

FEBRUARY 2019
PIERRE MARTIN-DUSSAUD¹

A primer of group theory for Loop Quantum Gravity and Spin-foams

arXiv:1902.08439v1 [math-ph] 22 Feb 2019

The aim of a theory really is, to a great extent, that of systematically organizing past experience in such a way that the next generation, our students and their students and so on, will be able to absorb the essential aspects in as painless a way as possible, and this is the only way in which you can go on cumulatively building up any kind of scientific activity without eventually coming to a dead end.

M. F. Atiyah, [Ati74].

¹Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France
pmd@cpt.univ-mrs.fr

Foreword

Calculations in Loop Quantum Gravity (LQG) and spin-foams theory rely heavily on group theory of $SU(2)$ and $SL_2(\mathbb{C})$. Even though many monographs exist devoted to this theory, the different tools needed (e.g. representation theory, harmonic analysis, recoupling theory...) are often dispersed in different books and with different conventions and notations. This has initially motivated the compilation of the present document. Nowadays, these notes can serve three main purposes:

1. A concise introduction for students to the essential mathematical tools of LQG. It bridges a gap between the level of students at the end of a Master programme, and the minimum level required to start doing research in LQG. In case it is going too fast, paragraphs have been inserted for a quick memory refreshing. They are written in small font size, and introduced by ‘♣ REMINDER’. Instead of introducing a new formula out of nowhere, we insist on motivations for doing it. The proofs are done if helpful for understanding, but are sometimes only sketched. They are written in small font size introduced by ‘► PROOF’, so that they can also be skipped easily.
2. A convenient compendium for researchers. Instead of having the formulas lost each in a different heavy and old book, the most useful ones are now gathered in a short toolbox.
3. A translational hub between the conventions of the main references. For many notions, no standard notations are universally used. Each author tends to use its own notations, which makes it difficult to switch easily from one reference to another. We have made some choices ourselves, but we show explicitly how they relate to various of the major references : such discussions are done in small font size introduced by ‘★ NOTA BENE’. We see this attempt as a step towards a more widely use of common notations. In particular, we give the conventions of the Wolfram Language which are helpful for implementing numerical computations.

Although most of the content is not new, we also offer some new derivations of results, simpler than what can be found elsewhere, or sometimes not written anywhere else. A commented bibliography is provided at the end to give a panorama of the existing literature and to help readers looking for more details.

These notes are aimed both at physicists, caring about their tools to be mathematically well grounded, and at mathematicians, curious about how some of their familiar abstract structures can reveal the beauty of quantum gravity. For the latter, we have introduced specific short paragraphs in small font size, introduced by ‘♥ PHYSICS’, that provides general ideas and references on how the mathematics have been exploited by theoretical physicists, especially in quantum gravity.

The first chapter introduces $SU(2)$ and $SL_2(\mathbb{C})$ the two Lie groups of main interest for quantum gravity. It is aimed at gathering the main ingredients which will be extensively used later, fixing notations and refreshing memories of the reader. Chapter 2 deals with representations of $SU(2)$, and present various possible realisations which are used in the literature. Chapter 3 condenses the main results of recoupling theory of $SU(2)$. Chapter 4 deals with the representations of $SL_2(\mathbb{C})$. Finally chapter 5 wraps everything up into a mathematical presentation of quantum gravity (with loops and spin-foams).

In writing these notes, I received much help from Simone Speziale, Giorgio Sarno, Fabio D’Ambrosio, Alexander Thomas and Carlo Rovelli, who I warmly thank for this.

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Chapter 1

Warmup

The central role of group theory in physics has been largely revealed in the modern theories of the XXth century. Quantum gravity makes no exception. The two main groups of interest for quantum gravity are $SL_2(\mathbb{C})$ and its subgroup $SU(2)$. This may seem natural since $SL_2(\mathbb{C})$ is, in some sense, the 'quantum version' (more precisely the universal cover) of the restricted Lorentz group $SO^+(3, 1)$, which is the symmetry group of Minkowski spacetime, and $SU(2)$ is the subgroup obtained when a preferred time-slice is chosen; but the reason why these groups come out in quantum gravity is actually more subtle (see section 5).

This first chapter is a quick and dense summary of the very basic tools and facts about $SL_2(\mathbb{C})$ and $SU(2)$, which will be later used extensively. It also fixes many of the notations. If you already feel warmed up, you would do well skipping this chapter. If you have never seen these notions in your life, you would do better to first learn them with an introductory book (see the references in the commented bibliography, appendix C).

1.1 Basics of $SL_2(\mathbb{C})$

The algebra of 2×2 complex matrices is denoted $\mathcal{M}_2(\mathbb{C})$. The linear group

$$GL_2(\mathbb{C}) \stackrel{\text{def}}{=} \{M \in \mathcal{M}_2(\mathbb{C}) \mid \det M \neq 0\} \quad (1.1)$$

is a Lie group. Its associated Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ is actually isomorphic to $\mathcal{M}_2(\mathbb{C})$ endowed with the Lie product $[M, N] = MN - NM$.

♣ REMINDER. The set $\mathcal{M}_2(\mathbb{C})$ is an *algebra* because it is a vector space (for addition of matrices) endowed with a bilinear product (the usual matrix product). $GL_2(\mathbb{C})$ is not a linear subspace, but it is a complex differential manifold of dimension 4. What's more, the inversion and the multiplication are both analytical maps, which makes $GL_2(\mathbb{C})$ a complex *Lie group*. A complex Lie group of dimension n is also a real Lie group of dimension $2n$. Therefore $GL_2(\mathbb{C})$ is a real Lie group of dimension 8. The *associated Lie algebra* is the tangent space over the neutral element of the Lie group, usually denoted with Gothic letters. It is indeed a *Lie algebra*, that is to say an algebra where the bilinear product, called the *Lie bracket*, is antisymmetric and satisfies the Jacobi identity.

The subset of invertible uni-modular matrices,

$$SL_2(\mathbb{C}) \stackrel{\text{def}}{=} \{M \in GL_2(\mathbb{C}) \mid \det M = 1\}, \quad (1.2)$$

is a complex Lie sub-group of dimension 3, called the special linear group $SL_2(\mathbb{C})$. Therefore, it is also a real Lie group of dimension 6. Topologically, $SL_2(\mathbb{C})$ is not compact, but it is simply connected.

The Lie algebra of $SL_2(\mathbb{C})$ is

$$\mathfrak{sl}_2(\mathbb{C}) = \{M \in \mathcal{M}_2(\mathbb{C}) \mid \text{Tr } M = 0\}. \quad (1.3)$$

It is a 3-dimensional complex Lie sub-algebra of $\mathfrak{gl}_2(\mathbb{C})$. A basis is given by the three *Pauli matrices*:

$$\sigma_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.4)$$

Interestingly, they satisfy¹

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k. \quad (1.5)$$

With the identity matrix $\mathbb{1}$, the Pauli matrices also provide a basis for $\mathcal{M}_2(\mathbb{C})$: any $a \in \mathcal{M}_2(\mathbb{C})$ can be written uniquely

$$a = a_0 \mathbb{1} + \sum_{k=1}^3 a_k \sigma_k \quad \text{with} \quad a_0, a_1, a_2, a_3 \in \mathbb{C}. \quad (1.6)$$

Note that in this basis, the determinant reads $\det a = a_0^2 - a_1^2 - a_2^2 - a_3^2$. Moreover, the identity matrix and the Pauli matrices also provide a basis to the *real* vector space of 2×2 hermitian matrices, defined by

$$H_2(\mathbb{C}) \stackrel{\text{def}}{=} \{M \in \mathcal{M}_2(\mathbb{C}) \mid M^\dagger = M\}. \quad (1.7)$$

Any $h \in H_2(\mathbb{C})$ can be written uniquely

$$h = h_0 \mathbb{1} + \sum_{k=1}^3 h_k \sigma_k \quad \text{with} \quad h_0, h_1, h_2, h_3 \in \mathbb{R}. \quad (1.8)$$

1.2 The restricted Lorentz group $SO^+(3, 1)$

The spacetime of special relativity is *Minkowski spacetime* \mathbb{M} . Mathematically, it is the vector space \mathbb{R}^4 , endowed with a *Lorentzian inner product*, whose signature is either $(-, +, +, +)$ (general relativists convention) or $(+, -, -, -)$ (particle physicists convention). The group of all isometries (distance-preserving transformations) of Minkowski spacetime is called the *Poincaré group* (sometimes the *inhomogeneous Lorentz group*). The isometries that leave the origin fixed form a linear subgroup, called the *Lorentz group* (sometimes the *homogeneous Lorentz group*), and denoted $O(3, 1)$ (or $O(1, 3)$). It is composed of four connected components related to each other by the operators of parity (space inversion) and time-reversal. The identity component forms a subgroup of $O(3, 1)$, made of transformations that preserves orientation and the direction of time. It is called the *proper orthochronous Lorentz group*, or the *restricted Lorentz group*, denoted $SO^+(3, 1)$.

As a real vector space, Minkowski spacetime \mathbb{M} is isomorphic to $H_2(\mathbb{C})$, with the map

$$X = (t, x, y, z) \mapsto h = t\mathbb{1} + x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}. \quad (1.9)$$

The inverse map is given by

$$h \mapsto X = \frac{1}{2}(\text{Tr } h, \text{Tr } h\sigma_1, \text{Tr } h\sigma_2, \text{Tr } h\sigma_3), \quad (1.10)$$

and the pseudo-scalar product (with convention $(+, -, -, -)$)

$$X \cdot X' = tt' - xx' - yy' - zz' = -\frac{1}{4} \text{Tr} (hh' - h\sigma_1 h' \sigma_1 - h\sigma_2 h' \sigma_2 - h\sigma_3 h' \sigma_3) \quad (1.11)$$

Note that the pseudo-norm of \mathbb{M} is mapped to the determinant over $H_2(\mathbb{C})$:

$$X \cdot X = \det h. \quad (1.12)$$

From the latter property, we see that the action of $a \in SL_2(\mathbb{C})$ upon $h \in H_2(\mathbb{C})$, given by

$$h \mapsto aha^\dagger, \quad (1.13)$$

defines a linear isometry on \mathbb{M} . Thus, it defines an homomorphism between $SO^+(3, 1)$ and $SL_2(\mathbb{C})$, and it is easy to show the following isomorphism of groups

$$SL_2(\mathbb{C})/\{\mathbb{1}, -\mathbb{1}\} \cong SO^+(3, 1). \quad (1.14)$$

$SL_2(\mathbb{C})$ is said to be the double cover, or the universal cover, of $SO^+(3, 1)$, or sometimes also the *Lorentz spin group*. This gives a first glimpse on the role of $SL_2(\mathbb{C})$ in fundamental physics.

¹Here and everywhere else, Einstein notation is understood over repeated indices.

1.3 The sub-groups of $SL_2(\mathbb{C})$

There are many sub-groups of $SL_2(\mathbb{C})$. We describe below the main ones. The figure 1 shows the relations of inclusion between them.

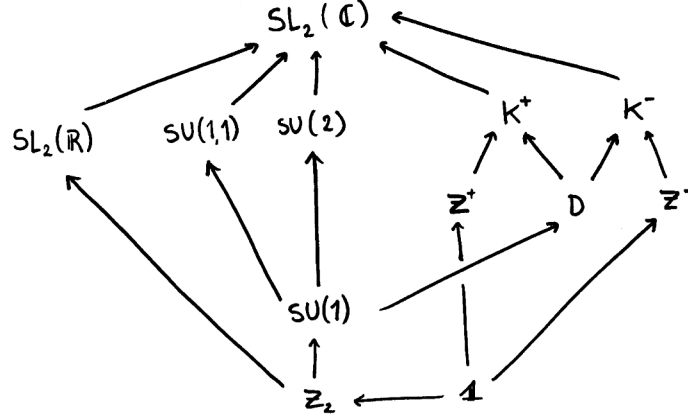


Figure 1: This graph represents the relations of inclusions between the subgroups of $SL_2(\mathbb{C})$.

■ **$SU(2)$, the unitary special group**, is defined by:

$$SU(2) \stackrel{\text{def}}{=} \left\{ u \in SL_2(\mathbb{C}) \mid u^\dagger u = \mathbb{1} \right\}. \quad (1.15)$$

Any $u \in SU(2)$ can be uniquely written as

$$u = u_0 e + i \sum_{k=1}^3 u_k \sigma_k \quad \text{with} \quad u_0, u_1, u_2, u_3 \in \mathbb{R} \quad \text{and} \quad \sum_{k=0}^3 u_k^2 = 1. \quad (1.16)$$

♥ PHYSICS. Through the isomorphism 1.9 and the action 1.13, the definition 1.15 enables to see $SU(2)$ as the stabilizer of the unit time vector $(1, 0, 0, 0)$. Sometimes, the stabilizer is also called the *little group* or the *isotropy group*. Physically, it means that $SU(2)$ only acts over the space, and not in the time direction. Choosing another time direction, related to $(1, 0, 0, 0)$ by a boost Λ , would have defined another stabilizer, isomorphic to $SU(2)$, which makes physicists sometimes talk of a $SU(2)$, as if there were several.

■ **$SU(1, 1)$** is defined by:

$$SU(1, 1) \stackrel{\text{def}}{=} \left\{ v \in SL_2(\mathbb{C}) \mid v^\dagger \sigma_3 v = \sigma_3 \right\}. \quad (1.17)$$

Any $v \in SU(1, 1)$ can be uniquely written as:

$$v = v_0 e + v_1 \sigma_1 + v_2 \sigma_2 + i v_3 \sigma_3 \quad \text{with} \quad v_0, v_1, v_2, v_3 \in \mathbb{R} \quad \text{and} \quad v_0^2 - v_1^2 - v_2^2 + v_3^2 = 1. \quad (1.18)$$

♥ PHYSICS. Similarly to the $SU(2)$ case, $SU(1, 1)$ can be understood by its action in Minkowski spacetime as the stabilizer of $(0, 0, 0, 1)$.

■ **$SL_2(\mathbb{R})$, the real linear special group**, is defined by

$$SL_2(\mathbb{R}) \stackrel{\text{def}}{=} \{ a \in \mathcal{M}_2(\mathbb{R}) \mid \det a = 1 \}, \quad (1.19)$$

and interestingly it is also

$$SL_2(\mathbb{R}) = \left\{ a \in SL_2(\mathbb{C}) \mid a^\dagger \sigma_2 a = \sigma_2 \right\}. \quad (1.20)$$

Any $a \in SU(1, 1)$ can be uniquely written as:

$$a = a_0 e + a_1 \sigma_1 + i a_2 \sigma_2 + a_3 \sigma_3 \quad \text{with} \quad a_0, a_1, a_2, a_3 \in \mathbb{R} \quad \text{and} \quad a_0^2 - a_1^2 + a_2^2 - a_3^2 = 1. \quad (1.21)$$

♥ PHYSICS. Similarly to the $SU(2)$ case, $SL_2(\mathbb{R})$ can be understood by its action in Minkowski spacetime as the stabilizer of $(0, 0, 1, 0)$.

★ NOTA BENE. Following the previous sequence, it would be fair to expect next sub-group to be the one defined by

$$\{b \in SL(2, \mathbb{C}) \mid b^\dagger \sigma_1 = \sigma_1 b^{-1}\}. \quad (1.22)$$

To the knowledge of the author, this group has no name and is not much studied in the literature.

■ **The upper K_+ and lower K_- triangular group** are defined by:

$$K_+ \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}^* \text{ and } \mu \in \mathbb{C} \right\} \quad K_- \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} \lambda^{-1} & 0 \\ \mu & \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}^* \text{ and } \mu \in \mathbb{C} \right\}.$$

They are also called the Borel sub-groups or the parabolic sub-groups.

■ **The subgroups Z_+ et Z_-** are defined by:

$$Z_+ \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\} \quad Z_- \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

■ **The diagonal group D** is defined by:

$$D \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \mid \delta \in \mathbb{C}^* \right\}. \quad (1.23)$$

■ **$SU(1)$, the uni-dimensional unitary group**, is defined by:

$$SU(1) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}. \quad (1.24)$$

It is isomorphic to $U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. Be aware that the notation ‘ $SU(1)$ ’ is not broadly used. $SU(1)$ is also the maximal torus (biggest compact, connected, abelian Lie subgroup) of $SU(2)$.

■ **Z_2 , the center**, defined as the subset of matrices which commute with all $SL_2(\mathbb{C})$, is shown to be

$$Z_2 = \{\mathbb{1}, -\mathbb{1}\}. \quad (1.25)$$

Since it is a normal subgroup (as any center of any group), the quotient $SL_2(\mathbb{C})/Z_2$ is also a group, which can be shown to be isomorphic to the restricted Lorentz group $SO^+(3, 1)$, as was already said in 1.14.

1.4 Decomposing $SL_2(\mathbb{C})$

The structural properties of a matrix group can be grasped through the study of its decompositions. We are going to present four different decompositions of $SL_2(\mathbb{C})$. First of all, let us introduce the notation $S_p(M)$ for the set of the eigenvalues of M , and

$$H_2^{++}(\mathbb{C}) \stackrel{\text{def}}{=} \{M \in H_2(\mathbb{C}) \mid \forall \lambda \in S_p(H), \lambda > 0\}, \quad (1.26)$$

for the set of 2×2 hermitian positive-definite matrices.

♣ REMINDER. Recall that any hermitian matrix is diagonalizable. It is the content of the spectral theorem, which says that H is an hermitian matrix if and only if there exists a unitary matrix U and a diagonal matrix D with real coefficients such that

$$H = U^\dagger D U.$$

The coefficients of D are the eigenvalues of H .

■ **Polar decomposition.** For all $M \in GL_2(\mathbb{C})$, there exists a unique unitary matrix $U \in U(2)$ and a unique positive-definite hermitian matrix $H \in H_2^{++}(\mathbb{C})$ such that:

$$M = H U. \tag{1.27}$$

► PROOF. The polar decomposition actually works for the set $H_n^{++}(\mathbb{C})$ of $n \times n$ positive-definite hermitian matrices. Recall that if $H \in H_n^{++}(\mathbb{C})$, then there exists a unique matrix $S \in H_n^{++}(\mathbb{C})$, called the square root of H , such that $H = S^2$.

Existence. Let $M \in GL_n(\mathbb{C})$ be the matrix to be decomposed, then $M^\dagger M$ is hermitian and positive. Thus we have a unique matrix $S \in H_n^{++}(\mathbb{C})$ such that $M^\dagger M = S^2$. We check finally that MS^{-1} is unitary.

Uniqueness. If $M = QS$ with Q unitary and S definite-positive hermitian, then $M^\dagger = SQ^{-1}$, so $M^\dagger M = S^2$. But $M^\dagger M$ is positive hermitian and has therefore a unique positive hermitian square root, so that S is this square root and Q is equal to MS^{-1} .

For $n = 1$, it is the decomposition $z = re^{i\theta}$ of a non-zero complex number. It is the reason why we call it polar decomposition (kind of generalisation of polar coordinates). □

Remarks:

1. The order does not matter, and the theorem would also be true with $M = UH$.
2. If $M = HU \in SL_2(\mathbb{C})$, then $U \in SU(2)$ and $\det H = 1$.

♥ PHYSICS. The polar decomposition has been used notably by Thiemann and Winkler in their analysis of the coherent states of quantum gravity (see [Thi01, TW01b, TW01c, TW01a]).

■ **Cartan decomposition.** For all $g \in SL_2(\mathbb{C})$, there exists $u, v \in SU(2)$ and $r \in \mathbb{R}_+$ such that:

$$g = ue^{r\sigma_3/2}v^{-1}. \tag{1.28}$$

► PROOF. The proof is essentially the same as previously. Existence. Let $g \in SL_2(\mathbb{C})$ to be decomposed. $g^\dagger g$ is positive-definite hermitian. With the spectral theorem, we have $v \in U(2)$ and d a real diagonal matrix with strictly positive coefficients such that $g^\dagger g = v^\dagger d v$. If $\det v = e^{i\theta}$, then $u = e^{-i\theta}v \in SU(2)$ and $g^\dagger g = u^\dagger d u$. Since $\det d = 1$, one can write $d = e^{r\sigma_3}$. Note that

$$e^{r\sigma_3/2} = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}, \tag{1.29}$$

Then we show $g u e^{r\sigma_3/2} \in SU(2)$. □

Remark:

1. The number r is called *the rapidity of the boost along the axis z* .
2. This theorem can be generalized to the case $SL_n(\mathbb{C})$.
3. Given the decomposition $g = ue^{r\sigma_3/2}v^{-1}$, r is uniquely determined, but u and v are not. The other possible choices are obtained by $(u, v) \mapsto (ue^{i\theta\sigma_3}, ve^{i\theta\sigma_3})$, with $\theta \in \mathbb{R}$.
4. The polar decomposition of $SL_2(\mathbb{C})$ is a particular case of the Cartan decomposition where $v^{-1} = \mathbb{1}$.

♥ PHYSICS. This decomposition has been used notably for the asymptotics of spin-foams amplitude[Spe17], and also for twisted geometries in [LS16].

■ **Gauss decomposition.** Let $g \in SL_2(\mathbb{C})$ such that $g_{22} \neq 0$. There exists a unique triplet $(z_+, d, z_-) \in Z_+ \times D \times Z_-$ such that

$$g = z_+ d z_- \quad (1.30)$$

► **PROOF.** Explicit computation. If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{C})$, then one can write:

$$g = \begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma\delta^{-1} & 1 \end{pmatrix} \quad (1.31)$$

□

■ **Iwasawa decomposition.** According to the Japanese mathematician (1917 - 1998), for any matrix $M \in SL_2(\mathbb{C})$, there exists a unique triplet $(Z, D, U) \in Z_+ \times D_{\mathbb{R}_+} \times SU(2)$ that decomposes M :

$$M = ZDU = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \quad (1.32)$$

with $(z, \lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}_+^* \times \mathbb{C}^2$.

♥ **PHYSICS.** This decomposition has been used notably for the study of covariant twisted geometries [LST12].

1.5 Basics of $SU(2)$

Let us now focus on the special unitary subgroup

$$SU(2) \stackrel{\text{def}}{=} \left\{ u \in SL_2(\mathbb{C}) \mid u^\dagger u = \mathbb{1} \right\}. \quad (1.33)$$

It is a 3-dimensional real Lie subgroup of the 6-dimensional real Lie group $SL_2(\mathbb{C})$. Any $u \in SU(2)$ can be uniquely written as

$$u = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \quad \text{with} \quad (\alpha, \beta) \in \mathbb{C}^2, |\alpha|^2 + |\beta|^2 = 1, \quad (1.34)$$

or equivalently

$$u = \begin{pmatrix} a + ib & -c + id \\ c + id & a - ib \end{pmatrix} \quad \text{with} \quad (a, b, c, d) \in \mathbb{R}^4, a^2 + b^2 + c^2 + d^2 = 1. \quad (1.35)$$

The latter expression shows that $SU(2)$ is diffeomorphic to S^3 (the unit sphere of \mathbb{R}^4). Therefore it is connected, simply connected and compact. The center of $SU(2)$ is $Z_2 = \{\mathbb{1}, -\mathbb{1}\}$, and the quotient $SU(2)/Z_2$ is also a group, which happens to be isomorphic to $SO(3)$ (see section 1.6).

♥ **PHYSICS.** The group $SU(2)$ is central in quantum physics. First, it appears for the theory of the angular momentum (spin). Historically, it was also used as an approximate symmetry group for the isospin that relates protons and neutrons. Then it reappeared to describe the electro-weak interaction. In LQG, $SU(2)$ comes with the holonomies, which are obtained by exponentiation of the Ashtekar variables, used for the quantization (see section 5).

The real Lie algebra of $SU(2)$ is

$$\mathfrak{su}(2) = \left\{ M \in \mathcal{M}_2(\mathbb{C}) \mid M^\dagger = -M \text{ and } \text{Tr } M = 0 \right\}. \quad (1.36)$$

It is a real vector space, of which a basis is given by $(i\sigma_1, i\sigma_2, i\sigma_3)$. Since $SU(2)$ is a compact Lie group, any element of $SU(2)$ can be written (non uniquely) as the exponential of an element of the associated Lie algebra $\mathfrak{su}(2)$ (it is a general theorem for compact Lie group).

■ **Exponential decomposition.** If $u \in SU(2)$, there exists a (non-unique) $\vec{\alpha} \in \mathbb{R}^3$ such that

$$u = e^{i\vec{\alpha} \cdot \vec{\sigma}} = \cos \|\vec{\alpha}\| I + i \sin \|\vec{\alpha}\| \frac{\vec{\alpha}}{\|\vec{\alpha}\|} \cdot \vec{\sigma}. \quad (1.37)$$

► PROOF. Let $M = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \in SU(2)$. The equality is easy to check for $r = \arccos(\operatorname{Re} a)$ and $\alpha_1 = \frac{r}{\sin r} \operatorname{Im} b$, $\alpha_2 = -\frac{r}{\sin r} \operatorname{Re} b$, $\alpha_3 = \frac{r}{\sin r} \operatorname{Im} a$. □

■ **Euler angles decomposition.** For all $u \in SU(2)$, there exists $\alpha, \beta, \gamma \in \mathbb{R}$ (called Euler angles) such that:

$$u = e^{-\frac{i\alpha}{2}\sigma_3} e^{-\frac{i\beta}{2}\sigma_2} e^{-\frac{i\gamma}{2}\sigma_3} \quad (1.38)$$

The choice can be made unique by restricting the domain of definition of the angles, with for instance $\alpha \in]-2\pi, 2\pi[$, $\beta \in [0, \pi]$ and $\gamma \in [|\alpha|, 4\pi - |\alpha|]$.

