On the Uniqueness of Kinematics of Loop Quantum Cosmology

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Abstract.
    The holonomy-flux algebra $\mathfrak{A}$ of loop quantum gravity is known to admit a natural representation that is uniquely singled out by the requirement of covariance under spatial diffeomorphisms. In the cosmological context, the requirement of spatial homogeneity naturally reduces $\mathfrak{A}$ to a much smaller algebra, $\mathfrak{A}_{\mathrm{Red}}$, used in loop quantum cosmology. In Bianchi I models, it is shown that the requirement of covariance under *residual* diffeomorphism symmetries again uniquely selects the representation of $\mathfrak{A}_{\mathrm{Red}}$ that has been commonly used. We discuss the close parallel between the two uniqueness results and also point out a difference.

1. Setting the Stage

In loop quantum gravity (LQG) one begins with a Hamiltonian framework in which the basic canonical pair consists of an SU(2) connection A_a^i and its momentum, a Lie algebra-valued vector density E_i^a of weight one, both defined on a 3-dimensional manifold M. To construct quantum kinematics, as usual, one has to select a class of elementary functions which are to have unambiguous quantum analogs. In LQG these are given by matrix elements of holonomies $h_\alpha(A)$ of connections A along suitable curves α in M and fluxes $E_{S,f}$ of E across suitable 2-surfaces S, smeared with Lie algebra valued fields f_i. The kinematical algebra $\mathfrak{A}$ —called the *holonomy flux algebra*— is then generated by the operators $\hat{h}_\alpha$ and $\hat{E}_{S,f}$ [1]. The algebra $\mathfrak{A}$ is 'background independent' in the sense that it uses only the manifold structure of M. To complete the construction of quantum kinematics, then, it remains to find a suitable Hilbert space $\mathcal{H}_{\text{kin}}$ and represent elements of $\mathfrak{A}$ by concrete operators on it. Motivated by background independence, $\mathcal{H}_{\text{kin}}$ was taken to be the space $L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_o)$ of square-integrable functions on the space $\bar{\mathcal{A}}$ of (suitably generalized) connections on M with respect to a natural *diffeomorphism invariant measure* μ_o [2, 3]. The configuration operators $\hat{h}_\gamma$ were represented by multiplication and the momentum operators $\hat{E}_{S,f}$ by Lie derivatives w.r.t. certain 'vector fields' on $\bar{\mathcal{A}}$. This representation of $\mathfrak{A}$ admits a cyclic vector Ψ_o which is is invariant under the action of *Diff*, the group of suitable diffeomorphisms of M [2]. This kinematics was constructed in the mid nineties and led to a specific quantum Riemannian geometry that underlies LQG [4].

However, a natural question arose: Is this representation of $\mathfrak{A}$ *uniquely* selected by some physical requirements? This was answered in the affirmative some 10 years later through a powerful theorem [5]: The physical requirement is precisely the existence of a cyclic state invariant under *Diff*, which in turn implies that the group *Diff* of symmetries is unitarily implemented on $\mathcal{H}_{\text{kin}}$. (See also [6]). This unitary implementation plays a crucial role in the subsequent imposition of the diffeomorphism constraint [4].

Let us now turn to cosmology. In the Bianchi I models we will focus on, spatial homogeneity causes a drastic reduction in the number of degrees of freedom. To obtain a simple description of those that survive, one commonly introduces and fixes some fiducial structures: a flat metric $\mathring{q}_{ab}$, an associated set of Cartesian coordinates x^i on M, the associated orthonormal co-frames $\mathring{\omega}^i := dx^i$ and the dual frames $\mathring{e}_i^a$. One then restricts oneself to pairs (A_a^i, E_j^a) of the form:‡

$$A_a^i = c^i \, \mathring{\omega}_a^i, \qquad E_i^a = p_i \, \sqrt{\mathring{q}} \, \mathring{e}_i^a \tag{1}$$

where $\mathring{q}$ is the determinant of the fiducial metric $\mathring{q}_{ab}$. Thus, because of spatial homogeneity, there are only three, global configuration degrees of freedom c^i, and three momenta p_i. However, if one naively evaluates the symplectic structure of the full

