Fermion coupling to loop quantum gravity: canonical formulation.

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In the model of fermion field coupled to loop quantum gravity, we consider the Gauss and the Hamiltonian constraints. According to the explicit solutions to the Gauss constraint, the fermion spins and the gravitational spin networks intertwine with each other so that the fermion spins contribute to the volume of the spin network vertices. For the Hamiltonian constraint, the regularization and quantization procedures are presented in detail. By introducing an adapted vertex Hilbert space to remove the regulator, we propose a diffeomorphism covariant graph-changing Hamiltonian constraint operator. This operator shows how fermions move in the loop quantum gravity spacetime and simultaneously influences the background quantum geometry.

I. INTRODUCTION

The real physical world consists of spacetime and matter. According to general relativity (GR), *"spacetime tells matter how to move; matter tells spacetime how to curve"-John Archibald Wheeler*, which should also be carried out in the quantum theory. Loop quantum gravity (LQG) [1–5], as a background-independent and non-perturbative quantum gravity theory, sets a stage for incorporating matters into quantum spacetime. In [6–10], the Brown-Kuchar model of gravity coupled to dust as well as the Rovelli-Smolin model of gravity coupled to massless Klein-Gordon field is quantized. In [11, 12], a minimal coupling of fermions and Yang-Mills fields to covariant LQG dynamics is proposed. The quantum theory of spinor fields coupled to LQG is well understood [13–19] In [20], a systematic procedure to couple standard model to the canonical LQG is proposed and further developed in [16, 21, 22]. With the present paper, by employing the procedure in [20], we investigate the Gauss and the Hamiltonian constraint in the model of fermion field coupled to LQG. In particular, the Gauss constraint is solved explicitly and, the Hamiltonian constraint is regularized and quantized in detail by introducing the so-called vertex Hilbert space to remove the regulator.

In the classical model of gravity coupled to the fermion field. The action of the gravitational sector can be formulated optionally with the first-order formulation (see, e.g., [1, 23] for the Palatini-Host action) or the second-order formulation (see, e.g., [4, 24] for Hilbert-Einstein action). In the pure gravity case, these two formulations are equivalent to each other up to boundary terms, while for the case with the fermion field coupled, the equivalency is no longer valid. For the first-order formulation, the action is a functional of some $\mathrm{SL}(2,\mathbb{C})$ connection, and the fermion field will be coupled to it directly. As a consequence, the fermion field will result in an on-shell torsion term in the connection. However, for the second-order formulation, the fermion field will be coupled to the torsion-free spin connection compatible with the tetrad. Thus, there is no torsion involved in this formulation. In the current paper, even though we adopt the second-order formulation for discussion, the results for the first-order formulation can be obtained analogously. Moreover, since no extra field is introduced for deparametrization in our model, the dynamics will be governed by the Hamiltonian constraint $H[N]$ with lapse functions N rather than the physical Hamiltonian. Then a problem arises that the Hamiltonian constraint operator cannot be defined in the diffeomorphism invariant space. This problem will be solved by, for instance, the master constraint framework [25, 26] or the deparametrization framework [9, 10]. In these frameworks, one finally needs to consider the Hamiltonian constraint operator with a constant lapse function or some dynamical lapse function such as the volume. These operators can be constructed directly with the Hamiltonian constraint operators $\widehat{H[N]}$. In other words, without loss of generality, one can only focus on how to define well the Hamiltonian constraint operators $\widehat{H[N]}$ which is a main task of the current work.

The phase space of the fermion field coupled to gravity is composed of fields $(A_a^i, E_j^b, \Psi, \Pi)$ on the spatial manifold Σ, where A_a^i is an SU(2) connection, E_j^b, the canonical conjugate to A_a^i, is a densitized triad field, Ψ denotes the fermion field, and Π is the canonical momentum conjugate to Ψ [1, 27, 28]. With the variables (A_a^i, E_j^b), the Hilbert space of the gravitational sector in LQG is constructed with the spin networks [29–31]. Thereon, one quantizes a family of operators representing the geometric observables, such as the 3-volume, 2-surface area, inverse metric, etc [31–36]. The spectra of these operators take discrete values, which implies the fundamental discreteness of the spacetime. The purely gravitational Hamiltonian constraint is regularized and promoted to an operator in the kinematic Hilbert space by [25, 26, 37–45]. The Hamiltonian constraint comprises of the curvature of the connection A. The curvature,

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as an operator, will attach loops on graphs of the spin network states. Based on how to attach loops, the quantum Hamiltonian operators in literature can be roughly classified into the following two categories: (i) graph preserving, (ii) graph changing. The graph preserving action is naturally from the lattice discretization point of view, while a continuum field theory approach leads directly to the second category, i.e., the graph changing action. Several proposals on the graph changing Hamiltonian operator are considered in works [2, 4, 8, 40, 41, 46, 47]. A helpful concept learned from these works is the vertex Hilbert space. A vertex Hilbert space is a Hilbert space group averaged with diffeomorphisms preserving some specific vertices. In other words, elements in a vertex Hilbert space are partially diffeomorphism invariant. The vertex Hilbert spaces are necessarily introduced to remove regulators and define limit [40, 46, 47]. Indeed, a graph-changing Hamiltonian operator is usually defined as the limit of some regularized Hamiltonian operators as the regulator approaches 0.

For the fermion sector, Π is related to Ψ by $\Pi=\sqrt{q}\,\Psi^\dagger$ with $q\equiv|\det(E)|$. In the quantum theory, this equation is expected to be realized in an appropriate form. In other words, one might require that the adjoint of the operator $\hat{\Pi}$ is related to the operator $\hat{\Psi}$ via $\hat{\Pi}^\dagger=\widehat{\sqrt{q}}\hat{\Psi}$. Then, considering a non-trivial real-valued function $f(A)$ of the connection A_a^i, one has, on one hand, $0=[\hat{\Pi},\widehat{f(A)}]$, but, on the other hand, $[\hat{\Pi},\widehat{f(A)}]^\dagger=[\widehat{f(A)},\hat{\Pi}^\dagger]=[\widehat{f(A)},\widehat{\sqrt{q}}\hat{\Psi}]\neq 0$. The two results do not coincide. To overcome this inconsistency, the author in [20] introduces the Grassman-valued half-densities $\tilde{\Psi}:=\sqrt[4]{q}\Psi$ and $\tilde{\Psi}^\dagger$ to coordinatize fermion sector of the phase space. Moreover, in order to do quantization, the smeared version of $\tilde{\Psi}$ by $\sqrt{\delta(x,y)}$, i.e., $\int \mathrm{d}^3y\sqrt{\delta(x,y)}\tilde{\Psi}(y)$, is also introduced by [20]. With the smeared variables, the fermion sector is quantized and the resulting quantum theory carries out the dieffeomorphsm-invariance feature.

This paper is organized as follows. In Sec. II the classical theory of gravity coupled to fermion field is introduced briefly. In Sec. III we introduce some basic notations of the kinematical Hilbert space of vacuum gravity and revisit the construction of the fermion kinematical Hilbert space. In Sec. IV, the Gauss constraint and the Hamiltonian constraint are regularized and quantized, where the adapted vertex Hilbert space is introduced and some physical results are discussed. Finally, in V, we summarize the remarkable results and propose some outlooks for further works.

II. CLASSICAL THEORY OF GRAVITY COUPLED TO FERMION

Let $\mathcal{M}$ denote the spacetime manifold, which is homeomorphism to $\mathbb{R}\times\Sigma$ with Σ being the spatial manifold. Given a 4-dimensional vector space V, let η_{IJ} be the Minkowski metric on it. A tetrad field e^I_μ gives the metric $g_{\mu\nu}=\eta_{IJ}e^I_\mu e^J_\nu$ on $\mathcal{M}$. The curvature of $g_{\mu\nu}$ defines the Einstein-Hilbert action in terms of the tetrad fields,

$$S_H[e]=\frac{1}{2\kappa}\int_{\mathcal{M}}\mathrm{d}^4xR[e]. \tag{2.1}$$

where $\kappa=8\pi G$. Let $\Gamma^I_{\mu J}$ denote the spin connection compatible with the tetrad so that

$$\mathrm{d}e^I+\Gamma^I{}_J\wedge e^J=0. \tag{2.2}$$

The model of gravity coupled to the fermion field is described by the action

$$S[e,\Psi]=S_H[e]-\frac{i}{2}\int_{\mathcal{M}}\mathrm{d}^4xe\,\left(\overline{\Psi}\gamma^Ie^\mu_I\nabla_\mu\Psi-c.c\right) \tag{2.3}$$

where γ^I denotes the gamma matrices satisfying $\gamma^I\gamma^J+\gamma^J\gamma^I=2\eta^{IJ}\mathbb{1}$, and the covariant derivative of Ψ is

$$\nabla_\mu\Psi=\partial_\mu\Psi-\frac{1}{4}\Gamma^I_{\mu J}\gamma^I\gamma^J\Psi. \tag{2.4}$$

Performing the 3+1-decomposition, we get the gravitational canonical pair (A^i_a,E^a_i). The Poisson brackets between them are

$$\{A^i_a(x),E^b_j(y)\}=\kappa\beta\delta^b_a\delta^i_j\delta(x,y), \tag{2.5}$$

where β is the Barbero-Immirzi parameter. For the fermion field, we split the Dirac fermion Ψ into its chiral components and follow the argument in [20] to introduce the half densities on Σ,

$$\xi:=\sqrt[4]{q}\Psi_-,\ \nu=\sqrt[4]{q}\Psi_+ \tag{2.6}$$

with $\Psi_\pm = \frac{1\pm\gamma^5}{2}\Psi$. Detailed Hamiltonian analysis (see Appendix A) tells that the conjugate moments to ξ and ν are their complex conjugates, and the anti-Poisson brackets are

$$\begin{aligned}\{\xi_A(x), \xi_B^\dagger(y)\}_+ &= -i\delta_{AB}\delta(x,y),\\ \{\nu_A(x), \nu_B^\dagger(y)\}_+ &= -i\delta_{AB}\delta(x,y).\end{aligned} \tag{2.7}$$

for all $A, B = \pm 1/2$.

The dynamics of this model is encoded in the Gauss constraint G_m, the diffeomorphism constraint H_a and the Hamiltonian constraint H, which are

$$\begin{aligned}G_m =& \frac{1}{\kappa\beta} D_a E_l^a + \frac{1}{2}(\xi^\dagger\sigma_m\xi + \nu^\dagger\sigma_m\nu),\\ H_a =& \frac{1}{\kappa\beta} E_i^b F_{ab}^i + \frac{i}{2}\Big\{\xi^\dagger D_a\xi - (D_a\xi)^\dagger\xi + \nu^\dagger D_a\nu - (D_a\nu)^\dagger\nu\Big\} + \beta K_a^m G_m,\\ H =& H_G + \frac{1}{\sqrt{q}}\Big[i(\xi^\dagger E_i^a\sigma^i D_a\xi - (D_a\xi)^\dagger E_i^a\sigma^i\xi) - \beta E_i^a K_a^i\xi^\dagger\xi - \frac{1}{\beta}(1+\beta^2)D_a E_i^a\xi^\dagger\sigma^i\xi - \beta E_i^a D_a\Big(\xi^\dagger\sigma^i\xi\Big)\\ &- i(\nu^\dagger E_i^a\sigma^i D_a\nu - (D_a\nu)^\dagger E_i^a\sigma^i\nu) + \beta E_i^a K_a^i\nu^\dagger\nu - \frac{1}{\beta}(1+\beta^2)D_a E_i^a\nu^\dagger\sigma^i\nu - \beta\frac{1}{\sqrt{q}} E_i^a D_a\Big(\nu^\dagger\sigma^i\nu\Big)\Big].\end{aligned} \tag{2.8}$$

Here, H_G denotes the scalar constraint of pure gravity,

$$H_G = \frac{1}{\kappa\sqrt{q}} E_i^a E_j^b \left(F_{ab}^m \epsilon_m{}^{ij} - 2(1+\beta^2)K_{[a}^i K_{b]}^j\right). \tag{2.9}$$

III. LOOP QUANTIZATION OF THE THEORY: KINEMATICS

A. the kinematical Hilbert space of vacuum gravity

In LQG, besides a fixed differentiability class C^m with $m \geq 1$, a semianalytic structure on Σ is also necessary [48].Then all local maps, differmorphisms, submanifold and function thereon are assumed to be C^m and semianalytic. Particularly, an edge is a semianalytic curve embedded in Σ. A graph is a collection of edges $\{e_1, \cdots, e_n\}$ where these e_k intersect each other at most at the ending points. Given a graph $\gamma \subset \Sigma$, let $E(\gamma)$ denote the set of its edges and $V(\gamma)$, its vertices. The number of elements in $E(\gamma)$ is denoted by $|E(\gamma)|$. A cylindrical function Ψ of the Ashtekar connection A is a function that can be written in the form

$$\Psi(A) = \psi_\gamma(h_{e_1}(A), \cdots, h_{e_n}(A)) \tag{3.1}$$

where $\psi_\gamma : \mathrm{SU}(2)^{|E(\gamma)|} \to \mathbb{C}$ is a complex function on $\mathrm{SU}(2)^{|E(\gamma)|}$ and $h_e(A) \in \mathrm{SU}(2)$ is the parallel transport along an edge e with respect to a given connection A,

$$h_e(A) = \mathcal{P}\exp\left(-\int_e A\right). \tag{3.2}$$

Given a cylindrical function Ψ with respect to a graph γ, it can always be expressed via another graph $\gamma' \supset \gamma$. Therefore, for two cylindrical functions $\Psi^{(1)}$ and $\Psi^{(2)}$ with respect to graphs γ_1 and γ_2 respectively, one can always find another graph γ_3 with $\gamma_3 \supset \gamma_1, \gamma_3 \supset \gamma_2$ and express $\Psi^{(i)}$ for $i = 1, 2$ by functions $\psi_{\gamma_3}^{(i)}$ on $\mathrm{SU}(2)^{|E(\gamma_3)|}$. Then the inner product of $\Psi^{(1)}$ and $\Psi^{(2)}$ is

$$\langle\Psi^{(1)}|\Psi^{(2)}\rangle = \int_{\mathrm{SU}(2)^n} \mathrm{d}\mu_H(g)\overline{\psi_{\gamma_3}^{(1)}(g_1, \cdots, g_n)}\psi_{\gamma_3}^{(2)}(g_1, \cdots, g_n) \tag{3.3}$$

where $n = |E(\gamma_3)|$ and $\mathrm{d}\mu_H$ is the Haar measure on $\mathrm{SU}(2)^n$. (3.3) defines the Ashtekar-Lewansowski measure $\mathrm{d}\mu_{\mathrm{AL}}$ on the quantum configuration space $\mathcal{A}$. Thus (3.3) is always rewritten as

$$\langle\Psi^{(1)}|\Psi^{(2)}\rangle = \int_{\mathcal{A}} \mathrm{d}\mu_{\mathrm{AL}}(A)\,\overline{\Psi^{(1)}(A)}\Psi^{(2)}(A) \tag{3.4}$$

The space of cylindrical function is denoted by **Cyl**. The Hilbert space $\mathcal{H}_G$ of the vacuum gravity is the completion of **Cyl** with the inner product define in (3.4).

