# Loop Quantum Gravity

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### Abstract

The problem of describing the quantum behavior of gravity, and thus understanding *quantum spacetime*, is still open. Loop quantum gravity is a well-developed approach to this problem. It is a mathematically well-defined background-independent quantization of general relativity, with its conventional matter couplings. Today research in loop quantum gravity forms a vast area, ranging from mathematical foundations to physical applications. Among the most significant results obtained so far are: (i) The computation of the spectra of geometrical quantities such as area and volume, which yield tentative quantitative predictions for Planck-scale physics. (ii) A physical picture of the microstructure of quantum spacetime, characterized by Planck-scale discreteness. Discreteness emerges as a standard quantum effect from the discrete spectra, and provides a mathematical realization of Wheeler's "spacetime foam" intuition. (iii) Control of spacetime singularities, such as those in the interior of black holes and the cosmological one. This, in particular, has opened up the possibility of a theoretical investigation into the very early universe and the spacetime regions beyond the Big Bang. (iv) A derivation of the Bekenstein–Hawking black-hole entropy. (v) Low-energy calculations, yielding *n*-point functions well defined in a background-independent context. The theory is at the roots of, or strictly related to, a number of formalisms that have been developed for describing background-independent quantum field theory, such as spin foams, group field theory, causal spin networks, and others. I give here a general overview of ideas, techniques, results and open problems of this candidate theory of quantum gravity, and a guide to the relevant literature.

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# Contents

# 1 Introduction

The loop approach to quantum gravity is more than twenty years old.[1] Today, it forms a wide research area around a well-defined tentative theory of quantum spacetime. The approach provides a candidate theory of quantum gravity, a physical picture of Planck-scale quantum geometry, calculation techniques, definite quantitative predictions, and a tool for discussing classical problems such as black-hole thermodynamics and the physics of the Big Bang.

We still do not know if the theory is *physically* correct. Direct or indirect experimental support is lacking. This is the case, unfortunately, for *all* present approaches to quantum gravity. The reason, of course, is the minuteness of the scale at which (presumably) quantum properties of spacetime manifest. Waiting for experiments, a theory must be evaluated and compared with alternatives only in terms of its consistency with what we do know about Nature, internal coherence, and its capacity to produce unambiguous novel predictions. But sound scientific standards demand that no definitive conclusion be drawn.

Although fairly well developed, loop quantum gravity (or "loop gravity") is not yet a complete theory, nor has its full consistency with classical general relativity been rigorously established yet. The sector of the theory, which has not yet solidified, is the dynamics, which exists in several variants that are presently under investigation. The strength of the theory is its compelling capacity to describe quantum spacetime in a background-independent and nonperturbative fashion, and, especially, its genuine attempt to synthesize the conceptual novelties introduced by quantum mechanics with the ones introduced by general relativity: loop quantum gravity offers a possible conceptual framework in which general relativity and quantum field theory make sense together and consistently.

The other large research program for a quantum theory of gravity besides loop gravity, is string theory, which is a tentative theory as well. String theory is more ambitious than loop gravity, since it also aims at unifying all known fundamental physics into a single theory. In Section 2.3, I compare strengths and weaknesses of these two competing approaches to quantum gravity.

This "living review" is intended to be a tool for orienting the reader in the field of loop gravity. Here is the plan for the review:

- Section 2, **"Quantum Gravity: Where Are We?"**, is an introduction to the problem, the reason for its relevance, and the present state of our knowledge.

- Section 3, **"The Development of Loop Gravity"**, is a short overview of the historical development of the theory.

- Section 4, **"Resources"**, contains pointers to introductory literature, institutions at which loop gravity is studied, web pages, and other information that may be of use to students and researchers.

- Section 5, **"Ideas and Physical Inputs"**, discusses the physical and mathematical ideas on which loop quantum gravity is based.

- The actual theory is introduced in Section 6, **"Formalism"**, at a simple, technical level.

- Section 7, **"Physical Results"**, is a list of the main physical results that have been derived from the theory.

- Section 8, **"Open Problems and Current Lines of Investigation"**, illustrates the main open problems and some currently active lines of research.

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[1]The first talk on "a loop space representation of quantum general relativity" was given at a conference in India in 1987 [264].

- Section 9, **"Short Summary and Conclusion"**, summarizes very briefly the state and the results of the theory.