‡ Throughout this communication, there is no summation over repeated contravariant or covariant indices. Contracted covariant and contravariant indices by contrast are summed over. Our c^i, p_j have been generally denoted by $\tilde{c}^i, \tilde{p}_j$ in the LQC literature.

theory for these *homogeneous* A^i_a, E^a_i, it diverges. Therefore, to obtain a well-defined phase space formulation and subsequent quantum kinematics, one must introduce an infra-red cutoff (to be removed at the end to obtain physical results). This is done by introducing a cell $\mathcal{C}$ whose edges are parallel to the fiducial $\mathring{e}^a_i$. Then, the non-vanishing Poisson brackets are given by $\{c^i, p_j\} = (8\pi\gamma G/\mathring{V})\,\delta^i_j$, where $\mathring{V}$ is the volume of the cell $\mathcal{C}$ with respect to the fiducial metric $\mathring{q}_{ab}$.

To construct quantum kinematics, one begins by noting that it is natural to restrict the holonomy and flux phase space functions using spatial homogeneity. For fluxes, it suffices to choose the surfaces to be the three faces of the cell (and smearing fields f_i to be $f_i = n_a \mathring{e}^a_i$ where n_a is the unit normal to the face with respect to $\mathring{q}_{ab}$). Then, the three flux functions $E_{S,f}$ turn out to be (multiples of the) p_i. For holonomies h_α, it suffices to choose the curves α to be aligned with the three edges of the cell and label them with numbers μ_i, the lengths of the (oriented) edges in units of the edge lengths of the cell. Then, if α is along the jth edge, $h_\alpha = (\cos(\mu^j c^j)/2)I + 2(\sin(\mu^j c^j)/2)\tau^j$ where τ^j are the Pauli matrices and I the unit matrix. Note that the dependence on c^j is completely captured by the functions $e^{i\mu^j c^j}$, with $\mu^j \in \mathbb{R}$. To summarize, then, spatial homogeneity naturally reduces the holonomy flux algebra $\mathfrak{A}$ to the much smaller, reduced algebra $\mathfrak{A}_{\mathrm{Red}}$, generated by the phase space functions $e^{i\mu^j c^j}$ and p_i [9, 10].

While the reduction from $\mathfrak{A}$ to $\mathfrak{A}_{\mathrm{Red}}$ is systematic, the construction of the representation of $\mathfrak{A}_{\mathrm{Red}}$ used in LQC has not descended so directly from LQG. For, while in full LQG the representation was uniquely selected by asking for a cyclic state which is invariant under *Diff*, it was generally believed that the ansatz (1) freezes all diffeomorphisms. Thus the key requirement that selected the unique representation in LQG seemed to have disappeared in LQC whence it seemed impossible to prove an uniqueness theorem along the lines of [5, 6]. Instead, one 'mimicked' the form of the unique cyclic state of LQG in a precise sense to obtain a cyclic state on $\mathfrak{A}_{\mathrm{Red}}$ and used it to construct the representation [7]. In particular, the LQC Hilbert space $\mathcal{H}_{\mathrm{kin}}$ is again the space $L^2(\mathbb{R}^3_{\mathrm{Bohr}}, \mathrm{d}\mu_o)$ of square integrable functions on the space $\mathbb{R}^3_{\mathrm{Bohr}}$ of (generalized) homogeneous Bianchi I connections with respect to a natural measure $\mathrm{d}\mu_o$ thereon, the holonomy operators $\widehat{\exp i\mu^j c^j}$ act by multiplications and the flux operators $\hat{p}_i$ by derivation [9, 10].

But the question has remained: Can we systematically arrive at this representation in LQC as was done in LQG in [5, 6]? The goal of this communication is to answer it in the affirmative. The key new observation is that *the ansatz (1) does not eliminate the diffeomorphism freedom completely and the residual diffeomorphism freedom can be used to select a cyclic state on $\mathfrak{A}_{\mathrm{Red}}$ uniquely.* Not surprisingly this is precisely the state that was arrived at by 'mimicking' full LQG.