Given a graph γ, the space of cylindrical functions with respect to γ is denoted by $\mathbf{Cyl}_\gamma$. The Cauchy completion of $\mathbf{Cyl}_\gamma$ with respect to (3.3) is denoted by $\mathcal{H}_{G,\gamma}$. Then according to (3.1), $\mathcal{H}_{G,\gamma}$ is naturally identical to $L^2(\mathrm{SU}(2),\mathrm{d}\mu_H)^{|E(\gamma)|}$ where the isometric map is

$$\mathbf{i}: L^2(\mathrm{SU}(2),\mathrm{d}\mu_H)^{|E(\gamma)|}\to\mathcal{H}_{G,\gamma}.$$

Moreover, $L^2(\mathrm{SU}(2),\mathrm{d}\mu_H)$ can be uniquely decomposed as

$$L^2(\mathrm{SU}(2),\mathrm{d}\mu_H)=L'^2(\mathrm{SU}(2),\mathrm{d}\mu_H)\oplus\mathbb{C}. \tag{3.5}$$

Then a subspace $\mathcal{H}^{\mathrm{irr}}_{G,\gamma}\subset\mathcal{H}_{G,\gamma}$ is given by the image of $L'^2(\mathrm{SU}(2),\mathrm{d}\mu_H)^{|E(\gamma)|}$ under $\mathbf{i}$, i.e.,

$$\mathcal{H}^{\mathrm{irr}}_{G,\gamma}=\mathbf{i}(L'^2(\mathrm{SU}(2),\mathrm{d}\mu_H)^{|E(\gamma)|}). \tag{3.6}$$

With $\mathcal{H}^{\mathrm{irr}}_{G,\gamma}$, **Cyl** can be decomposed as[1]

$$\mathbf{Cyl}=\bigoplus_\gamma\mathcal{H}^{\mathrm{irr}}_{G,\gamma}\oplus\mathbb{C}. \tag{3.7}$$

Given a cylindrical function Ψ with

$$\Psi(A)=\psi_\gamma(h_{e_1}(A),\cdots,h_{e_n}(A)),$$

the action of $D^\iota_{ab}(h_e)$ on Ψ is

$$(D^\iota_{ab}(h_e)\Psi)(A)=D^\iota_{ab}(h_e(A))\psi_\gamma(h_{e_1}(A),\cdots,h_{e_n}(A)), \tag{3.8}$$

$D^\iota_{ab}(h_e(A))$ denotes the Wigner-D matrix of $h_e(A)\in\mathrm{SU}(2)$. Another operators are the derivative operators denoted by $\hat{J}^{v,e}_i$ with $i=1,2,3$, where v is either the source point of the edge e, i.e., $v=s_e$, or the target point of e, i.e. $v=t_e$. For a cylindrical function Ψ, one can always express it with a graph γ satisfying $e\in\gamma$. Then the action of $\hat{J}^{v,e}_i$ on Ψ reads

$$(\hat{J}^{v,e}_j\Psi)(A)=\begin{cases} i\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\psi_\gamma(h_{e_1},\cdots,h_e e^{t\tau_j},\cdots,h_{e_n}),\ v=s_e,\\ i\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\psi_\gamma(h_{e_1},\cdots,e^{-t\tau_j}h_e,\cdots,h_{e_n}),\ v=t_e,\end{cases} \tag{3.9}$$

where $\tau_j=-i\sigma_j/2$ with σ_j being the Pauli matrices. With the operator $\hat{J}^{v,e}_j$, we can define an operator $\hat{J}^{x,[e]}_j$ as

$$\hat{J}^{x,[e]}_j\Psi=\sum_{e'\in[e]}\hat{J}^{x,e'}_j\Psi, \tag{3.10}$$

where $[e]$ is a maximal family of curves beginning at $x\in\Sigma$ such that each two curves overlap on a connected initial segment containing x.

B. loop quantization of the fermion field

From now on, we will only focus on the single Weyl component ξ. However, everything works similarly for the other chiral component ν. To quantize the fermion field, we follow [20] to use the modified symplectic structure

$$\{\theta_A(x),\theta^\dagger_B(y)\}_+=-i\delta_{AB}\delta_{x,y},\ A,B=\pm\frac{1}{2} \tag{3.11}$$

[1] For elements in an infinite direct sum, we require that all but finitely many components are zero.

where, with comparison to (2.7), the Dirac delta $\delta(x,y)$ is changed to the Kronecker delta $\delta_{x,y}$. This change indicates a canonical transformation from ξ to θ,

$$\begin{aligned}\theta(x) &= \int_\Sigma \mathrm{d}^3y\sqrt{\delta(x,y)}\xi(y),\\ \xi(x) &= \sum_{y\in\Sigma}\sqrt{\delta(x,y)}\theta(y).\end{aligned} \tag{3.12}$$

To prove the relation (3.12), one used the function $f_\epsilon(x,y) := \chi_\epsilon(x,y)/\epsilon^3$ to regularize the Dirac delta function [20], where χ_ϵ is

$$\chi_\epsilon(x,y) := \begin{cases} 1, \ \sum_{a=1}^{3}(x^a)^2+(y^a)^2 \le \left(\frac{\epsilon}{2}\right)^2, \\ 0, \text{ otherwise.}\end{cases}$$

According to (3.12),$\xi(x)$ will be singular for regular $\theta(x)$. This scenario contradicts that $\xi(x)$ is smooth as a classical field. To have a consistent understanding of this formulation, one interprets the singular fields $\xi(x)$ as defining the quantum configuration space of the fermion field so that $\theta(x)$ is a regular-field coordinate of this quantum configuration space.

As in vacuum LQG, the quantization starts by introducing the cylindrical functions. Before doing so, we will first introduce a convenient field ζ_A

$$\zeta_A(x) = \frac{1}{\sqrt{\hbar}}\theta_A(x), \ A = \pm\frac{1}{2}. \tag{3.13}$$

1. *the cylindrical functions of the fermion field*

A fermionic graph γ_F is a finite subset of Σ with $|\gamma_F|$ elements. Elements in γ_F are called (fermion) vertices. An orientation of γ_F is a surjection $n\mapsto v_F^{(n)}\in\gamma_F$ with $1\le n\le|\gamma_F|$. The surjection endows the elements in γ_F with an order. Given an oriented graph γ_F, we have a family of Grassmann numbers $\{\zeta^\dagger_{\pm\frac{1}{2}}(v_F)\}_{v_F\in\gamma_F}$, which will be renamed to $\rho_n^\dagger$ with $1\le n\le 2|\gamma_F|$ as

$$\rho^\dagger_{2i-1}\equiv\zeta^\dagger_{\frac{1}{2}}(v_F^{(i)}), \ \rho^\dagger_{2i}\equiv\zeta^\dagger_{-\frac{1}{2}}(v_F^{(i)}), \ 1\le i\le|\gamma_F|. \tag{3.14}$$

Then a function Ψ of $\zeta^\dagger$ with respect to γ_F takes the general form

$$\Psi(\zeta^\dagger) = f_0 + \sum_{n=1}^{2|\gamma_F|}\sum_{1\le i_1<i_2<\cdots<i_n\le N} f_{i_1\cdots i_n}\rho^\dagger_{i_1}\rho^\dagger_{i_2}\cdots\rho^\dagger_{i_n}, \tag{3.15}$$

where f_0 and $f_{i_1\cdots i_n}$ are complex numbers. Functions taking the form (3.15) are called the cylindrical functions of $\zeta^\dagger$ with respect to the graph γ_F. The space of cylindrical functions of $\zeta^\dagger$ will be denoted by $\mathbf{Cyl_F}$.

According to (3.15), each cylindrical function with respect to γ_F can be identical with a vector

$$f_{\gamma_F} = (f_0, \{f_{i_1\cdots i_n}\}_{1\le i_1<i_2<\cdots<i_n\le N})\in\mathbb{C}^{2^{2|\gamma_F|}}.$$

Moreover, for a cylindrical function Ψ expressed via $f_{\gamma_F}\in\mathbb{C}^{2^{2|\gamma_F|}}$, there always exists a lager graph $\gamma'_F\supset\gamma_F$ such that Ψ can also be expressed via $f'_{\gamma'_F}\in\mathbb{C}^{2^{2|\gamma'_F|}}$. Thus, given two functions $\Psi_1,\Psi_2\in\mathbf{Cyl_F}$ expressed with respect to γ_{F1} and γ_{F2} respectively, we can find another graph γ'_F containing both γ_{F1} and γ_{F2} to re-express Ψ_i for $i=1,2$ via $f^{(i)}_{\gamma'_F}\in\mathbb{C}^{2^{2|\gamma'_F|}}$. Then the inner product of Ψ_1 and Ψ_2 is defined as

$$\langle\Psi_1,\Psi_2\rangle = \int \mathrm{d}\mu(\rho_1\rho_1^\dagger)\mathrm{d}\mu(\rho_2\rho_2^\dagger)\cdots\mathrm{d}\mu(\rho_{2|\gamma'_F|}\rho^\dagger_{2|\gamma_F|'})\Psi_1^\dagger\Psi_2 \tag{3.16}$$

where

$$\mathrm{d}\mu(\zeta_n\zeta_n^\dagger) = \mathrm{d}\zeta_n^\dagger\mathrm{d}\zeta_n e^{\zeta_n\zeta_n^\dagger} \tag{3.17}$$

By (3.16), one can verify

$$\langle\Psi_1,\Psi_2\rangle=(f^{(1)}_{\gamma'_F})^\dagger f^{(2)}_{\gamma'_F}. \tag{3.18}$$

Moreover, even though the graphs γ'_F containing both γ_{F1} and γ_{F2} are not unique, Eq. (3.16) is independent of the choice of γ'_F since

$$\int \mathrm{d}\mu(\rho_n\rho_n^\dagger)=1. \tag{3.19}$$

Indeed, Eq. (3.16) defines a measure $\mathrm{d}\mu_H(\zeta^\dagger\zeta)$ on $\mathbf{Cyl_F}^\dagger\otimes\mathbf{Cyl_F}$ and will be rewritten as

$$\langle\Psi_1,\Psi_2\rangle=\int \mathrm{d}\mu_H(\zeta\zeta^\dagger)\Psi_1^\dagger\Psi_2. \tag{3.20}$$

The fermion Hilbert space $\mathcal{H}_F$ is the completion of $\mathbf{Cyl_F}$ with the inner product defined by (3.20), i.e.,

$$\mathcal{H}_F=\overline{\mathbf{Cyl_F}}. \tag{3.21}$$

On $\mathbf{Cyl_F}$, a type of operators are the multiplication operators $\widehat{\zeta^\dagger_{v_F,A}}$,

$$(\widehat{\zeta^\dagger_{v_F,A}}\Psi)(\zeta^\dagger)=\zeta^\dagger_A(v_F)\Psi(\zeta^\dagger),\ \forall\Psi\in\mathbf{Cyl_F}. \tag{3.22}$$

Another type of operators are the derivative operators $\hat{\zeta}_{v_F,A}$,

$$(\hat{\zeta}_{v_F,A}\Psi)(\zeta^\dagger)=\left(\frac{\partial}{\partial\zeta^\dagger_A(v_F)}\Psi\right)(\zeta^\dagger),\ \forall\Psi\in\mathbf{Cyl_F}. \tag{3.23}$$

It is easy to verify that $\widehat{\zeta^\dagger_{v_F,A}}$ and $\hat{\zeta}_{v_F,A}$ are adjoint to each other,

$$\widehat{\zeta^\dagger_{v_F,A}}=\hat{\zeta}^\dagger_{v_F,A}, \tag{3.24}$$

which realizes the real condition, and

$$[\zeta_{v_F,A},\zeta^\dagger_{v'_F,B}]_+=\zeta_{v_F,A},\zeta^\dagger_{v'_F,B}+\zeta^\dagger_{v'_F,B}\zeta_{v_F,A}=\delta_{AB}\delta_{v_Fv'_F}, \tag{3.25}$$

which implements the Poisson brackets (3.11) by defining

$$\hat{\theta}_A(v_F)=\sqrt{\hbar}\,\hat{\zeta}_{v_F,A}. \tag{3.26}$$

2. the spin network states of fermion field

Given a graph γ_F, the space of cylindrical functions with respect to γ_F is a finite-dimensional Hilbert space, denoted by $\mathcal{H}_{\gamma_F}$. Considering a graph $\gamma_F=\{v_F\}$ comprised of a single vertex, one has the the space $\mathcal{H}_{\{v_F\}}\equiv\mathcal{H}_{v_F}$ consisting of functions

$$\Psi(\zeta^\dagger)=a_{00}+a_{10}\zeta^\dagger_{\frac{1}{2}}(v_F)+a_{01}\zeta^\dagger_{-\frac{1}{2}}(v_F)+a_{11}\zeta^\dagger_{\frac{1}{2}}(v_F)\zeta^\dagger_{-\frac{1}{2}}(v_F) \tag{3.27}$$

The inner product of $\Psi^{(i)}(\zeta^\dagger)=a^{(i)}_{00}+a^{(i)}_{10}\zeta^\dagger_{\frac{1}{2}}(v_F)+a^{(i)}_{01}\zeta^\dagger_{-\frac{1}{2}}(v_F)+a^{(i)}_{11}\zeta^\dagger_{\frac{1}{2}}(v_F)\zeta^\dagger_{-\frac{1}{2}}(v_F)$ with $i=1,2$ is

$$\langle\Psi^{(1)},\Psi^{(2)}\rangle=\sum_{i,j\in\{0,1\}}(a^{(1)}_{ij})^*a^{(2)}_{ij} \tag{3.28}$$

For convenience, we introduce the Dirac bra-ket notation $|i,j\rangle_{v_F}$ to denote the state Ψ_{ij} with $i,j\in\{0,1\}$, where Ψ_{ij} are defined by $\Psi_{00}(\zeta^\dagger)=1$, $\Psi_{10}(\zeta^\dagger)=\zeta^\dagger_{\frac{1}{2}}(v_F)$, $\Psi_{01}(\zeta^\dagger)=\zeta^\dagger_{-\frac{1}{2}}(v_F)$ and $\Psi_{11}(\zeta^\dagger)=\zeta^\dagger_{\frac{1}{2}}(v_F)\zeta^\dagger_{-\frac{1}{2}}(v_F)$. Then the states $|i,j\rangle_{v_F}$ form an orthonormal basis of $\mathcal{H}_{v_F}$,

$${}_{v_F}\langle i_1,j_1|i_2,j_2\rangle_{v_F}=\delta_{i_1i_2}\delta_{j_1j_2}. \tag{3.29}$$

The action of $\hat{\zeta}_{v_F,A}$ and $\hat{\zeta}^\dagger_{v_F,A}$ for $A=\pm\frac{1}{2}$ on $\mathcal{H}_{v_F}$ is

$$\begin{array}{lll}\hat{\zeta}^\dagger_{v_F,\frac{1}{2}}|0,i_2\rangle_{v_F}=|1,i_2\rangle_{v_F}, & \hat{\zeta}^\dagger_{v_F,\frac{1}{2}}|1,i_2\rangle_{v_F}=0, & \forall i_2=0,1,\\ \hat{\zeta}_{v_F,\frac{1}{2}}|0,i_2\rangle_{v_F}=0, & \hat{\zeta}_{v_F,\frac{1}{2}}|1,i_2\rangle_{v_F}=|0,i_2\rangle_{v_F}, & \forall i_2=0,1,\\ \hat{\zeta}^\dagger_{v_F,-\frac{1}{2}}|i_1,0\rangle_{v_F}=(-1)^{i_1}|i_1,1\rangle_{v_F}, & \hat{\zeta}^\dagger_{v_F,-\frac{1}{2}}|i_1,1\rangle_{v_F}=0, & \forall i_1=0,1,\\ \hat{\zeta}_{v_F,-\frac{1}{2}}|i_1,0\rangle_{v_F}=0, & \hat{\zeta}_{v_F,-\frac{1}{2}}|i_1,1\rangle_{v_F}=(-1)^{i_1}|i_1,0\rangle_{v_F}, & \forall i_1=0,1.\end{array} \tag{3.30}$$

For the general case where $\gamma_F\subset\Sigma$ consists of more than one elements, we first associate to each $v_F\in\gamma_F$ the Hilbert space $\mathcal{H}_{v_F}$. Then with an orientation of γ_F, one has the tensor product space

$$\mathcal{H}_{F,\gamma_F}=\mathcal{H}_{v_F^{(1)}}\otimes\mathcal{H}_{v_F^{(2)}}\otimes\cdots\otimes\mathcal{H}_{v_F^{|\gamma_F|}}. \tag{3.31}$$