At the cost of several repetitions, the structure of this review is modular: to a large extent sections are independent of one another, have different style, and can be combined according to the interest of the reader. A reader interested only in a very brief overview of the theory and its results can find this in Section 9. Graduate students and nonspecialists may get a general idea of what goes on in this field and its main ideas from Sections 2 and 7. If interested only in the technical aspects of the theory and its physical results, one can read Sections 6 and 7 alone. Scientists working in this field can use Sections 6 and 7 as a reference, and I hope they will find Sections 2, 5 and 8 stimulating. I will not enter into technical details. I will point to the literature where these details are discussed. I have tried to be as complete as possible in indicating all relevant aspects and potential difficulties of the issues discussed.

The literature in this field is vast, and I am sure that there are works whose existence or relevance I have failed to recognize. I sincerely apologize to the authors whose contributions I have neglected or under-emphasized, and I strongly urge them to contact me to help me make this review more complete. The "living reviews" are constantly updated, and I can correct errors and omissions.

# 2  Quantum Gravity: Where Are We?

This is a nontechnical section in which I illustrate the problem of quantum gravity in general, its origin, its importance, and the present state of our knowledge in this regard.

The problem of describing the quantum regime of the gravitational field is still open. There are tentative theories and competing research directions. For an overview, see [258, 155]. The book [218] presents a large and interesting spectrum of viewpoints and opinions. The two largest research programs are string theory and loop gravity. Examples of other directions explored are noncommutative geometry [89], causal dynamical triangulations [13], causal sets [279], twistor theory [225], doubly-special relativity [166], and Euclidean quantum gravity [141, 144]. Research directions are variously related; in particular, formalisms such as spin foams (Section 6.7), group field theory (Section 6.8), or uniform discretizations (Section 6.10) are variously viewed as strictly related to loop gravity or independent research directions.

String theory and loop gravity differ not only because they explore distinct physical hypotheses, but also because they are expressions of two separate communities of scientists, which have sharply distinct prejudices, and who view the problem of quantum gravity in surprisingly different manners.[2]

## 2.1  What is the problem? The view of a high-energy physicist

High-energy physics has obtained spectacular successes during the last century, culminating with the laborious establishment of quantum field theory as the general form of dynamics and with the extraordinary and unexpected success of the $SU(3) \times SU(2) \times U(1)$ standard model. This success is now several decades old. Thanks to it, physics is in a position in which it has been rarely: there are virtually no experimental results that clearly challenge, or clearly escape, the present fundamental theory of the world. The theory we have encompasses virtually everything – except gravitational phenomena.[3] From the point of view of a particle physicist, gravity is then simply the last and the weakest of the interactions. The problem of quantum gravity is perceived as a last step in the path towards unification. It is then natural to try to understand the quantum properties of gravity using the strategy that has been so successful for the rest of microphysics, or variants of this strategy.

The search for a conventional quantum field theory capable of embracing gravity has spanned several decades and, through an adventurous sequence of twists, moments of excitement and bitter disappointments, has lead to string theory. The foundations of string theory are not yet well understood; and it is not entirely clear how the current versions of the theory, which are supersymmetric and formulated in 10 or 11 dimensions, can be concretely used for deriving comprehensive univocal predictions about *our* world. But string theory may claim remarkable theoretical successes and is today the most widely studied candidate theory of quantum gravity.

In string theory, gravity is just one of the excitations of a string or other extended object, living on some metric space. The existence of such background spaces, in which a theory is defined, is the key technical tool for the formulation and the interpretation of the theory, at least in the case of the perturbative definition of the theory. In tentative nonperturbative definitions, such as aiming to define the physical theory indirectly via a boundary quantum field theory, the theory relies only on the background "at infinity", needed for the definition of the boundary quantum field theory.

In all cases, for a physicist with a high-energy background, the central problem of quantum gravity is reduced to an aspect of the problem of understanding the still mysterious nonperturbative theory that has the various perturbative theories as its perturbation expansion.

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[2]The relative size of the two communities can be estimated from the fact that there were 452 registered participants at the STRINGS 07 conference in Madrid, EU, and 156 at the LOOPS 07 conference in Morelia, Mexico.

[3]This situation might change with the data from the LHC, expected soon.

## 2.2   What is the problem? The view of a relativist

For a relativist, on the other hand, the idea of a fundamental description of gravity in terms of physical excitations over a background space sounds physically wrong. The key lesson learned from general relativity is that there is no background metric space *over* which physics happens (except, of course, in approximations). The world is more complicated, or perhaps simpler, than that. For a relativist, in fact, general relativity is much more than the field theory of one particular force. Rather, it is the discovery that certain classical notions about space and time are inadequate at the fundamental level: they require modifications, which are possibly as basic as those introduced by quantum mechanics. One of these inadequate notions is precisely the notion of a background space (flat or curved), in which physics happens. This profound conceptual shift, which has led to the understanding of relativistic gravity, the discovery of black holes, relativistic astrophysics and modern cosmology, is now considered by relativists to be acquired knowledge about the world.