2. Residual diffeomorphism symmetries

We have fixed the fiducial fields $\mathring{q}_{ab}, \mathring{e}^a_i, \mathring{\omega}^i_a$ on M and the Bianchi type I phase space variables are the connections A^i_a and conjugate momenta E^a_i of the form (1). The

question is: Are there diffeomorphisms on M which preserve this form and have non-trivial action on the coefficients c^i, p_j? To preserve the form (1), the diffeomorphisms must map each of the three $\mathring{\omega}^i_a$ to a constant multiple of itself. Since $\mathring{\omega}^i = dx^i$, it follows that the most general vector field ξ^a generating such a diffeomorphism is a linear combination of anisotropic dilations and translations:

$$\xi^a = \lambda_1 x_1 \mathring{e}^a_1 + \lambda_2 x_2 \mathring{e}^a_2 + \lambda_3 x_3 \mathring{e}^a_3 + k^i \mathring{e}^a_i \qquad (2)$$

where λ_i, k^i are real constants. The action of translations $k^i \mathring{e}^a_i$ leaves each of the c^i, p_j invariant. Therefore it is just the 3-dimensional Abelian group G generated by the three anisotropic dilations, $x^1 \mapsto e^{\lambda_1} x^1$, etc, that has a nontrivial action on c^i, p_j:

$$c^1 \mapsto e^{\lambda_1} c^1, \quad p_1 \mapsto e^{\lambda_2 + \lambda_3} p_1 \quad \text{and cyclic permutations.} \qquad (3)$$

Are these phase space symmetries? A trivial calculation shows that while the vanishing Poisson brackets between the three c^i and those among the three p_i are preserved, the non-vanishing ones are preserved if an only if $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This is precisely the 2-dimensional group G_o of volume preserving anisotropic dilations. In the main part of this communication we will focus just on G_o.

3. The Weyl Algebra

The holonomy flux algebra is generated by $U(\vec{\mu}) := \widehat{\exp i\mu_j c^j}$ and $\hat{p}_i$. As usual, since it is mathematically more convenient to deal with (the bounded) unitary operators rather than (the unbounded) self-adjoint ones, let us exponentiate p_i and set $V(\vec{\eta}) := \exp i\eta^j \hat{p}_j$ with $\vec{\mu} \in \mathbb{R}^3$ and work with the pairs $U(\vec{\mu}), V(\vec{\eta})$. However, in the final picture we need $\hat{p}_j$ to be well defined self adjoint operators. This is easily achieved by demanding that *in the final representation the operators $V(\vec{\eta})$ should be continuous in the parameters $\vec{\eta}$.* There is no such a priori requirement on $U(\vec{\mu})$ because in full LQG there is no operator corresponding to the connections; only holonomies are well defined operators. The classical Poisson brackets dictate the algebraic structure of these operators:

$$U(\vec{\mu}_1)U(\vec{\mu}_2) = U(\vec{\mu}_1 + \vec{\mu}_2); \quad V(\vec{\eta}_1)V(\vec{\eta}_2) = V(\vec{\eta}_1 + \vec{\eta}_2);$$
$$U(\vec{\mu})V(\vec{\eta}) = e^{-ik\vec{\mu}\cdot\vec{\eta}} V(\vec{\eta})U(\vec{\mu}), \qquad \text{where } k = 8\pi\gamma\ell^2_{\text{Pl}}/\mathring{V}. \qquad (4)$$

It is often convenient to work with a combination

$$W(\vec{\mu}, \vec{\eta}) := e^{\frac{ik}{2}\vec{\mu}\cdot\vec{\eta}} U(\vec{\mu})V(\vec{\eta}) \qquad (5)$$

called the *Weyl operators* satisfying the following star relations and product rule:

$$[W(\vec{\mu}, \vec{\eta})]^\star = W(-\vec{\mu}, -\vec{\eta}),$$
$$W(\vec{\mu}_1, \vec{\eta}_1)\, W(\vec{\mu}_2, \vec{\eta}_2) = e^{-\frac{ik}{2}(\vec{\mu}_1\cdot\vec{\eta}_2 - \vec{\mu}_2\cdot\vec{\eta}_1)}\, W(\vec{\mu}_1 + \vec{\mu}_2, \vec{\eta}_1 + \vec{\eta}_2)\,. \qquad (6)$$

Note that the vector space $\mathfrak{W}$ generated by finite linear combinations of Weyl operators is closed under both operations and is a $\star$-algebra. This is the Weyl algebra for the Bianchi I model, the symmetry reduced version of the algebra used in [6] for LQG.

