints on quantum BF theory. The spin foam model of $\mathrm{SL}(2,\mathbb{C})$ BF theory has the partition function

$$Z_{\mathrm{BF}} = \int\limits_{\mathrm{SL}(2,\mathbb{C})} \prod_{ev} \mathrm{d}g_{ev} \sum_{n_f} \int\limits_{-\infty}^{\infty} \prod_f \mathrm{d}\rho_f \, \prod_f (n_f^2+\rho_f^2)\, A_f\left((\rho_f,n_f); g_{ev}\right) \,, \tag{129}$$

where

$$A_f((\rho,n); g_{ev}) = \mathrm{tr}\left[\prod_{e\subset f} D^{(\rho,n)}(g_{ve}) \mathbb{1}_{(\rho,n)} D^{(\rho,n)}(g_{ev'})\right] \,. \tag{130}$$

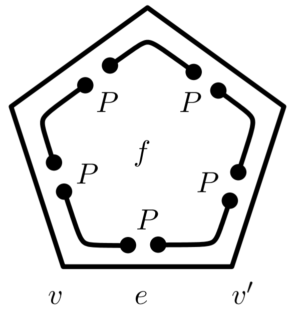

Figure 1: Graphical representation of face amplitude A_f: lines stand for representation matrices $D^{(\rho,n)}(g_{ve})$ and pairs of dots symbolize projectors $P_{(\rho,n),\zeta,U_e}$.

The transition to (126) consists of the following steps: first we impose the simplicity constraint on (ρ_f, n_f), so we introduce the variable $\zeta_f = \pm 1$ for encoding space– or timelikeness of a face and set $(\gamma^{\zeta_f} n_f, n_f)$. Then, we include a sum over normal vectors U_e for each edge e, taking the values $(1,0,0,0)$ or $(0,0,0,1)$. The identity operators $\mathbb{1}_{(\rho,n)}$ in the contraction (130) decompose into irreps of SU(2) or SU(1,1) according to formulas (18) and (67). Depending on the value of ζ_f and U_e we impose simplicity constraints on these irreps. This is done by the replacement

$$\mathbb{1}_{(\rho,n)} \quad \longrightarrow \quad P_{(\rho,n),\zeta,U}(\delta)\,. \tag{131}$$

C. Coherent state vertex amplitude

There are several equivalent ways in which a spin foam sum can be written down. One possibility is to express the total amplitude as a product of vertex amplitudes. Each vertex amplitude can be interpreted as the amplitude for a single 4–simplex. Furthermore, there are different definitions of the vertex amplitude, depending on the choice of boundary data for the 4–simplex. In references [11, 12, 14] vertex amplitudes were defined in terms of coherent states that encode the geometry of boundary tetrahedra.

Below we specify this type of vertex amplitude for the present model. For this we will use the coherent states

$$|j\,g\rangle \equiv D^j(g)|j\,j\rangle \in \mathcal{D}_j\,,$$

$$|j\,g\rangle_\pm \equiv D^j(g)|j\,\pm j\rangle \in \mathcal{D}_j^\pm\,,$$

$$|j\,g\rangle_s \equiv D^j(g)|j\,s\,\sigma\rangle \in \mathcal{C}_s^\epsilon\,,$$

which already appeared in the derivation of the simplicity constraint in sec. II and sec. III. With the parametrization in terms of quotient spaces (see appendix A) one can write such states in a uniform way as[10]

$$|j\,\vec{N}\rangle\,,\quad \text{where } \vec{N}\in S^2,\, \mathbb{H}_+,\, \mathbb{H}_- \text{ or } \mathbb{H}_s. \tag{132}$$