► PROOF. Explicit computation. The right hand side (RHS) gives

$$\begin{pmatrix} e^{-\frac{i(\alpha+\gamma)}{2}} \cos \beta/2 & -e^{\frac{i(\gamma-\alpha)}{2}} \sin \beta/2 \\ e^{-\frac{i(\gamma-\alpha)}{2}} \sin \beta/2 & e^{\frac{i(\alpha+\gamma)}{2}} \cos \beta/2 \end{pmatrix} \quad (1.39)$$

and for any $u \in SU(2)$, it is clearly possible to find $\alpha, \beta, \gamma \in \mathbb{R}$ to write u in this form. Note that there are other conventions for the definition of Euler angles. The definition we have chosen is the one of Varshalovich ([VMK87] p. 27) and Sakurai ([SN11] p. 177). Instead, Rühl ([Rüh70] p. 43) and the Wolfram Language have chosen the convention $u = e^{\frac{i\alpha}{2}\sigma_3} e^{\frac{i\beta}{2}\sigma_2} e^{\frac{i\gamma}{2}\sigma_3}$. □

1.6 The rotations $SO(3)$

As we have said in section 1.3, the action of $SU(2)$ over Minkowski space, given by 1.13, preserves the time direction. Then, $SU(2)$ acts on the spatial dimensions as the group of rotations over the Euclidean space \mathbb{R}^3 . The group of rotations of \mathbb{R}^3 is

$$SO(3) \stackrel{\text{def}}{=} \{M \in \mathcal{M}_3(\mathbb{R}) \mid M^T M = \mathbf{1} \text{ and } \det M = 1\}. \quad (1.40)$$

It is a Lie group whose Lie algebra is

$$\mathfrak{so}(3) = \{M \in \mathcal{M}_3(\mathbb{R}) \mid M^T + M = 0 \text{ and } \operatorname{Tr} M = 0\}. \quad (1.41)$$

We can show the following isomorphism of Lie algebra

$$\mathfrak{so}(3) \cong \mathfrak{su}(2). \quad (1.42)$$

Besides we have the following group isomorphism:

$$SO(3) \cong SU(2)/Z_2. \quad (1.43)$$

This can be seen with the map

$$g : \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(\alpha^2 + \alpha^{*2} - \beta^2 - \beta^{*2}) & \frac{i}{2}(\alpha^2 - \alpha^{*2} - \beta^2 + \beta^{*2}) & \alpha\beta^* + \alpha^*\beta \\ \frac{i}{2}(-\alpha^2 + \alpha^{*2} - \beta^2 + \beta^{*2}) & \frac{1}{2}(\alpha^2 + \alpha^{*2} + \beta^2 + \beta^{*2}) & i(-\alpha\beta^* + \alpha^*\beta) \\ -\alpha\beta - \alpha^*\beta^* & i(-\alpha\beta + \alpha^*\beta^*) & \alpha\alpha^* - \beta\beta^* \end{pmatrix} \quad (1.44)$$

which is a 2-to-1 onto homomorphism from $SU(2)$ to $SO(3)$. It satisfies notably $g(u) = g(-u)$. Topologically, $SO(3)$ is homeomorphic to the sphere S^3 where the antipodal points have been identified. It is notably connected, but not simply connected.

The action of the homomorphism 1.44 over the Euler decomposition 1.38, shows that any rotation $r \in SO(3)$, can be decomposed as

$$r = r_z(\alpha)r_y(\beta)r_z(\gamma) \quad \text{with } r_z(\phi) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and } r_y(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}. \quad (1.45)$$

where (α, β, γ) are (any choice of) Euler angles of (any choice of) one of the two antecedents of r by g . The unicity of the decomposition can be obtained for instance with the restriction $\alpha \in]-\pi, \pi[$, $\beta \in [0, \pi]$ and $\gamma \in [|\alpha|, 2\pi - |\alpha|]$.

1.7 Integrable functions over $SU(2)$

Haar measure. It can be shown that there exists a unique quasi-regular Borel measure μ over $SU(2)$ which satisfies

1. Invariant: $\mu(u) = \mu(gu) = \mu(ug)$;
2. Normalised: $\mu(SU(2)) = 1$.

It is called the (two-sided normalised) Haar measure of $SU(2)$.

♣ REMINDER. A Borel set in $SU(2)$ is any subset of $SU(2)$ obtained from open sets through countable union, countable intersection, or taking the complement. All Borel sets form an algebra called the Borel algebra $\mathcal{B}(SU(2))$. A Borel $SU(2)$ -measure μ is a non-negative function over $\mathcal{B}(SU(2))$ for which $\mu(\emptyset) = 0$, and which is countable additive (the measure of a disjoint union is the sum of the disjoint sets). A Borel measure is said to be quasi-regular if it is outer regular ($\mu(S) = \inf\{\mu(U) \mid S \subseteq U, U \text{ open}\}$) and inner regular: $\mu(S) = \sup\{\mu(K) \mid K \subseteq S, K \text{ compact}\}$.

The Haar measure enables to define integrals of functions f over $SU(2)$:

$$\int_{SU(2)} f(u) d\mu(u) \quad \text{also denoted} \quad \int_{SU(2)} f(u) du \quad (1.46)$$

The Hilbert space $L^2(SU(2))$. The space of complex functions over $SU(2)$ satisfying

$$\int_{SU(2)} |f(u)|^2 du < \infty, \quad (1.47)$$

is denoted $L^2(SU(2))$. It is an infinite-dimensional Hilbert space with the scalar product

$$(f_1, f_2) \stackrel{\text{def}}{=} \int_{SU(2)} f_1^*(u) f_2(u) du. \quad (1.48)$$

1.8 Integrable functions over $SL_2(\mathbb{C})$

In Vilenkin ([GGV66] pp. 214-215), an invariant measure da over $SL_2(\mathbb{C})$ is defined, so that for any $g \in SL_2(\mathbb{C})$, we have

$$da = d(ga) = d(ag) = d(a^{-1}). \quad (1.49)$$

It is given explicitly by

$$da = \left(\frac{i}{2}\right)^3 |a_{12}|^{-2} da_{11} d\bar{a}_{11} da_{12} d\bar{a}_{12} da_{22} d\bar{a}_{22}. \quad (1.50)$$

In Rühl [Rüh70], the invariant measure is given in terms of the coefficients in the decomposition 1.6, by

$$da = \pi^{-4} \delta\left(a_0 - \sum_{k=1}^3 a_k^2 - 1\right) da_0 d\bar{a}_0 da_1 d\bar{a}_1 da_2 d\bar{a}_2 da_3 d\bar{a}_3. \quad (1.51)$$

It is normalised so that the induced measure over $SU(2)$ is the same Haar measure defined in the previous section. In Rühl ([Rüh70] p. 285), it is shown that using the Cartan decomposition $a = ue^{r\sigma_3/2}v^{-1}$, we have:

$$d\mu(a) = \frac{1}{4\pi} \sinh^2 r dr dv. \quad (1.52)$$

Chapter 2

Representation theory of $SU(2)$

Motivated by their omnipresence in quantum physics, we are going to study the representations of $SU(2)$ over finite-dimensional Hilbert spaces¹. Now, due to Peter-Weyl's theorem (see appendix A), a complex finite representations of a compact group can be decomposed into a direct sum of irreps. We thus focus on irreps of $SU(2)$.

2.1 Irreps of $SU(2)$

To start with, we make use of the following correspondence, which is a particular case of the so-called *Weyl's unitary trick*. For finite dimensional representations, we can show that the following sets of representations are in one-to-one correspondence:

1. Holomorphic representations² of $SL_2(\mathbb{C})$,
2. Representations of $SU(2)$,
3. Representations of $\mathfrak{su}(2)$,
4. \mathbb{C} -linear representations³ of $\mathfrak{sl}_2(\mathbb{C})$.

Moreover, this correspondence preserves invariant subspaces and equivalences of representations.

► **PROOF.** We can show the following bijections [Kna86]:

- (1) \Rightarrow (2) Restriction of the action of $SL_2(\mathbb{C})$ to that of its subgroup $SU(2)$.
- (2) \Rightarrow (3) Differentiation as shown in appendix A.
- (3) \Rightarrow (4) Use the fact that $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$.
- (4) \Rightarrow (1) If G and H are two analytical groups with G simply connected, and if $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is an homomorphism between there Lie algebras, then there exists a smooth homomorphism $\Phi : G \rightarrow H$ whose differential at the identity is ϕ . In our case, $SL_2(\mathbb{C})$ is simply connected, and since ϕ is assumed \mathbb{C} -linear, Φ is holomorphic.

□

We can now describe all the \mathbb{C} -linear irreps of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem. *For all $n \in \mathbb{N}$, there exists a n -dimensional \mathbb{C} -linear irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, unique up to equivalence.*

► **PROOF.** The proof is constructive. Analysis: assume the existence of a $(n + 1)$ -dimensional representation over a vector space \mathcal{H} . One shows then that there exists a basis (v_i) with $i = 0..n$ such that the action of the elements $h, e, f \in \mathfrak{sl}_2(\mathbb{C})$, defined by

$$h \stackrel{\text{def}}{=} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e \stackrel{\text{def}}{=} (\sigma_1 + i\sigma_2)/2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f \stackrel{\text{def}}{=} (\sigma_1 - i\sigma_2)/2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

¹Hilbert spaces are of main interest for quantum physics. But many physicists may not be aware of the real representations of $SU(2)$, which are much less studied (see [IRS91] for more about them).

²'Holomorphic representation' means that the map defined by the representation over the vector space is holomorphic.

³As we will see later, $\mathfrak{sl}_2(\mathbb{C})$ can be seen both as a complex or as a real vector space. 'C-linear representations' means that we care about the complex structure of $\mathfrak{sl}_2(\mathbb{C})$. We will care about the \mathbb{R} -linear representations of $\mathfrak{sl}_2(\mathbb{C})$ in section 4.1.

and which satisfy the commuting relations

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h,$$

is given by:

$$h \cdot v_k = 2(j - k)v_k, \qquad e \cdot v_k = k(n - k + 1)v_{k-1}, \qquad f \cdot v_k = v_{k+1}.$$

Synthesis: one choose any basis (v_i) and defines the action of h , e , and f by the previous formulas. One checks easily that the obtained representation is irreducible. \square

The 3-dimensional complex vector space $\mathfrak{sl}_2(\mathbb{C})$ can also be seen as a 6-dimensional real vector space, which has $\mathfrak{su}(2)$ as subspace. Thus, by restriction of the action of $\mathfrak{sl}_2(\mathbb{C})$ to $\mathfrak{su}(2)$, the previously found \mathbb{C} -linear irreps of $\mathfrak{sl}_2(\mathbb{C})$, define also irreps of $\mathfrak{su}(2)$. Finally, by exponentiating with 1.37, we find all irreps of $SU(2)$ over complex vector spaces.

2.2 Angular momentum realisation

In physic textbooks, the representations of $\mathfrak{sl}_2(\mathbb{C})$ are indexed by an half-integer, called a *spin*. To each spin $j \in \mathbb{N}/2$ is associated an Hilbert space \mathcal{Q}_j of dimension $2j + 1$. The canonical basis, also called *magnetic basis* is composed of the vectors (or ‘kets’ in the Dirac language) denoted $|j, m\rangle$ with $m \in \{-j, \dots, 0, \dots, j\}$. It is made orthonormal by choosing the scalar product that satisfies the property (in Dirac notations):

$$\langle j, m | j, n \rangle = \delta_{mn}. \tag{2.1}$$

We now define the *angular momentum observables* $J_i \stackrel{\text{def}}{=} \frac{1}{2}\sigma_i$, sometimes called simply *generators of $SU(2)$* or *generators of rotations*. They satisfy:

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{2.2}$$

We then define their linear action over \mathcal{Q}_j by

$$\begin{aligned} J_1 |j, m\rangle &= \frac{1}{2}\sqrt{(j-m)(j+m+1)}|j, m+1\rangle + \frac{1}{2}\sqrt{(j+m)(j-m+1)}|j, m-1\rangle, \\ J_2 |j, m\rangle &= \frac{1}{2i}\sqrt{(j-m)(j+m+1)}|j, m+1\rangle - \frac{1}{2i}\sqrt{(j+m)(j-m+1)}|j, m-1\rangle, \\ J_3 |j, m\rangle &= m|j, m\rangle. \end{aligned} \tag{2.3}$$

♥ PHYSICS. In some textbooks, the generators are defined as $J_i \stackrel{\text{def}}{=} \frac{\hbar}{2}\sigma_i$ where \hbar is the reduced Planck constant, which has the dimension of an action. Indeed, this realisation originally comes from atomic physics, where the J_i represent ‘observables’ of angular momentum. For simplicity, we are working in the units where $\hbar = 1$, keeping in mind the possibility to restore \hbar explicitly at any moment by dimensional analysis. By the way, notice also that since ‘observables’ are required to be hermitian operators, the J_i are elements of $i\mathfrak{su}(2)$ and not of $\mathfrak{su}(2)$.

The action of the generators over \mathcal{Q}_j defines a $(2j + 1)$ -dimensional irrep of $SU(2)$, called the *spin- j representation*. This is shown by exhibiting the following equivalence⁴ with the irreps defined in the previous section: $|jm\rangle \cong v_{j-m}$. The generators are related by $J_3 \cong h/2$, $J_+ \cong e$ and $J_- \cong f$, where we define the *ladder operators* $J_+ \stackrel{\text{def}}{=} J_1 + iJ_2$ (up) and $J_- \stackrel{\text{def}}{=} J_1 - iJ_2$ (down). Their action is

$$\begin{aligned} J_+ |j, m\rangle &= \sqrt{(j-m)(j+m+1)}|j, m+1\rangle, \\ J_- |j, m\rangle &= \sqrt{(j+m)(j-m+1)}|j, m-1\rangle. \end{aligned} \tag{2.4}$$

Sometimes the action of J_+ over $|jm\rangle$ is written with a constant phase $e^{i\delta}$. This choice defines an equivalent representation, but the choice of a null phase (called the *Condon-Shortley convention* from [CS59]) is the most widespread.

⁴Two representations are equivalent if there exists a bijective intertwiner, i.e. a bijective map T between the two vector spaces, so that it commutes with the action of the group (see appendix A). It would be too heavy to write explicitly $|jm\rangle = T(v_{j-m})$, so that we choose to write rather $|jm\rangle = T(v_{j-m})$. Similarly, for the operators we write $J_+ \cong e$, rather than $J_+ = T \circ e \circ T^{-1}$.

Finally notice that $|jm\rangle$ is also an eigenvector of the *total angular momentum* $\vec{J}^2 \stackrel{\text{def}}{=} J_1^2 + J_2^2 + J_3^2$:

$$\vec{J}^2 |jm\rangle = j(j+1) |jm\rangle. \quad (2.5)$$

In fact, the $|jm\rangle$ form the unique orthonormal basis that diagonalises simultaneously the commuting operators J_3 and \vec{J}^2 . We say that J_3 and \vec{J}^2 form a *complete set of commuting operators (CSCO)*. From a mathematical perspective, notice also that \vec{J}^2 is not a element of the algebra $\mathfrak{isu}(2)$, but an element of the *universal enveloping algebra* $\mathcal{U}(\mathfrak{isu}(2))$ whose action can be easily computed by successive action of $\mathfrak{su}(2)$. Since \vec{J}^2 has the property to be a quadratic element that commutes with all of $\mathcal{U}(\mathfrak{isu}(2))$, it is called the *Casimir operator* of $\mathcal{U}(\mathfrak{isu}(2))$.

Wigner matrix. The action of the generators of $SU(2)$ over \mathcal{Q}_j defines a linear action of the group $SU(2)$ by exponentiation (see appendix A). The *Wigner matrix* $D^j(g)$, is the matrix that represents the action of $g \in SU(2)$ in the $|j, m\rangle$ basis. It is thus a square matrix of size $2j+1$. By definition, the coefficients of the Wigner matrix are the functions

$$D_{mn}^j(g) \stackrel{\text{def}}{=} \langle j, m | g | j, n \rangle. \quad (2.6)$$

★ **NOTA BENE.** One should be aware of a small ambiguity in the notation ' $\langle j, m | g | j, n \rangle$ ' that arises when g is a matrix that belongs simultaneously to $SU(2)$ and to $\mathfrak{su}(2)$. Then it should be said explicitly if one considers the group action or the algebra action when computing $\langle j, m | g | j, n \rangle$, because it gives a different result. This ambiguity comes from the fact that physicists do not usually write explicitly if they consider the group representation ρ , or its differential $D\rho$. Mathematicians would write $\langle j, m | \rho(g) | j, n \rangle$, or $\langle j, m | D\rho(g) | j, n \rangle$. From equation 1.37, if $g = e^a \in SU(2) \cap \mathfrak{su}(2)$, with $a \in \mathfrak{su}(2)$, then we know from A, that $\rho(g) = e^{D\rho(a)}$, but $\rho(g) \neq D\rho(e^a) = D\rho(g)$. In the definition of the Wigner matrix above, it is the group action which is considered.

The functions D_{mn}^j form an orthogonal family of $L^2(SU(2))$:

$$\int_{SU(2)} dg \overline{D_{m'n'}^{j'}(g)} D_{mn}^j(g) = \frac{1}{2j+1} \delta_{jj'} \delta_{mm'} \delta_{nn'}. \quad (2.7)$$

► **PROOF.** The left hand side (LHS) is the coefficient $\langle j'n' | A | jn \rangle$ of the operator $A : \mathcal{Q}_j \rightarrow \mathcal{Q}_{j'}$ defined by

$$A \stackrel{\text{def}}{=} \int du u^\dagger |j'n'\rangle \langle jm| u. \quad (2.8)$$

We first show that A is an intertwiner. If $j \neq j'$, then, by Schur's lemmas (cf. section A), $A = 0$. Otherwise, $j' = j$, and A is bijective, and there exists $\lambda \in \mathbb{C}$ so that $A = \lambda \mathbb{1}$. Taking the trace on both sides, we see that $\lambda = \frac{\delta_{mm'}}{2j+1}$. □

In fact, Peter-Weyl's theorem (see A) even asserts that the functions D_{mn}^j form a basis of $L^2(SU(2))$: any function $f \in L^2(SU(2))$ can be written

$$f(g) = \sum_{j \in \mathbb{N}/2} \sum_{m=-j}^j \sum_{n=-j}^j f_{mn}^j D_{mn}^j(g), \quad (2.9)$$

with coefficients $f_{mn}^j \in \mathbb{C}$. It implies notably an equivalence between the following Hilbert spaces

$$L^2(SU(2)) \cong \bigoplus_{j \in \mathbb{N}/2} (\mathcal{Q}_j \otimes \mathcal{Q}_j). \quad (2.10)$$

We are going to derive explicit expressions for computing $D_{mn}^j(g)$, but we first need to introduce another realisation of the spin- j irreps.

2.3 Homogeneous realisation

Let $\mathbb{C}_{2j}[z_0, z_1]$ be the vector space of polynomials of two complex variables, homogeneous of degree $2j \in \mathbb{N}$. If $P(z_0, z_1) \in \mathbb{C}_{2j}[z_0, z_1]$, it can be written as

$$P(z_0, z_1) = \sum_{k=0}^{2j} a_k z_0^k z_1^{2j-k}, \quad (2.11)$$

with coefficients $a_0, \dots, a_{2j} \in \mathbb{C}$. The action of $SU(2)$ given by

$$g \cdot P(\mathbf{z}) = P(g^T \mathbf{z}) \quad (2.12)$$

defines a $(2j + 1)$ -dimensional group representation.

► PROOF. The action satisfies

$$(g_1 g_2) \cdot P = g_1 \cdot (g_2 \cdot P) \quad \text{and} \quad e \cdot P = P,$$

which defines a group action over the vector space $\mathbb{C}_{2j}[z_0, z_1]$. ◻

★ NOTA BENE. We could also have defined the action by $P(a^{-1}\mathbf{z})$, $P(a^\dagger\mathbf{z})$ or $P(\mathbf{z}a)$. In fact $P(\mathbf{z}a) = P(a^T\mathbf{z})$, defines the same action. The convention that we have chosen here is the one of Rovelli ([RV14] p. 173). Moreover our convention is consistent with the choice we have made later for the representations of the principal series (cf. 4.3.1). In Bernard ([BLR12] p. 128) the convention $P(a^{-1}\mathbf{z})$ is used.

This group action induces the following action of the generators⁵

$$J_+ \cong z_0 \frac{\partial}{\partial z_1} \quad J_- \cong z_1 \frac{\partial}{\partial z_0} \quad J_3 \cong \frac{1}{2} \left(z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1} \right). \quad (2.13)$$

► PROOF. Let $g(t) = e^{tM}$. The differential of the representation is given by (see A):

$$\begin{aligned} M \cdot P(z_0, z_1) &= \frac{d}{dt} (g(t) \cdot P(z_0, z_1)) \Big|_{t=0} \\ &= \frac{d}{dt} (P(g_{11}(t)z_0 + g_{21}(t)z_1, g_{12}(t)z_0 + g_{22}(t)z_1)) \Big|_{t=0} \\ &= (M_{11}z_0 + M_{21}z_1) \frac{\partial P}{\partial z_0}(z_0, z_1) + (M_{12}z_0 + M_{22}z_1) \frac{\partial P}{\partial z_1}(z_0, z_1). \end{aligned}$$

Then it suffices to apply to J_3, J_+, J_- . ◻

This representation of is equivalent to the spin- j irrep through the correspondence:

$$|j, m\rangle \cong \left(\frac{(2j)!}{(j+m)!(j-m)!} \right)^{1/2} z_0^{j+m} z_1^{j-m}, \quad (2.14)$$

The RHS is sometimes denoted with Dirac notations $\langle z_0 z_1 | jm \rangle$.

► PROOF. One look for an intertwiner $\Phi : \mathcal{Q}_j \rightarrow \mathbb{C}_{2j}[z_0, z_1]$, such that for all i , $J_i \cdot \Phi(|j, m\rangle) = \Phi(J_i |j, m\rangle)$. We show easily the necessary condition:

$$\Phi(|j, m\rangle) = c \left(\frac{(2j)!}{(j+m)!(j-m)!} \right)^{1/2} z_0^{j+m} z_1^{j-m}, \quad (2.15)$$

We choose $c = 1$. ◻

The homogeneous realisation is very convenient to derive an explicit expression for the Wigner matrix coefficients, as we show now.

⁵See the footnote of page 13 for the notation \cong .

Wigner matrix formula.

$$D_{mn}^j(g) = \left(\frac{(j+m)!(j-m)!}{(j+n)!(j-n)!} \right)^{1/2} \sum_k \binom{j+n}{k} \binom{j-n}{j+m-k} g_{11}^k g_{21}^{j+n-k} g_{12}^{j+m-k} g_{22}^{k-m-n}. \quad (2.16)$$

The sum is done over the integers $k \in \{\max(0, m+n), \dots, \min(j+m, j+n)\}$.

► **PROOF.** First of all remark that action 2.12 is well defined for any $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in GL_2(\mathbb{C})$. Explicitly, it acts over the canonical basis of $\mathbb{C}_{2j}[z_0, z_1]$ like

$$\begin{aligned} g \cdot z_0^k z_1^{2j-k} &= P \left(\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right) = P \left(\begin{pmatrix} g_{11}z_0 + g_{21}z_1 \\ g_{12}z_0 + g_{22}z_1 \end{pmatrix} \right) \\ &= (g_{11}z_0 + g_{21}z_1)^k (g_{12}z_0 + g_{22}z_1)^{2j-k} \\ &= \left(\sum_{i=0}^k \binom{k}{i} (g_{11}z_0)^i (g_{21}z_1)^{k-i} \right) \left(\sum_{l=0}^{2j-k} \binom{2j-k}{l} (g_{12}z_0)^l (g_{22}z_1)^{2j-k-l} \right) \quad [\text{formule du binôme}] \\ &= \sum_{i=0}^k \sum_{l=0}^{2j-k} \binom{k}{i} \binom{2j-k}{l} g_{11}^i g_{21}^{k-i} g_{12}^l g_{22}^{2j-k-l} z_0^{i+l} z_1^{2j-i-l} \\ &= \sum_{i=0}^k \sum_{n=i}^{2j-k+i} \binom{k}{i} \binom{2j-k}{n-i} g_{11}^i g_{21}^{k-i} g_{12}^{n-i} g_{22}^{2j-k-n+i} z_0^n z_1^{2j-n} \quad [n := i+l] \\ &= \sum_{n=0}^{2j} \left(\sum_{i=\max(0, n+k-2j)}^{\min(k, n)} \binom{k}{i} \binom{2j-k}{n-i} g_{11}^i g_{21}^{k-i} g_{12}^{n-i} g_{22}^{2j-k-n+i} \right) z_0^n z_1^{2j-n} \end{aligned}$$

Up to there, nothing but simple computation. Now, a subtle change of variables:

$$m := n - j \quad q := k - j. \quad (2.17)$$

(The subtlety is that j is a spin, and we generalise the notation \sum for half-integers bounds with still a step of 1. A general polynomial is written

$$P(\mathbf{z}) = \sum_{m=-j}^j a_{j+m} z_0^{j+m} z_1^{j-m}. \quad (2.18)$$

The action of $GL_2(\mathbb{C})$ is:

$$g \cdot z_0^{j+q} z_1^{j-q} = \sum_{m=-j}^j \left(\sum_{i=\max(0, m+q)}^{\min(j+m, j+q)} \binom{q+j}{i} \binom{j-q}{m+j-i} g_{11}^i g_{21}^{q+j-i} g_{12}^{m+j-i} g_{22}^{-q-m+i} \right) z_0^{j+m} z_1^{j-m}.$$

□

This explicit formula is useful to show some symmetry properties like

$$\overline{D_{mn}^j(u)} = (-1)^{m-n} D_{-m, -n}^j(u). \quad (2.19)$$

♥ **PHYSICS.** This formula is much used for numerical computations with spin-networks and spin-foams, like in [BDF⁺10].