As in the full theory, it is convenient to use the Gel'fand, Naimark, Segal (GNS) construction [8] to find its representation. This requires us to choose a normalized

positive linear functional (PLF) $\mathbf{F}$ on $\mathfrak{W}$, i.e., a linear map, $\mathbf{F} : \mathfrak{W} \to \mathbb{C}$, from the Weyl algebra to the set of complex numbers, such that: i) $\mathbf{F}(a^\star a) \geq 0$ for all $a \in \mathfrak{W}$; and ii) $F(\mathbb{I}) = 1$, where $\mathbb{I}$ is the identity element of $\mathfrak{W}$. The choice made in LQC [7, 9, 10],

$$\mathbf{F}(W(\vec{\mu}, \vec{\eta})) = \delta_{\vec{\mu}, \vec{0}}, \quad \text{and extends to } \mathfrak{W} \text{ by linearity,} \tag{7}$$

mimics the PLF used in full LQG [5, 6]. Since $\mathbf{F}$ is continuous in $\vec{\eta}$, in the resulting GNS Hilbert space $\mathcal{H}$ the unitary operators representing $V(\vec{\eta})$ are continuous in the parameters $\vec{\eta}$, and are therefore generated by self-adjoint operators $\hat{p}_i$. Thus, we have a representation of the reduced holonomy-flux algebra $\mathfrak{A}_{\text{Red}}$. The Hilbert space $\mathcal{H}$ is often described in terms of the orthonormal basis $|\vec{\mu}\rangle$ of eigenvectors of $\hat{p}_j$. The action of the basic operators is given by:

$$U(\vec{\mu})|\vec{\mu}_o\rangle = |\vec{\mu} + \vec{\mu}_o\rangle, \quad \text{and} \quad V(\vec{\eta})|\vec{\mu}_o\rangle = e^{ik\vec{\eta}\cdot\vec{\mu}_o}|\vec{\mu}_o\rangle. \tag{8}$$

We will now show that this representation is uniquely selected by the requirement that the PLF be invariant under the action of the group G_o of volume preserving anisotropic dilations.

4. Uniqueness of the representation: Direct method

Since the induced action of G_o on the phase space preserves the symplectic structure, it provides a 2 parameter family $\Lambda(\vec{\lambda})$ of automorphisms on the Weyl algebra:

$$\Lambda(\vec{\lambda})[W(\vec{\mu}, \vec{\eta})] = W(e^{\lambda_1}\mu_1, e^{\lambda_2}\mu_2, e^{\lambda_3}\mu_3; \ e^{\lambda_2\lambda_3}\eta_1, e^{\lambda_3\lambda_1}\eta_2, e^{\lambda_1\lambda_2}\eta_3) \tag{9}$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 0$. As in the uniqueness theorems of LQG kinematics [5, 6], we now seek a PLF $\mathbf{F}$ on $\mathfrak{W}$ which is invariant under these automorphisms. The cyclic state in the resulting GNS representation would then be invariant under these residual diffeomorphism symmetries, whence they would be represented by unitary transformations on the GNS Hilbert space [8]. In addition we require that $\mathbf{F}(W(\vec{\mu}, \vec{\eta}))$ be continuous in $\vec{\eta}$ so that operators $\hat{p}_i$ will be well-defined and the GNS Hilbert space will also carry a representation of the holonomy-flux algebra $\mathfrak{A}_{\text{Red}}$.

For notational simplicity, let us set $F(\vec{\mu}; \vec{\eta}) := \mathbf{F}(W(\vec{\mu}, \vec{\eta}))$. Then the two conditions imply in particular that F must satisfy: i) $F(\vec{0}; \vec{\eta}) = F(\vec{0}; \ e^{-\lambda_1}\eta_1, e^{-\lambda_2}\eta_2, e^{-\lambda_3}\eta_3)$ for any $\lambda_i \in \mathbb{R}$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 0$; and, ii) $F(\vec{0}; \vec{\eta})$ is continuous in $\vec{\eta}$. In addition, the normalization condition on $\mathbf{F}$ implies $F(\vec{0}; \vec{0}) = 1$. It follows immediately that:

$$F(\vec{0}; \eta_1, 0, 0) = 1; \ F(\vec{0}; 0, \eta_2, 0) = 1; \ F(\vec{0}; 0, 0, \eta_3) = 1. \tag{10}$$

We are now equipped to prove the main result.