Consider a single vertex v of the dual complex, and number the edges $a = 1, \ldots 5$. Accordingly, faces f are labelled by unordered pairs ab and pairs ef of edges and faces are encoded by ordered pairs ab. The vertex amplitude is given by the product

$$A_v = \prod_{a<b} A_{ab}\,, \tag{133}$$

where

$$A_{ab} = \int\limits_{\mathrm{SL}(2,\mathbb{C})} \prod_a \mathrm{d}g_a\; \langle j_{ab}\,\vec{N}_{ab}|D^{(\rho_{ab},n_{ab})}(g_{ab})|j_{ab}\,\vec{N}_{ba}\rangle \tag{134}$$

The variables $g_a \in \mathrm{SL}(2,\mathbb{C})$ represent the connection on half–edges adjacent to the vertex. ρ_{ab} and n_{ab} are subject to the constraint $\rho_{ab} = \gamma^{\pm 1} n_{ab}$ for spacelike/timelike triangles. The coherent states $|j_{ab}\,\vec{N}_{ab}\rangle$ are states of the type (132) and the allowed choices for j_{ab} and $\vec{N}_{ab}$ follow from the table in sec. IV.

VI. DISCUSSION

Let us summarize our results. We derived simplicity constraints for the quantization of general Lorentzian 4–geometries. The method for this derivation was based on coherent states.

The constraints operate at two levels[11]: on irreps of $\mathrm{SL}(2,\mathbb{C})$ and on irreps of subgroups of $\mathrm{SL}(2,\mathbb{C})$. Firstly, there arise two possible restrictions on unitary irreps of $\mathrm{SL}(2,\mathbb{C})$, one for spacelike triangles and one for timelike triangles. Secondly, we have constraints on unitary irreps of SU(2) and SU(1,1). These irreps appear in the decomposition of irreps of $\mathrm{SL}(2,\mathbb{C})$ and encode the geometry of triangles. Irreps of SU(2) correspond to triangles in tetrahedra with a timelike normal, while irreps of SU(1,1) refer to triangles in tetrahedra with a spacelike normal. In the latter case, spacelike and timelike triangles are described by states in the discrete and continuous series of SU(1,1) respectively[12].

The constraints of the EPRL model were reproduced in the special case, when tetrahedra have only timelike normals. In all cases, we obtained a discrete area spectrum. Thus, the discreteness of area in loop quantum gravity is extended to timelike surfaces.

The derivation of the constraints rested on the idea that coherent states should mimic classical simple bivectors as closely as possible. For this reason, we constructed coherent states with the following properties:

1. Expectation values of bivectors satisfy the classical simplicity constraint.

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[10] To be precise, there is the additional label $\sigma = \pm$ for coherent states in the continuous series. For simplicity, we omit this label in the following formula.

[11] See also the table in sec. IV.

[12] In the context of 3d gravity, the relation between geometry and irreps of SU(1,1) was analyzed in [17] and [28].

2. The uncertainty in bivectors is minimal.

It turned out that such states can only exist in certain irreps of $\mathrm{SL}(2,\mathbb{C})$, SU(2) and SU(1,1), and this is what gave us the conditions on irreps.

Based on these constraints, we defined a new spin foam model that provides a quantization of general Lorentzian 4–geometries.

In regard to future work, there is a whole range of results that apply to the EPRL model and could be extended to the new model: for instance, the semiclassical limit, given by the asymptotics in the large area limit [10–12], the derivation of graviton propagators [13] and the description of intertwiners in terms of tetrahedra [14] . This will require the use of new techniques, since SU(2) will be replaced by the less familiar SU(1,1) and its representation theory.

A point that we left open is the definition of the spin foam model as an integral over coherent states. We did specify a vertex amplitude with coherent states as boundary data, but we did not use it to define the spin foam sum as a whole. For this, we would need a completeness relation for all three types of coherent states that appear in our construction. For SU(2) and for the discrete series of SU(1,1) such completeness relations were given by Perelomov [18]. For the continuous series, however, we have introduced a new type of coherent state, and in this case we do not have any completeness relation so far.