Euler angles expression. Wigner proposed also another explicit expression for its matrix, in terms of the so-called Euler angles. If $u \in SU(2)$, and $\alpha, \beta, \gamma \in \mathbb{R}^3$ are the Euler angles of u , such that $u = e^{-\frac{i\alpha}{2}\sigma_3} e^{-\frac{i\beta}{2}\sigma_2} e^{-\frac{i\gamma}{2}\sigma_3}$, then

$$D_{m'm}^j(u) = e^{-i(\alpha m' + \gamma m)} d_{m'm}^j(\beta)$$

with the *reduced Wigner matrix*

$$\begin{aligned} d_{m'm}^j(\beta) &= \left(\frac{(j+m')!(j-m)!}{(j+m)!(j-m')!} \right)^{\frac{1}{2}} \sum_{k=\max(0, m'+m)}^{\min(j+m', j+m)} (-1)^{m'+j-k} \binom{j+m}{k} \binom{j-m}{j-k+m'} \\ &\quad \times \left(\cos \frac{\beta}{2} \right)^{2k-m-m'} \left(\sin \frac{\beta}{2} \right)^{m+m'+2j-2k}. \quad (2.20) \end{aligned}$$

► **PROOF.** The proof is very easy once given formula 2.16. It only consists in applying it to the matrix

$$e^{-\frac{i\alpha}{2}\sigma_3} e^{-\frac{i\beta}{2}\sigma_2} e^{-\frac{i\gamma}{2}\sigma_3} = \begin{pmatrix} e^{-\frac{i(\alpha+\gamma)}{2}} \cos \beta/2 & -e^{\frac{i(\gamma-\alpha)}{2}} \sin \beta/2 \\ e^{-\frac{i(\gamma-\alpha)}{2}} \sin \beta/2 & e^{\frac{i(\alpha+\gamma)}{2}} \cos \beta/2 \end{pmatrix}.$$

This is what Rühl is doing ([Rüh70] p. 43), except that in his convention, the Euler angles are defined with no minus sign in front, so $\alpha \mapsto -\alpha$, $\beta \mapsto -\beta$, $\gamma \mapsto -\gamma$, and so $d_{m'm}^j(\beta)|_{\text{Rühl}} = d_{m'm}^j(-\beta)$, which finally also equals the RHO of 2.20 provided the coefficient $(-1)^{m'+j-k}$ is changed into $(-1)^{m+j-k}$. It is also the convention chosen by the Wolfram Language to define its function `WignerD`, hence:

$$\text{WignerD}[\{j, m, n\}, \alpha, \beta, \gamma] = e^{i(\alpha m + \gamma n)} d_{m'm}^j(-\beta). \quad (2.21)$$

A proof that does not presupposed formula 2.16 can be found in Sakurai ([SN11] p. 236-238) who has the same convention as ours for Euler angles. It uses the Schwinger's oscillator model for angular momentum. He obtains the same formula as ours, but written in a slightly different way, changing the index of summation $k \rightarrow j + m - k$. Finally, Varshalovich ([VMK87] p. 76), who has also the same convention, gives a series of variations in the way of writing the above formula. □

2.4 Projective realisation

The spin- j irrep can be realised over $\mathbb{C}_{2j}[z]$, the vector space of complex polynomials of one variable z of degree at most $2j$. This realisation is obtained from the $\mathbb{C}_{2j}[z_0, z_1]$ realisation by the map:

$$\begin{cases} \mathbb{C}_{2j}[z_0, z_1] & \rightarrow \mathbb{C}_{2j}[z] \\ P(z_0, z_1) & \mapsto P(z, 1) \end{cases} \quad (2.22)$$

This map is constructed from a projection from \mathbb{C}^2 to \mathbb{C} , hence the name ‘projective’ we give to this realisation. Sometimes it is also named the ‘holomorphic’ realisation. From this we deduce the action of $SU(2)$

$$a \cdot f(z) = (a_{12}z + a_{22})^{2j} f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right), \quad (2.23)$$

and of the algebra $\mathfrak{su}(2)$

$$J_+ \cong -z^2 \frac{d}{dz} + 2jz \quad J_3 \cong z \frac{d}{dz} - j \quad J_- \cong \frac{d}{dz}. \quad (2.24)$$

The canonical basis becomes

$$|j, m\rangle \cong \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} z^{j+m}. \quad (2.25)$$

We can give the following explicit expression for the scalar product that makes the canonical basis orthonormal:

$$\langle f|g \rangle \stackrel{\text{def}}{=} \frac{i}{2} \frac{2j+1}{\pi} \int_{\mathbb{C}} \overline{f(z)} g(z) \frac{dz d\bar{z}}{(1+|z|^2)^{2j+2}}. \quad (2.26)$$

2.5 Spinorial realisation

The following realisation of the spin- j irreps relies on notations which have been much developed by Penrose [PR84]. It was found useful for *twistor theory* [PR86], and later in quantum gravity for the so-called *twisted geometries* [FS10, LS16].

Abstract indices. We are going to use the clever *conventions of abstract indices* of Penrose ([PR84] pp. 68-115). To start with we need a set of ‘abstract indices’ \mathcal{L} , that is to say a countable set of symbols. We use for instance capital letters:

$$\mathcal{L} \stackrel{\text{def}}{=} \{A, B, \dots, Z, A_0, \dots, Z_0, A_1, \dots\}. \quad (2.27)$$

Then we denote $\mathfrak{S}^\bullet \stackrel{\text{def}}{=} \mathbb{C}^2$, and for any abstract index $A \in \mathcal{L}$, $\mathfrak{S}^A \stackrel{\text{def}}{=} \mathfrak{S}^\bullet \times \{A\}$. Obviously \mathfrak{S}^A is isomorphic to \mathbb{C}^2 as a complex vector space. An element of \mathfrak{S}^A will be typically denoted $z^A = (z, A) \in \mathfrak{S}^A$. The abstract index A serves as a marker to ‘type’ the vector $z \in \mathbb{C}^2$ (thus $z^A \neq z^B$). This notation is very efficient to deal with several copies of the same space (here \mathbb{C}^2), like in tensor theory.

The vector space of linear forms from \mathbb{C}^2 to \mathbb{C} , is called the dual space, and denoted \mathfrak{S}_\bullet . Similarly, we denote $\mathfrak{S}_A \stackrel{\text{def}}{=} \mathfrak{S}_\bullet \times \{A\}$, which is trivially isomorphic to the dual space of \mathfrak{S}^A . Its elements, called covectors, are denoted with an abstract lower capital index, z_A . Then the evaluation of a covector $y_A = (y, A)$ on a vector $z^A = (z, A)$ (called a ‘contraction’ or a ‘scalar product’) is denoted $y_A z^A = y(z) \in \mathbb{C}$ (the order does not matter $y_A z^A = z^A y_A$).

Spinors. Consider the space of formal (commutative and associative) finite sums of formal (commutative and associative) products of elements, one from each $\mathfrak{S}^{A_1}, \dots, \mathfrak{S}^{A_p}, \mathfrak{S}_{B_1}, \dots, \mathfrak{S}_{B_q}$. A typical element can be written:

$$t^{A_1 \dots A_p}_{B_1 \dots B_q} = \sum_{i=1}^m z_{1,i}^{A_1} \dots z_{p,i}^{A_p} y^{1,i}_{B_1} \dots y^{q,i}_{B_q}. \quad (2.28)$$

Then impose the rules

1. (Homogeneity) $\forall \alpha \in \mathbb{C}, (\alpha z_1^{A_1}) z_2^{A_2} \dots z_p^{A_p} = z_1^{A_1} (\alpha z_2^{A_2}) \dots z_p^{A_p}$
2. (Distributivity) $(z_1^{A_1} + z_2^{A_2}) z_3^{A_3} \dots z_p^{A_p} = z_1^{A_1} z_3^{A_3} \dots z_p^{A_p} + z_2^{A_2} z_3^{A_3} \dots z_p^{A_p}$.

The resulting space is a vector space denoted $\mathfrak{S}_{B_1 \dots B_q}^{A_1 \dots A_p}$. Its elements are called *spinors* of type (p, q) , and its dimension is 2^{p+q} .

★ **NOTA BENE.** The spinor space $\mathfrak{S}_{B_1 \dots B_q}^{A_1 \dots A_p}$ is isomorphic, but not equal, to $\mathfrak{S}^{A_1} \otimes \dots \otimes \mathfrak{S}^{A_p} \otimes \mathfrak{S}_{B_1} \otimes \dots \otimes \mathfrak{S}_{B_q}$. The difference is the commutativity of the product. For instance the formal product of \mathfrak{S}^{AB} is commutative (by assumption) in the sense that, for $z^A \in \mathfrak{S}^A$ and $y^B \in \mathfrak{S}^B$, $z^A y^B = y^B z^A$, whereas the tensor product is not, $z^A \otimes y^B \neq y^B \otimes z^A$, simply because $z^A \otimes y^B \in \mathfrak{S}^A \otimes \mathfrak{S}^B$ and $y^B \otimes z^A \in \mathfrak{S}^B \otimes \mathfrak{S}^A$ do not belong to the same set. Heuristically it can be said that the abstract indices keep track of the position in the tensor product.

The spinor space is endowed with a bunch a basic operations defined by a set of rules. It would be utterly unpedagogical to state these rules in the most general case. On the contrary they are very intuitive for simple examples, and generalise without ambiguities for higher order spinors.

1. (Index substitution) If $z^A = (z, A) \in \mathfrak{S}^A$, we denote $z^B = (z, B) \in \mathfrak{S}^B$. Thus $z^A \neq z^B$.
2. (Index permutation) If $t^{AB} = \sum_i z_i^A y_i^B \in \mathfrak{S}^{AB}$, we denote $t^{BA} = \sum_i z_i^B y_i^A \in \mathfrak{S}^{AB}$.
3. (Symmetrisation) $t^{(AB)} \stackrel{\text{def}}{=} \frac{1}{2}(t^{AB} + t^{BA})$ or generally $z^{(A_1 \dots A_n)} \stackrel{\text{def}}{=} \frac{1}{n!} \sum_{\sigma \in S_n} z^{A_{\sigma(1)} \dots A_{\sigma(n)}}$.
4. (Anti-symmetrisation) $t^{[AB]} \stackrel{\text{def}}{=} \frac{1}{2}(t^{AB} - t^{BA})$ or generally $z^{[A_1 \dots A_n]} \stackrel{\text{def}}{=} \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon_\sigma z^{A_{\sigma(1)} \dots A_{\sigma(n)}}$, with ϵ_σ the signature of the permutation σ .
5. (Contraction) If $\sum_i z_i^A y_i^B = t^A_B$, then $t^A_A = \sum_i z_i^A y_i^A \in \mathbb{C}$.

Index dualisation We denote the canonical basis of \mathbb{C}^2 :

$$e_0 \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.29)$$

It is easy to show that there exists a unique normalised skew-symmetric spinor of type $(0, 2)$. It is denoted ϵ_{AB} , and satisfies by definition:

$$\epsilon_{AB} = -\epsilon_{BA}, \quad \epsilon_{AB} e_0^A e_1^B = 1. \quad (2.30)$$

★ **NOTA BENE.** ϵ_{AB} corresponds over \mathbb{C}^2 to the unique 2-form ϵ normalised by the condition $\epsilon(e_0, e_1) = 1$, which is nothing but the determinant over \mathbb{C}^2 .

For two vectors $z = (z_0, z_1)$ and $y = (y_0, y_1)$, we show easily that:

$$\epsilon_{AB} z^A y^B = z_0 y_1 - z_1 y_0. \quad (2.31)$$

Interestingly, ϵ_{AB} defines a canonical mapping between \mathfrak{S}^A and \mathfrak{S}_A , given by

$$z^A \mapsto z_A = z^B \epsilon_{BA}. \quad (2.32)$$

It is called *index dualisation*.

★ **NOTA BENE.** Actually the mapping could also have been defined by $z^A \mapsto z_A = z^B \epsilon_{AB}$. It gives the same mapping ‘up to a sign’, since ϵ_{AB} is skew-symmetric. This other convention is chosen by Rovelli ([RV14] p. 23). Our convention is the one of Penrose ([PR84] p. 104).

The covectors of the dual space \mathfrak{S}_\bullet can also be described by a pair of components in the dual basis. The index dualisation can then be expressed in components as $((z_0, z_1), A) \mapsto ((-z_1, z_0), A)$. Similarly to the usual Dirac notation $|z\rangle = (z_0, z_1)$, a notation is sometimes introduced for the dual $[z| = (-z_1, z_0)$. With this choice, the RHS of 2.31 reads $[z|y\rangle$.

Conjugation. The conjugation of $z \in \mathbb{C}$ is denoted \bar{z} or z^* . We define the conjugation over \mathfrak{S}^A by $\overline{z^A} = \overline{(z, A)} \stackrel{\text{def}}{=} (\bar{z}, \dot{A}) = \bar{z}^{\dot{A}} \in \mathfrak{S}^{\dot{A}}$. Thus we have introduced a new set of abstract indices, the dotted indices:

$$\dot{\mathcal{L}} \stackrel{\text{def}}{=} \{\dot{A}, \dot{B}, \dots, \dot{Z}, \dot{A}_0, \dots, \dot{Z}_0, \dot{A}_1, \dots\}. \quad (2.33)$$

We impose moreover that $\overline{\overline{z^A}} = z^A$, i.e. $\ddot{A} = A$, so that the conjugation is an involution. Importantly, we regard the set \mathcal{L} and $\dot{\mathcal{L}}$ as incompatible classes of abstract indices, meaning that we forbid index substitution between them two. In other words, dotted and undotted indices commute: for any $t^{A\dot{B}} \in \mathfrak{S}^{A\dot{B}}$, we have $t^{A\dot{B}} = t^{\dot{B}A}$.

★ **NOTA BENE.** One way to formalise this ‘incompatibility’ between dotted and undotted indices would be to define rather $z^A = (z, A, 0)$ and $z^{\dot{A}} = (z, A, 1)$. Thus the index substitution $z^{\dot{A}} = (z, A, 1) \mapsto z^A = (z, B, 0) = z^B$ clearly does not enable to translate from a dotted to an undotted index. Only the complex conjugation can through $z^{\dot{A}} = (z, A, 1) \mapsto (\bar{z}, A, 1) = \bar{z}^{\dot{A}}$.

Inner product. We define the map J by:

$$J \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} -\bar{z}_1 \\ \bar{z}_0 \end{pmatrix}. \quad (2.34)$$

Using the previously introduced generalised Dirac notation, we read $J|z\rangle = |z]$. Since $J^2 = -1$, the map J behaves over \mathbb{C}^2 very much as the imaginary number i behaves over \mathbb{C} . For this reason the map J is said to define a *complex structure* over \mathbb{C}^2 . A combination of ϵ_{AB} and J defines an inner product over \mathfrak{S}^A :

$$- \epsilon_{AB} (Jz)^A y^B = \bar{z}_0 y_0 + \bar{z}_1 y_1. \quad (2.35)$$

In generalised Dirac notations, we read $-[Jz|y\rangle = \langle z|y\rangle$, which is consistent with the usual Dirac notation for the scalar product.

★ **NOTA BENE.** With the matrix action over spinors, defined just below, equation 2.36, we can see that the inner product is invariant under the action of $SU(2)$: $\langle u \cdot z | u \cdot y \rangle = \langle z | y \rangle$. There is no surprise since it is actually one way of defining $SU(2)$. However the inner product is not invariant under $SL_2(\mathbb{C})$ (contrary to the determinant). So, to ‘another choice of $SU(2)$ ’, in the sense of a stabilizer of a time direction (see 1.3), would correspond another invariant inner product, and thus another complex structure J . For instance, in [LST12], they choose rather $(-J)$ for the complex structure. The choice we have made here is the one of Rovelli ([RV14] p. 24).

Representation. The vector space $\mathcal{M}_2(\mathbb{C})$ is isomorphic to \mathfrak{S}_B^A , through the isomorphism that associates to any $t \in \mathcal{M}_2(\mathbb{C})$ the unique spinor t^A_B such that:

$$\forall z \in \mathbb{C}^2, \quad (tz)^A = t^A_B z^B. \quad (2.36)$$

Then the groups $SL_2(\mathbb{C})$ and $SU(2)$ can be represented over $\mathfrak{S}^{A_1 \dots A_p}$ such as:

$$u \cdot z^{A_1 \dots A_p} = u^A_{B_1} \dots u^A_{B_p} z^{B_1 \dots B_p}. \quad (2.37)$$

Yet this representation is not irreducible, since it is stable over the subspace of completely symmetric spinors $\mathfrak{S}^{(A_1 \dots A_p)}$.

► **PROOF.** Let's see that the action is stable over this subspace. Suppose $z^{A_1 \dots A_p}$ is completely symmetric, i.e. $z^{(A_1 \dots A_p)} = z^{A_1 \dots A_p}$. Then (sketching the proof, the details are left to the reader):

$$\begin{aligned} u \cdot z^{(A_1 \dots A_p)} &= u^{(A_1}_{B_1} \dots u^{A_p)}_{B_p} z^{B_1 \dots B_p} \\ &= u^{A_1}_{(B_1} \dots u^{A_p)}_{B_p} z^{B_1 \dots B_p} \\ &= u^{A_1}_{B_1} \dots u^{A_p}_{B_p} z^{(B_1 \dots B_p)} \\ &= u^{A_1}_{B_1} \dots u^{A_p}_{B_p} z^{B_1 \dots B_p} \\ &= u \cdot z^{A_1 \dots A_p} \end{aligned}$$

thus $u \cdot z^{A_1 \dots A_p}$ is also completely symmetric. \square

Thus $SL_2(\mathbb{C})$ and $SU(2)$ can be represented of the vector space $\mathfrak{S}^{(A_1 \dots A_p)}$ of dimension $p + 1$. A basis is given by

$$\{e_{i_1}^{(A_1} \dots e_{i_p}^{A_p)} \mid i_1, \dots, i_p \in \{0, 1\}\} = \{e_0^{(A_1} \dots e_0^{A_m} e_1^{A_{m+1}} \dots e_1^{A_p)} \mid m \in \{0, \dots, p\}\}. \quad (2.38)$$

This representation is irreducible and equivalent to the spin $p/2$ representation through the intertwiner:

$$e_0^{(A_1} \dots e_0^{A_m} e_1^{A_{m+1}} \dots e_1^{A_p)} \cong z_0^m z_1^{p-m}. \quad (2.39)$$

► **PROOF.** To see this, it is simpler to write the spinors in the canonical basis of $\mathfrak{S}^{A_1 \dots A_p}$:

$$\xi^{A_1 \dots A_p} = \sum_{i_1, i_2, \dots, i_p=0}^1 c^{i_1, \dots, i_p} e_{i_1}^{A_1} \dots e_{i_p}^{A_p} \quad (2.40)$$

The total symmetry of $\xi^{A_1 \dots A_p}$ imposes a total symmetry of the coefficients c^{i_1, \dots, i_p} . Then we define the following bijection between $\mathfrak{S}^{(A_1 \dots A_p)}$ and $\mathbb{C}_p[z_0, z_1]$:

$$\xi^{A_1 \dots A_p} \cong \sum_{p_1, \dots, p_n=0}^1 c^{p_1 \dots p_n} z_{p_1} \dots z_{p_n} \quad (2.41)$$

This defines an intertwiner as can be checked by looking at the action of a group element. \square

Chapter 3

Recoupling theory of $SU(2)$

The $SU(2)$ irreps provide the fundamental building blocks of quantum space-time. From a mathematical perspective, irreps are the fundamental bricks from which other representations are built. Indeed any finite representation of $SU(2)$ is completely reducible (i.e. it can be written as a direct sum of irreps). In particular, a tensor product of irreps can be decomposed into a direct sum of irreps: in other words there exists a bijective intertwiner that maps the tensor product to a direct sum of irreps. Such an intertwiner is sometimes called a ‘*coupling tensor*’ (see Moussouris [Mou83] pp. 10-11). This naming comes from quantum physics: when two systems *couple* (i.e. interact), the total system is described by states of the tensor product of the Hilbert spaces of the subsystems. Notice that there may exist several coupling tensors between a tensor product and its corresponding sum of irreps. It is precisely the goal of ‘*recoupling theory*’ to describe these coupling tensors and to understand how one can translate from one decomposition to another.

3.1 Clebsch-Gordan coefficients

Given \mathcal{Q}_{j_1} and \mathcal{Q}_{j_2} , two irreps of $SU(2)$, the tensor representation is defined over $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2}$. The canonical basis of $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2}$ is given by the elements

$$|j_1 m_1; j_2 m_2\rangle \stackrel{\text{def}}{=} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad (3.1)$$

where m_1 and m_2 belong to the usual range of magnetic indices. Interestingly, this basis is the unique orthonormal basis that diagonalises simultaneously the commuting operators:

$$J_3 \otimes \mathbb{1}, \quad \mathbb{1} \otimes J_3, \quad \vec{J}^2 \otimes \mathbb{1}, \quad \mathbb{1} \otimes \vec{J}^2. \quad (3.2)$$

Another complete set of commuting operators is given by

$$J_3 \otimes \mathbb{1} + \mathbb{1} \otimes J_3, \quad \left(\vec{J} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J} \right)^2, \quad \vec{J}^2 \otimes \mathbb{1}, \quad \mathbb{1} \otimes \vec{J}^2. \quad (3.3)$$

Therefore, there exists an orthonormal basis that diagonalises them simultaneously. It is given by

$$|j_1 j_2; k; n\rangle \quad \text{with } k \in \{|j_1 - j_2|, \dots, j_1 + j_2\} \quad \text{and } n \in \{-k, \dots, k\}, \quad (3.4)$$

and characterised by the action of the operators:

$$(J_3 \otimes \mathbb{1} + \mathbb{1} \otimes J_3) |j_1 j_2; k; n\rangle = n |j_1 j_2; k; n\rangle \quad (3.5)$$

$$\left(\vec{J} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J} \right)^2 |j_1 j_2; k; n\rangle = k(k+1) |j_1 j_2; k; n\rangle \quad (3.6)$$

$$\vec{J}^2 \otimes \mathbb{1} |j_1 j_2; k; n\rangle = j_1(j_1+1) |j_1 j_2; k; n\rangle \quad (3.7)$$

$$\mathbb{1} \otimes \vec{J}^2 |j_1 j_2; k; n\rangle = j_2(j_2+1) |j_1 j_2; k; n\rangle. \quad (3.8)$$

A proof can be found in Sakurai ([SN11] pp. 217-231). This result proves that $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2}$ can be decomposed into a direct sum of irreps, namely we have the following equivalence of representations

$$\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \cong \bigoplus_{k=|j_1-j_2|}^{j_1+j_2} \mathcal{Q}_k. \quad (3.9)$$

The equivalence is given by the bijective intertwiner $\iota : \mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \rightarrow \bigoplus_{k=|j_1-j_2|}^{j_1+j_2} \mathcal{Q}_k$, which satisfies

$$\iota |j_1, j_2; km\rangle = |km\rangle. \quad (3.10)$$

We define the *Clebsch-Gordan coefficients* by the scalar product

$$C_{j_1 m_1 j_2 m_2}^{j m} \stackrel{\text{def}}{=} \langle j_1 m_1 j_2 m_2 | j_1 j_2; j m \rangle. \quad (3.11)$$

Say differently, we have

$$|j_1 j_2; j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C_{j_1 m_1 j_2 m_2}^{j m} |j_1 m_1 j_2 m_2\rangle. \quad (3.12)$$

The Clebsch-Gordan coefficients can be seen as the matrix coefficients of the intertwiner in the canonical bases.

♥ PHYSICS. The Clebsch-Gordan coefficients appear largely in the quantum theory of angular momentum. Two spin-systems are described as one single spin-system with a larger total angular momentum.

Remarks

1. Due to the Condon-Shortley convention for the $SU(2)$ -action, we have $C_{j_1 m_1 j_2 m_2}^{j m} \in \mathbb{R}$.
2. The coefficients $C_{j_1 m_1 j_2 m_2}^{j m}$ are well-defined and non-zero, only if the following *Clebsch-Gordan inequality* (aka *triangle inequality*) is satisfied

$$|j_1 - j_2| \leq j \leq j_1 + j_2. \quad (3.13)$$

Otherwise, we choose by convention, that $C_{j_1 m_1 j_2 m_2}^{j m} = 0$.

3. If $m \neq m_1 + m_2$, then $C_{j_1 m_1 j_2 m_2}^{j m} = 0$.
4. Since the $|jm\rangle$ form an orthonormal basis, we have the following ‘orthogonality relations’

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 m_1 j_2 m_2}^{j' m'} = \delta_{j, j'} \delta_{m, m'}. \quad (3.14)$$

The Clebsch-Gordan coefficients are numbers, but their definition is quite implicit. Hopefully, there are also explicit formulas to compute them!