Theorem: Let $\mathbf{F}$ be a normalized positive linear functional on the Weyl algebra $\mathfrak{W}$ satisfying (10). Then $\mathbf{F}(W(\vec{\mu}, \vec{\eta})) = \delta_{\vec{\mu}, \vec{0}}$.
Proof: Being a PLF, $\mathbf{F}$ satisfies:

$$|\mathbf{F}(a^\star b)|^2 \leq \mathbf{F}(a^\star a)\,\mathbf{F}(b^\star b) \quad \text{for all } a, b \in \mathfrak{W}. \tag{11}$$

The key idea is to use this property with two different choices of a and b. Let $\vec{\eta}_o$ be any $\vec{\eta}$ which lies along one of the three axes so that $F(\vec{0}; \vec{\eta}_o) = 1$. Set $b = V(\vec{\eta}_o) - \mathbb{I}$.

Then it is trivial to check that $\mathbf{F}(b^\star b) = 0$. Therefore $\mathbf{F}(a^\star b) = 0$ for all $a \in \mathfrak{W}$. Now let $a = V(\vec{\eta})$ for an arbitrary $\vec{\eta}$. Then we have $0 = \mathbf{F}(a^\star b) = F(\vec{0}; \vec{\eta}_o - \vec{\eta}) - F(\vec{0}; -\vec{\eta})$. Since $F(\vec{0}; \vec{\eta}_o) = 1$, it follows that $F(\vec{0}; \vec{\eta}) = 1$ for all $\vec{\eta}$.

Now let $b = V(\vec{\eta}) - \mathbb{I}$ for any $\vec{\eta} \in \mathbb{R}^3$. Since we have established that $F(\vec{0}; \vec{\eta}) = 1$, we again have $\mathbf{F}(b^\star b) = 0$, whence $\mathbf{F}(a^\star b) = 0 = \mathbf{F}(b^\star a)$ for all $a \in \mathfrak{W}$. Therefore $\mathbf{F}(a(V(\eta) - \mathbb{I})) = 0$ and $\mathbf{F}((V(\eta) - \mathbb{I})a) = 0$ for all $a \in \mathfrak{W}$. This implies

$$\mathbf{F}(a) = \mathbf{F}(aV(\vec{\eta})) = \mathbf{F}(V(\vec{\eta})a) \quad \forall a \in \mathfrak{W} \text{ and } \vec{\eta} \in \mathbb{R}^3. \tag{12}$$

Let us now set $a = U(\vec{\mu})$ for any $\vec{\mu} \in \mathbb{R}^3$. Then, using $W(\vec{\mu}, \vec{\eta}) = e^{\frac{i}{2} k \vec{\mu} \cdot \vec{\eta}} U(\vec{\mu}) V(\vec{\eta})$, we obtain

$$F(\vec{\mu}; \vec{\eta}) = e^{\frac{i}{2} k \vec{\mu} \cdot \vec{\eta}} F(\vec{\mu}; \vec{0}) = e^{-\frac{i}{2} k \vec{\mu} \cdot \vec{\eta}} F(\vec{\mu}; \vec{0}) \tag{13}$$

for all $\vec{\mu}, \vec{\eta}$. This implies $F(\vec{\mu}; \vec{\eta}) = 0$ if $\mu \neq 0$. But we have already established that $F(\vec{0}; \eta) = 1$. Therefore we conclude $F(\vec{\mu}; \vec{\eta}) = \delta_{\vec{\mu}, \vec{0}}$. $\quad \square$

Thus, the requirement that the PLF be invariant under the automorphisms on $\mathfrak{W}$ implementing the residual diffeomorphism symmetries G_o led us to a unique cyclic representation of $\mathfrak{W}$. Moreover, this is precisely the representation that has been used in LQC. Note, incidently, that G_o invariance was used only to arrive at the conclusion that $F(\vec{0}; \vec{\eta}_o) = 1$ for all $\vec{\eta}$ on the three axes in the 3-dimensional η-space. So, if another physical requirement were to lead us to this condition, uniqueness will follow. We will return to this point in section 6 in the discussion of more general Bianchi models.