An orthonormal basis of $\mathcal{H}_{F,\gamma_F}$ consists of the vectors

$$|i_1,i_2,\cdots,i_{2|\gamma_F|}\rangle:=|i_1,i_2\rangle_{v_F^{(1)}}\otimes|i_3,i_4\rangle_{v_F^{(2)}}\otimes\cdots\otimes|i_{2|\gamma_F|-1},i_{2|\gamma_F|}\rangle_{v_F^{(\gamma_F)}} \tag{3.32}$$

with $i_k\in\{0,1\}$ for all $1\le k\le 2|\gamma_F|$. $|i_1,i_2,\cdots,i_{2|\gamma_F|}\rangle$ refers to the cylindrical function $\Psi_{\vec{i}}=\rho^\dagger_{i_1}\rho^\dagger_{i_2}\cdots\rho^\dagger_{i_n}$ with respect to γ_F. It is worth noting that the definition of $\mathcal{H}_{F,\gamma_F}$ depends on the orientation of γ_F since $\zeta_A(v_F)$ for all $A=\pm\frac{1}{2}$ and $v_F\in\gamma_F$ are Grassmann numbers. This fact can be illustrated more explicitly with the following examples. Consider another orientation $n\to\tilde{v}_F^{(n)}$ of γ_F such that

$$\tilde{v}_F^{(1)}=v_F^{(2)},\tilde{v}_F^{(2)}=v_F^{(1)},\tilde{v}_F^{(k)}=v_F^{(k)},\ \forall k\ge 3. \tag{3.33}$$

With respect to the new orientation, we have the tensor-product Hilbert space $\tilde{\mathcal{H}}_{F,\gamma_F}$ with the basis

$$\widetilde{|i_1,i_2,\cdots,i_{2|\gamma_F|}\rangle}:=|i_1,i_2\rangle_{\tilde{v}_F^{(1)}}\otimes|i_3,i_4\rangle_{\tilde{v}_F^{(2)}}\otimes\cdots\otimes|i_{2|\gamma_F|-1},i_{2|\gamma_F|}\rangle_{\tilde{v}_F^{(\gamma_F)}} \tag{3.34}$$

By definition, $\widetilde{|i_1,i_2,\cdots,i_{2|\gamma_F|}\rangle}$ referes to the cylindrical funciton

$$\tilde{\Psi}_{\vec{i}}=\rho^\dagger_{i_3}\rho^\dagger_{i_4}\rho^\dagger_{i_1}\rho^\dagger_{i_2}\cdots\rho^\dagger_{i_n}=(-1)^{(i_1+i_2)(i_3+i_4)}\Psi_{\vec{i}'}. \tag{3.35}$$

where $\vec{i}'=\{i_3,i_4,i_1,i_2,\cdots,i_{2|\gamma_F|}\}$. (3.35) implies the isometry between $\mathcal{H}_{F,\gamma_F}$ and $\tilde{\mathcal{H}}_{F,\gamma_F}$ defined by

$$\widetilde{|i_1,i_2,\cdots,i_{2|\gamma_F|}\rangle}=(-1)^{(i_1+i_2)(i_3+i_4)}|i_3,i_4,i_1,i_2,\cdots,i_{2|\gamma_F|}\rangle. \tag{3.36}$$

Indeed, the extra sign in (3.36) can be determined systematically by introducing the notions of graded objects. One can refer to Appendix B and the reference therein for more details on the notions of graded vector space and graded algebra. In our work, the Hilbert spaces $\mathcal{H}_{v_F}$ is graded. The degree $\mathfrak{d}(i_1,i_2)$ of $|i_1,i_2\rangle_{v_F}$ is

$$\mathfrak{d}(i_1,i_2)=i_1+i_2\mod 2. \tag{3.37}$$

The operator algebra on $\mathcal{H}_{v_F}$ is also graded. By definition, the degrees of the operators $\hat{\zeta}_{v_F,A}$ and $\hat{\zeta}^\dagger_{v_F,A}$ are

$$\mathfrak{d}(\hat{\zeta}_{v_F,A})=1=\mathfrak{d}(\hat{\zeta}^\dagger_{v_F,A}). \tag{3.38}$$

A general principle to deal with these graded objects is as follows. Whenever we swap two items, an additional sign appears by the rule

$$xy=(-1)^{\mathfrak{d}(x)\mathfrak{d}(y)}yx. \tag{3.39}$$

Following this rule and noting (3.32) and (3.34), we can obtain the identity (3.36) easily.

Given a vertex v_F, let $\mathcal{H}^{\rm irr}_{v_F}$ be the subspace of $\mathcal{H}_{v_F}$ spanned by $|0,1\rangle_{v_F}$, $|1,0\rangle_{v_F}$ and $|1,1\rangle_{v_F}$. Then the Hilbert space $\mathcal{H}^{\rm irr}_{F,\gamma_F}$ with respect to γ_F is defined as

$$\mathcal{H}^{\rm irr}_{F,\gamma_F}=\bigotimes_{v_F\in\gamma_F}\mathcal{H}^{\rm irr}_{v_F}. \tag{3.40}$$

With $\mathcal{H}^{\rm irr}_{v_F}$, $\mathbf{Cyl}_F$ can be decomposed as

$$\mathbf{Cyl}_F = \bigoplus_{\gamma_F} \mathcal{H}^{\rm irr}_{F,\gamma_F} \oplus \mathbb{C}. \tag{3.41}$$

The kinematical Hilbert space $\mathcal{H}$ of the entire system is the tensor product of $\mathcal{H}_G$ and $\mathcal{H}_F$,

$$\mathcal{H} = \overline{\mathcal{H}_G \otimes \mathcal{H}_F}. \tag{3.42}$$

A densely subspace $\mathbf{Cyl_{tot}}$ of $\mathcal{H}$ is

$$\mathbf{Cyl_{tot}} = \mathbf{Cyl} \otimes \mathbf{Cyl_F}. \tag{3.43}$$

The states in $\mathbf{Cyl_{tot}}$ will be called the cylindrical states. To obtain a cylindrical state, one needs a graph $\gamma = \gamma_G \cup \gamma_F$, where the gravitational graph γ_G is constituted of edges e and their ending points as vertices, and the fermionic graph γ_F contains only vertices v_F. To define a state with respect to γ, besides the data for a (gauge variant) LQG spin network state, one also needs to assign to each fermion vertex v_F a state $|i_1(v_F), i_2(v_F)\rangle_{v_F}$ with $i_1(v_F), i_2(v_F) \in \{0,1\}$. Given $\gamma = \gamma_G \cup \gamma_F$, the Hilbert space $\mathcal{H}^{\rm irr}_\gamma$ is defined as

$$\mathcal{H}^{\rm irr}_\gamma = \mathcal{H}^{\rm irr}_{G,\gamma_G} \otimes \mathcal{H}^{\rm irr}_{F,\gamma_F}. \tag{3.44}$$

Then we have

$$\mathbf{Cyl_{tot}} = \bigoplus_{\gamma} \mathcal{H}^{\rm irr}_{\gamma} \oplus \mathbb{C}. \tag{3.45}$$

In principle, a fermion vertex $v_F \in \Sigma$ can be located anywhere, regardless of the given gravitational graph γ_G. However, if v_F is chosen as point in $e \in E(\gamma_G)$ but $v_F \notin V(\gamma_G)$, we can always split e at v_F to define a new graph $\tilde\gamma_G$. Then v_F becomes a vertex of $\tilde\gamma_G$. Moreover, because of $\gamma_G \subset \tilde\gamma_G$, every cylindrical function with respect to γ_G can be re-expressed with respect to $\tilde\gamma_G$. Thus, it is sufficient to consider those graphs $\gamma = \gamma_G \cap \gamma_F$ where each fermion vertex v_F satisfies either $v_F \in V(\gamma_G)$ or $v_F \notin \gamma_G$. The collection of all vertices of γ is denoted by $V(\gamma)$.

IV. THE CONSTRAINT OPERATORS FOR GRAVITY COUPLED TO FERMION FIELD

A. the Gauss constraint

Classically, the Gauss constraint $G[\lambda]$ reads

$$G[\lambda] = \int_\Sigma {\rm d}^3x \lambda^m \left(\frac{1}{\kappa\beta} D_a E^a_m + \frac{1}{2}\xi^\dagger \sigma_m \xi \right). \tag{4.1}$$

It is straightfoward to quantize it as the operator

$$\widehat{G[\lambda]} = \sum_v \lambda^m(v) \hat G_{v,m} \tag{4.2}$$

with

$$\hat G_{v,m} = \hbar \sum_{[e]} \hat J^{v,[e]}_m + \hbar\, \hat\zeta^\dagger_{v,A} \frac{(\sigma_m)_{AB}}{2} \hat\zeta_{v,B} \tag{4.3}$$

Let us use $\hat{\mathcal{J}}_{v,m}$ to denote the second term in (4.3), namely

$$\hat{\mathcal{J}}_{v,m} = \hat\zeta^\dagger_{v,A} \frac{(\sigma_m)_{AB}}{2} \hat\zeta_{v,B}. \tag{4.4}$$

On the fermionic Hilbert space $\mathcal{H}_v$ at vertex v, the action of $\hat{\mathcal{J}}_{v,m}$ reads

$$\begin{aligned} &\hat{\mathcal{J}}_{v,m}|0,0\rangle_v = 0,\ \ \hat{\mathcal{J}}_{v,m}|1,1\rangle_v = 0 \\ &\hat{\mathcal{J}}_{v,m}\left(|1,0\rangle_v, |0,1\rangle_v\right) = \left(|1,0\rangle_v, |0,1\rangle_v\right)\frac{\sigma_m}{2}. \end{aligned} \tag{4.5}$$

According to (4.5), the operators $\hat{\mathcal{J}}_{v,m}$ for all $m = 1, 2, 3$ behave the same as the angular moment operators. Thus the operator $\hat{\mathcal{J}}_{v,m}$ generates a natural SU(2) action on $\mathcal{H}_v$. For $|\phi\rangle_v = \sum_{ij} \phi_{ij}|i, j\rangle_v$ and $u \in$ SU(2), the SU(2)-action on $|\phi\rangle_v$ is

$$u \triangleright |\phi\rangle_v = \big(|1, 0\rangle_v, |0, 1\rangle_v\big)\, u \begin{pmatrix}\phi_{10}\\ \phi_{01}\end{pmatrix} + \phi_{00}|0, 0\rangle_v + \phi_{11}|1, 1\rangle_v. \tag{4.6}$$

Thus, $\mathcal{H}_v$ is a reducible representation space of SU(2). More precisely, the 1-dimensional space spanned by either $|0, 0\rangle_v$ or $|1, 1\rangle_v$ is the trivial representation space, and the 2-dimensional space spanned by $|0, 1\rangle_v$ and $|1, 0\rangle_v$ is the 1/2-representation space where $|0, 1\rangle_v$ and $|1, 0\rangle_v$ serve as the standard basis according to (4.5). This fact implies the decomposition

$$\mathcal{H}_v \equiv \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{1/2}, \tag{4.7}$$

where $\mathcal{H}_j$ denote the j-representation space of SU(2).

For a graph $\gamma = \gamma_G \cup \gamma_F$, a spin j_e is assigned to the edge $e \subset \gamma_G$. Then at each vertex $v \in V(\gamma)$ there is the Hilbert space

$$\mathcal{H}_v^{\rm tot} = \bigotimes_{e \text{ start from } v} \mathcal{H}_{j_e} \otimes \bigotimes_{e' \text{ target to } v} \mathcal{H}^*_{j_{e'}} \otimes \mathcal{H}_v \tag{4.8}$$

where $\mathcal{H}_j^*$ denotes the dual space of $\mathcal{H}_j$. On $\mathcal{H}_v^{\rm tot}$, the SU(2) action generates the Gauss constraint. Thus, the solution space to the Gauss constraint is

$$\mathcal{H}^{\rm Gau} = \bigotimes_{x \in V(\gamma)} {\rm Inv}\left(\mathcal{H}_v^{\rm tot}\right), \tag{4.9}$$

where ${\rm Inv}\left(\mathcal{H}_v^{\rm tot}\right) \subset \mathcal{H}_v^{\rm tot}$ is the SU(2)-invariant subspace. To see ${\rm Inv}(\mathcal{H}_v^{\rm tot})$ more precisely, let us assume all edges at v are outgoing without loss of generality. Then we have

$$\mathcal{H}_v^{\rm tot} = \bigotimes_{e \text{ at } v} \mathcal{H}_{j_e} \otimes \mathcal{H}_v. \tag{4.10}$$

Given an order of the edges at v, one can choose an orthonormal basis of $\bigotimes_{e \text{ at } v} \mathcal{H}_{j_e}$ composed of vectors $|k_2, k_3, \cdots, k_n, M\rangle$ satisfying

$$\begin{aligned}
\sum_{i=1}^{3} (L_i^{(l)})^2 |k_2, k_3, \cdots, k_n, M\rangle =& k_l(k_l + 1)|k_2, k_3, \cdots, k_n, M\rangle,\ \forall l = 2, \cdots, n\\
L_3^{(n)}|k_2, k_3, \cdots, k_n, M\rangle =& M|k_2, k_3, \cdots, k_n, M\rangle,\\
L_1^{(n)}|k_2, k_3, \cdots, k_n, M\rangle =& \sum_{s=\pm 1} \frac{1}{2}\sqrt{(k_n - sM)(k_n + sM + 1)}|k_2, k_3, \cdots, k_n, M + s\rangle\\
L_2^{(n)}|k_2, k_3, \cdots, k_n, M\rangle =& \sum_{s=\pm 1} \frac{-is}{2}\sqrt{(k_n - sM)(k_n + sM + 1)}|k_2, k_3, \cdots, k_n, M + s\rangle
\end{aligned} \tag{4.11}$$

Here $\hat{L}_i^{(l)}$ denotes $\hat{L}_i^{(l)} := \sum_{k=1}^{l} \hat{J}_i^{v,e_k}$. Let us define ${\rm Inv}\left(\mathcal{H}_G^{(v)}\right) \subset \bigotimes_{e \text{ at } v} \mathcal{H}_{j_e}$ as the subspace spanned by $|k_2, k_3, \cdots, k_{n-1}, 0, 0\rangle$ for all possible $k_2, k_3, \cdots, k_{n-1}$. One has

$$L_i^{(n)}|k_2, k_3, \cdots, k_{n-1}, 0, 0\rangle = 0,\ \forall i = 1, 2, 3. \tag{4.12}$$

Moreover, with the vectors $|k_2, k_3, \cdots, k_{n-1}, 1/2, M\rangle$, we define

$$|k_2, k_3, \cdots, k_{n-1}\rangle_{\rm tot} = \left|k_2, k_3, \cdots, k_{n-1}, \frac{1}{2}, \frac{1}{2}\right\rangle \otimes |0, 1\rangle_v - \left|k_2, k_3, \cdots, k_{n-1}, \frac{1}{2}, -\frac{1}{2}\right\rangle \otimes |1, 0\rangle_v. \tag{4.13}$$

Then one has

$$(L_i^{(n)} + \hat{\mathcal{J}}_{v,i})|k_2, k_3, \cdots, k_{n-1}\rangle_{\rm tot} = 0,\ \forall i = 1, 2, 3. \tag{4.14}$$

Let $\mathcal{H}_{\rm inv}\subset\mathcal{H}_v^{\rm tot}$ denote the subspace spanned by $|k_2,k_3,\cdots,k_{n-1}\rangle_{\rm tot}$ for all possible $k_2,k_3,\cdots,k_{n-1}$. Then $\mathrm{Inv}(\mathcal{H}_v^{\rm tot})$ can be decomposed as

$$\mathrm{Inv}(\mathcal{H}_v^{\rm tot})=\left(\mathrm{Inv}\left(\mathcal{H}_G^{(v)}\right)\otimes|0,0\rangle_v\right)\oplus\left(\mathrm{Inv}\left(\mathcal{H}_G^{(v)}\right)\otimes|1,1\rangle_v\right)\oplus\mathcal{H}_{\rm inv},\tag{4.15}$$

where $\mathrm{Inv}\left(\mathcal{H}_G^{(v)}\right)\otimes|i_1,i_2\rangle_v$ is the space composed of vectors $|\psi\rangle\otimes|i_1,i_2\rangle$ for all $|\psi\rangle\in\mathrm{Inv}\left(\mathcal{H}_G^{(v)}\right)$.