Explicit formula

$$C_{j_1 m_1 j_2 m_2}^{j m} = \delta_{m, m_1+m_2} \sqrt{2j+1} \sqrt{\frac{(j+m)!(j-m)!(-j+j_1+j_2)!(j-j_1+j_2)!(j+j_1-j_2)!}{(j+j_1+j_2+1)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} \\ \times \sum_k \frac{(-1)^{k+j_2+m_2} (j+j_2+m_1-k)!(j_1-m_1+k)!}{(j-j_1+j_2-k)!(j+m-k)!k!(k+j_1-j_2-m)!} \quad (3.15)$$

► **PROOF.** Other similar expressions can be found in Varshalovich ([VMK87] p. 238), with references to various proofs. It can be notably convenient to recognise in the sum above the so-called *hyper-geometrical function* ${}_3F_2$:

$$C_{j_1 m_1 j_2 m_2}^{j m} = \delta_{m, m_1 + m_2} \sqrt{2j+1} \frac{\sqrt{(j+j_1-j_2)!(j-j_1+j_2)!}}{\sqrt{(-j+j_1+j_2)!(j+j_1+j_2+1)!}} \frac{\sqrt{(j+m)!(j-m)!(j_1+m_1)!(j_2-m_2)!}}{\sqrt{(j_1-m_1)!(j_2+m_2)!}} \\ \times {}_3F_2(j-j_1-j_2, m_1-j_1, -j_2-m_2; j-j_2+m_1+1, j+j_1-m_2+1; 1). \quad (3.16)$$

□

In the Wolfram Language, they are implemented as

$$C_{j_1 m_1 j_2 m_2}^{j_3 m_3} = \text{ClebschGordan}[\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\}]. \quad (3.17)$$

.

Exercise. Show that

$$D_{m_1 n_1}^{j_1} (g) D_{m_2 n_2}^{j_2} (g) = \sum_{j \in \mathbb{N}/2} \sum_{m=-j}^j \sum_{m'=-j}^j C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 n_1 j_2 n_2}^{j m'} D_{m m'}^j (g). \quad (3.18)$$

► **PROOF.**

$$D_{m_1 n_1}^{j_1} (g) D_{m_2 n_2}^{j_2} (g) = \langle j_1 m_1 | g | j_1 n_1 \rangle \langle j_2 m_2 | g | j_2 n_2 \rangle \\ = \langle j_1 m_1; j_2 m_2 | g | j_1 n_1; j_2 n_2 \rangle \\ = \sum_{j \in \mathbb{N}/2} \sum_{m=-j}^j \sum_{j' \in \mathbb{N}/2} \sum_{m'=-j'}^{j'} \langle j_1 m_1 j_2 m_2 | j m \rangle \langle j m | g | j' m' \rangle \langle j' m' | j_1 n_1 j_2 n_2 \rangle \\ = \sum_{j \in \mathbb{N}/2} \sum_{m=-j}^j \sum_{m'=-j}^j C_{j_1 m_1 j_2 m_2}^{j m} C_{j_1 n_1 j_2 n_2}^{j m'} D_{m m'}^j (g)$$

□

3.2 Invariant subspace

A general tensor product of n irreps can be decomposed into a direct sum

$$\bigotimes_{i=1}^n \mathcal{Q}_{j_i} \cong \bigoplus_{k=0}^J \left(\underbrace{\mathcal{Q}_k \oplus \dots \oplus \mathcal{Q}_k}_{d_k \text{ times}} \right), \quad (3.19)$$

where $J = \sum_i j_i$ and d_k is the degeneracy of the irrep \mathcal{Q}_k . Here, ‘decomposing’ means ‘finding a bijective intertwiner between the two spaces’. Concretely, such a decomposition is obtained by applying successively the decomposition of only two, given by 3.9. The operator $\mathbb{1} \otimes \dots \otimes J_i \otimes \dots \otimes \mathbb{1}$ corresponding to J_i acting on the k^{th} Hilbert space of the product is denoted $(J_i)_k$. The three components $(J_1)_k, (J_2)_k, (J_3)_k$ form the vectorial operator \vec{J}_k .

♥ **PHYSICS.** In quantum gravity, such tensor spaces appear in the *kinematical Hilbert space* \mathcal{H} . The description of the dynamics requires to impose constraints that select subspaces of \mathcal{H} . One important constraint is the *Gauss constraint* which reduces \mathcal{H} to its $SU(2)$ -invariant subspace $\text{Inv}_{SU(2)} \mathcal{H}$ that we define below.

We define the $SU(2)$ -invariant subspace as

$$\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^n \mathcal{Q}_{j_i} \right) \stackrel{\text{def}}{=} \left\{ \psi \in \bigotimes_{i=1}^n \mathcal{Q}_{j_i} \mid \forall g \in SU(2), g \cdot \psi = \psi \right\}. \quad (3.20)$$

It can also be characterized rather by the action of the algebra:

$$\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^n \mathcal{Q}_{j_i} \right) = \left\{ \psi \in \bigotimes_{i=1}^n \mathcal{Q}_{j_i} \mid \forall s \in \mathfrak{su}(2), s \cdot \psi = 0 \right\}. \quad (3.21)$$

From this, it is easy to see that

$$\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^n \mathcal{Q}_{j_i} \right) \cong \underbrace{\mathcal{Q}_0 \oplus \dots \oplus \mathcal{Q}_0}_{d_0 \text{ times}}, \quad (3.22)$$

where $\mathcal{Q}_0 \cong \mathbb{C}$ is the trivial representation. Interestingly, we also have the following isomorphism:

$$\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^n \mathcal{Q}_{j_i} \right) \cong \text{Hom}_{SU(2)} \left(\bigotimes_{i=1}^n \mathcal{Q}_{j_i}, \mathcal{Q}_0 \right), \quad (3.23)$$

where the RHS is the vector space of $SU(2)$ -intertwiners between $\bigotimes_{i=1}^n \mathcal{Q}_{j_i}$ and \mathcal{Q}_0 .

► **PROOF.** Let $T : \bigotimes_{k=1}^n \mathcal{Q}_{j_k} \rightarrow \mathcal{Q}_0$ be an intertwiner. Since T is a linear form, there exists $\psi_T \in \bigotimes_{k=1}^n \mathcal{Q}_{j_k}$ such that $T(\phi) = \langle \phi | \psi_T \rangle$. Since T is also an intertwiner, we have for all $\psi \in \bigotimes_{k=1}^n \mathcal{Q}_{j_k}$ and $u \in SU(2)$, $\langle \phi | u \cdot \psi_T \rangle = \langle u^\dagger \cdot \phi | \psi_T \rangle = T(u^\dagger \cdot \phi) = u^\dagger \cdot T(\phi) = T(\phi) = \langle \phi | \psi_T \rangle$. So $u \cdot \psi_T = \psi_T$. We can check that the map $T \mapsto \psi_T$ is linear and bijective. QED. □

Orthogonal projector. By definition, the orthogonal projector P over $\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^n \mathcal{Q}_{j_i} \right)$ satisfies

$$P^2 = P \quad \text{and} \quad P^\dagger = P. \quad (3.24)$$

It is easy to show that

$$P = \int_{SU(2)} dg \bigotimes_{k=1}^n D^{j_k}(g). \quad (3.25)$$

Remember also that if $|j\rangle$ is an orthonormal basis of $\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^n \mathcal{Q}_{j_i} \right)$, then P can also be written as

$$P = \sum_j |j\rangle\langle j|. \quad (3.26)$$

3.3 Wigner's $3jm$ -symbol

We can decompose $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3}$ into a direct sum by applying 3.9 first on the left tensor product:

$$\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3} \rightarrow \left(\bigoplus_{j_{12}} \mathcal{Q}_{j_{12}} \right) \otimes \mathcal{Q}_{j_3} \rightarrow \bigoplus_{j_{12}=|j_1-j_2|}^{j_1+j_2} \bigoplus_{k=|j_{12}-j_3|}^{j_{12}+j_3} \mathcal{Q}_k \quad (3.27)$$

Thus we construct an orthonormal basis of $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3}$ given by the states

$$|(j_1 j_2) j_3; j_{12} k n\rangle = \sum_{m_1, m_2, m_3, m_{12}} C_{j_1 m_1 j_2 m_2}^{j_{12} m_{12}} C_{j_{12} m_{12} j_3 m_3}^{k n} \bigotimes_{i=1}^3 |j_i, m_i\rangle, \quad (3.28)$$

with $j_{12} \in \{|j_1 - j_2|, \dots, j_1 + j_2\}$ and $k \in \{|j_{12} - j_3|, \dots, j_{12} + j_3\}$ and $n \in \{-k, \dots, k\}$.

► **PROOF.** This orthonormal basis is obtained by applying two times the relation 3.28, applied first to the left tensor product. Notice that applying it first on the right would build rather the states:

$$|j_1(j_2 j_3); j_{23} k n\rangle = \sum_{m_1, m_2, m_3, m_{23}} C_{j_2 m_2 j_3 m_3}^{j_{23} m_{23}} C_{j_1 m_1 j_{23} m_{23}}^{k n} \bigotimes_{i=1}^3 |j_i, m_i\rangle. \quad (3.29)$$

□

Then we show

$$\text{Inv}_{SU(2)} \left(\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3} \right) = \text{Span} \{ |(j_1 j_2) j_3; j_3 0 0 \rangle \}. \quad (3.30)$$

► PROOF. First we show the equivalence:

$$\bar{J}^2 |(j_1 j_2) j_3; j_{12} k n\rangle = 0 \Leftrightarrow k = 0.$$

We conclude using the characterisation 3.21. \square

Thus if the Clebsch-Gordan condition is satisfied ($|j_1 - j_2| \leq j_3 \leq j_1 + j_2$), $\text{Inv}_{SU(2)}(\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3})$ is one dimensional. Otherwise $\text{Inv}_{SU(2)}(\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3}) = \{0\}$. Now supposing that the condition is satisfied, there exists a unique unit vector in $\text{Inv}_{SU(2)}(\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3})$,

$$|0\rangle = \sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \bigotimes_{k=1}^3 |j_k, m_k\rangle, \quad (3.31)$$

such that the coefficients $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ in the canonical basis are real and satisfy the symmetry properties:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix}. \quad (3.32)$$

The coefficients $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are called the *Wigner's 3jm-symbol* and are related to the Clebsch-Gordan coefficients by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}} C_{j_1 m_1 j_2 m_2}^{j_3, -m_3}. \quad (3.33)$$

In Mathematica, they are given by `ThreeJSymbol[{j1, m1}, {j2, m2}, {j3, m3}]`.

► PROOF. In $\text{Inv}_{SU(2)}(\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3})$, all vectors are proportional to

$$\begin{aligned} |(j_1 j_2) j_3; j_3 00\rangle &= \sum_{m_1, m_2, m_3, m} C_{j_1 m_1 j_2 m_2}^{j_3 m} C_{j_1 m_1; j_2 m_2}^{j_3 m} |j_1 m_1; j_2 m_2; j_3 m_3\rangle && \text{[applying 3.28]} \\ &= \sum_{m_1, m_2, m_3, m} \delta_{m, -m_3} \delta_{j, j_3} \frac{(-1)^{j_3 + m_3}}{\sqrt{2j_3 + 1}} C_{j_1 m_1; j_2 m_2}^{j_3 m} |j_1 m_1; j_2 m_2; j_3 m_3\rangle && \text{[computing 3.15]} \\ &= \sum_{m_1, m_2, m_3} \frac{(-1)^{j_3 + m_3}}{\sqrt{2j_3 + 1}} C_{j_1 m_1; j_2 m_2}^{j_3, -m_3} |j_1 m_1; j_2 m_2; j_3 m_3\rangle && \text{[simplifying].} \end{aligned}$$

The proportionality factor is chosen to be $(-1)^{j_1 - j_2 + j_3}$ to match the reality and the symmetry requirements. \square

Remarks.

1. These symbols satisfy nice symmetry properties such as:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \quad (3.34)$$

2. The orthogonality relations 3.14 become

$$\sum_{j m} (2j + 1) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ m'_1 & m'_2 & m \end{pmatrix} = \delta_{m_1 m'_1} \delta_{m_2 m'_2} \quad (3.35)$$

$$\sum_{m_1 m_2} (2j + 1) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} = \delta_{j j'} \delta_{m m'} \quad (3.36)$$

Exercise. Show that

$$\int_{SU(2)} D_{m_1 n_1}^{j_1}(u) D_{m_2 n_2}^{j_2}(u) D_{m_3 n_3}^{j_3}(u) du = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \quad (3.37)$$

► PROOF. The result can be directly obtained by equating 3.25 and 3.26, and expressing the equality in the magnetic basis. \square

3.4 Wigner's $4jm$ -symbol

Similarly to the previous section we can decompose $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3} \otimes \mathcal{Q}_{j_4}$ into a direct sum by applying 3.9 successively:

$$\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3} \otimes \mathcal{Q}_{j_4} \cong \bigoplus_{j_{12}=|j_1-j_2|}^{j_1+j_2} \bigoplus_{k=|j_{12}-j_3|}^{j_{12}+j_3} \bigoplus_{l=|k-j_4|}^{k+j_4} \mathcal{Q}_l \quad (3.38)$$

In particular, we can see that

$$\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^4 \mathcal{Q}_{j_i} \right) \cong \underbrace{\mathcal{Q}_0 \oplus \dots \oplus \mathcal{Q}_0}_{d_0 \text{ times}} \quad \text{with } d_0 = \max(|j_1 - j_2|, |j_3 - j_4|) - \min(j_1 + j_2, j_3 + j_4). \quad (3.39)$$

An orthonormal basis of $\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^4 \mathcal{Q}_{j_i} \right)$ is given by

$$|j\rangle_{12} = \sum_{m_1, m_2, m_3, m_4} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \bigotimes_{k=1}^4 |j_k, m_k\rangle, \quad \text{with } j \in \{\max(|j_1 - j_2|, |j_3 - j_4|), \dots, \min(j_1 + j_2, j_3 + j_4)\} \quad (3.40)$$

and

$$\begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \stackrel{\text{def}}{=} \sum_m (-1)^{j-m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j & j_3 & j_4 \\ -m & m_3 & m_4 \end{pmatrix} \quad (3.41)$$

► PROOF. First, we construct an orthonormal basis of $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2} \otimes \mathcal{Q}_{j_3} \otimes \mathcal{Q}_{j_4}$ given by the states

$$|(j_1 j_2 j_3 j_4; jklm)\rangle = \sum_{m_1, m_2, m_3, m, n, m_4} C_{j_1 m_1 j_2 m_2}^{j m} C_{j m j_3 m_3}^{k n} C_{k n j_4 m_4}^{l m} \bigotimes_{i=1}^4 |j_i, m_i\rangle, \quad \text{with } j \in \{|j_1 - j_2|, \dots, j_1 + j_2\} \text{ and } k \in \{|j - j_3|, \dots, j + j_3\} \\ \text{and } l \in \{|k - j_4|, \dots, k + j_4\} \text{ and } n \in \{-l, \dots, l\}. \quad (3.42)$$

$\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^4 \mathcal{Q}_{j_i} \right)$ is spanned by the vectors with $l = 0$. Similarly to the case $n = 3$, we compute

$$|((j_1 j_2 j_3 j_4; j j_4 0 0))\rangle = (-1)^{j_4 + j_1 - j_2 - j_3} \sum_{m_1, m_2, m_3, m_4} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \bigotimes_{i=1}^4 |j_i, m_i\rangle.$$

□

This basis has the interesting property that it diagonalises $(\vec{J}_1 + \vec{J}_2)^2$:

$$(\vec{J}_1 + \vec{J}_2)^2 |j\rangle_{12} = j(j+1) |j\rangle_{12}. \quad (3.43)$$

The $4jm$ -symbol also satisfy orthogonality relations:

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j_{12})} \begin{pmatrix} j_1 & j_2 & j_3 & l_4 \\ m_1 & m_2 & m_3 & n_4 \end{pmatrix}^{(l_{12})} = \frac{\delta_{j_{12} l_{12}} \delta_{j_4 l_4} \delta_{m_4 n_4}}{d_{j_{12}} d_{j_4}}. \quad (3.44)$$

Finally we can show, similarly to 3.37, that

$$\int_{SU(2)} D_{m_1 n_1}^{j_1}(u) D_{m_2 n_2}^{j_2}(u) D_{m_3 n_3}^{j_3}(u) D_{m_4 n_4}^{j_4}(u) du \\ = \sum_j (2j+1) \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ n_1 & n_2 & n_3 & n_4 \end{pmatrix}^{(j)}. \quad (3.45)$$

3.5 Wigner's 6j-symbol

In the previous section, we have exhibited an orthonormal basis for $\text{Inv}_{SU(2)} \left(\bigotimes_{i=1}^4 \mathcal{Q}_{j_i} \right)$. It is built from one possible decomposition of $\bigotimes_{i=1}^4 \mathcal{Q}_{j_i}$ into irreps. Another possible decomposition leads to another basis

$$|j\rangle_{23} = \sum_{m_1, m_2, m_3, m_4} \sqrt{2j+1} \begin{pmatrix} j_4 & j_1 & j_2 & j_3 \\ m_4 & m_1 & m_2 & m_3 \end{pmatrix}^{(j)} \bigotimes_{i=1}^4 |j_i, m_i\rangle \quad (3.46)$$

► PROOF. From

$$|(j_1(j_2j_3))j_4; jklm\rangle = \sum_{m_1, m_2, m_3, m, n, m_4} C_{j_2m_2j_3m_3}^{jm} C_{j_1m_1jm}^{kn} C_{knj_4m_4}^{lm} \bigotimes_{i=1}^4 |j_i, m_i\rangle, \quad (3.47)$$

we show that

$$|(j_1(j_2j_3))j_4; jj_400\rangle = (-1)^{j_1+j_2-j_3+j_4} \sum_{m_1, m_2, m_3, m_4} \sqrt{2j_{23}+1} \begin{pmatrix} j_4 & j_1 & j_2 & j_3 \\ m_4 & m_1 & m_2 & m_3 \end{pmatrix}^{(j_{23})} \bigotimes_{i=1}^4 |j_i, m_i\rangle.$$

□

The change of basis is given by

$${}_{12} \langle j|k\rangle_{23} = \sqrt{2j+1} \sqrt{2k+1} (-1)^{j_1+j_2+j_3-j_4-2j-2k} \begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & k \end{Bmatrix} \quad (3.48)$$

where we have defined a new symbol:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \stackrel{\text{def}}{=} \sum_{m_1, \dots, m_6} (-1)^{\sum_{i=1}^6 (j_i - m_i)} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ m_1 & -m_5 & m_6 \end{pmatrix} \\ \times \begin{pmatrix} j_4 & j_2 & j_6 \\ m_4 & m_2 & -m_6 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ m_3 & -m_4 & m_5 \end{pmatrix} \quad (3.49)$$

In the Wolfram Language, it is returned by the function `SixJSymbol[{j1, j2, j3}, {j4, j5, j6}]`. These symbols satisfy the symmetries

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_3 & j_2 & j_1 \\ j_6 & j_5 & j_4 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_2 & j_3 \\ j_1 & j_5 & j_6 \end{Bmatrix}. \quad (3.50)$$

Similarly one can define the symbols $9j$ and $15j$.

♥ PHYSICS. The $\{6j\}$ -symbol appeared in quantum gravity when Ponzano and Regge realised that the $\{6j\}$ -symbol approximate the action of general relativity in the semi-classical limit [PR68]. More precisely they have shown that

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \underset{j_i \rightarrow \infty}{\sim} \frac{1}{\sqrt{12\pi V}} \cos(S(j_i) + \pi/4), \quad (3.51)$$

where V is the volume of a tetrahedron whose edges have a length of $j_i + 1/2$, and $S(j_i)$ is the so-called *Regge action*, which is a discrete 3-dimensional version of the Einstein-Hilbert action. This result was a important source of inspiration for later development of spin-foams.

3.6 Graphical calculus

The recoupling theory of $SU(2)$ can be nicely implemented graphically. The underlying philosophy of it, is to take advantage of the 2-dimensional surface offered by our sheets of paper and our blackboards to literally *draw our calculations*, rather than restricting oneself to the usual one dimensional lines of calculations. If done properly, the method can help to understand the structure of analytical expressions, and make computations faster. Of course, the first principle of graphical calculus is that there should be a one-to-one correspondence between analytical expressions and diagrams. There exists many conventions for this correspondence in the literature, so we have chosen one that seems to be quite popular [SSS18], and which is described in details by Varshalovich ([VMK87], Chap. 11).

Definitions. The basic object of this graphical calculus is the 3-valent node, that represent the Wigner's $3jm$ symbol:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{array}{c} j_2 \\ \swarrow \quad \uparrow \quad \searrow \\ j_1 \quad \quad \quad j_3 \\ - \end{array} = \begin{array}{c} j_3 \\ \swarrow \quad \uparrow \quad \searrow \\ j_1 \quad \quad \quad j_2 \\ + \end{array}. \quad (3.52)$$

Remark that

1. The signs $+/-$ on the nodes indicate the sense of rotation (anticlockwise/clockwise) in which the spins must be read. To alleviate notations we can decide not to write them by choosing conventionally that the default sign of the nodes is minus, if not otherwise specified. The arrows on the wires will be used below to define the operation of summation.
2. Everywhere we implicitly assumed that the Clebsch-Gordan inequalities are satisfied.
3. The magnetic indices are implicit on the diagram, which creates no ambiguity, as long as we associate m_i to the spin j_i .
4. The symmetry properties 3.32 are naturally implemented on the diagram, which also guarantees the one-to-one correspondence between the analytical expression and the diagram.
5. Only the topology of the diagram matters, which means that all topological deformations are allowed.

$$\begin{array}{c} j_2 \\ \swarrow \quad \uparrow \quad \searrow \\ j_1 \quad \quad \quad j_3 \end{array} = \begin{array}{c} j_2 \\ \swarrow \quad \uparrow \quad \searrow \\ j_1 \quad \quad \quad j_3 \end{array} = \begin{array}{c} j_2 \\ \swarrow \quad \uparrow \quad \searrow \\ j_1 \quad \quad \quad j_3 \end{array} \quad (3.53)$$

This principle of topological equivalence is a strong principle of graphical calculus, that will hold for any other diagram constructed later.

Then we can define graphically the two basic operations of algebra: multiplication and summation. Multiplication is implemented simply by juxtaposition of two diagrams:

$$\begin{array}{c} j_2 \\ \swarrow \quad \uparrow \quad \searrow \\ j_1 \quad \quad \quad j_3 \end{array} \begin{array}{c} j_5 \\ \swarrow \quad \uparrow \quad \searrow \\ j_4 \quad \quad \quad j_6 \end{array} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ m_4 & m_5 & m_6 \end{pmatrix} \quad (3.54)$$

To define the summation, we shall first tell more about the orientation of external wires. As you may have noticed, the arrows on the wires are all outgoing. Now we define also the ingoing orientation with the general rule that inverting the orientation¹ of an external line ($j m$) amounts analytically to transforming m to $-m$ and multiplying the overall expression by a factor $(-1)^{j-m}$. For instance

$$\begin{array}{c} j_2 \\ \swarrow \quad \uparrow \quad \searrow \\ j_1 \quad \quad \quad j_3 \end{array} = (-1)^{j_1-m_1} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{pmatrix}. \quad (3.55)$$

¹It can be seen as the contraction with the 'metric tensor' $\epsilon_{mm'} = (-1)^{j-m} \delta_{m,-m'}$, as it is often said.

The summation over a magnetic index m (from $-j$ to j) is now represented by gluing two external wires with the same spin j and magnetic index m , but of opposite directions, like:

$$\begin{array}{c} j_1 & j_2 & j_3 & j_4 \\ & \nearrow & \nwarrow & \\ & & \text{---} & \\ & \nwarrow & \nearrow & \\ & j & & \end{array} = \sum_{m=-j}^j \begin{array}{c} j_1 & j_2 \\ & \nearrow & \nwarrow \\ & & \text{---} \\ & \nwarrow & \nearrow \\ & j & & \end{array} \begin{array}{c} j & j_3 & j_4 \\ \nwarrow & \nearrow & \\ \text{---} & & \\ \nearrow & \nwarrow & \\ j & & \end{array} \quad (3.56)$$

On the RHS, we recognise the definition of the $4jm$ -symbol, so that

$$\begin{array}{c} j_1 & j_2 & j_3 & j_4 \\ & \nearrow & \nwarrow & \\ & & \text{---} & \\ & \nwarrow & \nearrow & \\ & j & & \end{array} = \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \quad (3.57)$$

The line between two nodes, whose magnetic index is summed over, is called an internal line, in opposition to external lines, which have a free hand. Contrary to the previous rule of inversion for external lines, it is easy to show that changing the orientation of an internal line gives a phase:

$$\begin{array}{c} j_1 & j_2 & j_3 & j_4 \\ & \nearrow & \nwarrow & \\ & & \text{---} & \\ & \nwarrow & \nearrow & \\ & j & & \end{array} = (-1)^{2j} \begin{array}{c} j_1 & j_2 & j_3 & j_4 \\ & \nearrow & \nwarrow & \\ & & \text{---} & \\ & \nwarrow & \nearrow & \\ & j & & \end{array} .$$

A powerful aspect of graphical calculus comes from the representation of the Kronecker delta with a single line

$$\begin{array}{c} (j_1, m_1) \\ \uparrow \\ \downarrow \\ (j_2, m_2) \end{array} = \delta_{j_1 j_2} \delta_{m_1 m_2} \quad \text{or} \quad \begin{array}{c} m \\ \downarrow \\ \uparrow \\ j \\ \downarrow \\ n \end{array} = \delta_{mn} . \quad (3.58)$$

The rule of summation applied to it enables to compute its trace:

$$\begin{array}{c} j \\ \circlearrowright \end{array} = 2j + 1 \quad (3.59)$$

For instance, the orthogonality relation 3.36 now reads

$$\begin{array}{c} (j_1, m_1) \\ \downarrow \\ \circlearrowleft \\ \uparrow \\ (j_2, m_2) \end{array} \begin{array}{c} j_3 & j_4 \\ \circlearrowright \\ j_1 \end{array} = \begin{array}{c} (j_1, m_1) \\ \downarrow \\ \downarrow \\ (j_2, m_2) \end{array} \quad (3.60)$$

Lemmas. From all the rules described above, the following lemma can already be checked as an exercise.