5. Uniqueness of the representation: Conceptual underpinning

It is instructive to see an alternate proof of the second half of the uniqueness theorem because it makes the conceptual underpinning of the result and the parallel between the LQC and LQG representations transparent, and because it could extend to more general situations. We begin by assuming that, thanks to the symmetry condition, the PFL we are seeking must satisfy $\mathbf{F}(W(\vec{0}, \vec{\eta})) = 1$. Let us suppose that such a PLF exists and let $\mathcal{K}$ denote the kernel of the PLF, i.e., the subspace of $\mathfrak{W}$ defined by $\mathbf{F}(a^\star a) = 0$ for all $a \in \mathcal{K}$. The GNS construction then yields a Hilbert space $\mathcal{H}$ which is the Cauchy completion of the quotient $\mathfrak{W}/\mathcal{K}$.

The cyclic state $|\Psi_o\rangle \in \mathcal{H}$ is the equivalence class to which the identity operator $\mathbb{I}$ belongs. Since $\mathbf{F}(\mathbb{I}) = 1$, we have $\langle \Psi_o | \Psi_o \rangle = 1$. Set $|\Psi_{\vec{\eta}}\rangle = V(\vec{\eta})|\Psi_o\rangle$. Then, $\langle \Psi_{\vec{\eta}} | \Psi_{\vec{\eta}} \rangle = 1$ and furthermore $\langle \Psi_o | \Psi_{\vec{\eta}} \rangle = \langle \Psi_o | V(\vec{\eta}) \Psi_o \rangle = F(\vec{0}; \vec{\eta}) = 1$. Thus, $|\Psi_o\rangle$ and $|\Psi_{\vec{\eta}}\rangle$ are unit vectors and their scalar product is 1. Therefore they must coincide. Thus, $V(\vec{\eta})|\Psi_o\rangle = |\Psi_o\rangle$ for all $\vec{\eta}$.

Next, set $|\Psi_{\vec{\mu}}\rangle := U(\vec{\mu})|\Psi_o\rangle$. Then

$$V(\vec{\eta})|\Psi_{\vec{\mu}}\rangle = V(\vec{\eta}) U(\vec{\mu})|\Psi_o\rangle = e^{ik\vec{\mu} \cdot \vec{\eta}} U(\vec{\mu}) V(\vec{\eta})|\Psi_o\rangle = e^{ik\vec{\mu} \cdot \vec{\eta}} |\Psi_{\vec{\mu}}\rangle. \tag{14}$$

Thus, for all $\vec{\mu}, \vec{\eta}$, $|\Psi_{\vec{\mu}}\rangle$ is an eigenvector of $V(\vec{\eta})$ with eigenvalue $e^{ik\vec{\mu} \cdot \vec{\eta}}$. Therefore it follows that: i) If $\vec{\mu} \neq \vec{\mu}'$, $|\Psi_{\vec{\mu}}\rangle - |\Psi_{\vec{\mu}'}\rangle \notin \mathcal{K}$ so for each $\vec{\mu} \in \mathbb{R}^3$ there is a distinct ket $|\Psi_{\vec{\mu}}\rangle$; and, ii) $\langle \Psi_{\vec{\mu}} | \Psi_{\vec{\mu}'} \rangle = \delta_{\vec{\mu}, \vec{\mu}'}$. Consider the vector space $\mathcal{V} := \{\sum_{n=1}^{N} K_n |\Psi_{\vec{\mu}_n}\rangle\}$

spanned by finite but otherwise arbitrary linear combinations of $|\Psi_{\vec{\mu}}\rangle$. It contains the cyclic state $|\Psi_o\rangle$ and is left invariant by the Weyl algebra $\mathfrak{W}$. Therefore $\mathcal{V} = \mathfrak{W}/\mathcal{K}$, and its Cauchy completion is the GNS Hilbert space $\mathcal{H}$. Thus we have explicitly constructed the GNS representation. By inspection, $\mathbf{F}(W(\vec{\mu}, \vec{\eta})) = \langle \Psi_o | e^{\frac{ik}{2}\vec{\mu}\cdot\vec{\eta}} U(\vec{\mu}) V(\vec{\eta}) |\Psi_o \rangle = \delta_{\vec{\mu},\vec{0}}$. Furthermore by identifying kets $|\Psi_{\vec{\mu}}\rangle$ with the kets $|\vec{\mu}\rangle$ of section 3, we obtain an explicit isomorphism between this GNS representation and the one that has been used in LQC.