From Newton to the beginning of the last century, physics has had a solid foundation in a small number of key notions such as space, time, causality and matter. In spite of substantial evolution, these notions have remained rather stable and self-consistent. In the first quarter of the last century, quantum theory and general relativity have deeply modified this foundation. The two theories have obtained solid success and vast experimental corroboration, and can now be considered well established. Each of the two theories modifies the conceptual foundation of classical physics in a (more or less) internally-consistent manner, but we do not have a novel conceptual foundation capable of supporting *both* theories. This is why we do not yet have a theory capable of predicting what happens in the physical regime in which both theories are relevant, the regime of Planck-scale phenomena, $10^{-33}$ cm.

General relativity has taught us not only that space and time share the property of being dynamical with the rest of the physical entities, but also – more crucially – that spacetime location is relational (see Section 5.3). Quantum mechanics has taught us that any dynamical entity is subject to Heisenberg's uncertainty at small scale. Therefore, we need *a relational notion of a quantum spacetime* in order to understand Planck-scale physics.

Thus, for a relativist, the problem of quantum gravity is the problem of bringing a vast conceptual revolution, begun with quantum mechanics and general relativity, to a conclusion and to a new synthesis.[4] In this synthesis the notions of space and time need to be deeply reshaped in order to take into account what we have learned with both our present "fundamental" theories.

The difference between the formulation of the problem of quantum gravity given by a high-energy physicist and a relativist derives therefore from a different evaluation of general relativity. For the first, it is just one additional field theory with a funny gauge invariance; for the second, it is a complete modification in the way we think about space and time.

This issue is often confused with the issue of whether the Einstein equations are low-energy equations that need to be corrected at high energy. But the two issues are not related: many relativists expect that the Einstein equations may very well require corrections at high energy. However, they do not expect that the corrected theory will mean a return to the old pre-general-relativistic notions of space and time.

Unlike string theory, loop quantum gravity has a direct fundamental formulation, in which the degrees of freedom are clear, and which does not rely on a background spacetime. Loop quantum gravity is thus a genuine attempt to grasp what quantum spacetime is at the fundamental level. Accordingly, the notion of spacetime that emerges from the theory is profoundly different from the one on which conventional quantum field theory or string theory is based.

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[4]For a discussion of this idea, see [256], [276] and [261].

## 2.3   Strings and loops

Above I have pointed out the distinct cultural paths leading to string theory and loop gravity. Here I attempt to compare the actual achievements that the two theories have obtained so far in describing Planck-scale physics.

Once more, however, I want to emphasize that, whatever prejudices this or that physicist may have, both theories are *tentative*: as far as we truly know, either, or both, could very well turn out to be physically wrong. And I do not mean that they could be superseded: I mean that all their specific predictions could be disproved by experiments. Nature does not always share our aesthetic judgments, and the history of theoretical physics is full of great enthusiasms turned into disappointment. The arbiters in science are experiments, and *not a single experimental result supports directly any of the current theories that go beyond the standard model and general relativity* (say with neutrino mass and a cosmological constant).

On the contrary, a fact, which is perhaps not sufficiently emphasized, is that *all* predictions made so far by theories that go beyond the standard model and general relativity (proton decay, supersymmetric particles, exotic particles, anomalous solar-system dynamics, short-scale corrections to Newton's law. . . ) have for the moment been regularly falsified by experiment!

Comparing this situation with the astonishing experimental success of the standard model and classical general relativity should make us very cautious, I believe. The possibility that a large part of the current theoretical research is following a wrong direction is very concrete[5]. Lacking experiments, theories can only be compared on completeness and aesthetic criteria, but these criteria may be misleading. One should not forget that, according to many, for quite some time these criteria favored Ptolemy over Copernicus.

In this situation, the existence of competing ideas, competing prejudices and competing research programs is not a weakness of theoretical physics; to the contrary, it is a genuine strength. Science grows in debates and confrontation of ideas.