Let v be a n-valence gauge invariant fermion vertex, where the ith edge e_i is assigned to spin j_i. According to the decomposition (4.15), the gauge invariant Hilbert space $\mathrm{Inv}(\mathcal{H}_v^{\rm tot})$ contains a subspace $\mathcal{H}_{\rm inv}$, isometric to the gauge invariant Hilbert space of a $(n+1)$-valence vacuum-gravity vertex where the ith edge with $1\leq i\leq n$ is assigned to spin j_i and the $(n+1)$th, the spin 1/2. Then once we consider the volume operator at v, this extra spin 1/2 will also have contribution. According to that the extra spin 1/2 originates from the fermion filed, one gets an intuitive picture that fermion field contributes to the volume operator. Moreover, a n-valence vertex in vacuum-LQG is always regarded as a polyhedron whose faces are dual to the edges. The flux operators $\hat{J}_i^{v,e}$ associated to each edge e constitute the area vector of the dual face. Then the vacuum-LQG Gauss constraint is just the closer condition $\sum_e\hat{J}_i^{v,e}=0$ ensuring that the faces can form a closed polyhedron. Now, the fermion field is considered. Then the Gauss constraint (4.3) implies

$$\sum_e\hat{J}_i^{v,e}=-\hat{\mathcal{J}}_{v,i}\tag{4.16}$$

where the right hand side does not vanish in general. In other words, the faces dual to the edges could not give a closed polyhedron for states in $\mathcal{H}_{\rm inv}$. By (4.16), the area defect of this unclosed polyhedron can be filled by $\hat{\mathcal{J}}_{v,i}$, i.e. the fermion spin at the vertex (see [21] for more details on the fermion spin). A direct consequence of the above discussion is that the volume of a 3-valence vertex with fermion does not vanish any more for states in $\mathcal{H}_{\rm inv}$. Let j_i with $i=1,2,3$ be the spins on the edges. Then the states in $\mathcal{H}_{\rm inv}$ are spanned by $|k\rangle_{\rm tot}\equiv|k\rangle$ with $k=j_3\pm1/2$. The action of the operator $\hat{q}_{123}$ on $|k\rangle$ is

$$\begin{aligned}\langle k|\hat{q}_{123}|k+1\rangle=&\frac{-i}{4\sqrt{(2k+1)(2k+3)}}\sqrt{(j_1-j_2+k+1)(-j_1+j_2+k+1)(j_1+j_2-k)(j_1+j_2+k+2)}\\&\sqrt{(j_3-\frac{1}{2}+k+1)(-j_3+\frac{1}{2}+k+1)(j_3+\frac{1}{2}-k)(j_3+\frac{1}{2}+k+2)},\end{aligned}\tag{4.17}$$

where $\hat{q}_{123}$ is the operator proportional to the square of the volume operator [2]. Then we have

$$\begin{aligned}\left\langle j_3-\frac{1}{2}\middle|\hat{q}_{123}\middle|j_3+\frac{1}{2}\right\rangle=&i\frac{1}{16}\sqrt{(2j_1+2j_2-2j_3+1)(2j_1-2j_2+2j_3+1)}\\&\sqrt{(-2j_1+2j_2+2j_3+1)(2j_1+2j_2+2j_3+3)}.\end{aligned}\tag{4.18}$$

Since the associated Hilbert space is 2-dimensional, the whole Hilbert space is the eigenspace of the volume operator with eigenvalue

$$V_v=\frac{\kappa_0\ell_p^{3/2}}{16\sqrt{3}}\sqrt{\left|\left\langle j_3-\frac{1}{2}\middle|\hat{q}_{123}\middle|j_3+\frac{1}{2}\right\rangle\right|}.\tag{4.19}$$

B. The Hamiltonian constraint

As discussed in [16, 20], the smeared Hamiltonian constraint in terms of $\theta_A(x)$ reads

$$H_F[N]:=\sum_{x\in\Sigma}N(x)H_F(x)\tag{4.20}$$

where H_F is

$$H_F=i\frac{1}{\sqrt{q}}(\theta^\dagger E_i^a\sigma^iD_a\theta-(D_a\theta)^\dagger E_i^a\sigma^i\theta)-\beta\frac{1}{\sqrt{q}}E_i^aK_a^i\theta^\dagger\theta-\frac{1+\beta^2}{\beta}\frac{1}{\sqrt{q}}D_aE_i^a\theta^\dagger\sigma^i\theta-\beta\frac{1}{\sqrt{q}}E_i^aD_a\left(\theta^\dagger\sigma^i\theta\right).\tag{4.21}$$

Fix a coordinate system x^a on Σ and a positive number ϵ. Divide Σ into a family $\mathcal{C}_\epsilon$ of cells such that each cell $C\in\mathcal{C}_\epsilon$ is cubic with the coordinate volume less than ϵ^3 and different cells can only share points on their boundaries. Given a graph $\gamma=\gamma_G\cup\gamma_F$, for each cell $C\in\mathcal{C}_\epsilon$, let γ_C denote $\gamma\cap C$. Since the limit $\epsilon\to0$ will be considered eventually, we will assume that ϵ is small enough such that $\gamma_C\neq\emptyset$ is one of the following types:

(i) the type-A graph: γ_C is composed of a single edge;

(ii) the type-B graph: γ_C is composed of a single fermion vertex without connecting any edges;

(iii) the type-C graph: γ_C is composed of edges intersecting a single vertex.

For each cell $C \in \mathcal{C}_\epsilon$, let us define

$$\begin{aligned} H_C^{(1)} &= \int_C \mathrm{d}^3x N(x)\theta^\dagger(x)E_i^a(x)\sigma^i D_a\theta(x), \\ H_C^{(2)} &= \int_C \mathrm{d}^3x N(x)E_i^a(x)K_a^i(x)\theta^\dagger(x)\theta(x), \\ H_C^{(3)} &= \int_C \mathrm{d}^3x N(x)D_a E_i^a(x)\theta^\dagger(x)\sigma^i\theta(x), \end{aligned} \tag{4.22}$$

and introduce

$$H_{\mathcal{C}_\epsilon} = \sum_{C\in\mathcal{C}_\epsilon} \frac{1}{V_C}\left(i(H_C^{(1)} - H_C^{(1)\dagger}) - \beta H_C^{(2)} - \frac{1+\beta^2}{\beta}H_C^{(3)} - \beta(H_C^{(1)} + H_C^{(1)\dagger})\right), \tag{4.23}$$

where the volume V_C of C is

$$V_C = \int_C \mathrm{d}^3x\sqrt{|\det(E)|}.$$

Then $H_F[N]$ is the limit of $H_{\mathcal{C}_\epsilon}$ as $\epsilon \to 0$, i.e.

$$H_F[N] = \lim_{\epsilon\to 0} H_{\mathcal{C}_\epsilon}. \tag{4.24}$$

Thus, we need to quantize $H_{\mathcal{C}_\epsilon}$ to get an operator $\widehat{H_{\mathcal{C}_\epsilon}}$, and the operator of $H_F[N]$ is defined to be the limit of $\widehat{H_{\mathcal{C}_\epsilon}}$ as $\epsilon \to 0$. To this end, let us first quantize

$$\tilde{H}_C^{(i)} = \frac{1}{V_C}H_C^{(i)},\ i = 1,2,3. \tag{4.25}$$

for each $C \in \mathcal{C}_\epsilon$. We have

$$\widehat{\tilde{H}_C^{(i)}} = \sqrt{\widehat{V_C^{-1}}}\,\hat{H}_C^{(i)}\sqrt{\widehat{V_C^{-1}}}. \tag{4.26}$$

The operators $\widehat{\tilde{H}_C^{(i)}}$ act on states associated to the graphs $\gamma_C = \gamma \cap C$ aforementioned. Due to the operator $\sqrt{\widehat{V_C^{-1}}}$ at the most right, $\widehat{\tilde{H}_C^{(i)}}$ will annihilate the states with respect to the type-A and type-B graphs. Hence, only the states on type-C graphs is necessarily to be considered. From now on, γ_C will be used to specifically denote the type-C graphs until otherwise stated. The vertex where the edges in γ_C intersect will be denoted by v_C.

Let us begin with the operator $\widehat{\tilde{H}_C^{(1)}}$. Replacing $E_i^a(x)$ by $-i\kappa\hbar\beta\delta/\delta A_a^i(x)$ in $H_C^{(1)}$ in (4.22), we can quantize $H_C^{(1)}$ as

$$\hat{H}_C^{(1)} = -i\kappa\hbar\beta\int_C \mathrm{d}^3x N(x)\hat{\theta}^\dagger(x)\sigma^i\widehat{D_a\theta(x)}\frac{\delta}{\delta A_a^i(x)} \tag{4.27}$$

Given an edge $e : [0,\delta] \to C$ of γ_C with $e(0) = v_C$, let $U_e(t,0,A)$ denote the parallel transport from $e(0)$ to $e(t)$ along e. By definition, $U_e(t,0,A)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U_e(t,0;A) = -A_a(e(t))\dot{e}^a(t)U(t,0;A), \quad \text{and} \quad U(0,0;A) = I. \tag{4.28}$$

Defining $h_e := U(t,t;A)$, one has

$$\begin{aligned} &-i\kappa\hbar\beta\int_C \mathrm{d}^3x\, N(x)\hat{\theta}^\dagger(x)\sigma^i\widehat{D_a}\theta(x)\frac{\delta}{\delta A_a^i(x)}h_e \\ &= -\kappa\hbar\beta N(s_e)\hat{\theta}^\dagger(s_e)\sigma^i\left(h_e^{-1}\hat{\theta}(t_e) - \hat{\theta}(s_e)\right)\hat{J}_i^{v,e}h_e \end{aligned} \tag{4.29}$$

where we used

$$\delta \dot{e}^a(t_1) D_a \theta(e(t_1)) = U_e(t_1, t_1+\delta; A)\theta(e(t_1+\delta)) - \theta(e(t_1)) + O(\delta^2). \tag{4.30}$$

Hence, for a state Ψ_C with respect to γ_C, one has

$$\hat{H}_C^{(1)}\Psi_C = -\kappa\hbar\beta N(v_C) \sum_{e\in\gamma_C} \hat{\theta}^\dagger(v_C)\sigma^i \left(h_e^{-1}\hat{\theta}(t_e) - \hat{\theta}(v_C)\right) \hat{J}_i^{v_C,e}\Psi_C. \tag{4.31}$$

which gives $\widehat{\tilde{H}_C^{(1)}}$, acting on Ψ_C as

$$\widehat{\tilde{H}_C^{(1)}}\Psi_C = -\kappa\hbar\beta N(v_C) \sum_{e\in\gamma_C} \sqrt{\widehat{V_C^{-1}}}\,\hat{\theta}^\dagger(v_C)\sigma^i \left(h_e^{-1}\hat{\theta}(t_e) - \hat{\theta}(v_C)\right) \hat{J}_i^{v_C,e}\sqrt{\widehat{V_C^{-1}}}\Psi_C. \tag{4.32}$$

For the second term $\widehat{\tilde{H}_C^{(2)}}$, one has

$$\hat{H}_C^{(2)} = \int_C \mathrm{d}^3x N(x) E_i^a(x) K_a^i(x) \theta^\dagger(x)\theta(x). \tag{4.33}$$

Taking advantage of the Thiemann's trick to quantize pure gravity Hamiltonian constraint in LQG [4], one has

$$\int_C \mathrm{d}^3x f(x) E_i^a(x) K_a^i(x) = \frac{1}{2\kappa\beta^2}\left\{\int_C \mathrm{d}^3x f(x) H_E(x), V_C\right\}, \tag{4.34}$$

where $H_E(x)$ is the Euclidean part of the vacuum Hamiltonian constraint, i.e.

$$H_E(x) = \frac{\epsilon_{ijk}F_{ab}^i(x)E_j^a(x)E_k^b(x)}{\sqrt{q(x)}}.$$

Thus, $H_C^{(2)}$ is quantized as

$$\hat{H}_C^{(2)} = \frac{1}{2i\kappa\hbar\beta^2}\left[N(v_C)\hat{H}_{E,v_C}\hat{\theta}^\dagger(v_C)\hat{\theta}(v_C), \hat{V}_{v_C}\right] \tag{4.35}$$

where $\hat{H}_{E,v_C}$ denotes the Euclidean part of the vacuum Hamiltonian constraint operator at v_C and $\hat{V}_{v_C}$, the volume operator at v_C. Eq. (4.35) gives the operator $\widehat{\tilde{H}_C^{(2)}}$ as

$$\widehat{\tilde{H}_C^{(2)}} = \frac{1}{4i\kappa\hbar\beta^2} N(v_C)\sqrt{\widehat{V_{v_C}^{-1}}}\left(\hat{H}_{E,v_C}\hat{V}_{v_C} - \hat{V}_{v_C}\hat{H}_{E,v_C}\right)\hat{\theta}^\dagger(v_C)\hat{\theta}(v_C)\sqrt{\widehat{V_{v_C}^{-1}}}. \tag{4.36}$$

Finally, for the third term $\widehat{\tilde{H}_C^{(3)}}$, we have

$$\hat{H}_C^{(3)}\Psi_C = \kappa\hbar\beta N(v_C)\left(\sum_{e \text{ at } v_C} \hat{J}_i^{v_C,e}\right)\hat{\theta}^\dagger(v_C)\sigma^i\hat{\theta}(v_C)\Psi_C. \tag{4.37}$$

Thus $\widehat{\tilde{H}_C^{(3)}}$ reads

$$\widehat{\tilde{H}_C^{(3)}}\Psi_C = \kappa\hbar\beta N(v_C)\sqrt{\widehat{V_C^{-1}}}\left(\sum_{e \text{ at } v_C} \hat{J}_i^{v_C,e}\right)\hat{\theta}^\dagger(v_C)\sigma^i\hat{\theta}(v_C)\sqrt{\widehat{V_C^{-1}}}\Psi_C \tag{4.38}$$

With (4.32), (4.36) and (4.38), the operator $\widehat{H_{\mathcal{C}_\epsilon}}$ is defined as

$$\widehat{H_{\mathcal{C}_\epsilon}} = \sum_{C\in\mathcal{C}_\epsilon^{(3)}} i(\widehat{\tilde{H}_C^{(1)}} - \widehat{\tilde{H}_C^{(1)}}^\dagger) - \beta\widehat{\tilde{H}_C^{(2)}} - \frac{1+\beta^2}{\beta}\widehat{\tilde{H}_C^{(3)}} - \beta(\widehat{\tilde{H}_C^{(1)}} + \widehat{\tilde{H}_C^{(1)}}^\dagger) \tag{4.39}$$

where

$$\mathcal{C}_\epsilon^{(3)} = \{C \in \mathcal{C}_\epsilon, \gamma\cap C \text{ is type-C}\}.$$

Even though, $\widehat{H_{\mathcal{C}_\epsilon}}$ in (4.39) is defined with a partition structure $\mathcal{C}_\epsilon$ on Σ, this partition structure is indeed not necessary. In other words, one can define an operator equal to $\widehat{H_{\mathcal{C}_\epsilon}}$ with Σ endowed with another structure, more convenient for the further study. To this end, let us first introduce the following notations.

Definition IV.1 (removable vertex). *A vertex v of a graph γ is removable if it satisfies the following conditions.*

(i) v is a two-valence vertex connecting e_1 and e_2;

(ii) The composition of e_1 and e_2 as a curve is C^m and semianalytic.