1. Reversing all external lines has no effect:

$$\begin{array}{c} j_2 \\ \uparrow \\ j_1 \swarrow \quad \nearrow j_3 \end{array} = \begin{array}{c} j_2 \\ \downarrow \\ j_1 \swarrow \quad \nearrow j_3 \end{array} . \tag{3.61}$$

2. Changing the sign of the node gives a phase

$$\begin{array}{c} j_2 \\ \uparrow \\ j_1 \swarrow \quad \nearrow j_3 \\ - \end{array} = (-1)^{j_1+j_2+j_3} \begin{array}{c} j_2 \\ \uparrow \\ j_1 \swarrow \quad \nearrow j_3 \\ + \end{array} . \tag{3.62}$$

3. The evaluation of the so-called Θ -graph:

$$\begin{array}{c} j_1 \rightarrow \\ \circlearrowleft \\ j_2 \rightarrow \\ \circlearrowright \\ j_3 \rightarrow \end{array} = 1. \tag{3.63}$$

4. Similarly, 3.44 implies

$$\begin{array}{c} j_1 \\ \circlearrowleft \\ j_2 \\ \circlearrowright \\ j_3 \\ \circlearrowleft \\ j_4 \end{array} \begin{array}{c} i \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ k \end{array} = \frac{\delta_{i,k}}{d_i}. \tag{3.64}$$

Invariant functions. One nice thing about this graphical calculus is that it makes easy to represent and to remember the Wigner $6j$ -symbols:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} = \begin{array}{c} j_1 \swarrow \quad \searrow j_3 \\ \quad \nearrow j_2 \\ \quad \nwarrow j_4 \\ j_5 \swarrow \quad \searrow j_6 \end{array} \tag{3.65}$$

As we can see, the $6j$ -symbol looks like a tetrahedron. All magnetic indices are summed over, so that it is only a function of the spins j_i , what we can call an *invariant function*. It gives us the idea to define other invariant functions by finding other diagrams with no external links. For instance we define the $9j$ -symbol:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{array} \right\} = \begin{array}{c} j_1 \swarrow \quad \searrow j_3 \\ \quad \nearrow j_2 \\ \quad \nwarrow j_4 \\ j_9 \swarrow \quad \searrow j_6 \\ \quad \nearrow j_8 \\ \quad \nwarrow j_7 \end{array} \tag{3.66}$$

Notice that we could have also defined the $9j$ -symbol to be rather

$$(3.67)$$

but this one can be actually rewritten as the product of two $6j$ -symbol. Such a decomposition cannot be done with the $9j$ -symbol 3.66, so that it is said ‘irreducible’². We also have the $15j$ -symbol:

$$\begin{pmatrix} j_1 & j_2 & j_{11} \\ j_4 & j_5 & j_{15} \\ j_7 & j_3 & j_{14} \\ j_9 & j_6 & j_{13} \\ j_8 & j_{10} & j_{12} \end{pmatrix} = \text{Diagram} \quad (3.68)$$

which is the definition used by [SSS18]. It is different from the convention chosen in [Oog92], which is

$$\begin{pmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \\ j_1 & j_2 & j_3 & j_4 & j_5 \\ l_{10} & l_9 & l_8 & l_7 & l_6 \end{pmatrix} = \text{Diagram} \quad (3.69)$$

Contrary to the $6j$ -symbol, there is no consensus about what is called the $15j$ -symbol, but in all cases it corresponds to an invariant function associated to 3-valent graph with 15 links. Actually, we can build 5 topologically different $15j$ -symbols³. Here we see the power of graphical calculus: imagine if we had given the analytical formula for it... that is doable, but unreadable.

♥ PHYSICS. In the spirit of the result of Ponzano and Regge 3.51, Ooguri used the $15j$ -symbol to provide a model of quantum gravity [Oog92]. It still plays a major role in the EPRL model [Spe17].

²For more details on the $9j$ -symbol, see [Edm57] pp. 100-114.

³For more details on the $15j$ -symbol, see [YLV62] pp. 65-70.

Chapter 4

Representation theory of $SL_2(\mathbb{C})$

To put it in a nutshell, the kinematic aspects of loop quantum gravity deal with representations of $SU(2)$, while the dynamics, in its spin-foam formulation lies in the representation theory of $SL_2(\mathbb{C})$. The current models of spin-foams, like the EPRL one, use extensively the principal series of $SL_2(\mathbb{C})$.

Fortunately all the irreps of $SL_2(\mathbb{C})$ are known. Whether or not all representations, including non-reducible ones, have been classified is unknown to the author. In section 4.1, we present the finite-dimensional ones. In section 4.2, we summarize the infinite-dimensional ones. Finally, in section 4.3, we focus on the principal series which are of main interest for quantum gravity.

4.1 Finite irreps

The finite irreps of $SL_2(\mathbb{C})$ are well-known. They can be obtained from the irreps of $\mathfrak{sl}_2(\mathbb{C})$. In section 2.1, we have already seen the finite irreps of its 3-dimensional (complex) Lie algebra $\mathfrak{sl}_2(\mathbb{C})$: they are indexed by a spin $j \in \mathbb{N}/2$. It is also possible to see $\mathfrak{sl}_2(\mathbb{C})$ as a real Lie algebra of dimension 6, in which case, we will rather denote it $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$. In this section, we will describe the (real) linear representations of $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$. We have the following isomorphism between real vector spaces:

$$\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2). \quad (4.1)$$

The algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is the Lie algebra of $SU(2) \times SU(2)$. A consequence of Peter-Weyl's theorem is that the irreps of a cartesian product is a tensor product of the irreps of the factors ([Kna86] p. 32). Thus the irreps of $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$ are given by the usual tensor representation over $\mathcal{Q}_{j_1} \otimes \mathcal{Q}_{j_2}$, abbreviated by (j_1, j_2) . The action is given by:

$$a \cdot (|j_1, m_1\rangle \otimes |j_2, m_2\rangle) \stackrel{\text{def}}{=} (a |j_1, m_1\rangle) \otimes |j_2, m_2\rangle + |j_1, m_1\rangle \otimes (a |j_2, m_2\rangle). \quad (4.2)$$

The isomorphism 4.1 provides naturally a basis of $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$, given by the three Pauli matrices $\sigma_i \in i\mathfrak{su}(2)$ and the three matrices $i\sigma_i \in \mathfrak{su}(2)$. To match the earlier notations introduced in section 2.2, we often denote the *rotation generators* $J_i \stackrel{\text{def}}{=} \frac{1}{2}\sigma_i$ and the *boost generators* $K_i \stackrel{\text{def}}{=} \frac{i}{2}\sigma_i$. These generators satisfy the commutation relations:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \quad [J_i, K_j] = i\varepsilon_{ijk}K_k \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k.$$

★ **NOTA BENE.** Another basis is given by the *complexified generators*. Posing $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$, the commutation relations become:

$$[A_i, A_j] = i\varepsilon_{ijk}A_k \quad [B_i, B_j] = i\varepsilon_{ijk}B_k \quad [A_i, B_j] = 0.$$

We can also define the scale operators. Posing $K_{\pm} \stackrel{\text{def}}{=} K_1 \pm iK_2$ and $J_{\pm} \stackrel{\text{def}}{=} J_1 \pm iJ_2$, the scale operators satisfy:

$$\begin{aligned} [J_3, J_{\pm}] &= \pm J_{\pm} & [J_+, J_-] &= 2J_3 \\ [K_3, K_{\pm}] &= \mp J_{\pm} & [K_+, K_-] &= -2J_3 \\ [J_+, K_+] &= [J_-, K_-] = [J_3, K_3] = 0 & & \\ [K_3, J_{\pm}] &= \pm K_{\pm} & [J_{\pm}, K_{\mp}] &= \pm 2K_3 & [J_3, K_{\pm}] &= \pm K_{\pm} \end{aligned}$$

Then the various realisations, which were described in chapter 2 for the action of $\mathfrak{su}(2)$, can be adapted to $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$.

Homogeneous realisation. Let $m, n \geq 2$, and $\mathbb{C}_{(m,n)}[z_0, z_1; \bar{z}_0, \bar{z}_1]$ the vector space of homogeneous polynomials of degree m in (z_0, z_1) and homogeneous of degree n in (\bar{z}_0, \bar{z}_1) . The action of $SL_2(\mathbb{C})$ is given by

$$g \cdot P(z_0, z_1) = P\left(g^T \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}\right).$$

The associated action of the algebra $\mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}$ is given by

$$J_+ \cong z_0 \frac{\partial}{\partial z_1} + \bar{z}_0 \frac{\partial}{\partial \bar{z}_1} \quad J_- \cong z_1 \frac{\partial}{\partial z_0} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_0} \quad J_3 \cong \frac{1}{2} \left(z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1} + \bar{z}_0 \frac{\partial}{\partial \bar{z}_0} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right).$$

Projective realisation. As seen in section 2.4, the restriction of $P(z_0, z_1)$ to $P(z, 1)$ induces another realisation over the space of polynomials of degree at most m in z and at most n in \bar{z} . The action is given by

$$g \cdot \phi(\xi) = (g_{12}\xi + g_{22})^m \overline{(g_{12}\xi + g_{22})}^n \phi\left(\frac{g_{11}\xi + g_{21}}{g_{12}\xi + g_{22}}\right).$$

Spinorial realisation. Finally the (m, n) representation of $SL_2(\mathbb{C})$ can be realised over the space of totally symmetric spinors $\mathfrak{S}^{(A_1 \dots A_m)(\dot{A}_1 \dots \dot{A}_n)}$ such as (see Penrose [PR84] p. 142)

$$u \cdot z^{A_1 \dots A_m} \dot{A}_1 \dots \dot{A}_n = u^{A_1}_{B_1} \dots u^{A_m}_{B_m} \bar{u}^{\dot{A}_1}_{\dot{B}_1} \dots \bar{u}^{\dot{A}_n}_{\dot{B}_n} z^{B_1 \dots B_m} \dot{B}_1 \dots \dot{B}_n. \quad (4.3)$$

Finite representations of $SL_2(\mathbb{C})$ cannot be unitary (except the trivial one), because it is a simply connected non-compact Lie group. If we want unitary representations, we shall turn to infinite ones.

4.2 Infinite irreps

In this section, we describe all the infinite-dimensional irreps of $SL_2(\mathbb{C})$.

★ **NOTA BENE.** All the unitary irreps of the Lorentz group have been found simultaneously in 1946 by Gel'fand and Naimark [GN47], by Harish-Chandra [HC47] and by Bargmann [Bar47]. It seems nevertheless that Gel'fand and Naimark had the priority in publishing (unfortunately their article is only in Russian). The question remained to find all the irreps, unitary or not, and this was solved also by Naimark in [Nai54]. In 1963, Gel'fand, Minlos and Shapiro published a first book (with english translation) that review all these results [GMS63]. In 1964, Naimark wrote a more detailed and well-written book that wraps up the subject for mathematically-oriented physicists [Nai64].

The infinite irreps of $SL_2(\mathbb{C})$ are parametrised by $(m, \rho) \in \mathbb{Z} \times \mathbb{C}$ with $\text{Im } \rho \geq 0$ and $\rho^2 \neq -(|m| + 2n)^2 \in \mathbb{N}^*$. A realisation is given over the Hilbert space $L^2(\mathbb{C})$ endowed with the scalar product

$$\langle \varphi | \phi \rangle = \frac{i}{2} \int_{\mathbb{C}} \overline{\varphi(\xi)} \phi(\xi) (1 + |\xi|^2)^{-\text{Im } \rho} d\xi d\bar{\xi}, \quad (4.4)$$

and the action

$$a \cdot f(z) = (a_{12}z + a_{22})^{\frac{m}{2} + \frac{i\rho}{2} - 1} \overline{(a_{12}z + a_{22})}^{-\frac{m}{2} + \frac{i\rho}{2} - 1} f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right). \quad (4.5)$$

Remarks:

1. If $\rho^2 = -(|m| + 2n)^2$ with $n \in \mathbb{N}^*$, the Hilbert space and the action still defines a representation, but a reducible one. Then, if one restricts the action to the subspace of polynomials of degree at most $p = \frac{m}{2} + i\frac{\rho}{2} - 1$ in z and $q = -\frac{m}{2} + i\frac{\rho}{2} - 1$ in \bar{z} , the representation is irreducible and equivalent to the finite-dimensional representation (p, q) .
2. Not all the representations (m, ρ) are unitary. They are unitary in only two cases: when $\rho \in \mathbb{R}$ (principal series) and when $m = 0$ and $i\rho \in]-2, 0[$ (complementary series), provided another scalar product is chosen in the latter case (see below).
3. Among these infinite irreps, only the principal representations (ρ, k) and $(-\rho, -k)$ are equivalent.
4. A proof of the result above can be found in Naimark ([Nai64] pp. 294-295). A sketch of it in the case of the principal series can be found in appendix B.

Principal series. When $\rho \in \mathbb{R}$, the scalar product over $L^2(\mathbb{C})$ becomes the more usual

$$(f_1, f_2) \stackrel{\text{def}}{=} \frac{i}{2} \int_{\mathbb{C}} \overline{f_1(z)} f_2(z) dz d\bar{z}, \quad (4.6)$$

and the representation $(\rho, m) \in \mathbb{R} \times \mathbb{Z}$ is unitary. It is called the principal series. Moreover the representations (ρ, m) et $(-\rho, -m)$ are unitarily equivalent.

★ NOTA BENE. We have used below the convention of Naimark for indexing the representations of the principal unitary series, $(\rho, m) \in \mathbb{R} \times \mathbb{Z}$ (see [Nai64] p. 150). Vilenkin uses the same convention ([GGV66] p. 191). Rühl uses rather the convention $(\rho_R, m_R) = (\rho, -m)$ ([Rüh70] p. 54). Gel'fand uses the convention $(\rho_G, m_G) = (\rho/2, m)$ ([GMS63] p. 247). In LQG, it is common to use the conventions $(p, k) = (\rho/2, m/2)$, as Rovelli does ([RV14] p. 182) or Barrett [BDF⁺10]. Usually in LQG, the parameter k is even restricted to be non-negative, which actually does not suppress any representation since (p, k) and $(-p, -k)$ are equivalent.

Complementary series. When $m = 0$ and $i\rho \in]-2, 0[$, the action becomes

$$a \cdot \phi(\xi) = |a_{12}\xi + a_{22}|^{i\rho-2} \phi\left(\frac{a_{11}\xi + a_{21}}{a_{12}\xi + a_{22}}\right). \quad (4.7)$$

It also defines a unitary representation for the scalar product

$$\langle \varphi | \phi \rangle = \left(\frac{i}{2}\right)^2 \int_{\mathbb{C}^2} \frac{\overline{\varphi(\xi)} \phi(\eta)}{|\xi - \eta|^{2+i\rho}} d\xi d\bar{\xi} d\eta d\bar{\eta}. \quad (4.8)$$

4.3 Principal unitary series

In this section, we will present three realisation of the unitary principal series. The construction of the principal series by Gel'fand and Naimark is based on the induced representations method (see appendix B for a sketch of the proof). Though rigorous from the mathematical point of view, it is not very intuitive, especially for physicists. In 1962, Gel'fand, Graev and Vilenkin published a book (referred to as 'the Vilenkin' hereafter) where they build the principal series from a space of homogeneous functions, which may seem more natural ([GGV66] pp. 139-201). A beautiful and concise exposure can be found in the article of Dao and Nguyen ([DN67] pp. 18-21). We present the method in the following subsection.

4.3.1 Homogeneous realisation

Consider $\mathcal{F}(\mathbb{C}^2)$, the vector space of the complex functions over \mathbb{C}^2 . A function $F \in \mathcal{F}(\mathbb{C}^2)$ is said to be *homogeneous of degree* $(\lambda, \mu) \in \mathbb{C}^2$ if it satisfy for all $\alpha \in \mathbb{C}$:

$$F(\alpha \mathbf{z}) = \alpha^\lambda \bar{\alpha}^\mu F(\mathbf{z}). \quad (4.9)$$

To be consistently defined (when $\alpha = e^{2i\pi n}$), the degree should satisfy the condition :

$$\mu - \lambda \in \mathbb{Z}. \quad (4.10)$$

Instead of (λ, μ) , we will rather use in the following, the parameters $(p = \frac{\mu + \lambda + 2}{2i}, k = \frac{\lambda - \mu}{2})$ (same choice of parameters as Rovelli [RV14] p. 182). Define $\mathcal{D}^{(p,k)}[z_0, z_1]$ as the subspace of homogeneous functions of degree (λ, μ) infinitely differentiable over $\mathbb{C}^2 \setminus \{0\}$ in the variables z_0, z_1, \bar{z}_0 et \bar{z}_1 with a certain topology¹. We define a continuous representation $SL_2(\mathbb{C})$ over $\mathcal{D}^{(p,k)}[z_0, z_1]$ by

$$a \cdot F(\mathbf{z}) \stackrel{\text{def}}{=} F(a^T \mathbf{z}). \quad (4.11)$$

★ **NOTA BENE.** We could also have defined the action by $F(a^{-1}\mathbf{z})$, $F(a^\dagger\mathbf{z})$ or $F(\mathbf{z}a)$. In fact $F(\mathbf{z}a) = F(a^T\mathbf{z})$, defines the same action. The action with a^\dagger is obtained by the transformation $a_{ij} \mapsto \bar{a}_{i\bar{j}}$. The action with a^{-1} is obtained by the transformation $a_{ij} \mapsto (2\delta_{ij} - 1) \sum_{kl} (1 - \delta_{ik})(1 - \delta_{jl}) a_{kl}$. The convention that we have chosen here is the one of Vilenkin ([GGV66] p. 145), Dao and Nguyen ([DN67] p. 18), Rühl ([Rüh70] p. 53), Rovelli ([RV14] p. 182) and Barrett [BDF⁺10]. Knapp ([Kna86] p. 28) is using the convention $F(a^{-1}\mathbf{z})$.

Now define the following 2-form over \mathbb{C}^2 :

$$\Omega(z_0, z_1) = \frac{i}{2}(z_0 dz_1 - z_1 dz_0) \wedge (\bar{z}_0 d\bar{z}_1 - \bar{z}_1 d\bar{z}_0).$$

Interestingly, it is invariant for the action of $SL_2(\mathbb{C})$: $\Omega(a\mathbf{z}) = \Omega(\mathbf{z})$. Let Γ be a path in \mathbb{C}^2 that intersects each projective line exactly once. Then define the scalar product over $\mathcal{D}^{(p,k)}[z_0, z_1]$:

$$(F, G) = \int_{\Gamma} \overline{F(\mathbf{z})} G(\mathbf{z}) \Omega(\mathbf{z}).$$

Thus $\mathcal{D}^{(p,k)}[z_0, z_1]$ is an Hilbert space. Interestingly, the result does not depend on the path Γ provided $p \in \mathbb{R}$, which we consider to be the case in the following. This scalar product is invariant for $SL_2(\mathbb{C})$: $(a \cdot F, a \cdot G) = (F, G)$. Thus the representation is unitary. It could be also shown to be irreducible. In the next subsection, we will see that the representation $\mathcal{D}^{(p,k)}[z_0, z_1]$ is equivalent to the representation $(\rho = 2p, m = 2k)$ of the principal series described in section 4.2.

4.3.2 Projective realisation

Consider the map

$$\iota : \begin{cases} \mathbb{C} & \rightarrow \mathbb{C}^2 \\ \zeta & \mapsto (\zeta, 1) \end{cases}$$

ι is a diffeomorphism from \mathbb{C} to its range. It parametrises an horizontal straight line of the complex plane. The projective construction consists in restricting the domain of definition of the homogeneous function to this line. If $F \in \mathcal{F}(\mathbb{C}^2)$, define $\iota^*F \in \mathcal{F}(\mathbb{C})$ as

$$\iota^*F(z) \stackrel{\text{def}}{=} F \circ \iota(z) = F(z, 1). \quad (4.12)$$

The 2-form Ω becomes similarly

$$\iota^*\Omega(z) = \frac{i}{2} dz \wedge d\bar{z},$$

which is nothing but the usual Lebesgue measure over \mathbb{C} . Thus we define the Hilbert space $L^2(\mathbb{C})$ with the the scalar product

$$(f, g) = \frac{i}{2} \int_{\mathbb{C}} \overline{f(z)} g(z) dz \wedge d\bar{z}.$$

¹The topology is defined by the following property of convergence: a sequence $F_n(z_0, z_1)$ is said to converge to 0 if it converges to zero uniformly together with all its derivatives on any compact set in the (z_0, z_1) -plane which does not contain the $(0, 0)$ (see Vilenkin [GGV66] p. 142).

Thus we have a map $\iota^* : \mathcal{D}^{(p,k)}[z_0, z_1] \rightarrow L^2(\mathbb{C})$. In fact ι^* is bijective: for all $f \in L^2(\mathbb{C})$, there exists a unique $F \in \mathcal{D}^{(p,k)}[z_0, z_1]$ such that $f = \iota^* F$. F is given explicitly by

$$F(z_0, z_1) = z_1^{-1+ip+k} \bar{z}_1^{-1+ip-k} f\left(\frac{z_0}{z_1}\right). \quad (4.13)$$

Importantly, ι^* induces naturally an action of $SL_2(\mathbb{C})$ over $\mathcal{F}(\mathbb{C})$, such that ι^* becomes an intertwiner between two equivalent representations. After computation, we obtain:

$$a \cdot f(z) \stackrel{\text{def}}{=} (a_{12}z + a_{22})^{-1+ip+k} \overline{(a_{12}z + a_{22})}^{-1+ip-k} f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right). \quad (4.14)$$

This formula is exactly the same formula as 4.5, with the indices $(p, k) = (\rho/2, m/2)!$ Thus we have constructed explicitly the representations of the principal series, and we have shown the equivalence of the realisations $\mathcal{D}^{(p,k)}[z_0, z_1]$ and $L^2(\mathbb{C})$.

4.3.3 $SU(2)$ -realisation

We are going to build another realisation of the unitary principal representations. It is based on a space of ‘ $U(1)$ -covariant’ functions over $SU(2)$. The general idea lies over the observation that $SU(2)/SU(1) \cong \mathbb{C}P^1$. To be precise, consider the map

$$\kappa : \begin{cases} SU(2) & \rightarrow \mathbb{C}^2 \\ u & \mapsto (u_{21}, u_{22}) \end{cases}$$

κ is a diffeomorphism to its range. Since $SU(2)$ is homeomorphic to the unit circle of \mathbb{C}^2 , this construction can be seen as the injection of the circle in the plane \mathbb{C}^2 .