6. Discussion

We began by noting that the ansatz (1) used in the Bianchi I models does *not* completely fix the diffeomorphism freedom. There is a three parameter group G of anisotropic dilations that respects the ansatz but has non-trivial action on the symmetry reduced phase space. However it is only the 2 parameter subgroup G_o of volume preserving diffeomorphisms of G that preserve the symplectic structure. Therefore we focused on G_o. This G_o is faithfully represented by a group of automorphisms on the Weyl algebra $\mathfrak{W}$. As is usual in quantum mechanics and quantum field theory, we then seek cyclic representations of $\mathfrak{W}$. If we demand, as in full LQG [5, 6], that the required PLF on $\mathfrak{W}$ be invariant under the automorphisms induced by the diffeomorphism symmetries, we are led to a unique representation of $\mathfrak{W}$. Moreover this is precisely the representation that has been used in the LQC literature [7, 9]. Thus the situation in LQC has turned out to be completely parallel to that in LQG: *the representation is uniquely selected by the residual diffeomorphism symmetries.* In both cases the representation was first found and used extensively and the uniqueness was established much later.

We conclude with a few remarks:

i) While there is a conceptual parallel between LQG and LQC, there is also a difference. If the topology is $\mathbb{R}^3$, the group G_o of diffeomorphism we considered is included in the group *Diff* used in LQG [5]. However, the LQG uniqueness result would have held even if one had restricted oneself to diffeomorphisms which are asymptotically identity. The uniqueness theorem would have still picked the standard PLF and one could have just checked at the end that the PLF is also invariant under the action of G_o. This difference is directly related to the fact that we are now working with homogeneous fields which do not have local degrees of freedom.

ii) In more general Bianchi models with different spatial topologies, the analog of G_o may not exist. But the induced automorphisms continue to exist and can be interpreted as changes of the fiducial $\mathring{\omega}_a^i, \mathring{e}_i^a$. Demanding that the PLF be invariant under them would again lead to a unique cyclic representation of the Weyl algebra.

iii) What is the situation with elements of G with $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$ which are *not* in G_o? Because the induced action of these elements of G does not preserve the symplectic structure, they do not yield automorphisms on all of $\mathfrak{W}$. But they do induce automorphisms on the two Abelian sub-algebras of $\mathfrak{W}$ generated separately by $U(\vec{\mu})$ and $V(\vec{\eta})$. Our PLF is invariant under them.

iv) The spatially flat, isotropic Hilbert space of LQC states is naturally embedded

in our Bianchi I Hilbert space $\mathcal{H}$. In this sense, the uniqueness result naturally descends from the Bianchi I to the k=0 Friedmann model. However, what if one chose to work directly with the Friedmann model? Then, $\lambda_1 = \lambda_2 = \lambda_3$ and G reduces just to the 1-parameter group of dilations. The action of this diffeomorphism induces automorphisms only on the two Abelian sub-algebras as discussed above. However, the requirement that the PLF be invariant under this action suffices to select the PLF uniquely [11] and this is precisely the PLF that has been used in the Friedmann model of LQC [7]. By contrast in the Schrödinger representation, discussed below, this 1-parameter group of dilations is not unitarily implemented and, furthermore, the Friedmann Hilbert space is not a subspace of the Bianchi I Hilbert space.

v) What happens in the Schrödinger representation of the Weyl algebra $\mathfrak{W}$, where the Hilbert space is $L^2(\mathbb{R}^3, \mathrm{d}^3 c)$? Since $c^1 \mapsto e^{\lambda_1} c^1$, etc, with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, it follows that the Lesbegue measure is preserved, whence G_o is again unitarily represented. Furthermore, this representation is again cyclic but it does *not* admit any cyclic state that is invariant under the induced action of G_o! This raises an interesting question: Are there perhaps cyclic representations of the holonomy-flux algebra $\mathfrak{A}$ of LQG in which *Diff* is unitarily represented but none of the cyclic vectors is invariant under *Diff*? If they do, they could represent different phases of LQG kinematics, complementing the standard representation [2, 4] which captures the LQG quantum geometry at the Planck scale.

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