Given a graph γ, one can obtain another graph $\ker(\gamma)$ by removing all of its removable vertices. $\ker(\gamma)$ will be called the kernel of γ. Let $\Gamma_{\ker}(\Sigma)$ be the collection of kernels of all graphs in Σ. Fix once and for all a parametrization for each $\gamma \in \Gamma_{\ker}$, where a parametrization of a graph is an assignment to each edge $e \in E(\gamma)$ a parametrization $[0,1] \ni t \to e(t) \in \Sigma$. Then give a graph $\gamma \in \Gamma_{\ker}$, for its edge e taking v as an endpoint, we can define $e(v,\delta) \subset e$ as the segment starting from v and ending at either $e(\delta)$ for $v = e(0)$ or $e(1-\delta)$ for $v = e(1)$. Then given a graph $\gamma = \gamma_G \cup \gamma_F$, for each $v \in V(\gamma_G)$, mimicking the operator $\widehat{\tilde{H}_C^{(i)}}$ in (4.32), (4.36) and (4.38), we define

$$\widehat{\tilde{H}^{(i)}(v;\vec{\delta}_v)} = \sqrt{\widehat{V_v^{-1}}}\,\widehat{H^{(i)}(v;\vec{\delta}_v)}\sqrt{\widehat{V_v^{-1}}},\ i = 1,2,3 \tag{4.40}$$

with

$$\begin{aligned}
\widehat{H^{(1)}(v;\vec{\delta}_v)} =& -\kappa\hbar\beta N(v)\sum_{e\text{ at }v}\left(\hat{\theta}^\dagger(v)\sigma^i h^{-1}_{e(v,\delta_{v,e})}\hat{\theta}(t_{e(v,\delta_{v,e})}) - \hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\right)\hat{J}_i^{v,e},\\
\widehat{H^{(2)}(v;\vec{\delta}_v)} =& \frac{1}{4i\kappa\hbar\beta^2}N(v)\left(\hat{H}_{E,v}\hat{V}_v - \hat{V}_v\hat{H}_{E,v}\right)\hat{\theta}^\dagger(v)\hat{\theta}(v),\\
\widehat{H^{(3)}(v;\vec{\delta}_v)} =& \kappa\hbar\beta N(v)\left(\sum_{e\text{ at }v}\hat{J}_i^{v,e}\right)\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v).
\end{aligned} \tag{4.41}$$

where $\vec{\delta}_v = \{\delta_{v,e}\}_{e\text{ at }v}$. With these operators, we define

$$\begin{aligned}
\widehat{H(\delta)} :=& \sum_{v\in V(\gamma_G)} i(\widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)} - \widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)}^\dagger) - \beta\widehat{\tilde{H}^{(2)}(v;\vec{\delta}_v)} - \frac{1+\beta^2}{\beta}\widehat{\tilde{H}^{(3)}(v;\vec{\delta}_v)} - \beta(\widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)} + \widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)}^\dagger)\\
=& \sum_{v\in V(\gamma)} i(\widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)} - \widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)}^\dagger) - \beta\widehat{\tilde{H}^{(2)}(v;\vec{\delta}_v)} - \frac{1+\beta^2}{\beta}\widehat{\tilde{H}^{(3)}(v;\vec{\delta}_v)} - \beta(\widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)} + \widehat{\tilde{H}^{(1)}(v;\vec{\delta}_v)}^\dagger)
\end{aligned} \tag{4.42}$$

where $\delta \equiv \{\vec{\delta}_v\}_{v\in V(\gamma)}$ and the second step is a consequence of the fact that $\widehat{\tilde{H}^{(i)}(v;\vec{\delta}_v)}$ with $v \notin \gamma_G$ vanish for all $i = 1,2,3$, due to the operator $\sqrt{\widehat{V_v^{-1}}}$ in $\widehat{\tilde{H}^{(i)}(v;\vec{\delta}_v)}$. Comparing the operators $\widehat{H(\delta)}$ and $\widehat{H_{\mathcal{C}_\epsilon}}$, one can verify easily that $\widehat{H(\delta)} = \widehat{H_{\mathcal{C}_\epsilon}}$ for some suitable δ. Hence, we can use $\widehat{H(\delta)}$ instead of $\widehat{H_{\mathcal{C}_\epsilon}}$ for our further study. In the following context, without loss of generality, we will assume $\delta_{v,e}$ =constant for convenience. The discussion for non-constant δ can be discussed similarly.

In the operator $\widehat{H(\delta)}$, the most interesting term is the operator

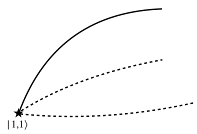
$$\hat{\mathfrak{h}}_1(\delta) := \hat{J}_i^{v,e}\hat{\theta}^\dagger(t_{e(v,\delta)})h_{e(v,\delta)}\sigma^i\hat{\theta}(v)$$

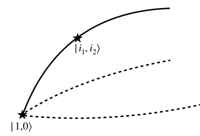
contained in $\widehat{H^{(1)}(v,\delta)}^\dagger$. By definition, one has

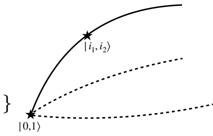
$$\begin{aligned}
&\hat{\mathfrak{h}}_1(\delta)\left(D^j_{mn}(h_{e(v,\delta)})\otimes|k_1,k_2\rangle_v\otimes|l_1,l_2\rangle_{t_{e(v,\delta)}}\right)\\
=&2W_jW_{\frac{1}{2}}\sum_{J=j\pm\frac{1}{2}}\sum_{A,C}(-1)^{m+A-n-C}\begin{bmatrix}J & j & \frac{1}{2}\\ \frac{1}{2} & 1 & J\end{bmatrix}\begin{pmatrix}J & j & \frac{1}{2}\\ -(m+A) & m & A\end{pmatrix}\begin{pmatrix}J & \frac{1}{2} & j\\ -(n+C) & C & n\end{pmatrix}\times\\
&D^J_{m+A,n+C}(h_{e(v,\delta)})\otimes\hat{\theta}_C(v)|k_1,k_2\rangle_v\otimes\hat{\theta}^\dagger_A(t_{e(v,\delta)})|l_1,l_2\rangle_{t_{e(v,\delta)}}
\end{aligned} \tag{4.43}$$

where $W_j = \sqrt{j(j+1)(2j+1)}$, $\begin{bmatrix}J & j & \frac{1}{2}\\ \frac{1}{2} & 1 & J\end{bmatrix}$ is the $6j$ symbol and $\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}$ denotes the $3j$ symbol. According to (4.43), the operator $\hat{\mathfrak{h}}_1(\delta)$ moves the fermion vertex at v to the vertex $t_{e(v,\delta)}$ in the edge e, and simultaneously change the spin on $e(v,\delta)$. Let us use $\star$ to represent a fermion vertex and a solid line to represent the edge e. Then, setting $|l_1l_2\rangle = |0,0\rangle$, we illustrate how the operator $\hat{\mathfrak{h}}_1(\delta)$ moves the fermion vertex at v as follows.

(1) For $k_1 = 1 = k_2$, one has

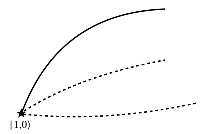
$\hat{\mathfrak{h}}_1(\delta)$ 

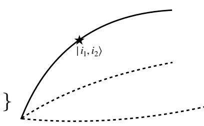
$\to \sum_{(i_1,i_2)\in\{(1,0),(0,1)\}}$ 

$+ \sum_{(i_1,i_2)\in\{(1,0),(0,1)\}}$ 

(4.44)

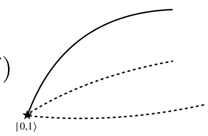
(2) For $k_1 = 1$ and $k_2 = 0$, one has

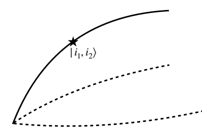
$\hat{\mathfrak{h}}_1(\delta)$ 

$\to \sum_{(i_1,i_2)\in\{(1,0),(0,1)\}}$ 

(4.45)

(3) For $k_1 = 0$ and $k_2 = 1$, one has

$\hat{\mathfrak{h}}_1(\delta)$ 

$\to \sum_{(i_1,i_2)\in\{(1,0),(0,1)\}}$ 

(4.46)

There are two problems on the operator $\hat{\mathfrak{h}}_1(\delta)$. At first, according to (4.44), (4.45) and (4.46), the limit of $\hat{\mathfrak{h}}_1(\delta)$ as $\delta \to 0$ does not exist. As a consequence, the limit of $\widehat{H(\delta)}$ as $\delta \to 0$ does not exist, too. Second, given a state $|\psi\rangle$, $\hat{\mathfrak{h}}_1(\delta)|\psi\rangle$ and $\hat{\mathfrak{h}}_1(\delta')|\psi\rangle$ for $\delta \neq \delta'$ are diffeomorphism equivalent. However, $\hat{\mathfrak{h}}_1(\delta)^\dagger(\hat{\mathfrak{h}}_1(\delta)|\psi\rangle)$ and $\hat{\mathfrak{h}}_1(\delta)^\dagger(\hat{\mathfrak{h}}_1(\delta')|\psi\rangle)$ could be no longer diffeomorphism equivalent because we could have $\hat{\mathfrak{h}}_1(\delta)^\dagger(\hat{\mathfrak{h}}_1(\delta')|\psi\rangle) = 0$ but $\hat{\mathfrak{h}}_1(\delta)^\dagger(\hat{\mathfrak{h}}_1(\delta)|\psi\rangle) \neq 0$ for some $|\psi\rangle$. Therefore, the operator $\hat{\mathfrak{h}}_1(\delta)^\dagger$ is not diffeomorphism covariant. As a result, $H(\delta)$ is not diffeomorphism covariant. A solution to this two problems is to introduce the vertex Hilbert space $\mathcal{H}_{\rm vtx}$ which is in the dual space of $\mathbf{Cyl_{tot}}$. The procedure to use $\mathcal{H}_{\rm vtx}$ to solve the two problems is sketched as follows. We split $\widehat{H(\delta)}$ into $\widehat{H(\delta)} = \widehat{H_1(\delta)} + \widehat{H_1(\delta)}^\dagger$ with

$$\widehat{H_1(\delta)} := \sum_{v\in V(\gamma)} -i\widehat{\tilde{H}^{(1)}(v;\vec{\delta})}^\dagger - \frac{\beta}{2}\widehat{\tilde{H}^{(2)}(v;\vec{\delta})} - \frac{1+\beta^2}{2\beta}\widehat{\tilde{H}^{(3)}(v;\vec{\delta})} - \beta\widehat{\tilde{H}^{(1)}(v;\vec{\delta})}^\dagger. \tag{4.47}$$

Since $\widehat{H_1(\delta)}$ contains $\hat{\mathfrak{h}}_1(\delta)$ rather than $\hat{\mathfrak{h}}_1(\delta)^\dagger$, $\widehat{H_1(\delta)}$ can be promoted to a diffeomorphism covariant operator in $\widehat{H_1(\delta)}^\star$ in $\mathcal{H}_{\rm vtx}$ by the duality such that $\widehat{H_1(\delta)}^\star = \widehat{H_1(\delta')}^\star$ for $\delta \neq \delta'$. Hence the limit of $\widehat{H_1(\delta)}^\star$ as $\delta \to 0$ exists. The limit will be denoted by $\hat{A}_F$ which is manifestly diffeomorphism covariant. Moreover, for the operator $\widehat{H_1(\delta)}^\dagger$, rather than promoting it to an operator in $\mathcal{H}_{\rm vtx}$ by the duality, we will just define $(\widehat{H_1(\delta)}^\star)^\dagger$ as its correspondence in $\mathcal{H}_{\rm vtx}$. Consequently, the operator referring to the limit of $\widehat{H(\delta)}$ as $\delta \to 0$ in $\mathcal{H}_{\rm vtx}$ is just $\hat{A}_F + \hat{A}_F^\dagger$.

1. the vertex Hilbert space $\mathcal{H}_{\rm vtx}$

Given a graph γ, we will consider the subgroup of C^m semianalytic diffeomorphisms which acts on $V(\ker(\gamma))$ trivially, i.e. which preserves every vertex of $\ker(\gamma)$. This subgroup will be denoted by $\mathrm{Diff}_{V(\ker(\gamma))}$. There are two subgroups of $\mathrm{Diff}_{V(\ker(\gamma))}$. The first one, denoted by Diff_γ, preserves γ. The other one, denoted by Diff^0_γ, preserves every edge of $\ker(\gamma)$. Hence Diff^0_γ is a subgroup of Diff_γ. The quotient

$$\mathrm{GS}_\gamma = \mathrm{Diff}_\gamma/\mathrm{Diff}^0_\gamma \tag{4.48}$$

is the group of graph symmetries of γ. GS_γ is a finite group with order $|\mathrm{GS}(\gamma)|$. Given a state Ψ_γ with respect to γ and a diffeomorphism ϕ, $\phi \star \Psi_\gamma$ denotes the pullback of Ψ under ϕ. The averaging with respect to $\mathrm{GS}(\gamma)$ defines a projection $P_\gamma : \mathcal{H}^{\rm irr}_\gamma \to \mathcal{H}^{\rm irr}_\gamma$,

$$P_\gamma : \Psi_\gamma \mapsto \sum_{\phi\in\mathrm{GS}_\gamma} \phi \star \Psi_\gamma. \tag{4.49}$$

The averaging with respect to the remaining diffeomorphisms $\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}_\gamma$ defines

$$\eta(\Psi_\gamma) := \sum_{\phi\in\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}_\gamma} \phi\star P_\gamma(\Psi_\gamma) = \sum_{\phi\in\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}^0_\gamma} \phi\star\Psi_\gamma. \tag{4.50}$$

Obviously, $\eta(\Psi_\gamma)$ belongs to the algebraic dual space $\mathbf{Cyl_{tot}}^*$ of $\mathbf{Cyl_{tot}}$. Taking advantage the decomposition (3.45), one can extend η to a well-defined operation on $\mathbf{Cyl_{tot}}$. Let $\eta(\mathbf{Cyl_{tot}})\subset\mathbf{Cyl_{tot}}^*$ denote the image of $\mathbf{Cyl_{tot}}$ under η. The group averaging procedure naturally endows $\eta(\mathbf{Cyl_{tot}})$ with an inner product

$$\big(\eta(\Psi_\gamma),\eta(\Psi'_{\gamma'})\big) = \frac{1}{|\mathrm{GS}_\gamma|}(\eta(\Psi_\gamma)|\Psi'_{\gamma'}\rangle, \tag{4.51}$$

where $1/|\mathrm{GS}_\gamma|$ is chosen for latter convenience (see (4.57)) and $(\eta(\Psi_1)|\Psi_2\rangle$ denotes the action of $\eta(\Psi_1)\in\mathbf{Cyl_{tot}}^*$ on Ψ_2, defined by

$$(\eta(\Psi_\gamma)|\Psi'_{\gamma'}\rangle = \sum_{\phi\in\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}^0_\gamma} \langle\phi\star\Psi_\gamma,\Psi'_{\gamma'}\rangle. \tag{4.52}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathcal{H}$. Then the vertex Hilbert space $\mathcal{H}_{\mathrm{vtx}}$ is

$$\mathcal{H}_{\mathrm{vtx}} = \overline{\eta(\mathbf{Cyl_{tot}})}. \tag{4.53}$$

Given a finite subset $W\subset\Sigma$, let $\Gamma_{\ker}(W)$ be the collection of graphs γ satisfying $V(\ker(\gamma)) = W$. Two graphs $\gamma,\gamma'\in\Gamma_{\ker}(W)$ are said to be equivalent, denoted by $\gamma\sim_d\gamma'$, if there exists a diffeomorphism $\phi\in\mathrm{Diff}_W$ such that $\phi(\gamma)=\gamma'$, where Diff_W is the set of diffeomorphism preserving W. Let $[\Gamma_{\ker}(W)]$ be the quotient space $[\Gamma_{\ker}(W)] = \Gamma_{\ker}(W)/\sim_d$, and $[\gamma]\in[\Gamma_{\ker}(W)]$ denote the equivalence class of γ. Let $\mathcal{S}^{\mathrm{irr}}_\gamma$ denote the image of P_γ. Then $\eta:\mathcal{S}^{\mathrm{irr}}_\gamma\to\mathcal{H}^{\mathrm{irr}}_{\mathrm{vtx}}$ is isometric. By $\eta(\mathcal{S}^{\mathrm{irr}}_\gamma)$ denoting the image of $\mathcal{S}^{\mathrm{irr}}_\gamma$ under η, one has

$$\eta(\mathcal{S}^{\mathrm{irr}}_\gamma) = \eta(\mathcal{S}^{\mathrm{irr}}_{\gamma'}),\ \forall\gamma,\gamma'\in[\gamma]. \tag{4.54}$$

Thus, we can define

$$\eta(\mathcal{S}^{\mathrm{irr}}_{[\gamma]}) := \eta(\mathcal{S}^{\mathrm{irr}}_{\gamma'}) \tag{4.55}$$

by choosing arbitrary $\gamma'\in[\gamma]$. Let $\mathrm{FS}(\Sigma)$ denote the set of finite subsets of Σ. Then we have

$$\eta(\mathbf{Cyl_{tot}}) = \bigoplus_{W\in\mathrm{FS}(\Sigma)}\bigoplus_{[\gamma]\in\Gamma_{\ker}(W)} \eta(\mathcal{S}^{\mathrm{irr}}_{[\gamma]})\oplus\mathbb{C}. \tag{4.56}$$

The factor $1/|\mathrm{GS}_\gamma|$ in (4.51) ensures that η is an isometric from $\mathcal{S}^{\mathrm{irr}}_\gamma$ and $\eta(\mathcal{S}^{\mathrm{irr}}_\gamma)$, i.e.,

$$\eta(\mathbf{Cyl_{tot}}) \cong \bigoplus_{W\in\mathrm{FS}(\Sigma)}\bigoplus_{[\gamma]\in\Gamma_{\ker}(W)} \mathcal{S}^{\mathrm{irr}}_{\sigma([\gamma])}\oplus\mathbb{C}. \tag{4.57}$$

where $\sigma([\gamma])\in[\gamma]$ is a representative of $[\gamma]$ fixed once.