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Appendix A: Parametrization of coherent states

In this appendix, we explain the parametrization of coherent states in terms of hyperboloids. Consider first the discrete series and the coherent states

$$|j\,g\rangle_\pm \equiv D^j(g)|j\,\pm j\rangle \in \mathcal{D}^\pm_j\,. \tag{A1}$$

Up to phase, the states are determined by cosets in $\mathrm{SU}(1,1)/\mathrm{U}(1)$. Elements $g \in \mathrm{SU}(1,1)$ can be parametrized by

$$g = \mathrm{e}^{-\mathrm{i}\varphi J_3}\,\mathrm{e}^{-\mathrm{i}uK_1}\,\mathrm{e}^{\mathrm{i}\psi J_3}\,, \qquad -\pi < \varphi \leq \pi\,, \quad -2\pi < \psi \leq 2\pi\,, \quad 0 < u < \infty\,, \tag{A2}$$

so the cosets have representatives

$$g = \mathrm{e}^{-\mathrm{i}\varphi J_3}\,\mathrm{e}^{-\mathrm{i}uK_1}\,. \tag{A3}$$

The parameters φ and u coordinatize the upper/lower hyperboloid via

$$\vec{N} = \pm(\cosh u, \sinh u \sin\varphi, \sinh u \cos\varphi)\,, \qquad N^2 = 1\,, \tag{A4}$$

giving the isomorphism $\mathrm{SU}(1,1)/\mathrm{U}(1) \simeq \mathbb{H}_\pm$. Thus, the representatives (A3) can be expressed as functions $g(\vec{N})$ of $\vec{N} \in \mathbb{H}_\pm$, and the coherent states may be defined by

$$|j\,\vec{N}\rangle \equiv |j\,g(\vec{N})\rangle\,. \tag{A5}$$

Formula (A4) was chosen such that

$$\langle j\,\vec{N}|\vec{J}|j\,\vec{N}\rangle = j\vec{N}\,. \tag{A6}$$

For the continuous series, we employ coherent states

$$|j\,g\rangle_s \equiv D^j(g)|j\,s\,\sigma\rangle \in \mathcal{C}^\epsilon_s\,. \tag{A7}$$

In this case, we consider the quotient $\mathrm{SU}(1,1)/(G_1\otimes\mathbb{Z}_2)$, where G_1 is the one–parameter subgroup generated by K_1. Now, it is convenient to choose the following parametrization [21]:

$$g = \mathrm{e}^{-\mathrm{i}\varphi J_3}\,\mathrm{e}^{-\mathrm{i}tK_2}\,\mathrm{e}^{\mathrm{i}uK_1}\,,\qquad -2\pi < \varphi \le 2\pi\,,\quad -\infty < t,u < \infty\,. \tag{A8}$$

Representatives of cosets are given by elements

$$g = \mathrm{e}^{-\mathrm{i}\varphi J_3}\mathrm{e}^{-\mathrm{i}tK_2}\,,\qquad -\pi < \varphi \le \pi\,,\quad -\infty < t < \infty\,. \tag{A9}$$

Using

$$\vec{N} = (-\sinh t, \cosh t\cos\varphi, \cosh t\sin\varphi)\,,\qquad N^2 = -1\,, \tag{A10}$$

we parametrize the single–sheeted hyperboloid $\mathbb{H}_s$ by (φ,t), and it follows that $\mathrm{SU}(1,1)/(G_1\otimes\mathbb{Z}_2)\simeq\mathbb{H}_s$. Hence we write representatives (A9) as functions $g(\vec{N})$ of $\vec{N}\in\mathbb{H}_s$. The corresponding coherent states are specified by

$$|j\,\vec{N}\,\sigma\rangle \equiv |j\,g(\vec{N})\,\sigma\rangle\,, \tag{A11}$$

where

$$\langle j\,\vec{N},\sigma|\vec{F}|j\,\vec{N}\,\sigma\rangle = s\vec{N}\,. \tag{A12}$$

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