The main merits of string theory are (i) its elegant unification of different theories used in known fundamental physics, (ii) its well-defined perturbation expansion, expected to be finite order-by-order, and (iii) its theoretical and mathematical richness and complexity. The main incompleteness is that its nonperturbative regime is very poorly understood, and we do not know the background-independent formulation: in a sense, we do not really know what the theory is, and how to describe its basics degrees of freedom. Thus, we control the theory only in sectors that (because of the numbers of dimensions or the unbroken super-symmetry) are neither Planck-scale physics, nor low-energy physics. More precisely: (i) There is not much Planck-scale physics derived from string theory so far. Exceptions are the investigation of the Bekenstein–Hawking entropy, including Hawking radiation spectrum and greybody factors, for certain peculiar kinds of black holes (classical references are [282, 148, 147, 146]; see a string review for references on recent developments in this topic), and some very-high-energy scattering amplitudes [9, 10, 11, 12, 295, 283]. An intriguing aspect of these scattering amplitudes is that they appear to indicate that geometry below the Planck scale cannot be probed – and thus in a sense does not exist – in the theory. (ii) We are *not* able to recover the correct low-energy physics, namely the full standard model in 4D, without unbroken supersymmetry, three generations, and the full standard-model phenomenology, from string theory. We do not even know for sure if correct low-energy physics is really predicted by the theory, and, if so, if it is predicted uniquely or as one out of many possibilities.

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[5]I have heard the following criticism from a well-known high-energy physicist: "Loop gravity is certainly physically wrong because it is not supersymmetric, and because it is in four dimensions". This is silly. For the moment the world still insists on looking four-dimensional and non-supersymmetric in all experiments. Science is to adapt our ideas to what we find in Nature; it is not to disbelieve Nature because we like our ideas. We must not mistake very interesting hypotheses (such as supersymmetry and higher dimensions) for established truths.

The main merit of loop quantum gravity is that it provides a mathematically-rigorous formulation of a background-independent, nonperturbative generally-covariant quantum field theory. It provides a physical picture of the world, and quantitative predictions, at the Planck scale. This has allowed, for instance, explicit investigations of the physics of the Big Bang, and the derivation of black-hole entropy for physical black holes. The main incompleteness of the theory regards the formulation of the dynamics, which is studied along different directions, and in several variants. In particular, the recovery of low-energy physics is under investigation, but no convincing derivation of classical GR from loop gravity is yet available. Finally, recall that the aim of loop quantum gravity is to unify gravity and quantum theory, and not to achieve a complete unified theory of all interactions.

Strings and loop gravity may not necessarily be competing theories: there might be a sort of complementarity, at least methodological, between the two. Indeed, the open problems of string theory mostly concern its background-independent formulation, while loop quantum gravity is precisely a set of techniques for dealing with background-independent theories. Perhaps the two approaches might even, to some extent, converge. The possibility has been explored, for instance, in [277]. Undoubtedly, there are similarities between the two theories: first of all the obvious fact that both theories utilize the idea that the relevant excitations at the Planck scale are one-dimensional objects – loops and strings.

But there are also key differences: in an image, strings are one-dimensional objects *moving in space*, while loops are one-dimensional objects *forming space*.

# 3   The Development of Loop Gravity

The following chronology does not exhaust the history of loop gravity, it only indicates some steps in the construction of the theory, in order to provide orientation.

**1986 Connection formulation of general relativity**
Loop gravity is based on the "Ashtekar formulation" of classical general relativity. (Abhay Ashtekar calls it "connectio-dynamics", in contrast to Wheeler's "geometro-dynamics".) [271, 16, 17]. Many recent works in loop gravity are based on a *real* variant of the original Ashtekar connection whose utility for Lorentzian general relativity has been emphasized by Barbero [60, 61, 62, 63].

**1986 Wilson-loop solutions of the Hamiltonian constraint**
Soon after the introduction of the Ashtekar variables, Ted Jacobson and Lee Smolin realize that the Wheeler–DeWitt equation, reformulated in terms of the new variables, admits a simple class of exact solutions: the traces of the holonomies of the Ashtekar connection around smooth non-self-intersecting loops [161]. In other words: the Wilson loops of the Ashtekar connection solve the Wheeler–DeWitt equation if the loops are smooth and non self-intersecting.