2. the Hamiltonian operator on $\mathcal{H}_{\mathrm{vtx}}$

Given $\Psi_\gamma\in\mathcal{H}^{\mathrm{irr}}_\gamma$ with respect to $\gamma=\gamma_F\cup\gamma_G$, by definition, the operator $\widehat{H_1(\delta)}$ takes the form

$$\widehat{H_1(\delta)}\Psi_\gamma = \sum_{v\in V(\gamma)} N(v)\hat{O}(v)\Psi_\gamma, \tag{4.58}$$

with some operator $\hat{O}(v)$ satisfying that $\hat{O}(v)=0$ for all removable vertices $v\in V(\gamma)$. Hence, one has

$$\widehat{H_1(\delta)}\Psi_\gamma = \sum_{v\in V(\ker(\gamma))} N(v)\hat{O}(v)\Psi_\gamma. \tag{4.59}$$

By definition, the operator $\hat{A}_F$ on $\mathcal{H}_{\rm vtx}$ is given by

$$\left(\hat{A}_F\eta(\Psi_\gamma)\Big|\Phi_{\gamma'}\right\rangle := \lim_{\delta\to 0}\left(\widehat{H_1(\delta)}^\star\eta(\Psi_\gamma)\Big|\Phi_{\gamma'}\right\rangle, \tag{4.60}$$

where the operator $\hat{O}^\star$ dual to $\hat{O}$ reads

$$\left(\hat{O}^\star\eta(\Psi_\gamma)\Big|\Phi_{\gamma'}\right\rangle = \left(\eta(\Psi_\gamma)\Big|\hat{O}\Phi_{\gamma'}\right\rangle \tag{4.61}$$

According to the definition of $\widehat{H_1(\delta)}$ in (4.47), let us now consider the operator $(\widehat{H^{(1)}(v,\delta)^\dagger})^\star$ at first. The operator $\widehat{H^{(1)}(v;\delta)}^\dagger$ with $v\in V(\ker(\gamma))$ can be splitted into two parts as

$$\widehat{H^{(1)}(v;\delta)}^\dagger = -\kappa\hbar\beta N(v)\hat{\mathfrak{h}}_1(\delta) + \kappa\hbar\beta N(v)\hat{\mathfrak{h}}_2 \tag{4.62}$$

with $\hat{\mathfrak{h}}_2$ defined as

$$\hat{\mathfrak{h}}_2 = \sum_{e\text{ at }v}\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e} \tag{4.63}$$

By (4.60), let us consider

$$\lim_{\delta\to 0}\left(\hat{\mathfrak{h}}_1(\delta)^\star\eta(\Psi_\gamma)\Big|\Phi_{\gamma'}\right\rangle = \lim_{\delta\to 0}\left(\eta(\Psi_\gamma)\Bigg|\sum_{e\text{ at }v}\hat{\mathfrak{h}}_1(\delta)\Phi_{\gamma'}\right\rangle. \tag{4.64}$$

At first, the right hand side vanishes if

$$\ker(\gamma')\notin\{\phi(\ker(\gamma)),\phi\in{\rm Diff}_{V(\ker(\gamma))}\}.$$

Moreover, for $\ker(\gamma')\in\{\phi(\ker(\gamma)),\phi\in{\rm Diff}_{V(\ker(\gamma))}\}$ but $\ker(\gamma')\neq\ker(\gamma)$, one can alway fine another graph $\tilde{\gamma}$ with $\tilde{\gamma}\in\{\phi(\gamma),\phi\in{\rm Diff}_{V(\ker(\gamma))}/{\rm Diff}_\gamma\}$ and $\ker(\tilde{\gamma})=\ker(\gamma')$ to replace $\eta(\Psi_\gamma)$ in the right hand of (4.64) by $\eta(\tilde{\Psi}_{\tilde{\gamma}})$, where $\tilde{\Psi}_{\tilde{\gamma}}$ is the state associated to $\tilde{\gamma}$ and satisfying $\eta(\Psi_\gamma)=\eta(\tilde{\Psi}_{\tilde{\gamma}})$. Hence, without loss of generality, we can set $\ker(\gamma')=\ker(\gamma)$ in (4.64). This setting simplifies (4.64) to be

$$\lim_{\delta\to 0}\left(\hat{\mathfrak{h}}_1(\delta)^\star\eta(\Psi_\gamma)\Big|\Phi_{\gamma'}\right\rangle = \lim_{\delta\to 0}\left(\sum_{\phi\in{\rm Diff}_{\ker(\gamma)}/{\rm Diff}^0_\gamma}\phi\star\Psi_\gamma\Bigg|\hat{\mathfrak{h}}_1(\delta)\Phi_{\gamma'}\right\rangle \tag{4.65}$$

where ${\rm Diff}_{\ker(\gamma)}$ is comprised of the diffeomorphisms preserving $\ker(\gamma)$. Add a fermion vertex at the target of $e(v,\delta)\subset\gamma'$ to get the new graph $\dot{\gamma}'(e,v,;\delta)$. By (4.44), (4.45) and (4.46), $\hat{\mathfrak{h}}_1(\delta)\Phi_{\gamma'}$ will be some state in $\mathcal{H}^{\rm irr}_{\dot{\gamma}'(e,v;\delta)}$. Using $\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)}$ to denote $\hat{\mathfrak{h}}_1(\delta)\Phi_{\gamma'}$, we have

$$\lim_{\delta\to 0}\left(\hat{\mathfrak{h}}_1(\delta)^\star\eta(\Psi_\gamma)\Big|\Phi_{\gamma'}\right\rangle = \lim_{\delta\to 0}\left(\sum_{\phi\in{\rm Diff}_{\ker(\gamma)}/{\rm Diff}^0_\gamma}\phi\star\Psi_\gamma\Bigg|\sum_{e\text{ at }v}\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)}\right\rangle. \tag{4.66}$$

Because the limit of $\delta\to 0$ will considered, we can set δ small enough such that there is no fermion vertex between $t_{e(v,\delta)}$ and v. By (4.44), (4.45) and (4.46), the fermion state $t_{e(v,\delta)}$ in $\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)}$ is $|0,1\rangle$ or $|1,0\rangle$, i.e. $\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)}$ takes the form

$$\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)} = \tilde{\Phi}^{(1)}_{\gamma'}\otimes|0,1\rangle_{t_{e(v,\delta)}} + \tilde{\Phi}^{(2)}_{\gamma'}\otimes|1,0\rangle_{t_{e(v,\delta)}}. \tag{4.67}$$

Thus, the right hand side of (4.66) will vanish if there do not exist any edges $e\in\ker(\gamma)$ at v such that

$$\dot{\gamma}'(e,v;\delta)\in\{\phi(\gamma),\phi\in{\rm Diff}_{\ker(\gamma)}/{\rm Diff}^0_\gamma\}. \tag{4.68}$$

Let $E_0(\ker(\gamma))\subset E(\ker(\gamma))$ be the collection of the edges satisfying (4.68). Given an edge $e\in E_0(\ker(\gamma))$, let $\phi_0^{(e,v;\delta)}\in{\rm Diff}_{\ker(\gamma)}/{\rm Diff}^0_\gamma$ be the diffeomorphism such that $\phi_0^{(e,v;\delta)}(\gamma)=\dot{\gamma}'(e,v;\delta)$. One has

$$\left(\sum_{\phi\in{\rm Diff}_{\ker(\gamma)}/{\rm Diff}^0_\gamma}\phi\star\Psi_\gamma\Bigg|\sum_{e\text{ at }v}\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)}\right\rangle = \sum_{e\in E_0(\ker(\gamma))}\left\langle\phi_0^{(e,v;\delta)}\star\Psi_\gamma\Big|\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)}\right\rangle \tag{4.69}$$

By definition of $\phi_0^{(e,v;\delta)}$ with $e\in E_0(\ker(\gamma))$, in the graph $\phi_0^{(e,v;\delta)}(\gamma)$ which the state $\phi_0^{(e,v;\delta)}\star\Psi_\gamma$ is associated to, there must exist a non-negligible fermion vertex at $t_{e(v,\delta)}$, i.e. the fermion state at $t_{e(v,\delta)}$ is not $|0,0\rangle$. Thus, according to (4.67), the right hand of (4.69) will vanish for all $\Phi_{\gamma'}$ (refer to (4.65) for $\Phi_{\gamma'}$) if the fermion state of $\phi_0^{(e,v;\delta)}\star\Psi_\gamma$ at $t_{e(v,\delta)}$ is $|1,1\rangle$. Thus, let us define a projection $\hat{\mathbb{P}}_{v_F}$ on the fermion Hilbert space $\mathcal{H}_{v_F}$ by

$$\hat{\mathbb{P}}_{v_F}|i_1,i_2\rangle=\begin{cases}0, & i_1=1=i_2,\\ |i_1,i_2\rangle, & \text{otherwise.}\end{cases}\tag{4.70}$$

With $\hat{\mathbb{P}}_{t_{e(v,\delta)}}$, we have

$$\left(\sum_{\phi\in\mathrm{Diff}_{\ker(\gamma)}/\mathrm{Diff}_\gamma}\phi\star\Psi_\gamma\middle|\sum_{e\text{ at }v}\dot{\Phi}_{\dot\gamma'(e,v;\delta)}\right\rangle=\sum_{e\in E_0(\ker(\gamma))}\left\langle\hat{\mathfrak{h}}_1(e,\delta)^\dagger\hat{\mathbb{P}}_{t_{e(v,\delta)}}\phi_0^{(e,v;\delta)}\star\Psi_\gamma\middle|\Phi_{\gamma'}\right\rangle\tag{4.71}$$

Defining δ_e by

$$\phi_0^{(e,v;\delta)}(e(v,\delta_e))=e(v,\delta),\tag{4.72}$$

one has

$$\left\langle\hat{\mathfrak{h}}_1(e,\delta)^\dagger\hat{\mathbb{P}}_{t_{e(v,\delta)}}\phi_0^{(e,v;\delta)}\star\Psi_\gamma\middle|\Phi_{\gamma'}\right\rangle=\left\langle\phi_0^{(e,v;\delta)}\star(\hat{\mathfrak{h}}_1(e_\phi,\delta_e)^\dagger\hat{\mathbb{P}}_{t_{e_\phi(v,\delta_e)}}\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle\tag{4.73}$$

with $e_\phi\equiv\phi_0^{(e,v;\delta)}(e)$. Since δ is supposed to be sufficiently small, $\hat{\mathfrak{h}}_1(e_\phi,\delta)\hat{\mathbb{P}}_{t_{e_\phi(v,\delta)}}$ will annihilate the closet fermion vertex to v in $\phi_0^{(e,v;\delta)}(\gamma_F\cap e_\phi-\{t_{e_\phi},s_{e_\phi}\})$. Here $\gamma_F\cap e_\phi-\{t_e,s_e\}$ denotes the set of fermion vertices v_F with $v_F\in\gamma_F\cap e_\phi$ and $v_F\notin\{s_{e_\phi},t_{e_\phi}\}$. Let $v_F^0(e_\phi,v)\in V(\gamma)$ be the one closest to v. Then δ_e satisfies

$$e_\phi(|p-\delta_e|)=v_F^0(e_\phi,v)\tag{4.74}$$

where $p=0$ or 1 is given by $v=e_\phi(p)$. Moreover, let $\gamma_\star(e_\phi)$ be the graph by dropping $v_F^0(e_\phi,v)$ from γ. One has

$$\hat{\mathfrak{h}}_1(e_\phi,\delta_e)^\dagger\hat{\mathbb{P}}_{t_{e_\phi(v,\delta)}}\Psi_\gamma\ \in\ \mathcal{H}^{\mathrm{irr}}_{\gamma_\star(e_\phi)}\tag{4.75}$$

$$\phi_0^{(e,v,\delta)}(\gamma_\star(e_\phi))\ =\ \gamma',\tag{4.76}$$

$$\phi_0^{(e,v,\delta)}(t_{e_\phi(v,\delta_e)})\ =\ t_{e(v,\delta)}.\tag{4.77}$$

This means that $\phi_0^{(e,v,\delta)}$ can be factorized as

$$\phi_0^{(e,v,\delta)}=\tilde{\phi}_0^{(e,v,\delta)}\circ\bar{\phi}_0^{(e,v,\delta)}\tag{4.78}$$

where $\tilde{\phi}_0^{(e,v,\delta)}\in\mathrm{Diff}_{\ker(\gamma)}/\mathrm{Diff}^0_{\gamma_\star(e_\phi)}$ is determined by (4.76) and $\bar{\phi}_0^{(e,v,\delta)}\in\mathrm{Diff}^0_{\gamma_\star(e_\phi)}/\mathrm{Diff}^0_\gamma$, by (4.77). With this factorization, one has

$$\left\langle\phi_0^{(e,v;\delta)}\star(\hat{\mathfrak{h}}_1(e_\phi,\delta_e)\hat{\mathbb{P}}_{t_{e_\phi(v,\delta_{e_0})}}\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle=\left\langle\tilde{\phi}_0^{(e,v;\delta)}\star(\hat{\mathfrak{h}}_1(e_\phi,\delta_e)\hat{\mathbb{P}}_{t_{e_\phi(v,\delta_e)}}\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle\tag{4.79}$$

Combining (4.71), (4.73) and (4.79), we thus have

$$\left(\sum_{\phi\in\mathrm{Diff}_{\ker(\gamma)}/\mathrm{Diff}^0_\gamma}\phi\star\Psi_\gamma\middle|\sum_{e\text{ at }v}\dot{\Phi}_{\dot\gamma'(e,v;\delta)}\right\rangle=\sum_{e\in E_0(\ker(\gamma))}\left\langle\tilde{\phi}_0^{(e,v;\delta)}\star(\hat{\mathfrak{h}}_1(e_\phi,\delta_e)\hat{\mathbb{P}}_{t_{e_\phi(v,\delta_e)}}\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle.\tag{4.80}$$