Then, define $\kappa^* : \mathcal{F}(\mathbb{C}^2) \rightarrow \mathcal{F}(SU(2))$ such that

$$\kappa^* F(u) \stackrel{\text{def}}{=} F \circ \kappa(u) = F(u_{21}, u_{22}). \quad (4.15)$$

If $F \in \mathcal{D}^{(p,k)}[z_0, z_1]$, then we show easily that $\kappa^* F$ satisfies the covariance property

$$\kappa^* F(e^{i\theta\sigma_3} u) = e^{-2i\theta k} \kappa^* F(u). \quad (4.16)$$

We denote $\mathcal{D}^{(p,k)}[u] \stackrel{\text{def}}{=} \kappa^* \mathcal{D}^{(p,k)}[z_0, z_1]$. Thus $\kappa^* : \mathcal{D}^k[z_0, z_1] \rightarrow \mathcal{D}^{(p,k)}[u]$ is a bijection. Its inverse is given explicitly by

$$F(z_0, z_1) = (|z_0|^2 + |z_1|^2)^{-1+ip} \phi\left(\frac{1}{\sqrt{|z_0|^2 + |z_1|^2}} \begin{pmatrix} z_1^* & -z_0^* \\ z_0 & z_1 \end{pmatrix}\right). \quad (4.17)$$

► **PROOF.** We have to check that 4.17 is the inverse expression of 4.15. Then we check the homogeneity property $(\kappa^*)^{-1}\phi(\alpha(z_0, z_1)) = \alpha^\lambda \bar{\alpha}^\mu (\kappa^*)^{-1}\phi(z_0, z_1)$. □

We could also translate the measure $\kappa^* \Omega$, and thus endow $\mathcal{D}^{(p,k)}[u]$ with the structure of an Hilbert space. Interestingly, it is a subspace of $L^2(SU(2))$. As previously, one can translate the action of $SL_2(\mathbb{C})$ over $\mathcal{D}^{(p,k)}[u]$ such that κ^* become a bijective intertwiner, and we obtain

$$a \cdot \phi(u) = (|\beta_{a,u}|^2 + |\alpha_{a,u}|^2)^{-1+ip} \phi\left(\frac{1}{\sqrt{|\beta_{a,u}|^2 + |\alpha_{a,u}|^2}} \begin{pmatrix} \alpha_{a,u} & -\beta_{a,u}^* \\ \beta_{a,u} & \alpha_{a,u}^* \end{pmatrix}\right), \quad (4.18)$$

with $\alpha_{a,u} \stackrel{\text{def}}{=} (u_{21}a_{12} + u_{22}a_{22})^*$ and $\beta_{a,u} \stackrel{\text{def}}{=} u_{21}a_{11} + u_{22}a_{21}$. Thus $\mathcal{D}^{(p,k)}[u]$ is a third equivalent realisation of the unitary principal series. The equivalence with $L^2(\mathbb{C})$ is made through $(\kappa \circ \iota^{-1})^*$ which gives explicitly

$$\phi(u) = u_{22}^{-1+ip+k} \bar{u}_{22}^{-1+ip-k} f\left(\frac{u_{21}}{u_{22}}\right), \quad (4.19)$$

and conversely

$$f(z) = (1 + |z|^2)^{-1+ip} \phi\left(\frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & -z^* \\ z & 1 \end{pmatrix}\right). \quad (4.20)$$

4.3.4 Canonical basis

The advantage of the $SU(2)$ -realisation is that we already know interesting function over $SU(2)$, namely the coefficients of the Wigner matrix D_{pq}^j . Are they elements of $\mathcal{D}^{(p,k)}[u]$? They are indeed, provided they satisfy the covariance property 4.16. We compute easily

$$D_{pq}^j \left(e^{i\theta\sigma_3} u \right) = \sum_{k=-j}^j \langle j, p | e^{i\theta\sigma_3} | j, k \rangle D_{kq}^j (u) = \sum_{k=-j}^j \delta_{pk} e^{2ik\theta} D_{kq}^j (u) = e^{2ip\theta} D_{pq}^j (u).$$

Thus, the covariance property it is satisfied if $p = -k$, and so

$$\forall j \in \{|k|, |k| + 1, \dots\}, \quad \forall q \in \{-j, \dots, j\}, \quad D_{-k,q}^j \in \mathcal{D}^{(p,k)}[u]. \quad (4.21)$$

Since the $D_{mn}^j(u)$ form a basis of $L^2(SU(2))$, we show easily that the subset exhibited in 4.21 form a basis of $\mathcal{D}^{(p,k)}[u]$. Another consequence is the following decomposition of $\mathcal{D}^{(p,k)}[u]$ into irreps of $SU(2)$:

$$\mathcal{D}^{(p,k)}[u] \cong \bigoplus_{j=|k|}^{\infty} \mathcal{Q}_j. \quad (4.22)$$

We then call *canonical basis* of $\mathcal{D}^{(p,k)}[u]$ the set of functions:

$$\phi_{jm}^{(p,k)}(u) \stackrel{\text{def}}{=} \sqrt{\frac{2j+1}{\pi}} D_{-k,m}^j(u), \quad \text{with } j = |k|, |k| + 1, \dots \text{ and } -j \leq m \leq j. \quad (4.23)$$

From 2.7, we see that they satisfy the orthogonality relations

$$\int_{SU(2)} du \overline{\phi_{jm}^{(p,k)}(u)} \phi_{ln}^{(p,k)}(u) = \frac{1}{\pi} \delta_{jl} \delta_{mn}. \quad (4.24)$$

★ **NOTA BENE.** In Rühl ([Rüh70] p. 59), the factor $\frac{1}{\sqrt{\pi}}$ is absent from the definition of the $\phi_{jm}^{(p,k)}$. Thus, the orthogonality relations do not show a factor $\frac{1}{\pi}$ on the RHS. We have chosen this factor so that the canonical basis $f_{jm}^{(p,k)}$ of $L^2(\mathbb{C})$ (see below 4.25) is orthonormal for the usual scalar product with the Lebesgue measure dz (for Rühl the measure is dz/π).

Moreover $\phi_{jm}^{(p,k)}$ could have been defined with a phase factor $e^{i\psi(p,j)}$. This is set to zero in some literature including [Rüh70, BDF⁺10], and we follow that convention here. An alternative phase convention leading to real $SL_2(\mathbb{C})$ -Clebsch-Gordan coefficients is obtained for the choice [KVM78, Spe17] $e^{i\psi(p,j)} = (-1)^{-\frac{j}{2}} \frac{\Gamma(j+i\rho+1)}{|\Gamma(j+i\rho+1)|}$. This phase convention is the one used in the approach of [Spe17] to compute the coefficients and vertex amplitudes and to the numerical analysis [DS18, Don18, FMDS] that implement this approach. From the perspective of the large spin asymptotics this additional phase plays no role and can be taken into account independently, if so desired.

An intermediate choice of phase is the one of [DN67, Ras03], which has the advantage of simplifying the recursion relations satisfied by the Clebsch-Gordan coefficients [ARRW70b, ARRW70a]. The latter are now either real or purely imaginary.

The intertwiner κ^* enables to translate this basis in $\mathcal{D}^{(p,k)}[z_0, z_1]$, and we obtain the canonical basis:

$$F_{jm}^{(p,k)}(z_0, z_1) = \sqrt{\frac{2j+1}{\pi}} (|z_0|^2 + |z_1|^2)^{ip-1} D_{-k,m}^j \left(\frac{1}{\sqrt{|z_0|^2 + |z_1|^2}} \begin{pmatrix} z_1^* & -z_0^* \\ z_0 & z_1 \end{pmatrix} \right),$$

where an explicit expression for $D_{-k,m}^j$ is given by equation 2.16. The same is done with the intertwiner ι^* to $L^2(\mathbb{C})$, and we obtain the canonical basis:

$$f_{jm}^{(p,k)}(z) = \sqrt{\frac{2j+1}{\pi}} (1 + |z|^2)^{ip-1-j} D_{-k,m}^j \begin{pmatrix} 1 & -z^* \\ z & 1 \end{pmatrix} \quad (4.25)$$

♣ **REMINDER.** From 2.16 we have

$$D_{-k,m}^j \begin{pmatrix} 1 & -z^* \\ z & 1 \end{pmatrix} = \left(\frac{(j-k)!(j+k)!}{(j+m)!(j-m)!} \right)^{1/2} \sum_{i=\max(0, m-k)}^{\min(j-k, j+m)} \binom{j+m}{i} \binom{j-m}{j-k-i} z^{j+m-i} (-z^*)^{j-k-i}. \quad (4.26)$$

The constant factors of 4.23 have been chosen so that

$$\frac{i}{2} \int_{\mathbb{C}} \overline{f_{jm}^{(p,k)}(z)} f_{ln}^{(p,k)}(z) dz d\bar{z} = \delta_{jl} \delta_{mn}.$$

Finally, in ket notations, the canonical basis is denoted $|p, k, jm\rangle$.

4.3.5 Action of the generators

Similarly to 2.13, the action of the $SL_2(\mathbb{C})$ -generators can be computed from the action of the group (see Naimark [Nai64] pp. 104-117). The generators of the rotations ‘stays inside’ the same $SU(2)$ -irreps:

$$\begin{cases} J_3 |p, k, j, m\rangle = m |p, k, j, m\rangle \\ J_+ |p, k, j, m\rangle = \sqrt{(j+m+1)(j-m)} |p, k, j, m+1\rangle \\ J_- |p, k, j, m\rangle = \sqrt{(j+m)(j-m+1)} |p, k, j, m-1\rangle \end{cases} \quad (4.27)$$

The generators of the boost spread over the neighbouring subspaces:

$$K_3 |p, k, j, m\rangle = \alpha_j \sqrt{j^2 - m^2} |p, k, j-1, m\rangle + \gamma_j m |p, k, j, m\rangle - \alpha_{j+1} \sqrt{(j+1)^2 - m^2} |p, k, j+1, m\rangle, \quad (4.28)$$

$$\begin{aligned} K_+ |p, k, j, m\rangle &= \alpha_j \sqrt{(j-m)(j-m-1)} |p, k, j-1, m+1\rangle \\ &+ \gamma_j \sqrt{(j-m)(j+m+1)} |p, k, j, m+1\rangle \\ &+ \alpha_{j+1} \sqrt{(j+m+1)(j+m+2)} |p, k, j+1, m+1\rangle \end{aligned} \quad (4.29)$$

$$\begin{aligned} K_- |p, k, j, m\rangle &= -\alpha_j \sqrt{(j+m)(j+m-1)} |p, k, j-1, m-1\rangle \\ &+ \gamma_j \sqrt{(j+m)(j-m+1)} |p, k, j, m-1\rangle \\ &- \alpha_{j+1} \sqrt{(j-m+1)(j-m+2)} |p, k, j+1, m-1\rangle \end{aligned} \quad (4.30)$$

with $\gamma_j \stackrel{\text{def}}{=} \frac{kp}{j(j+1)}$ and $\alpha_j \stackrel{\text{def}}{=} i \sqrt{\frac{(j^2-k^2)(j^2+p^2)}{j^2(4j^2-1)}}$. From these expressions, it is possible to compute the action of the two Casimir operators:

$$\begin{aligned} (\vec{K}^2 - \vec{L}^2) |p, k, j, m\rangle &= (p^2 - k^2 + 1) |p, k, j, m\rangle, \\ \vec{K} \cdot \vec{L} |p, k, j, m\rangle &= pk |k, p, j, m\rangle. \end{aligned} \quad (4.31)$$

4.3.6 $SL_2(\mathbb{C})$ Wigner’s matrix

We define the $SL_2(\mathbb{C})$ Wigner’s matrix by its coefficients

$$D_{j_1 q_1 j_2 q_2}^{(p,k)}(a) \stackrel{\text{def}}{=} \langle p, k; j_1 q_1 | a | p, k; j_2 q_2 \rangle. \quad (4.32)$$

These coefficients satisfy the orthogonality relations:

$$\int_{SL_2(\mathbb{C})} dh D_{j_1 m_1 l_1 n_1}^{(p_1, k_1)}(h) D_{j_2 m_2 l_2 n_2}^{(p_2, k_2)}(h) = \frac{1}{4(p_1^2 + k_1^2)} \delta(p_1 - p_2) \delta_{k_1 k_2} \delta_{j_1 j_2} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{n_1 n_2}. \quad (4.33)$$

Explicit expressions. To compute explicitly, we use a concrete realisation, for instance

$$D_{jplq}^{(p,k)}(g) = \frac{i}{2} \int_{\mathbb{C}} d\omega d\bar{\omega} \overline{f_{jp}^{(p,k)}(\omega)} (g_{12}\omega + g_{22})^{-1+ip+k} \overline{(g_{12}\omega + g_{22})}^{-1+ip-k} f_{lq}^{(p,k)} \left(\frac{g_{11}\omega + g_{21}}{g_{12}\omega + g_{22}} \right). \quad (4.34)$$

Cartan decomposition states that any $g \in SL_2(\mathbb{C})$ can be written (non uniquely) as $g = ue^{r\sigma_3/2}v^{-1}$, with $u, v \in SU(2)$ and $r \in \mathbb{R}_+$ (see 1.4). Then, we show that

$$D_{jmln}^{(p,k)}(g) = \sum_{q=-\min(j,l)}^{\min(j,l)} D_{mq}^j(u) d_{jlq}^{(p,k)}(r) D_{qn}^l(v^{-1}). \quad (4.35)$$

with the *reduced $SL_2(\mathbb{C})$ Wigner’s matrix* defined as

$$d_{jlm}^{(p,k)}(r) \stackrel{\text{def}}{=} D_{jmlm}^{(p,k)}(e^{r\sigma_3/2}). \quad (4.36)$$

► PROOF.

$$\begin{aligned}
D_{jmln}^{(p,k)}(g) &= \sum_{p \geq |k|} \sum_{q=-p}^p \sum_{p' \geq |k|} \sum_{q'=-p'}^{p'} D_{jmq}^{(p,k)}(u) D_{pq'q'}^{(p,k)}(e^{r\sigma_3/2}) D_{p'q'l'n}^{(p,q)}(v^{-1}) \\
&= \sum_{p \geq |k|} \sum_{q=-p}^p \sum_{p' \geq |k|} \sum_{q'=-p'}^{p'} \delta_{jp} D_{mq}^j(u) D_{pq'q'}^{(p,k)}(e^{r\sigma_3/2}) \delta_{p'l} D_{q'n}^l(v^{-1}) \\
&= \sum_{q=-j}^j \sum_{q'=-l}^l D_{mq}^j(u) D_{jq'lq'}^{(p,k)}(e^{r\sigma_3/2}) D_{q'n}^l(v^{-1}) \\
&= \sum_{q=-j}^j \sum_{q'=-l}^l D_{mq}^j(u) \langle p, k, jq | e^{r\sigma_3/2} | p, k, lq' \rangle D_{q'n}^l(v^{-1}) \\
&= \sum_{q=-\min(j,l)}^{\min(j,l)} D_{mq}^j(u) D_{jq'lq'}^{(p,k)}(e^{r\sigma_3/2}) D_{q'n}^l(v^{-1})
\end{aligned}$$

□

We have the following symmetry properties:

$$d_{jlm}^{(p,k)}(r) = d_{lij}^{(-p,k)}(-r) = d_{j\bar{l}, -m}^{(p,-k)}(r) = (-1)^{j-l} d_{lij}^{(-p,-k)}(r) = \overline{d_{lij}^{(p,k)}(-r)}. \quad (4.37)$$

■ Integral formula 1.

$$\begin{aligned}
d_{jlm}^{(p,k)}(r) &= \sqrt{(2j+1)(2l+1)} \left(\frac{(j-k)!(j+k)!}{(j+p)!(j-p)!} \right)^{1/2} \left(\frac{(l-k)!(l+k)!}{(l+p)!(l-p)!} \right)^{1/2} \\
&\times \sum_{i=\max(0,p-k)}^{\min(j-k,j+p)} \sum_{i'=\max(0,p-k)}^{\min(l-k,l+p)} \left[(-1)^{j+l-2k-i-i'} \binom{j+p}{i} \binom{j-p}{j-k-i} \binom{l+p}{i'} \binom{l-p}{l-k-i'} \right] \\
&\times e^{r(ip-1+p-k-2i')} \int_0^\infty 2|\omega| (1+|\omega|^2)^{-ip-1-j} (e^{-2r} + |\omega|^2)^{ip-1-l} |\omega|^{2(j+l-i'-i+p-k)} d|\omega| \quad (4.38)
\end{aligned}$$

► PROOF.

$$\begin{aligned}
D_{jplp}^{(p,k)}(e^{r\sigma_3/2}) &= \int_{\mathbb{C}} d\omega \overline{f_{jp}^{(p,k)}(\omega)} e^{r(1-ip)} f_{lp}^{(p,k)}(e^r \omega) \\
&= \int_{\mathbb{C}} d\omega \sqrt{\frac{2j+1}{\pi}} (1+|\omega|^2)^{-ip-1-j} \overline{D_{-k,p}^j} \begin{pmatrix} 1 & -\omega^* \\ \omega & 1 \end{pmatrix} e^{r(1-ip)} \\
&\times \sqrt{\frac{2l+1}{\pi}} (1+e^{2r}|\omega|^2)^{ip-1-l} D_{-k,p}^l \begin{pmatrix} 1 & -e^r \omega^* \\ e^r \omega & 1 \end{pmatrix}.
\end{aligned}$$

We conclude using the explicit expression of equation 4.26. □

■ Integral formula 2.

$$\begin{aligned}
d_{jlm}^{(p,k)}(r) &= \sqrt{(2j+1)(2l+1)} \sqrt{\frac{(j-k)!(j+k)!(l-k)!(l+k)!}{(j+m)!(j-m)!(l+m)!(l-m)!}} \\
&\times \sum_{n_1, n_2} \left[(-1)^{j+l+2m-n_1-n_2} \binom{j+m}{n_1} \binom{j-m}{j-k-n_1} \binom{l+m}{n_2} \binom{l-m}{l-k-n_2} \right] \\
&\times e^{r(ip-1-2n_2-k+m)} \int_0^1 dt [1 - (1 - e^{-2r}t)]^{ip-1-l} t^{n_1+n_2+k-m} (1-t)^{j+l-n_1-n_2-k+m}. \quad (4.39)
\end{aligned}$$

► PROOF. Change of variables: $|\omega|^2 = \frac{1-t}{t} \Leftrightarrow t = \frac{1}{1+|\omega|^2}$ et $d\omega^2 = -\frac{1}{t^2} dt$. □

■ **Hyper-geometrical formula.**

$$\begin{aligned}
d_{jlm}^{(p,k)}(r) &= \frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!} \sqrt{\frac{(j-k)!(j+k)!(l-k)!(l+k)!}{(j-m)!(j+m)!(l-m)!(l+m)!}} \\
&\times \sum_{n_1, n_2} (-1)^{n_1+n_2} \binom{j+m}{n_1} \binom{j-m}{n_1-m-k} \binom{l+m}{n_2} \binom{l-m}{n_2-m-k} \\
&\times (j+l-n_1-n_2+m+k)!(n_1+n_2-m-k)! e^{r(m+k-ip-1-2n_1)} \\
&\times {}_2F_1(n_1+n_2-m-k+1, j+ip+1, j+l+2; 1-e^{-2r}). \quad (4.40)
\end{aligned}$$

► **PROOF.** We have used the integral expression of the hyper-geometrical function ${}_2F_1$,

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a}. \quad (4.41)$$

□

■ **Rühl's formula.** ([Rüh70] p. 64)

$$\begin{aligned}
d_{j_l q}^{(p,k)}(r) &= (2j+1)^{1/2} (2l+1)^{1/2} \int_0^1 dt ((1-t)e^r + te^{-r})^{-1+ip} d_{-k,q}^j(\arccos(2t-1)) \\
&\times d_{-k,q}^l \left(\arccos \left(\frac{te^{-r} - (1-t)e^r}{te^{-r} + (1-t)e^r} \right) \right) \quad (4.42)
\end{aligned}$$

4.4 Recoupling of $SL_2(\mathbb{C})$

$SL_2(\mathbb{C})$ -Clebsch-Gordan coefficients. Similarly to the $SU(2)$ case, the tensor product of two irreps of $SL_2(\mathbb{C})$ can be decomposed into a direct sum of irreps:

$$\mathcal{D}^{(p_1, k_1)} \otimes \mathcal{D}^{(p_2, k_2)} \cong \int_{\mathbb{R}} dp \bigoplus_{\substack{k \in \mathbb{Z}/2 \\ k_1 + k_2 + k \in \mathbb{N}}} \mathcal{D}^{(p, k)}. \quad (4.43)$$

Kerimov and Verdiev first got interested in the generalisation of the Clebsch-Gordan coefficients to the irreps of $SL_2(\mathbb{C})$ [KVM78]. The $SL_2(\mathbb{C})$ -Clebsch-Gordan coefficients are defined by the relation

$$|p, k; j, m\rangle = \int dp_1 dp_2 \sum_{k_1 j_1 m_1} \sum_{k_2 j_2 m_2} C_{p_1 k_1 j_1 m_1, p_2 k_2 j_2 m_2}^{p k j m} |p_1, k_1; j_1 m_1\rangle \otimes |p_2, k_2; j_2 m_2\rangle. \quad (4.44)$$

The coefficients are non-zero only when $k_1 + k_2 + k_3 \in \mathbb{N}$, in addition to the usual triangle inequality $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$.

We have explicit expression for the $SL_2(\mathbb{C})$ -Clebsch-Gordan coefficients but they are a bit tough. First of all, remark that the magnetic part factorises as

$$C_{p_1 k_1 j_1 m_1, p_2 k_2 j_2 m_2}^{p_3 k_3 j_3 m_3} = \chi(p_1, p_2, p_3, k_1, k_2, k_3; j_1, j_2, j_3) C_{j_1 m_1 j_2 m_2}^{j_3 m_3}. \quad (4.45)$$

χ is a function of 9 variables which can be computed by the following expression (found initially in

[KVM78] but corrected slightly in [Spe17]):

$$\begin{aligned}
\chi(p_1, p_2, p_3, k_1, k_2, k_3; j_1, j_2, j_3) &= \kappa(-1)^{(j_1+j_2+j_3+k_1+k_2+k_3)/2} (-1)^{-k_2-k_1} N_{p_1}^{j_1} N_{p_2}^{j_2} \overline{N_{p_3}^{j_3}} \\
&\times \frac{1}{4\sqrt{2\pi}} \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)} \left(\frac{(j_1-k_1)!(j_2+k_2)!}{(j_1+k_1)!(j_2-k_2)!} \right)^{1/2} \\
&\times \Gamma(1-\nu_3+\mu_3)\Gamma(1-\nu_3-\mu_3) \sum_{n=-j_1}^{j_1} \left(\frac{(j_1-n)!(j_2+k_3-n)!}{(j_1+n)!(j_2-k_3+n)!} \right)^{1/2} C_{j_1 n; j_2, k_3-n}^{j_3 k_3} \\
&\times \sum_{l_1=\max(k_1, n)}^{\min(j_1, k_3+j_2)} \sum_{l_2=\max(-k_2, n-k_3)}^{j_2} \frac{(j_1+l_1)!(j_2+l_2)!(-1)^{l_1-k_1+l_2+k_2}}{(j_1-l_1)!(l_1-k_1)!(l_1-n)!(j_2-l_2)!(l_2+k_2)!(l_2-n+k_3)!} \\
&\times \frac{\Gamma(2-\nu_1-\nu_2-\nu_3+\mu_1+l_1+l_2-n)\Gamma(1-\nu_1+\mu_3+l_1)\Gamma(1-\nu_2-\mu_3+l_2)}{\Gamma(2-\nu_1-\nu_2+l_1+l_2)\Gamma(1-\nu_3+\mu_1-n)\Gamma(2-\nu_1-\nu_3+l_1)\Gamma(2-\nu_3-\nu_2+l_2)} \quad (4.46)
\end{aligned}$$

with

$$\begin{aligned}
\nu_1 &= \frac{1}{2}(1+ip_1-ip_2-ip_3) \\
\nu_2 &= \frac{1}{2}(1-ip_1+ip_2-ip_3) \\
\nu_3 &= \frac{1}{2}(1+ip_1+ip_2+ip_3) \\
\mu_1 &= \frac{1}{2}(-k_1+k_2+k_3) \\
\mu_2 &= \frac{1}{2}(k_1-k_2+k_3) \\
\mu_3 &= \frac{1}{2}(-k_1-k_2-k_3)
\end{aligned} \quad (4.47)$$

and a phase

$$\kappa = \frac{\Gamma(\nu_1+\mu_1)\Gamma(\nu_2+\mu_2)\Gamma(\nu_3+\mu_3)\Gamma(-1+\nu_1+\nu_2+\nu_3+\mu_1+\mu_2+\mu_3)}{|\Gamma(\nu_1+\mu_1)\Gamma(\nu_2+\mu_2)\Gamma(\nu_3+\mu_3)\Gamma(-1+\nu_1+\nu_2+\nu_3+\mu_1+\mu_2+\mu_3)|}, \quad (4.48)$$

and

$$N_p^j = \frac{\Gamma(1+j+ip)}{|\Gamma(1+j+ip)|}, \quad (4.49)$$

and the usual gamma function defined over \mathbb{C} by analytic continuation of

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad \text{with } \operatorname{Re} z > 0. \quad (4.50)$$

★ **NOTA BENE.** The phase κ satisfies $|\kappa| = 1$ was chosen to make the $SL_2(\mathbb{C})$ -Clebsch-Gordan coefficients real (equivalent to the Condon-Shortley convention in the $SU(2)$ case). Contrary to the usual $SU(2)$ -Clebsch-Gordan coefficients, there is no consensual convention for this phase. The choice of Kerimov differs from that of Anderson [ARRW70b] or Speziale [Spe17].

These seemingly intricate expressions have nevertheless been used with much efficiency in [Spe17] to compute numerically spin-foam amplitudes. The formula is indeed interesting because expressed with only finite sums.

Graphical calculus. When willing to define a graphical calculus for $SL_2(\mathbb{C})$ one encounters the difficulty of finding a good $SL_2(\mathbb{C})$ -equivalent to the $3jm$ -symbol of $SU(2)$ recoupling theory, so that it would satisfy the good symmetry relations to be well-represented by a 3-valent vertex. This issue is investigated in [ARRW70b], but the symmetry relations are intricate and depends on the convention chosen for the phase κ . As a result there is no consensus about the definition of the rules of graphical calculus for $SL_2(\mathbb{C})$. Following the phase convention of [Spe17], we define then

$$\left(\begin{array}{ccc} (p_1, k_1) & (p_2, k_2) & (p_3, k_3) \\ (j_1, m_1) & (j_2, m_2) & (j_3, m_3) \end{array} \right) \stackrel{\text{def}}{=} (-1)^{2j_1-j_2+j_3-m_3} C_{p_1 k_1 j_1 m_1, p_2 k_2 j_2 m_2}^{p_3 k_3 j_3, -m_3}. \quad (4.51)$$

Graphically it corresponds to the 3-valent vertex

$$\begin{pmatrix} (p_1, k_1) & (p_2, k_2) & (p_3, k_3) \\ (j_1, m_1) & (j_2, m_2) & (j_3, m_3) \end{pmatrix} = \begin{array}{c} (p_1, k_1) \quad (p_2, k_2) \quad (p_3, k_3) \\ \swarrow \quad \uparrow \quad \searrow \\ \end{array} \quad (4.52)$$

With the same rules of orientation and summation as that of section 3.6, we can then fully develop the graphical calculus of $SL_2(\mathbb{C})$. For instance, we can define $SL_2(\mathbb{C})$ -invariant functions, like the $(6p, 6k)$ -symbol. The $SL_2(\mathbb{C})$ -15j-symbol can be used to define the spin-foam amplitude (see section 5.2).