**1987 The loop representation**
The discovery of the Jacobson–Smolin Wilson-loop solutions suggests that one "change basis in the Hilbert space of the theory", choosing the Wilson loops as the new basis states for quantum gravity [264, 240, 263, 265]. Quantum states can be represented in terms of their expansion on the loop basis, namely as functions on a space of loops. This idea is well known in the context of canonical lattice Yang–Mills theory [164]. Its application to continuous Yang–Mills theory had been explored by Gambini and Trias [132, 133]. The difficulties of the loop representation in the context of Yang–Mills theory are cured by the diffeomorphism invariance of GR (see Section 6.4). The immediate results are two: (i) the diffeomorphism constraint is completely solved by knot states (loop functionals that depend only on the knotting of the loops), making earlier suggestions by Smolin on the role of knot theory in quantum gravity [273] concrete; and (ii) knot states with support on non-self-intersecting loops are proven to be solutions of all quantum constraints, namely exact physical states of quantum gravity.

**1988 Exact states of quantum gravity**
The investigation of exact solutions of the quantum constraint equations, and their relation to knot theory (in particular to the Jones polynomial and other knot invariants), started soon after the formulation of the theory [149, 78, 79, 80, 81, 233, 127, 129, 163, 122, 107].

**1989 Model theories**
The years immediately following the discovery of the loop formalism are mostly dedicated to understanding the loop representation by studying it in simpler contexts, such as 2+1 general relativity [26, 195, 36], Maxwell [37], linearized gravity [38], and, later, 2D Yang–Mills theory [35].

**1992 Discreteness: I. Weaves**
The first indication that the theory predicts Planck-scale discreteness derives from studying the states that approximate flat geometries on large scale [39]. These states, called "weaves", can be viewed as a formalization of Wheeler's "spacetime foam". Surprisingly, these states turn out *not* to require that the average spacing of the loops go to zero.

**1992 $C^*$ algebraic framework**
Abhay Ashtekar and Chris Isham show that the loop transform can be given a rigorous mathematical foundation, and lay the foundation for a mathematical systematization of the loop ideas, based on $C^*$ algebra ideas [27].

**1994 Fermions**
Fermion coupling is explored in [206, 207]. Later, matter's kinematics is studied by Baez and Krasnov [168, 54], while Thiemann extends his results on dynamics to the coupled Einstein Yang–Mills system in [290].

**1994 Ashtekar–Lewandowski measure and scalar product**
Abhay Ashtekar and Jerzy Lewandowski lay the foundation of the differential formulation of loop quantum gravity by constructing a diffeomorphism-invariant measure on the space of (generalized) connections [28, 29, 30]. They give a mathematically-rigorous construction of the state space of the theory. They define a consistent scalar product and prove that the quantum operators in the theory are consistently defined. Key contributions to the understanding of the measure are given by John Baez, Don Marolf and Josè Mourão [43, 44, 42, 198]. Don Marolf clarifies the use of formal group integration for solving the constraints [194, 196, 197]. The definitive setting of the two versions of the formalism is completed shortly after for the loop formalism (the actual loop representation) [98] and for the differential formalism (the connection representation) [34]. Roberto DePietri proves the equivalence of the two formalisms [96], using ideas from Thiemann [284] and Lewandowski [175].

**1994 Discreteness: II. Area and volume eigenvalues**
Certain geometrical quantities, in particular area and volume, are represented by operators that have discrete eigenvalues. The first set of these eigenvalues is obtained in [268]. The result is confirmed and extended using a number of different techniques. Renate Loll [183, 184] uses lattice techniques to analyze the volume operator and correct a numerical error in [268]. Ashtekar and Lewandowski [174, 31] recover and complete the computation of the spectrum of the area using the connection representation and new regularization techniques. In turn, the full spectrum of the area is then recovered using the loop representation [121]. The general eigenvalues of the volume are computed [98]. Lewandowski clarifies the relation between different versions of the volume operator [175].

**1995 Spin networks**
A long standing problem with the loop basis was its overcompleteness. A technical, but crucial step in understanding the theory is the discovery of the spin-network basis, which solves this overcompleteness [269]. The idea derives from the work of Roger Penrose [224, 223], from analogous bases used in lattice gauge theory, and from ideas by Lewandowski [173] and Jorge Pullin on the relevance of graphs and nodes for the theory. The spin network formalism is cleaned up and clarified by John Baez [47, 48].

**1996 Hamiltonian constraint**
The first version of the loop Hamiltonian constraint [263, 265] is studied and repeatedly modified in a number of works [149, 71, 81, 79, 78, 80, 233, 127, 74]. A key step is the realization that certain regularized loop operators have finite limits on diffeomorphism-invariant states [266]. The search culminates with the work of Thomas Thiemann, who is able to construct a fully well-defined anomaly-free Hamiltonian operator [285, 289, 291].