It will be convenient to introduce the following notations to rewrite (4.80). At first, we generalize the notation $v_F^0(e_\phi,v)$ and $\gamma_\star(e_\phi)$ to arbitrary edge e. Define $v_F^0(e,v)\in\{v_F\in\gamma_F\cap e,v_F\neq t_e,v_F\neq s_e\}\subset V(\gamma)$ as the fermion vertex closest to v , and define $\gamma_\star(e)$ as the graph obtained by dropping $v_F^0(e,v)$ from γ. Second, let $e(v)$ be the segment of e from v to $v_F^0(e,v)$. Finally, let $\mathcal{D}(e,v)$ be the collection of $\phi\in\mathrm{Diff}_{\ker(\gamma)}/\mathrm{Diff}^0_{\gamma_\star(e)}$ satisfying

$$\phi(\gamma_\star(e))=\gamma'.\tag{4.81}$$

With these notations, we introduce the operator $\widehat{\mathfrak{H}}_v(e)$

$$\widehat{\mathfrak{H}}_v(e) = \hat{\theta}^\dagger(v)\sigma^i h^{-1}_{e(v)}\hat{\theta}(v^0_F(e,v))\hat{J}^{v,e}_i\hat{\mathbb{P}}_{v^0_F(e,v)}, \tag{4.82}$$

and rewrite (4.80) as

$$\begin{aligned}
&\left(\sum_{\phi\in \mathrm{Diff}_{\ker(\gamma)}/\mathrm{Diff}^0_\gamma} \phi\star\Psi_\gamma \middle| \sum_{e \text{ at } v}\dot{\Phi}_{\dot{\gamma}'(e,v;\delta)}\right\rangle\\
=&\sum_{e \text{ at } v}\sum_{\phi\in\mathcal{D}(e,v)}\left\langle \phi\star(\hat{\mathfrak{H}}_v(e)\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle\\
=&\sum_{e \text{ at } v}\sum_{\phi\in \mathrm{Diff}_{\ker(\gamma)}/\mathrm{Diff}^0_{\gamma_\star(e)}}\left\langle \phi\star(\hat{\mathfrak{H}}_v(e)\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle
\end{aligned} \tag{4.83}$$

where in the first step we used that the terms corresponding to edges e with $\mathcal{D}(e,v)=\emptyset$ vanish automatically, and the second step employs

$$\left\langle \phi\star(\hat{\mathfrak{H}}_v(e)\hat{\mathbb{P}}_{t_{e(v,\delta_e)}}\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle = 0,\ \forall\phi\notin\mathcal{D}(e,v). \tag{4.84}$$

Substituting this result (4.83) into (4.66), we have

$$\lim_{\delta\to 0}\left(\hat{\mathfrak{h}}_1(\delta)^\star\eta(\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle = \sum_{e \text{ at } v}\sum_{\phi\in \mathrm{Diff}_{\ker(\gamma)}/\mathrm{Diff}^0_{\gamma_\star(e)}}\left\langle \phi\star(\hat{\mathfrak{H}}_v(e)\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle, \tag{4.85}$$

due to the right hand side of (4.83) is independent of δ. Finally, releasing the assumption $\ker(\gamma)=\ker(\gamma')$, one has

$$\lim_{\delta\to 0}\left((\hat{\mathfrak{h}}_1(\delta)^\dagger)^\star\eta(\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle = \left(\eta(\hat{\mathfrak{H}}_v\Psi_\gamma)\middle|\Phi_{\gamma'}\right\rangle \tag{4.86}$$

where $\hat{\mathfrak{H}}_v$ is

$$\hat{\mathfrak{H}}_v = \sum_{e \text{ at } v}\hat{\mathfrak{H}}_v(e). \tag{4.87}$$

Eq. (4.86) implies

$$\lim_{\delta\to 0}\hat{\mathfrak{h}}_1(\delta)^\star\eta(\Psi_\gamma) = \eta(\hat{\mathfrak{H}}_v\Psi_\gamma). \tag{4.88}$$

Therefore, by (4.62) we have

$$\lim_{\delta\to 0}(\widehat{H^{(1)}(v;\delta)}^\dagger)^\star\eta(\Psi_\gamma) = -\kappa\hbar\beta N(v)\eta\left[\left((\hat{\mathfrak{H}}_v - \sum_{e \text{ at } v}\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\hat{J}^{v,e}_i\right)\Psi_\gamma\right] \tag{4.89}$$

where we used the manifest fact that

$$\lim_{\delta\to 0}\hat{\mathfrak{h}}_2^\star\eta(\Psi_\gamma) = \eta(\hat{\mathfrak{h}}_2\Psi_\gamma) \tag{4.90}$$

Now let us consider the operator $\widehat{H^{(2)}(v;\delta)}$. An issue on defining $\lim\limits_{\delta\to 0}\widehat{H^{(2)}(v;\delta)}$ is the operator $\hat{H}_{E,v}$ comprising $\widehat{H^{(2)}(v;\delta)}$. In this paper, we will employ the the work [47] to define $\hat{H}_{E,v}$ in $\mathcal{H}_{\rm vtx}$. A subtlety here is that the vertex Hilbert space defined in [47] is a little different from ours. In [47], the authors defines the vertex Hilbert space with diffeomorphisms preserving $V(\gamma)$ while our work considers the diffeomorphisms preserving $V(\ker(\gamma))$. Regardless of this difference, the operator $\hat{H}_{E,v}$ introduced in [47] , for $v\in V(\ker(\gamma))$, is well-defined in our vertex Hilbert space $\mathcal{H}_{\rm vtx}$. Indeed, due to the operator $\sqrt{\widehat{V_v^{-1}}}$ in $\widehat{\tilde{H}^{(2)}(v;\delta)}$, we do not need to consider $\hat{H}_{E,v}$ with v being the removable. According to this discussion, for $v\in V(\ker(\gamma))$, we have

$$\lim_{\delta\to 0}\widehat{H^{(2)}(v;\delta)}\eta(\Psi_\gamma) = \frac{\kappa_1(v)}{4i\kappa\hbar\beta^2}N(v)\eta\left[\left(\hat{H}_{E,v}\hat{V}_v - \hat{V}_v\hat{H}_{E,v}\right)\hat{\theta}^\dagger(v)\hat{\theta}(v)\Psi_\gamma\right], \tag{4.91}$$

where $\kappa_1(v)$ is introduced in [47] to remove the dependence on the partition. Here $\hat{H}_{E,v}$ is defined as $\hat{H}_{E,v} = \hat{F}_v \hat{V}_v^{-1} + V_v^{-1} \hat{F}_v^\dagger$ with

$$\hat{F}_v = -2 \sum_{e,e' \text{ at } v} \epsilon(\dot{e}, \dot{e}') \epsilon^{ijk} \operatorname{tr}\big(h_{\alpha_{ee'}} \tau_k\big) \hat{J}_i^{v,e} \hat{J}_k^{v,e'} \tag{4.92}$$

where $\alpha_{ee'}$ is a loop tangent to the two edges e and e' at the vertex v up to orders k_e+1 and $k_{e'}+1$ respectively with k_e and $k_{e'}$ being respectively the orders of tangentiality of e and e' at v (see [47] for more details.) By this definition, $\hat{F}_v$ will change the graph by adding a loop at v and, thus, $\hat{F}_v^\dagger$, by removing a loop. Finally, for the operator $\widehat{H^{(3)}(v;\delta)}$, since $\widehat{H^{(3)}(v;\delta)}$ for $v \in V(\ker(\gamma))$ is independent of δ and $\mathrm{Diff}_{V(\ker(\gamma))}$ invariant. we have

$$\lim_{\delta \to 0} \Big(\widehat{H^{(3)}(v;\delta)}^\star \eta(\Psi_\gamma)\Big| = \Big(\eta(\hat{H}_F^{(3)}(v)\Psi_\gamma)\Big| \tag{4.93}$$

with

$$\hat{H}_F^{(3)}(v) := \kappa\hbar\beta N(v) \left(\sum_{e \text{ at } v_C} \hat{J}_i^{v_C,e}\right) \hat{\theta}^\dagger(v)\sigma^i \hat{\theta}(v). \tag{4.94}$$

Let us summarize our results. According to the above discussion, we have

$$\hat{A}_F \eta(\Psi_\gamma) := \lim_{\delta \to 0} \widehat{H_1(\delta)}^\star \eta(\Psi_\gamma) = \sum_{v \in V(\gamma)} \eta\left(\sqrt{\widehat{V_v^{-1}}}\hat{H}_v \sqrt{\widehat{V_v^{-1}}}\Psi_\gamma\right) \tag{4.95}$$

where $\hat{H}_v$ is defined as

$$\hat{H}_v = -i\widehat{H_1(v)} - \frac{\beta}{2}\widehat{H_2(v)} - \frac{1+\beta^2}{2\beta}\widehat{H_3(v)} - \beta\widehat{H_1(v)}, \tag{4.96}$$

with

$$\begin{aligned}
\widehat{H_1(v)} &= -\kappa_2(v)\ell_p^2 \beta N(v) \left(\hat{\mathfrak{H}}_v - \sum_{e \text{ at } v} \hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e}\right), \\
\widehat{H_2(v)} &= \frac{\kappa_1(v)}{4i\ell_p^2\beta^2} N(v) \left(\hat{H}_{E,v}\hat{V}_v - \hat{V}_v \hat{H}_{E,v}\right) \hat{\theta}^\dagger(v)\hat{\theta}(v), \\
\widehat{H_3(v)} &= \kappa_3(v)\ell_p^2\beta N(v) \left(\sum_{e \text{ at } v} \hat{J}_i^{v,e}\right) \hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v),
\end{aligned} \tag{4.97}$$

according to (4.89), (4.91) and (4.93) respectively. Here we again introduce the parameters κ_2 and κ_3 as in [34] in order to remove the dependence on the partition. Taking advantage of these results, we finally define the fermion Hamiltonian operator $\hat{H}_F$ on $\mathcal{H}_{\rm vtx}$ as

$$\hat{H}_F = \hat{A}_F + \hat{A}_F^\dagger. \tag{4.98}$$

It is worth discussing the term in $\hat{A}_F^\dagger$ originating from the operator $\hat{\mathfrak{H}}_v$. Let $\hat{\mathfrak{H}}'_v$ on $\mathcal{H}_{\rm vtx}$ be the operator defined by

$$\hat{\mathfrak{H}}'_v \eta(\Psi_\gamma) = \eta(\hat{\mathfrak{H}}_v \Psi_\gamma). \tag{4.99}$$

Then, we are concerned about the operator $(\hat{\mathfrak{H}}'_v)^\dagger$ in $\hat{A}_F^\dagger$. By the definition (4.87), $\hat{\mathfrak{H}}_v$ is comprised of $\hat{\mathfrak{H}}_v(e)$ which annihilates the fermion vertex contained in e and closet to v. Thus, the operator $(\hat{\mathfrak{H}}'_v)^\dagger$ will be comprised with some operators, each of which is associated to an edge e at v and creates a fermion vertex $v_F \in e$ such that v_F becomes the closest vertex to v among the vertices contained in e. To be more precise, let us define the operator $\hat{\mathfrak{H}}_v^+(e)$ (refer to (4.82)) as

$$\hat{\mathfrak{H}}_v^+(e) = \hat{J}_i^{v,e} \hat{\theta}^\dagger(\mathfrak{v}_{e,v}) h_{e(\mathfrak{v}_{e,v})} \sigma^i \hat{\theta}(v) \tag{4.100}$$

where $\mathfrak{v}_{e,v} \in e$ is a point in e such that $\mathfrak{v}_{e,v} \in e$ is closest to v among $(\gamma_F \cup \mathfrak{v}_{e,v}) \cap e - \{s_e, t_e\}$, and $e(\mathfrak{v}_{e,v}) \subset e$ is the segment starting from v and ending at $\mathfrak{v}_{e,v}$. It is noted that even though $\mathfrak{v}_{e,v}$ is not uniquely determined by the definition, the operator $(\hat{\mathfrak{H}}'_v)^\dagger$ on $\mathcal{H}_{\rm vtx}$ defined via $\widehat{\mathfrak{H}}^+_v(e)$ will be independent of the precise position of $\mathfrak{v}_{e,v}$, because of the averaging operation η with respect to diffeomorphisms. Indeed, $(\hat{\mathfrak{H}}'_v)^\dagger$ is given by

$$(\hat{\mathfrak{H}}'_v)^\dagger\eta(\Psi_\gamma)=\eta\left(\sum_{e\text{ at }v}\widehat{\mathfrak{H}}^+_v(e)\Psi_\gamma\right). \tag{4.101}$$

Let us complete this section with a discussion on the intuitive picture led by the action of $\widehat{\mathfrak{H}}_v(e)$ and $\widehat{\mathfrak{H}}^+_v(e)$. By definition, $\widehat{\mathfrak{H}}^+_v(e)$ creates a fermion vertex $v_F\in e$. v_F is next to v and carries states $a|1,0\rangle+b|0,1\rangle$. Simultaneously, $\widehat{\mathfrak{H}}^+_v(e)$ changes the fermion state $|i_1,i_2\rangle_v$ at v in such a way that $|1,1\rangle_v\mapsto c|1,0\rangle_v+d|1,0\rangle_v$ for some constant c and d, and $\alpha|1,0\rangle_v+\beta|0,1\rangle_v\mapsto|0,0\rangle$ for arbitrary constants α and β. Moreover, because of the holonomy operator $h_{e(v)}$ and flux operator $\hat{J}^{v,e}_i$ in $\widehat{\mathfrak{H}}^+_v(e)$, the spin on the segment $e(v)\subset e$ and the intertwiner at v are changed. These results can be summarized in one word that the operator $\widehat{\mathfrak{H}}^+_v(e)$ moves a fermion at v to v_F, and changes the geometry around v simultaneously. For the operator $\widehat{\mathfrak{H}}_v(e)$, it reverses this procedure, that is, $\widehat{\mathfrak{H}}_v(e)$ moves a fermion at the fermion vertex in e and closet to v, to the vertex v, and changes the geometry around v simultaneously. Moreover, because of the projection operator $\hat{\mathbb{P}}_{v_F}$ in $\widehat{\mathfrak{H}}_v(e)$ (see (4.82)), when the fermion vertex v_F in e and closet to v carries a fermion state $|1,1\rangle_{v_F}$, the fermion at v_F cannot be moved by the operator $\widehat{\mathfrak{H}}_v(e)$. To see the consequence of this fact, let us first imagine an edge e with the source s_e and middle point v_F being the fermion vertices, where v_F carries a fermion state, saying, $|1,0\rangle_{v_F}$. Then acted by $\widehat{\mathfrak{H}}^+_{s_e}(e)$, the fermion state at s_e will be moved to some point v'_F between s_e and v_F. Then, acted by $\widehat{\mathfrak{H}}_{t_e}(e)$ twice, the fermion at v'_F will be moved to t_e. Now, imagine that v_F carries the state $|1,1\rangle_{v_F}$. Then, acted by $\widehat{\mathfrak{H}}^+_{s_e}(e)$, the fermion state at s_e will be again moved to v'_F. Then, no matter for how many times the resulting state is acted by $\widehat{\mathfrak{H}}_{t_e}(e)$, the fermion at v'_F cannot be moved to t_e, due to the operator $\hat{\mathbb{P}}_{v_F}$ in in $\widehat{\mathfrak{H}}_{t_e}(e)$. Intuitively, in the second situation, the fermion at s_e is confined around s_e by the fermion state $|1,1\rangle_{v_F}$ at v_F. This picture on how a fermion moves in LQG spacetime and influences the background geometry in the LQG framework recovers somehow the classical picture that *"spacetime tells matter how to move; matter tells spacetime how to curve"-John Archibald Wheeler.*

V. SUMMATION AND OUTLOOK

This work is concerned about the model of fermion field coupled to loop quantum gravity. The Gauss and Hamiltonian constraints in this model are studied in details. In the solution to the Gauss constraint, fermion spins and the gravitational spin network intertwine with each other so that the fermion spins contribute to the volume of the spin network at vertices. Consequently, the closure condition encoded in the Gauss constraint will no longer satisfied for the gauge invariant state with non-vanishing fermion spin. In other words, the faces dual to the edges at a fermion vertex with non-vanishing fermion spin could not form a closed polyhedron, and the area defect of this unclosed polyhedron is filled by the fermion spin. Thus, a 3-valence gauge invariant vertex with non-vanishing fermion spin does not have vanishing volume again. The volume of this type of vertices is computed in details.