Chapter 5

Loops and Foams in a nutshell

Loop Quantum Gravity (LQG) is a good candidate theory for quantum gravity. It is obtained by the canonical quantization of general relativity and describes the quantum states of space with the so-called spin-networks. Spin-foam theory is a later spinoff of both LQG and the sum-over-histories approach to quantum gravity. It describes quantum space-time, seen as the time evolution of spin-networks.

Most of the main textbooks provide a derivation of the theory, following more or less its historical developments through the process of quantization [RV14, Bae00, DS10]. Here we will only introduce the general mathematical framework of the theory, trying to be as concise as possible, since we believe that a full-fledged fundamental theory should come to a point where it stands on its own, with its mathematical framework and physical principles, without any reference to older approximate theories like general relativity or non-relativistic quantum mechanics.

5.1 Spin-network

As any good quantum theory, LQG comes with an Hilbert space. It is the mathematical space of the various possible states of physical space. A very convenient basis is parametrised by the so-called spin-networks that we first define.

Spin-network. An *abstract*¹ *oriented graph* Γ is an ordered pair $\Gamma = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N} = \{n_1, \dots, n_N\}$ is a finite set of N nodes, and $\mathcal{L} = \{l_1, \dots, l_L\}$ a finite set of L links, endowed with a target map $t : \mathcal{L} \rightarrow \mathcal{N}$ and a source map $s : \mathcal{L} \rightarrow \mathcal{N}$, assigning each link to its endpoints (respectively the head or the tail, defined by the orientation). We denote \mathcal{L}_n the set of links attached to a given node n . The valency of a node n is the number of links which have n as an endpoint. A graph is said to be p -valent if the valency of each node is p . Given an oriented graph Γ , we denote Λ_Γ the set of labellings j that assign to any link $l \in \mathcal{L}$, an $SU(2)$ -irrep $j_l \in \mathbb{N}/2$. Given a labelling $j \in \Lambda_\Gamma$, we denote

$$\text{Inv}(n, j) \stackrel{\text{def}}{=} \text{Inv}_{SU(2)} \left(\bigotimes_{l \in \mathcal{L}_n} \mathcal{Q}_{j_l} \right). \quad (5.1)$$

The tensor product above assumes that an ordering of the links around a node, i.e. a sense of rotation and a starting link, has been prescribed. A *spin-network* is a triple $\Sigma = (\Gamma, j, \iota)$, where Γ is an oriented graph, $j \in \Lambda_\Gamma$, and ι is a map that assigns to any $n \in \mathcal{N}$ an intertwiner $|\iota_n\rangle \in \text{Inv}(n, j)$. Figure 1 shows a pictorial representation of a 4-valent spin-network.

Hilbert space. The Hilbert space of LQG is given by

$$\mathcal{H} = \bigoplus_{\Gamma} \mathcal{H}_\Gamma \quad (5.2)$$

¹Strictly speaking ‘LQG’ refers to the canonical approach for which the spin-networks are embedded. Here, we adopt a more abstract point of view, sometimes called ‘covariant LQG’, which is motivated by spin-foams. This alternative construction raises difficulties for defining the hamiltonian, but they are circumvented by the spin-foam formalism.

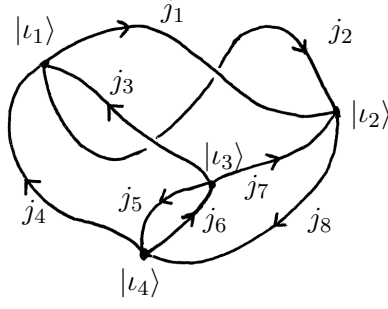


Figure 1: A 4-valent spin-network.

where the direct sum is made over all possible oriented 4-valent graphs Γ , and \mathcal{H}_Γ is

$$\mathcal{H}_\Gamma = \bigoplus_{j \in \Lambda_\Gamma} \bigotimes_{n \in \mathcal{N}} \text{Inv}(n, j) \quad (5.3)$$

It is spanned by the set of *spin-networks states*

$$|\Gamma, j, \iota\rangle = \bigotimes_{n \in \mathcal{N}} |\iota_n\rangle \quad (5.4)$$

where Γ range over all possible 4-valent graphs, j over Λ_Γ , and $|\iota_n\rangle$ over an orthonormal basis of $\text{Inv}(n, j)$. By definition of the invariant space $\text{Inv}(n, j)$, it is straightforward to see that ‘the action of any $g_n \in SU(2)$ over a node n ’, i.e. over $\text{Inv}(n, j)$, let the spin-network states invariant:

$$g_n \cdot |\Gamma, j, \iota\rangle = |\Gamma, j, \iota\rangle. \quad (5.5)$$

With this property, the spin-network states are said to satisfy the *Gauss constraint*² at each node.

Since we only consider 4-valent graphs, an orthonormal basis of $\text{Inv}(n, j)$ is given by the states of equation 3.40. Thus, instead of writing the abstract states $|\iota\rangle$, it is equivalent to split each 4-valent node (according to the prescribed ordering of the links around the nodes), like

$$\quad (5.6)$$

and then associate to the virtual link the spin $\iota \in \{\max(|j_1 - j_2|, |j_3 - j_4|), \dots, \min(j_1 + j_2, j_3 + j_4)\}$, which parametrises the basis $|\iota\rangle_{12}$ of equation 3.40. By metonymy the spin ι is also called an *intertwiner*. Thus the spin-network of Figure 1, becomes

$$\quad (5.7)$$

²This designation comes from an analogy with Maxwell theory of electromagnetism.

Spin-network wave function. The isomorphism 2.10, deduced from Peter-Weyl's theorem, offers another possible realisation of \mathcal{H}_Γ , as a subspace of $L^2(SU(2)^N)$, denoted $L^2(SU(2)^N)_\Gamma$. A spin-network state $|\Gamma, j, \iota\rangle$ becomes a function $\Psi_{(\Gamma, j, \iota)} \in L^2(SU(2)^N)_\Gamma$, obtained with the following procedure:

1. Associate to each link l

$$\begin{array}{c} n_l \\ \curvearrowright \\ j_l \\ \curvearrowleft \\ m_l \end{array} \cong D_{m_l n_l}^{j_l}(g_l) \quad (5.8)$$

with the magnetic indices m_l or n_l , depending of the orientation, and the variable $g_l \in SU(2)$.

2. Associate to each (splitted) node a $4jm$ symbol, like

$$\begin{array}{c} j_1 \swarrow \\ \downarrow \\ j_2 \searrow \\ \vdots \\ j_4 \swarrow \\ \downarrow \\ j_3 \searrow \end{array} \cong (-1)^{j_4 - n_4} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & -n_4 \end{pmatrix}^{(\iota)} \quad (5.9)$$

with an index $-n$ and a phase $(-1)^{j-n}$ for outgoing links.

3. Finally multiply all together, and sum over all the magnetic indices.

Thus, we obtain a set of *spin-network wave functions* $\Psi_{(\Gamma, j, \iota)}(g_{l_1}, \dots, g_{l_L})$ that span a peculiar subspace of $L^2(SU(2)^N)$, denoted³ $L^2(SU(2)^N)_\Gamma$. For instance, the spin-network

$$\begin{array}{c} \curvearrowright \\ j_1 \swarrow \\ \downarrow \\ j_2 \searrow \\ \downarrow \\ j_3 \swarrow \\ \downarrow \\ j_4 \searrow \\ \curvearrowright \end{array} \quad (5.10)$$

encodes the function

$$\begin{aligned} & \Psi(u_1, u_2, u_3, u_4) \\ &= \sum_{m_i, n_i} (-1)^{\sum_i (j_i - n_i)} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ -n_1 & -n_2 & -n_3 & -n_4 \end{pmatrix}^{(\iota)} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(\kappa)} \prod_{i=1}^4 D_{m_i n_i}^{j_i}(u_i). \end{aligned} \quad (5.11)$$

From the isomorphism 2.10, we can express the Gauss constraint 5.5 as an invariance of the functions $\Psi_{(\Gamma, j, \iota)}(g_{l_1}, \dots, g_{l_N})$: for all set $(u_n) \in SU(2)^L$, parametrised by the nodes $n \in \mathcal{N}$, we have

$$\Psi_{(\Gamma, j, \iota)}(g_{l_1}, \dots, g_{l_N}) = \Psi_{(\Gamma, j, \iota)}(u_{s(l_1)} g_{l_1} u_{t(l_1)}^{-1}, \dots, u_{s(l_N)} g_{l_N} u_{t(l_N)}^{-1}) \quad (5.12)$$

with s and t the source and target map of the graph. In fact, $L^2(SU(2)^N)_\Gamma$ can be characterized as the subspace of functions of $L^2(SU(2)^N)$ that satisfy this property.

Notice finally that evaluating the function at the identity on all links result in the graphical calculus previously defined in section 3.6.

³This subspace is sometimes denoted $L^2(SU(2)^L/SU(2)^N)$, but this is not mathematically rigorous.

Algebra of observables. In fact, there is not much information in the Hilbert space itself. What really matters physically is the algebra of observables \mathcal{A} acting upon it. The observables of LQG are obtained by the principle of correspondence. Thus, they come with a geometrical interpretation: they correspond notably to measurements of area or measurement of volume. The Hilbert space \mathcal{H} is built from the building block spaces \mathcal{Q}_{j_l} , where j_l labels a link l . Similarly, the algebra of observables is built from the action of $\mathfrak{su}(2)$ (the flux) and $SU(2)$ (the holonomy) over \mathcal{Q}_{j_l} . Notice that an observable should not ‘go out’ of \mathcal{H} : in other words, an observable should commute with the Gauss constraint.

Given a graph Γ , the observable of area associated to a link l is

$$\hat{A}_l \stackrel{\text{def}}{=} 8\pi \frac{\hbar G}{c^3} \gamma \sqrt{\vec{J}_l^2}, \quad (5.13)$$

where γ is a real parameter called the *Immirzi parameter*, and \vec{J}_l are the generators of $SU(2)$ acting over \mathcal{Q}_{j_l} . The spin-network basis diagonalise \hat{A}_l :

$$\hat{A}_l |\Gamma, j, \iota\rangle \stackrel{\text{def}}{=} j_l(j_l + 1) |\Gamma, j, \iota\rangle. \quad (5.14)$$

It also diagonalise the observable $(\vec{J}_1 + \vec{J}_2)^2$, acting over a node n ,



$$, \quad (5.15)$$

so that

$$(\vec{J}_1 + \vec{J}_2)^2 |\Gamma, j, \iota\rangle \stackrel{\text{def}}{=} \iota_{12}(\iota_{12} + 1) |\Gamma, j, \iota\rangle. \quad (5.16)$$

The latter observable encodes a notion of ‘angle’ between the links j_1 and j_2 . Given a graph Γ , the set of area observables associated to each link and a set of ‘angle operators’ like $(\vec{J}_1 + \vec{J}_2)^2$ (one per each node), define a complete set of commuting observables over \mathcal{H}_Γ , diagonalised by the spin-network basis.

On each node like 5.15, we can also define the volume operator

$$\hat{V}_n = \frac{\sqrt{2}}{3} (8\pi G \hbar \gamma)^{3/2} \sqrt{|\vec{J}_1 \cdot (\vec{J}_2 \times \vec{J}_3)|}. \quad (5.17)$$

It is not diagonalised by the spin-network basis, but its eigenvalues can be computed numerically. It does not commute with $(\vec{J}_1 + \vec{J}_2)^2$ but it does with the areas, so that the areas \hat{A}_l and the volumes \hat{V}_n form another complete set of commuting observables (diagonalised by another basis than that of spin-networks).

These geometric operators of area, volume or angle, built from the principle of correspondance, suggest a vision of the ‘quantum geometry’. It is obtained as the dual picture of a graph Γ : a tetrahedron is associated to each node, and they glue together along faces (whose area is given by the eigenvalue of \hat{A}_l) dual to links.

5.2 Spin-foam

Dynamics. The latter mathematical framework of loop quantum gravity is obtained through the canonical quantization of general relativity: the spin-network states represent quantum states of space. The time evolution of these states should be found by finding the subspace formed by the solution to the hamiltonian constraint $\hat{H} |\Psi\rangle = 0$, where $|\Psi\rangle$ is a superposition of spin-network states, \hat{H} the quantized hamiltonian. This hard path of finding the dynamics was followed notably by Thiemann [Thi07]. Below we present a way to short-circuit the issue, called spin-foams, which takes inspiration from former sum-over-histories approaches to quantum gravity.

Spin-foams. Spin-foams can be seen both as higher dimensional version of Feynman diagrams propagating the gravitational field, and as the time evolution of spin-networks. Spin-foams are built out of combinatorial objects, which generalises graphs to higher dimensions, called *piecewise linear cell complexes*, often abbreviated as *complexes*. A *oriented 2-complex* is an ordered triple $\kappa = (\mathcal{E}, \mathcal{V}, \mathcal{F})$, with a finite set $\mathcal{E} = \{e_1, \dots, e_E\}$ of edges, a finite set $\mathcal{V} = \{v_1, \dots, v_V\}$ of vertices, and a finite set $\mathcal{F} = \{f_1, \dots, f_F\}$ of faces, such that they all ‘glue consistently’⁴. The orientation is given on the edges by a target map $t : \mathcal{E} \rightarrow \mathcal{V}$ and a source map $s : \mathcal{E} \rightarrow \mathcal{V}$, and the orientation of each faces gives a cyclic ordering of its bounding vertices.

Given an oriented 2-complex κ , we denote Λ_κ the set of labellings j that assign an $SU(2)$ -irrep $j_f \in \mathbb{N}/2$ to any face $f \in \mathcal{F}$. Similarly we denote I_κ the set of labellings ι that assign to each edge e an intertwiner ι_e ,

$$\iota_e \in \text{Inv}(e, j) \stackrel{\text{def}}{=} \text{Inv}_{SU(2)} \left(\bigotimes_{f \in \mathcal{F}(e)} \mathcal{Q}_{j_f} \right), \quad (5.18)$$

where $\mathcal{F}(e)$ is the set of faces adjacent to the edge e . A *spin-foam* is a triple $F = (\kappa, j, \iota)$. where κ is an oriented 2-complex, $j \in \Lambda_\kappa$, and $\iota \in I_\kappa$. We can stick to a purely ‘abstract’ combinatorial definition of 2-complexes, but we can also adopt a geometrical ‘hypostasis’ that represents ‘faces’ as polygons. For instance, Figure 2 shows a spin-foam embedded into 3-dimensional euclidean space. Notice that such

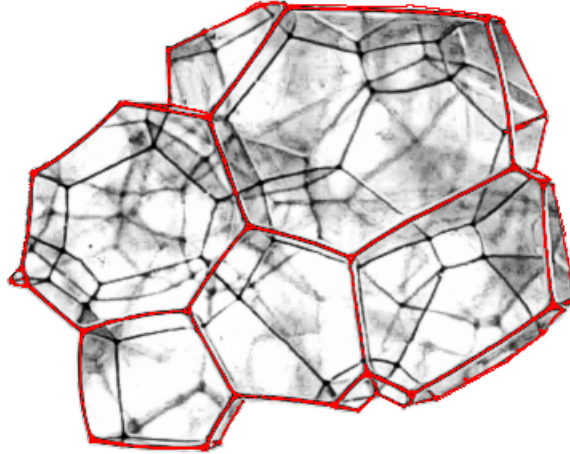


Figure 2: A 2-complex embedded in 3-dimensional euclidean space. Its boundary is a graph (in red).

a graphical representation is not always possible in 3 dimensions, and sometimes a fourth dimension can be required. Interestingly, the boundary of a 2-complex⁵ is a graph, as can be seen on Figure 2. Thus, the boundary of a spin-foam is spin-network. The vertices and the edges of the boundary are called respectively nodes and links. Each link bounds an inside face, so that the spin of the link is also the spin of the face. Similarly, each node is an endpoint of an inside edge, so that the associated intertwiners match.

Spin-foam amplitude. To each spin-foam we associate an amplitude, which is like the propagator associated to a Feynman diagram. Its interpretation is made precise below. Given a spin-foam (κ, j, ι) , we define the spin-foam amplitude as

$$\mathcal{A}(\kappa, j, \iota) = \left(\prod_{f \in \mathcal{F}} (2j_f + 1) \right) \left(\prod_{e \in \mathcal{E}} (2\iota_e + 1) \right) \left(\prod_{v \in \mathcal{V}} A_v(j, \iota) \right). \quad (5.19)$$

⁴There is a way to give a precise meaning to this gluing, but it will be sufficient to keep it intuitive below, and to avoid these technicalities.

⁵The notion of boundary of an abstract 2-complex requires a formal definition, but we keep it intuitive below for simplicity. We can admit that any 2-complex comes with a boundary.

A_v is called the vertex amplitude. In the short history of spin-foam amplitudes there has already been many various formula proposed for the vertex amplitude. First, let us say that for quantum gravity, it is sufficient to consider spin-foams whose vertices are 5-valent (5 edges attached to it) and whose edges are 4-valent (4 faces attached to it). This restriction comes from the fact that the 2-complex of quantum gravity are built by dualising the triangulation of a 4-dimensional manifold. Unfortunately there is no possible nice picture as Figure 2 to visualise such a 2-complex since it cannot be embedded into the 3-dimensional euclidean space. However it is sufficient to get an idea of the combinatorial structure of each vertex by representing the adjacent edges with dots and the faces with lines, so that we draw the *vertex graph*

The orientation and the spin of the links, and the intertwiners of the nodes are naturally inherited from the underlying spin-foam, so that the vertex graph is a spin-network.

★ **NOTA BENE.** To avoid confusion, let us recap. Each spin-foam comes with a *boundary spin-network*, and also with a *vertex graph* for each of its vertex. If the spin-foam is made of only one vertex, then the boundary spin-network and the vertex graph coincides. Contrary to the the boundary spin-network, there is in general no interpretation of the vertex graphs in terms of quantum states of space.

The combinatorial shape of each vertex suggests to define the amplitude A_v as the value obtained with the rules of graphical calculus of $SU(2)$ recoupling theory, defined in section 3.6. This is precisely what Ooguri did in [Oog92] by defining the vertex amplitude as the $\{15j\}$ -symbol, but it later appear not to be a good candidate for quantum gravity. Since then many other models were suggested [Per13]. They all consist in finding other rules than that of $SU(2)$ recoupling theory to assign a value to the vertex graph 5.20.

The EPRL model, introduced in [ELPR08], is a model that is still considered as a good candidate for quantum gravity. The vertex amplitude is computed from the vertex graph 5.20 with the following rules:

1. Compute the spin-network wave function as shown in the previous section. We obtain a function of $L^2(SU(2)^{10})$ which satisfies the Gauss constraint 5.12:

$$\Psi_{(\Gamma,j,\iota)}(g_{l_1}, \dots, g_{l_{10}}) \quad (5.21)$$

2. Apply the so-called Y_γ -map, which is the linear map $Y_\gamma : L^2(SU(2)^{10}) \rightarrow \mathcal{F}(SL_2(\mathbb{C})^{10})$ defined over the canonical basis of Wigner matrix coefficients by

$$Y_\gamma \left(\prod_i D_{m_i n_i}^{j_i} \right) = \prod_i D_{j_i m_i j_i n_i}^{(\gamma j_i j_i)} \quad (5.22)$$

where γ is the Immirzi parameter. We thus obtain a function of $\mathcal{F}(SL_2(\mathbb{C})^{10})$

$$Y_\gamma \Psi_{(\Gamma,j,\iota)}(h_{l_1}, \dots, h_{l_{10}}). \quad (5.23)$$

It still satisfies the invariance of the Gauss constraint 5.12 for $SU(2)$ action, be not for $SL_2(\mathbb{C})$.

3. Project down to the $SL_2(\mathbb{C})$ -invariant subspace on each node with the projector $P_{SL_2(\mathbb{C})}$ acting as

$$P_{SL_2(\mathbb{C})} Y_\gamma \Psi_{(\Gamma, j, \iota)}(h_{l_1}, \dots, h_{l_{10}}) = \int_{SL_2(\mathbb{C})} \left(\delta(a_{n_5}) \prod_{n \in \mathcal{N}} da_n \right) \Psi_{(\Gamma, j, \iota)}(a_{s(l_1)} h_{l_1} a_{t(l_1)}^{-1}, \dots, a_{s(l_{10})} h_{l_{10}} a_{t(l_{10})}^{-1}). \quad (5.24)$$

with n_5 any of the 5 nodes. The delta function $\delta(a)$ (only non-vanishing when $a = \mathbb{1}$) is required to avoid the divergence of the integration, but the final result does not depend on the choice of node n_5 . To put it differently the integration is only effective over (any) four nodes, while the fifth a_{n_5} is fixed to the identity $\mathbb{1}$.

4. Evaluate all the variables h_l to $\mathbb{1}$. So if (Γ, j, ι) is the vertex graph of a vertex v in a spin-foam (κ, j, ι) , we can finally write in a nutshell

$$A_v(j, \iota) = (P_{SL_2(\mathbb{C})} Y_\gamma \Psi_{(\Gamma, j, \iota)})(\mathbb{1}). \quad (5.25)$$

Thus we have fully defined the spin-foam amplitude $\mathcal{A}(\kappa, j, \iota)$ of the EPRL model. The specificity of this model is the Y_γ -map which selects only the irreps ($p = \gamma j, k = j$) among the principal series of $SL_2(\mathbb{C})$. It implements the so-called *simplicity constraints*, which enable to formulate general relativity as a BF theory [Bae00]. Besides, the apparent sophisticated procedure should not hide the fact that the value of $A_v(j, \iota)$ is the same than that obtained from the $SL_2(\mathbb{C})$ graphical calculus, defined in section 4.4, when the simplicity constraint is applied.

★ **NOTA BENE.** For those only interested in the actual computation of the amplitude of a given vertex graph, we can summarise the previous procedure with the following algorithm:

1. Associate a variable $h_p \in SL_2(\mathbb{C})$ to each intertwiner ι_p .
2. Associate to each link

$$\iota_q \begin{array}{c} \curvearrowright \\ \xrightarrow{j} \\ \curvearrowleft \end{array} \iota_p \cong D_{j m j n}^{(\gamma j, j)}(h_p^{-1} h_q). \quad (5.26)$$

3. Associate a $3jm$ -symbol to each node as in usual graphical calculus (equation 3.52).
4. Multiply everything together and sum over all the magnetic indices m and n .
5. Integrate over (any) four of the five $SL_2(\mathbb{C})$ variables h_p , and fix the fourth to the identity $\mathbb{1}$.

Interpretation. The interpretation of spin-foams relies on the general boundary formulation of quantum mechanics which was introduced by Oeckl [Oec03, Oec08]. Consider a finite region of spacetime. Its boundary Σ is a 3-dimensional hypersurface which constitutes the quantum system under consideration. Its space of states is the Hilbert space of LQG, \mathcal{H} , spanned by the spin-network states. An observer \mathcal{O} may know some partial information about the state ψ of Σ , which can be expressed by the fact that $\psi \in \mathcal{S}$, where \mathcal{S} is a linear subspace of \mathcal{H} . Then \mathcal{O} can carry on measurements with the operators of the algebra to know more about ψ . If \mathcal{A} is a linear subspace of \mathcal{S} , then the probability to find $\psi \in \mathcal{A}$ is

$$P(\mathcal{A}|\mathcal{S}) = \frac{\sum_{i \in I} |\rho(\xi_i)|^2}{\sum_{j \in J} |\rho(\zeta_j)|^2}, \quad (5.27)$$

where ξ_i (resp. ζ_j) is an orthonormal basis of \mathcal{A} (resp. \mathcal{S}). $\rho : \mathcal{H} \rightarrow \mathbb{C}$ is a linear map, called the *transition amplitude* defined for a spin-network state Ψ by

$$\rho(\Psi) \stackrel{\text{def}}{=} \sum_{\sigma} W_{\sigma}(\Psi) \quad (5.28)$$

where the sum is done over all possible spin-foams σ which have Ψ as a boundary, and $W_{\sigma}(\Psi)$ is the 2-complex amplitude defined as

$$W_{\sigma}(\Psi) = \sum_j \sum_{\iota} \mathcal{A}(\sigma, j, \iota) \quad (5.29)$$

where the sum is done over all the possible spin labelling $j \in \Lambda_\kappa$, and intertwiner labellings $\iota \in I_\kappa$, that are compatible with the spin-network Ψ at the boundary.

This completes the mathematical formulation of the theory and its probabilistic interpretation. Of course, much remain to be discovered. In particular, the theory has yet to meet the benchmark of the experimental evidence, but this would be another story to tell!

Appendix A

Representations and intertwiners

Representation of groups. A good way to understand the structural properties of a group is to look how it can act on vector spaces. By ‘action’, I mean a linear action that preserves the group product: it is called a ‘representation’.

In physics, notably in quantum mechanics, we often focus on representations over Hilbert spaces. Let G be a locally compact group, and $GL(\mathcal{H})$ the group of bounded linear operators over a Hilbert space \mathcal{H} that admit a bounded inverse. A (bounded continuous) *representation* ρ of G over \mathcal{H} is an homomorphism $\rho : G \rightarrow GL(\mathcal{H})$, such that the resulting map $G \times \mathcal{H} \rightarrow \mathcal{H}$ is continuous. In the case of a finite dimensional Hilbert space \mathcal{H} , $GL(\mathcal{H})$ is just the space of invertible linear maps, and a representation is any linear action of G over \mathcal{H} . It is said *unitary* if it preserves the scalar product.

Representation of Lie algebras. There are also representations of Lie algebra, which are linear action preserving the Lie bracket. Any representation of a Lie group defines by differentiation a representation of its Lie algebra. Precisely, if $\rho : G \rightarrow GL(\mathcal{H})$ is a representation of a Lie group G , the *differential of the representation* ρ , is the linear map $D\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathcal{H})$ defined for all $X \in \mathfrak{g}$ by:

$$(D\rho)(X) = \left. \frac{d}{dt} \rho(e^{tX}) \right|_{t=0}. \quad (\text{A.1})$$

It is shown to be a Lie algebra representation. Moreover, for all $X \in \mathfrak{g}$,

$$\rho(e^X) = e^{D\rho(X)}. \quad (\text{A.2})$$

One shows

1. If $F \subset \mathcal{H}$ is stable for ρ , then F is also stable for $D\rho$.
2. If $D\rho$ is irreducible, then ρ is also irreducible.
3. If G is connected, the converse of (1) and (2) are also true.

Conversely, given a Lie algebra \mathfrak{g} , there is no unique Lie group associated to it, but there is a unique simply connected one G , which is obtained by exponentiation of \mathfrak{g} . Then given any morphism of Lie algebra ϕ , there exists a morphism of Lie group ρ such that $\phi = D\rho$. Thus a representation of \mathfrak{g} will entail a representation on each of its associated Lie groups.

Irrep. A representation is *irreducible* if it admits no other closed stable subspace than $\{0\}$ and \mathcal{H} . To go faster, we commonly say ‘irrep’ instead of ‘irreducible representation’. They can be seen as the building blocks of the other representations. From two representations, one can build others: the direct sum and the tensor product notably. If V and W are two vector spaces of representation for a group G and its algebra \mathfrak{g} , we define a representation over the direct sum $V \oplus W$ by

$$\forall g \in G, \quad g \cdot (v + w) = g \cdot v + g \cdot w \quad (\text{A.3})$$

$$\forall X \in \mathfrak{g}, \quad X \cdot (v + w) = X \cdot v + X \cdot w. \quad (\text{A.4})$$

We also define a representation over the tensor product $V \otimes W$

$$\forall g \in G, \quad g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w) \quad (\text{A.5})$$

$$\forall X \in \mathfrak{g}, \quad X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot w). \quad (\text{A.6})$$

Intertwiners. If V and W are two vector spaces of representation for a group G and its algebra \mathfrak{g} , an *intertwiner* (or equivariant map or intertwining operator) is a linear map $T : V \rightarrow W$ satisfying:

$$T(g \cdot v) = g \cdot T(v). \quad (\text{A.7})$$

The space of intertwiners, denoted $\text{Hom}_G(V, W)$, is a subspace of the vector space of linear maps $\text{Hom}(V, W)$. Two representations are *equivalent* if there is an invertible intertwiner between them two. An invertible intertwiner is a way to *identify* two representations, as if there were only a change of notation between them. In the language of category theory, an intertwiner is nothing but a *natural transformation* between two *functors*, each functor being a representation of the group. In the main text, we have chosen to alleviate the notations by making the intertwiner implicit, so that we write for instance (see section 2.3)

$$J_+ \cong z_0 \frac{\partial}{\partial z_1} \quad \text{and} \quad |j, m\rangle \cong \left(\frac{(2j)!}{(j+m)!(j-m)!} \right)^{1/2} z_0^{j+m} z_1^{j-m}, \quad (\text{A.8})$$

where the symbol of congruence ‘ \cong ’ should be understood as ‘equal from the perspective of the group representation’. Thus, two equivalent representations will often be presented as two *realisations* of the same representation. But of course \cong is not a strict equality ‘ $=$ ’ in the mathematical sense since for instance $\mathbb{C}_{2j}[z_0, z_1]$ carries other mathematical structures to which the intertwiner is blind.

Schur’s Lemma. If $T : V \rightarrow W$ is an intertwiner between two finite irreps of G , then either $T = 0$, or T is bijective. Moreover, if the irreps are unitary and T is bijective, then for any other bijective intertwiner T' there exists $\lambda \in \mathbb{C}$ such that $T' = \lambda T$.

Peter-Weyl’s theorem. Any important case is when the group G is compact (like $SU(2)$, but not like $SL_2(\mathbb{C})$). In this case we have the following properties:

1. Any (complex) finite representation of G can be endowed with an hermitian product which makes the representation unitary.
2. Any unitary irrep of G is finite-dimensional.
3. Any unitary representation can be decomposed into a direct sum of irreps.

Theses results justify notably that focusing on unitary irreps of $SU(2)$, as we do in chapter 2, is sufficient to describe all possible finite or unitary representations of $SU(2)$. Finally, the compactness of G enables to define the space of square-integrable functions $L^2(G)$ with the Haar measure, and we have

4. the linear span of all matrix coefficients of all finite unitary irreps of G is dense in $L^2(G)$.

A proof can be found in Knapp ([Kna86] pp. 17-20).

Appendix B

Induced representation

There is a well-known method to build a representation of group, *induced* from a representation of one of its subgroup. We present below two possible formal definitions of the method (see the book [Mau97] for more details). Then we apply it to the case of $SL_2(\mathbb{C})$.

Let K be a subgroup of G , and ρ a representation of K over a vector space V .

Definition 1. To build a representation of G starting from ρ , we first build a vector space \mathcal{H}^ρ , then a group homomorphism $U^\rho : G \rightarrow GL(\mathcal{H}^\rho)$. Let \mathcal{H}^ρ be the vector space of functions $f : G \rightarrow V$ such that

$$\forall g \in G, \quad \forall k \in K, \quad f(gk) = \rho(k)f(g). \quad (\text{B.1})$$

For all $g \in G$, we define the linear map $U^\rho(g) : \mathcal{H}^\rho \rightarrow \mathcal{H}^\rho$ by

$$\forall f \in \mathcal{H}^\rho, \quad \forall x \in G, \quad U^\rho(g)f(x) = f(g^{-1}x). \quad (\text{B.2})$$

Thus $(U^\rho, \mathcal{H}^\rho)$ is the representation of G *induced* from the representation (ρ, V) of the subgroup K .

Definition 2. Denote the quotient $M \stackrel{\text{def}}{=} G/K$. Let $P(M, K)$ be a K -principal bundle. Denote $P \times_\rho V \rightarrow M$ the associated vector bundle. It is a bundle of base M and fibre V . Let

$$\mathcal{H}^\rho = \{\text{sections } f \text{ of the bundle } P \times_\rho V \rightarrow M\}. \quad (\text{B.3})$$

For all $g \in G$, we define the linear map $U^\rho(g)$ by

$$\forall f \in \mathcal{H}^\rho, \quad \forall x \in G/K, \quad (U^\rho(g)f)(x) = f(g^{-1}x). \quad (\text{B.4})$$

Thus $(U^\rho, \mathcal{H}^\rho)$ is the representation of G *induced* from the representation (ρ, V) of the subgroup K . It is equivalent to the first definition.

Example. Consider the trivial subgroup $\{e\}$ of a Lie group G , and its trivial representation over \mathbb{C} . The induced representation is then given by the Hilbert space $L^2(G)$, endowed with an left-invariant (resp. right-invariant) measure, and the linear action $g \cdot f(h) = f(g^{-1}h)$ (resp. $g \cdot f(h) = f(hg)$). It is also called the left (resp. right) regular representation.

Application to $SL_2(\mathbb{C})$. Naimark has built the unitary representations of $SL_2(\mathbb{C})$ induced by the uni-dimensional representations of the upper-triangular subgroup K_+ [Nai64]. First he shows he following diffeomorphism between differentiable manifolds:

$$SL_2(\mathbb{C})/K_+ \cong \bar{\mathbb{C}}. \quad (\text{B.5})$$

► **PROOF.** Let H be a subgroup of G . The quotient G/H is the set of all sets Hg with $g \in G$. From Gauss decomposition, equation 1.30, if $g \in SL_2(\mathbb{C})$ such that $g_{22} \neq 0$, there exists a unique $(k, z) \in K_+ \times Z_-$ such that:

$$g = kz. \quad (\text{B.6})$$

The group Z_- is isomorphic to \mathbb{C} . If $a \in SL_2(\mathbb{C})$ satisfies $a_{22} = 0$ then it belongs to the class of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ since

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} = \begin{pmatrix} a_{21}^{-1} & a_{11} \\ 0 & a_{21} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{B.7})$$

Thus, identifying the class of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with $\{\infty\}$, we see that $SL_2(\mathbb{C})/K_+ \cong \bar{\mathbb{C}}$. \square

Then we compute the expression of the induced linear action of $SL_2(\mathbb{C})$ over $\bar{\mathbb{C}}$:

$$a \cdot z = \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}. \quad (\text{B.8})$$

It is nothing but the so-called Möbius transformation.

► **PROOF.** An action of $SL_2(\mathbb{C})$ over $SL_2(\mathbb{C})/K_+$, is naturally given by:

$$a \cdot Kg = Kga. \quad (\text{B.9})$$

It leads to the given expression over $\bar{\mathbb{C}}$. \square

Consider the Hilbert space of square integrable complex functions $L^2(\mathbb{C})$ with the scalar product:

$$(f_1, f_2) \stackrel{\text{def}}{=} \frac{i}{2} \int_{\mathbb{C}} \overline{f_1(z)} f_2(z) dz \wedge d\bar{z}. \quad (\text{B.10})$$

Here we use the usual Lebesgue measure over \mathbb{C} . We look for a unitary representation over $L^2(\mathbb{C})$ of the form:

$$a \cdot f(z) = \alpha(z, a) f(a \cdot z). \quad (\text{B.11})$$

Then it can be shown (after lines of computation) that for all $(\rho, m) \in \mathbb{R} \times \mathbb{Z}$, there exists a unitary representation of $SL_2(\mathbb{C})$ over $L_2(\mathbb{C})$ given by

$$V_a f(z) = (a_{12}z + a_{22})^{\frac{m}{2} + \frac{i\rho}{2} - 1} \overline{(a_{12}z + a_{22})^{-\frac{m}{2} + \frac{i\rho}{2} - 1}} f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right). \quad (\text{B.12})$$

The set of representations is called the principal unitary series of $SL_2(\mathbb{C})$. It can be shown these representations are irreducible!

Variation. Rühl constructs an induced representation in [Rüh70] (p. 57). Formally, he first observes the following diffeomorphism between manifolds:

$$SL_2(\mathbb{C})/K_+ \cong SU(2)/U(1). \quad (\text{B.13})$$

► **PROOF.** A convenient way to see it is to decompose $a \in SL_2(\mathbb{C})$ with $k \in K_+$ and $u \in SU(2)$ such that:

$$a = ku. \quad (\text{B.14})$$

Such a decomposition exists since for all $\theta \in \mathbb{R}$:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{e^{-i\theta}}{\sqrt{|a_{21}|^2 + |a_{22}|^2}} & \frac{a_{12}a_{22}^* + a_{11}a_{21}^* e^{i\theta}}{\sqrt{|a_{21}|^2 + |a_{22}|^2}} \\ 0 & \sqrt{|a_{21}|^2 + |a_{22}|^2} e^{i\theta} \end{pmatrix} \begin{pmatrix} \frac{a_{22}^* e^{i\theta}}{\sqrt{|a_{21}|^2 + |a_{22}|^2}} & -\frac{a_{21}^* e^{i\theta}}{\sqrt{|a_{21}|^2 + |a_{22}|^2}} \\ \frac{a_{21} e^{-i\theta}}{\sqrt{|a_{21}|^2 + |a_{22}|^2}} & \frac{a_{22} e^{-i\theta}}{\sqrt{|a_{21}|^2 + |a_{22}|^2}} \end{pmatrix}. \quad (\text{B.15})$$

But there is no uniqueness of the decomposition. If $u \in SU(2)$ decomposes a , then also does for all $\theta \in \mathbb{R}$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} u. \quad (\text{B.16})$$

Conversely, it is easy to show that these are the only possible matrices of $SU(2)$ decomposing a . \square

Instead of constructing a space of functions over $SL_2(\mathbb{C})/K_+$, it is then equivalent to consider functions over $SU(2)$ satisfying a covariance condition for the group $U(1)$ (in the spirit of definition 1). Thus, we consider functions over $SU(2)$ satisfying the condition:

$$\phi \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} u \right) = e^{in\theta} \phi(u) \quad (\text{B.17})$$

with $n \in \mathbb{Z}$. The choice of the factor $e^{in\theta}$ corresponds to uni-dimensional representations of $U(1)$. This way another representation of $SL_2(\mathbb{C})$ can be built. It is equivalent to the previous of Naimark provided that $n = -2k$.

Appendix C

Commented bibliography

This commented bibliography gathers the main textbooks that will provide more details than this primer.

Warmup

[Kna86] A. W. Knap, *Representation Theory of Semisimple Groups*, Princeton University Press, 1986.

This book is a clear and exhaustive introduction to the representation theory of semi-simple groups, with specific focus on $SL_2(\mathbb{C})$ and its subgroup. It requires nevertheless to have followed a first semester course on Lie groups and algebras.

[Hal03] Brian C. Hall, *Lie Groups, Lie Algebras, and Representations*, Springer, 2003.

This book provides a beautifully written introduction for physicists.

[BLR12] D. Bernard, Y. Laszlo and D. Renard, *Éléments de théorie des groupes et symétries quantiques*, cours de l'École polytechnique, 2012.

This very pedagogical introduction to groups gets inspiration from physics. It is taking on a wide series of subjects in a concise manner. Unfortunately, there is only a French version.

Representation and recoupling of $SU(2)$

[SN11] J.J Sakurai and Jim Napolitano, *Modern Quantum Mechanics*, Addison-Wesley, 2011.

This classic book is an introduction to quantum mechanics. The chapter 3 deals with the theory of angular momentum. A numbers of basic formulas can be found there. Here, we have used it for the Euler angles decomposition. Many other classical textbooks cover the angular momentum with small (but interesting) variations like [Edm57], [CS59] and [YLV62].

[VMK87] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, *Quantum theory of angular momentum*, World Scientific, 1987.

As the title suggests, this book could be looked at as the bible for the quantum aspects of angular momentum. It is supposed to be exhaustive in terms of formulas. So it is not really the kind of book you read, but rather something like a directory when you need something specific and not very memorable.

[Mou83] John P. Moussouris, *Quantum Models of Space-Time based on Recoupling Theory*, PhD. thesis (Oxford), 1983.

This is a beautifully written PhD thesis by Moussouris, under the supervision of Roger Penrose. It deals notably with the recoupling theory of $SU(2)$, and its link to space-time. Unfortunately, the document is not easily accessible.

Representation of $SL_2(\mathbb{C})$

[Rüh70] W. Rühl, *The Lorentz Group and Harmonic Analysis*, W. A. Benjamin, Inc, 1970.

This old book was written by a physicist and is maybe too sloppy in the mathematical exposure. It is nevertheless a classic textbook with a lot of useful formulas. It focuses on the study of $SL_2(\mathbb{C})$ and $SL_2(\mathbb{R})$.

[GMS63] I. M. Gel'fand, R. A. Minlos and Z. Ya. Shapiro, *Representations of the rotation and Lorentz groups and their applications*, Pergamon Press, 1963.

This book proposes a self-contained presentation of the representations of the rotation and Lorentz groups. However, its rudimentary page layout makes it a bit hard to read. From that respect, the book of Naimark, one year later, is a better introduction (and is also probably more detailed in its content).

[Nai64] M. A. Naimark, *Linear Representations of the Lorentz Group*, Pergamon Press, 1964.

This book introduces the subject to physicists. It is well-written, very introductory in the beginning, complete on the subject and quite rigorous (through not reaching the usual purely mathematical standards). Unfortunately the formalism and the notation start getting old and sometimes look a bit clumsy, which make the reading a bit bumpy.

[GGV66] I. M. Gel'fand, M. I. Graev and N. Ya. Vilenkin, *Generalized Functions: Volume 5, Integral Geometry and Representation Theory*, Academic Press, 1966.

This book is the English translation of the Russian version, published in 1962. The chapters of interest for us are chapter III devoted to the representations of $SL_2(\mathbb{C})$ and chapter IV for its harmonic analysis.

Loop Quantum Gravity and Spin-Foams

[Rov04] C. Rovelli, *Quantum Gravity*, Cambridge University Press, 2004.

This major textbook is recommended for its insistence on underlying physical ideas. The mathematical formulas are also present but some of the tools of representation and recoupling theories are assumed to be already known.

[RV14] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity*, Cambridge University Press, 2014.

This book is a concise exposition of the covariant formulation of LQG, also known as the spin-foam formalism. It gathers all the main achievements of the theory. It can alternatively be used as a technical toolbox ready for use or as a general introduction that sketches the programme and the physical ideas upon which it relies. Nevertheless, the mathematics are not explained in details (through lots of formulas are found) and it is sometimes a bit sloppy with the mathematical accuracy.

Bibliography

- [ARRW70a] R. L. Anderson, R. Raczka, M. A. Rashid, and P. Winternitz. Clebsch-Gordan coefficients for the coupling of $SL(2, \mathbb{C})$ principal-series representations. *Journal of Mathematical Physics*, 11(3):1050–1058, 1970.
- [ARRW70b] R. L. Anderson, R. Raczka, M. A. Rashid, and P. Winternitz. Recursion and symmetry relations for the Clebsch-Gordan coefficients of the homogeneous Lorentz group. *Journal of Mathematical Physics*, 11(3):1059–1068, 1970.
- [Ati74] M. F. Atiyah. How research is carried out. *Bull. IMA*, 10:232–234, 1974.
- [Bae00] J C Baez. An Introduction to Spin Foam Models of Quantum Gravity and BF Theory. In H Gausterer, H Grosse, and L Pittner, editors, *Geometry and quantum physics*, volume 543 of *Lecture Notes in Physics*, pages 25–94, Berlin, Germany; New York, U.S.A., 2000. Springer.
- [Bar47] V. Bargmann. Irreducible Unitary Representations of the Lorentz Group. *Annals of Mathematics*, 48(3):568–640, 1947.
- [BDF⁺10] John W Barrett, R J Dowdall, Winston J Fairbairn, Frank Hellmann, and Roberto Pereira. Lorentzian spin foam amplitudes: graphical calculus and asymptotics. *Classical and Quantum Gravity*, 27(16):165009, aug 2010.
- [BLR12] D Bernard, Y Laszlo, and D Renard. *Éléments de théorie des groupes et symétries quantiques*. cours de l'École polytechnique, 2012.
- [CS59] E. U. Condon and G. H. Shortley. *The Theory of Atomic Spectra*. Cambridge at the University Press, Cambridge, U.K., 1959.
- [DN67] Vong Duc Dao and Van Hieu NGuyen. On the theory of unitary representations of the $SL(2, \mathbb{C})$ group. *Ann. Inst. Henri Poincaré*, VI(1):17–37, 1967.
- [Don18] Pietro Donà. Infrared divergences in the EPRL-FK spin foam model. *Classical and Quantum Gravity*, 35(17):175019, 2018.
- [DS10] Pietro Donà and Simone Speziale. Introductory lectures to loop quantum gravity. In Abdelhafid Bounames and Abnenacer Makhoulouf, editors, *TVC 79. Gravitation : théorie et expérience*. Hermann, 2010.
- [DS18] Pietro Donà and Giorgio Sarno. Numerical methods for EPRL spin foam transition amplitudes and Lorentzian recoupling theory. *General Relativity and Gravitation*, 50(10):1–24, 2018.
- [Edm57] A. R. Edmonds. *Angular Momentum in Quantum Mechanics*. Princeton University Press, Princeton, 1957.
- [ELPR08] Jonathan Engle, Etera Livine, Roberto Pereira, and Carlo Rovelli. LQG vertex with finite Immirzi parameter. *Nuclear Physics B*, B799(1-2):136–149, 2008.

- [FMSD] Marco Fanizza, Pierre Martin-Dussaud, and Simone Speziale. Asymptotics of $SL(2, \mathbb{C})$ tensor invariants. *in progress*.
- [FS10] Laurent Freidel and Simone Speziale. From twistors to twisted geometries. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, D82(8):84041, 2010.
- [GGV66] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin. *Generalized Functions: Volume 5, Integral Geometry and Representation Theory*. Academic Press, 1966.
- [GMS63] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro. *Representations of the rotation and Lorentz groups and their applications*. Pergamon Press, 1963.
- [GN47] I. M. Gel'fand and M. A. Naimark. Unitary representations of the Lorentz group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 11(5):411–504, 1947.
- [Hal03] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations*. Springer, 2003.
- [HC47] Harish-Chandra. Infinite Irreducible Representations of the Lorentz Group. *Proceedings of the Royal Society A*, 189(1018):372–401, 1947.
- [IRS91] Gerald Itzkowitz, Sheldon Rothman, and Helen Strassberg. A note on the real representations of $SU(2, \mathbb{C})$. *Journal of Pure and Applied Algebra*, 69(3):285–294, 1991.
- [Kna86] Anthony W. Knapp. *Representation Theory of Semisimple Groups*. Princeton University Press, 1986.
- [KVM78] G. A. Kerimov, Yi. A. Verdiev, and O N Mathematical. Clebsch-Gordan coefficients of the $SL(2, \mathbb{C})$ group. *Reports on Mathematical Physics*, 13(3), 1978.
- [LS16] Miklos Långvik and Simone Speziale. Twisted geometries, twistors, and conformal transformations. *Physical Review D*, 94(2):20, 2016.
- [LST12] Etera R. Livine, Simone Speziale, and Johannes Tambornino. Twistor Networks and Covariant Twisted Geometries. *Physical Review D*, 85(6):064002, 2012.
- [Mau97] Krzysztof Maurin. *The Riemann Legacy*. Kluwer Academic Publishers, 1997.
- [Mou83] John P. Moussouris. *Quantum Models of Space-Time based on Recoupling Theory*. PhD thesis, Oxford, 1983.
- [Nai54] M. A. Naimark. On linear representations of the proper Lorentz group. *Dokl. Akad. Nauk SSSR*, 97:969–972, 1954.
- [Nai64] M A Naimark. *Linear Representations of the Lorentz Group*. Pergamon Press, 1964.
- [Oec03] Robert Oeckl. A 'general boundary' formulation for quantum mechanics and quantum gravity. *Phys. Lett.*, B575:318–324, 2003.
- [Oec08] Robert Oeckl. General boundary quantum field theory: Foundations and probability interpretation. *Adv. Theor. Math. Phys.*, 12:319–352, 2008.
- [Oog92] Hiroshi Ooguri. Topological lattice models in four-dimensions. *Mod. Phys. Lett.*, A7:2799–2810, 1992.
- [Per13] Alejandro Perez. The Spin Foam Approach to Quantum Gravity. *Living Reviews in Relativity*, 16:3, 2013.
- [PR68] G Ponzano and Tullio Regge. Semiclassical limit of Racah coefficients. In F Bloch, editor, *Spectroscopy and group theoretical methods in Physics*, Amsterdam, 1968. North-Holland.

- [PR84] Roger Penrose and Wolfgang Rindler. *Spinors and Space-time, Volume 1*. Cambridge University Press, 1984.
- [PR86] Roger Penrose and Wolfgang Rindler. *Spinors and Space-time, Volume 2*. Cambridge University Press, 1986.
- [Ras03] M. A. Rashid. Boost matrix elements of the homogeneous Lorentz group. *Journal of Mathematical Physics*, 20(7):1514–1519, 2003.
- [Rov04] Carlo Rovelli. *Quantum Gravity*. Cambridge University Press, 2004.
- [Rüh70] W Rühl. *The Lorentz Group and Harmonic Analysis*. W. A. Benjamin, Inc, 1970.
- [RV14] Carlo Rovelli and Francesca Vidotto. *Covariant Loop Quantum Gravity*. Cambridge University Press, 2014.
- [SN11] J. J. Sakurai and Jim Napolitano. *Modern Quantum Mechanics*. Addison-Wesley, 2 edition, 2011.
- [Spe17] Simone Speziale. Boosting Wigner’s nj-symbols. *Journal of Mathematical Physics*, 58:032501, 2017.
- [SSS18] Giorgio Sarno, Simone Speziale, and Gabriele V. Stagno. 2-vertex Lorentzian spin foam amplitudes for dipole transitions. *General Relativity and Gravitation*, 50(4):43, apr 2018.
- [Thi01] Thomas Thiemann. Gauge field theory coherent states (GCS): I. General properties. *Classical and Quantum Gravity*, 18(11):2025–2064, 2001.
- [Thi07] T Thiemann. *Modern Canonical Quantum General Relativity*. Cambridge University Press, Cambridge, U.K., 2007.
- [TW01a] Thomas Thiemann and Oliver Winkler. Gauge field theory coherent states (GCS). II: Peakedness properties. *Class. Quant. Grav.*, 18:2561–2636, 2001.
- [TW01b] Thomas Thiemann and Oliver Winkler. Gauge field theory coherent states (GCS) III: Ehrenfest theorems. *Class. Quant. Grav.*, 18:4629–4682, 2001.
- [TW01c] Thomas Thiemann and Oliver Winkler. Gauge field theory coherent states (GCS). IV: Infinite tensor product and thermodynamical limit. *Class. Quant. Grav.*, 18:4997–5054, 2001.
- [VMK87] D A Varshalovich, A N Moskalev, and V K Khersonskii. *Quantum theory of angular momentum*. World Scientific, 1987.
- [YLV62] A. P. Yutsis, I. B. Levinson, and V. V. Vanagas. *Mathematical Apparatus of the Theory of Angular Momentum*. Israel Program for Scientific Translations, Jerusalem, Israel, 1962.