For the Hamiltonian constraint, the regularization and quantization procedures are presented in detail. There are several remarkable issues on the definition of the Hamiltonian constraint operator. At first, in order to take the limit of the regularized expression as the regulator approaches 0 and define a diffeomorphism covariant Hamiltonian constraint operator, we introduce the vertex Hilbert space. By definition, the vertex Hilbert space is comprised of the cylindrical function group averaged with the diffeomorphisms preserving the unremovable vertices of graphs. Thus, the states in vertex Hilbert space are partially diffeomorphism invariant. Due to this diffeomorphism invariant feature, on the vertex Hilbert space, the regularized Hamiltonian operators $\widehat{H(\delta)}$ with different values of the regulator δ are identical. Consequently, the limit of $\widehat{H(\delta)}$ as δ approaches 0 can be taken. Moreover, the operator $\widehat{H(\delta)}$ can be divided into two parts $\widehat{H_1(\delta)}$ and $\widehat{H_1(\delta)}^\dagger$ where $\widehat{H_1(\delta)}^\dagger$ is diffeomorphism covariant but $\widehat{H_1(\delta)}$ is not. It is problematic that $\widehat{H_1(\delta)}$ is not diffeomorphism covariant so that the diffeomorphism-equivalence feature of two states can not be always kept by the action of $\widehat{H_1(\delta)}$. To solve this problem, we also need to introduce the vertex Hilbert space. By definition, the vertex Hilbert space is a dual space of the cylindrical function space. Thus the action of $\widehat{H_1(\delta)}^\dagger$ can be promoted to it by the duality. The operator in the vertex Hilbert space dual to $\widehat{H_1(\delta)}^\dagger$ is denoted by $\hat{A}_F$. Since $\widehat{H_1(\delta)}^\dagger$ is diffeomorphism covariant, so do $\hat{A}_F$ and $\hat{A}^\dagger_F$. Then the operator referring to $\widehat{H_1(\delta)}$ in the vertex Hilbert space is proposed as $\hat{A}^\dagger_F$ rather than the dual operator to $\widehat{H_1(\delta)}$, so that the diffeomorphism-covariance problem is solved. Finally, in the Hamiltonian constraint operator, there are contained the concerning the operators $\widehat{\mathfrak{H}}^+_v(e)$ and

$\widehat{\mathfrak{H}}_v(e)$. These two operators tells how the fermion moves in LQG spacetime and influences the background geometry in the LQG framework, which somehow recovers the classical picture that *"spacetime tells matter how to move; matter tells spacetime how to curve"-John Archibald Wheeler.* According to our results, the operator $\widehat{\mathfrak{H}}_v^+(e)$ moves a fermion at the vertex v to a vertex v_F in e and next to v, and simultaneously changes the spin on the segment of e from v to v_F as well as the intertwiner at v. This procedure will be reversed by the operator $\widehat{\mathfrak{H}}_v(e)$, which moves a fermion, located at the fermion vertex v_F in e and adjacent to v, to the vertex v, and simultaneously changes the spin on the segment of e from v to v_F as well as the intertwiner at v. In addition, $\widehat{\mathfrak{H}}_v(e)$ is defined to contain a projection operator $\hat{\mathbb{P}}_{v_F^0(e,v)}$. As a consequence of this operator, the fermion located at, says, s_e will be confined around s_e by the state $|1,1\rangle_{v_F}$ located at $v_F \in e$ (see the discussion at the end of Sec. IV B).

Even though the current work is concerned about the graph changing feature, the framework can be easily adapted to define a graph preserving version of the Hamiltonian constraint operator. Then one can apply this graph preserving operator to the lattice model of fermion coupled to LQG. Then some open issues in lattice fermion field theory can be employed and studied.

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Appendix A: Hamiltonian analysis for fermion field

Define $P_\pm = \frac{1\pm\gamma^5}{2}$, one has

$$P_\pm^2 = P_\pm,\ P_+P_- = P_-P_+ = 0,\ P_\pm\gamma^\mu = \gamma^\mu P_\mp. \tag{A1}$$

Therefore, we have

$$\overline{\Psi}\gamma^I e_I^\mu P_\pm \nabla_\mu \Psi = (\Psi_\pm)^\dagger \gamma^0\gamma^I e_I^\mu \Psi_\pm + \frac{1}{4}(\Psi_\pm)^\dagger\gamma^0\gamma^I e_I^\mu \omega_{\mu KL}\gamma^K\gamma^L\Psi_\pm \tag{A2}$$

with $\Psi_\pm := P_\pm\Psi$. Let us choose the Weyl basis of the γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ i\mathbb{1}_2 & 0\end{pmatrix},\quad \gamma^k = \begin{pmatrix} 0 & i\sigma^k \\ -i\sigma^k & 0\end{pmatrix},\quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2\end{pmatrix}. \tag{A3}$$

Then $\Psi_\pm$ take the form $\Psi_- = (\psi, 0)^T$ and $\Psi_+ = (0,\eta)^T$. We thus get

$$\overline{\Psi}\gamma^I e_I^\mu\nabla_\mu\Psi = -\psi^\dagger\overline{\sigma}^I e_I^\mu\partial_\mu\psi + \frac{1}{4}\psi^\dagger\overline{\sigma}^I e_I^\mu\omega_{\mu KL}\sigma^K\overline{\sigma}^L\psi - \eta^\dagger\sigma^I e_I^\mu\eta + \frac{1}{4}\eta^\dagger\sigma e_I^\mu\omega_{\mu KL}\overline{\sigma}^K\sigma^L\eta \tag{A4}$$

with $\sigma^I = (\mathbb{1}, \text{Pauli matrax}^i)$ and $\overline{\sigma}^I = (\mathbb{1}, -\text{Pauli matrax}^i)$.

Performing the 3+1-decomposition $\mathcal{M} = \mathbb{R}\times\Sigma$, one has

$$e_I^\mu = e_I^\nu q_\nu^\mu - n^\mu n_I \tag{A5}$$

where q_ν^μ is the projection to Σ and $n^\mu = (t^\mu - N^\mu)/N$ with N and N^μ being the lapse function and the shift vector respectively, and t^μ being some time evolution vector field given by $t^\mu\partial_\mu t = 1$. Substituting (A5) into (A4), we have

$$\overline{\Psi}e_I^\mu\gamma^I\nabla_\mu\Psi = \left(\psi^\dagger e_i^a\sigma^i\mathcal{D}_a^+\psi - \eta^\dagger e_i^a\mathcal{D}_a^-\eta\right) - \frac{1}{N}(t^\mu - N^\mu)\left(\psi^\dagger\mathcal{D}_\mu^+\psi + \eta^\dagger\mathcal{D}_\mu^-\eta\right), \tag{A6}$$

where we defined

$$\mathcal{D}_a^\pm = \partial_a + (\Gamma_a^m \mp K_a^m)\tau_m =: \partial_a + \mathcal{A}_a^\pm. \tag{A7}$$

Defining

$$\nabla_a = \partial_a + \Gamma_a^m\tau_m \tag{A8}$$

we can express the action of the fermion field in terms of K^i_a and ∇_a explicitly, which reads

$$
\begin{aligned}
S_F =& -\frac{i}{2}\int_{\mathcal{M}} \mathrm{d}^4x(\overline{\Psi}\gamma^I e^\mu_I \nabla_\mu \Psi - c.c) \\
=&\frac{i}{2}\int \mathrm{d}^4x\sqrt{q}\Bigg(\left(\psi^\dagger\partial_t\psi + \eta^\dagger\partial_t\eta - c.c\right) + 2\Gamma_{tm}\left(\psi^\dagger\tau^m\psi + \eta^\dagger\tau^m\eta\right) - N^a\left(\psi^\dagger\nabla_a\psi + \eta^\dagger\nabla_a\eta - (\nabla_a\psi)^\dagger\psi - (\nabla_a\eta)^\dagger\eta\right) \\
&- \frac{N}{\sqrt{q}}\Big[\psi^\dagger E^a_i\sigma^i\nabla_a\psi - (\nabla_a\psi)^\dagger E^a_i\sigma^i\psi + 2\psi^\dagger[E^a, K_a]\psi - \eta^\dagger E^a_i\sigma^i\nabla_a\eta + (\nabla_a\eta)^\dagger E^a_i\sigma^i\eta + 2\eta^\dagger[E^a, K_a]\eta\Big]\Bigg)
\end{aligned}
\tag{A9}
$$

Define $\xi_A = \sqrt[4]{q}\psi_A$ and $\nu_A = \sqrt[4]{q}\eta_A$ with $A = 1, 2$. Eq. (A9) implies the following non-vanishing anti-Poisson bracket,

$$
\begin{aligned}
\{\xi_A(x), \xi^\dagger_B(y)\}_+ &= -i\delta_{AB}\delta(x, y) \\
\{\nu_A(x), \nu^\dagger_B(y)\}_+ &= -i\delta_{AB}\delta(x, y)
\end{aligned}
\tag{A10}
$$

For the gravitational parts, the action is

$$
S_H = \frac{1}{\kappa}\int \mathrm{d}^4x\,(E^a_i\mathcal{L}_t K^i_a + \frac{1}{2\sqrt{q}}NE^a_iE^b_j\Omega^{ij}_{ab} + (t\cdot\Gamma)_m\epsilon^{klm}K_{ak}E^a_l + 2N^bE^a_i\nabla_{[a}K^i_{b]}) \tag{A11}
$$

Substituting the expression (A9) and (A11) into the total action $S = S_H + S_F$, one can obtain the constraints governing the classical dynamics which are expressed in terms of $\nabla_a = \partial_a + \Gamma^m_a\tau_m$. Then taking advantage of $A^i_a = \Gamma^i_a + \beta K^i_a$, one can simplify these constraints in terms of the derivative $D_a = \partial_a + A^i_a\tau_i$. The results are listed as follows. The total action reads

$$
S = S_G + S_F = \int \mathrm{d}^4x(\text{simplytic structure terms } - \lambda^m G_m - N^a H_a - \frac{1}{2}NH). \tag{A12}
$$

The Gaussian constraint is

$$
G_m = \frac{1}{\kappa\beta}D_aE^a_l + \frac{1}{2}\sqrt{q}(\psi^\dagger\sigma_m\psi + \eta^\dagger\sigma_m\eta) \tag{A13}
$$

The vector constraints is

$$
H_a = \frac{1}{\kappa\beta}E^b_iF^i_{ab} + \frac{i}{2}\sqrt{q}\Big\{\psi^\dagger D_a\psi - (D_a\psi)^\dagger\psi + \eta^\dagger D_a\eta - (D_a\eta)^\dagger\eta\Big\} + \beta K^m_a G_m. \tag{A14}
$$

The scalar constraint is

$$
\begin{aligned}
H =& H_G + \Big[i(\psi^\dagger E^a_i\sigma^i D_a\psi - (D_a\psi)^\dagger E^a_i\sigma^i\psi) - \beta E^a_iK^i_a\psi^\dagger\psi - \frac{1}{\beta}(1+\beta^2)D_aE^a_i\psi^\dagger\sigma^i\psi - \beta\frac{1}{\sqrt{q}}E^a_iD_a\Big(\sqrt{q}\psi^\dagger\sigma^i\psi\Big) \\
&- i(\eta^\dagger E^a_i\sigma^i D_a\eta - (\nabla_a\eta)^\dagger E^a_i\sigma^i\eta) + \beta E^a_iK^i_a\eta^\dagger\eta - \frac{1}{\beta}(1+\beta^2)D_aE^a_i\eta^\dagger\sigma^i\eta - \beta\frac{1}{\sqrt{q}}E^a_iD_a\Big(\sqrt{q}\eta^\dagger\sigma^i\eta\Big)\Big]
\end{aligned}
\tag{A15}
$$

where H_G denote the scalar constraint of pure gravity

$$
H_G = \frac{1}{\kappa\sqrt{q}}E^a_iE^b_j\left(F^m_{ab}\epsilon_m{}^{ij} - 2(1+\beta^2)K^i_{[a}K^j_{b]}\right). \tag{A16}
$$

Appendix B: graded vector space and graded algebra

We follow the notions given in [49]. A vector space V over $\mathbb{R}$ or $\mathbb{C}$ is graded (over $\mathbb{Z}_2$) if there are fixed subspaces V_0 and V_1 such that $V = V_0 \oplus V_1$. An element $v \in V$ is homogeneous if v is either in V_0 or in V_i. For all $v \in V_i$ with $v \neq 0$, we define their degree as

$$
\mathfrak{d}(v) = i. \tag{B1}
$$

Given two graded vector space V and W, the space $\mathrm{Hom}(V,W)$ of homomorphism from V to W is graded. An element $\alpha \in \mathrm{Hom}(V,W)$ is said to be homogeneous and of $\mathfrak{d}(\alpha)$ provided

$$\alpha[V_i] \subset W_{i+\mathfrak{d}(\alpha) \bmod 2}. \tag{B2}$$

with $\alpha[V_i]$ denotes the image of α acting on V_i.

An algebra $(A,\cdot)$ is a graded algebra if A is a graded vector space and $A_i \cdot A_j \subset A_{i+j \bmod 2}$ where $A_i \cdot A_j$ denotes the space of elements $a_i \cdot a_j$ for all $a_i \in A_i$ and $a_j \in A_j$. A graded algebra A is a graded commutative algebra if the product satisfies

$$x \cdot y = (-1)^{\mathfrak{d}(x)\mathfrak{d}(y)} y \cdot x \tag{B3}$$

where $x, y \in A$ are homogeneous. Any commutative algebra $\mathbb{A}$ is a graded commutative algebra with the grade $\mathbb{A}_1 = \mathbb{A}$ and $\mathbb{A}_0 = \{0\}$. An example of the graded commutative algebra is the exterior algebra of some finite vector space V, i.e.

$$A = \mathbb{R} \oplus V \oplus (V \wedge V) \oplus (V \wedge V \wedge V) \oplus \cdots \oplus \bigwedge^{n} V. \tag{B4}$$

A is graded as

$$A_0 = \bigoplus_{k=0} \bigwedge^{2k} V, \;\; A_0 = \bigoplus_{k=0} \bigwedge^{2k+1} V. \tag{B5}$$

A graded algebra $(\mathfrak{a}, [\cdot,\cdot])$ is a graded Lie algebra if the Lie bracket satisfies

(1) $[x,y] = (-1)^{1+\mathfrak{d}(x)\mathfrak{d}(y)}[y,x]$;

(2) $(-1)^{\mathfrak{d}(x)\mathfrak{d}(z)}[[x,y],z] + (-1)^{\mathfrak{d}(y)\mathfrak{d}(x)}[[y,z],x] + (-1)^{\mathfrak{d}(z)\mathfrak{d}(y)}[[z,x],y] = 0.$

An operation ∂ on a graded algebra A is called a derivative if it satisfies

$$\partial(xy) = (\partial x)y + (-1)^{\mathfrak{d}(\partial)\mathfrak{d}(x)} x(\partial y) \tag{B6}$$

where $\mathfrak{d}(\partial)$ is defined by thinking of it as a homomorphism on A. It can be checked that the operator $[x,\cdot]$ on a graded Lie algebra A for all $x \in A$ is a derivative.

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