**1996 Black-hole entropy**
The derivation of the Bekenstein–Hawking formula for the entropy of a black hole from loop quantum gravity is obtained [253], on the basis of the ideas of Kirill Krasnov [170, 171] and

Lee Smolin [274]. The theory is developed and made rigorous by Ashtekar, Baez, Corichi and Krasnov [22].

**1997 Spin foams**
A "sum over histories" spacetime formulation of loop quantum gravity is derived [257, 236] from the canonical theory. The resulting covariant theory turns out to be a sum over topologically non-equivalent surfaces, realizing suggestions by Baez [45, 42, 47, 41], Reisenberger [235, 234] and Iwasaki [156] that a covariant version of loop gravity should look like a theory of surfaces. Baez studies the general structure of theories defined in this manner [49] and introduces the term "spin foam".

**1997 The Barrett–Crane vertex**
John Barrett and Louis Crane introduce the Barrett–Crane vertex amplitude [66], which will become one of the main tools for exploring dynamics in loop gravity and in other approaches.

**1999 Group field theory**
The definition of the Barrett–Crane spin-foam model – in its different versions – is completed in [97, 115], where group-field-theory techniques are also introduced, deriving them from topological field theories.

**2000 Quantum cosmology**
The application of loop quantum gravity to cosmology is started by Martin Bojowald [72], to be later extensively developed by Ashtekar, Bojowald and others.

**2001 Spin-foam finiteness**
Alejandro Perez gives the first proof of finiteness of a spin-foam model [228].

**2003 Master constraint**
Thomas Thiemann introduces the idea of replacing the full set of quantum constraints with a single ("master") constraint [293].

**2004 Black hole singularity at $r = 0$**
Leonardo Modesto [203], and, independently, Ashtekar and Bojowald [24], apply techniques derived from quantum cosmology to explore the $r = 0$ singularity at the center of a black hole, showing that this is controlled by the quantum theory.

**2005 Loop/spin-foam equivalence in 3D**
Karim Noui and Alejandro Perez prove the equivalence of loop quantum gravity and the spin-foam formalism in three-dimensional quantum gravity [210].

**2005 The LOST theorem**
A key uniqueness theorem for the representation used in loop quantum gravity is proved by Lewandowski, Okolow, Sahlmann and Thiemann [176] and, independently and in a slightly different version, by Christian Fleischhack [111].

**2005 Noncommutative geometry from loop quantum gravity**
Laurent Freidel and Etera Livine show that the low-energy limit of quantum gravity coupled with matter in three dimensions is equivalent to a field theory on a noncommutative spacetime [116].

**2006 Graviton propagator**
Beginning of the computation of $n$-point functions from loop quantum gravity [205] and first computation of some components of the graviton propagator [262].

**2007 The new vertex and the loop/spin-foam relation in 4D**

A vertex amplitude correcting some difficulties of the Barrett–Crane model is introduced in [106], and gives rise to rapid development [105, 181, 182, 227] leading to the formulation of a class of spin-foam models that provide a covariant definition of the LQG dynamics [104, 114, 6].

# 4 Resources

The best way to learn a theory is to read different presentations of the same theory and appreciate different points of views. This is particularly true for a theory, which is tentative and in evolution, like loop quantum gravity. Here is a list of textbooks and introductions to the theory, and other resources.

- Textbooks:

    - *Rovelli's book* [261] is a general introduction to loop quantum gravity. It contains a first part in which the main conceptual problems of quantum gravity are discussed, and both general relativity and quantum mechanics are reformulated in a form suitable for their merge. The second part is a detailed introduction to the formalism, with the derivation of all basic results. The emphasis is more on the physical ideas and intuition than on mathematical completeness.

    - *Thiemann's book* [294] is more mathematically oriented and a bit harder to read. It is far more complete than my book, particularly on the definition of the Hamiltonian operator, matter, etc. Apart from some occasional divergence, the two books can be read as a two-part ("I: Conceptual basis and physical introduction" and "II: Mathematical formalism") text on loop quantum gravity.

    - *Ashtekar's old book* [18] may serve as a valuable basic introductory course on Ashtekar variables, particularly for relativists and mathematicians. (The part of the book on the loop representation is essentially a reprint of the original article [265]. For this quantum part, I recommend looking at the article, rather than the book, since the article is more complete.)

    - The *book by Gambini and Pullin* [128] is especially good on lattice techniques and on the variant of loop quantum gravity called the "extended loop representation" [100, 99].

    - An interesting collection of papers can be found in *the volume [46] edited by John Baez.* The other *book by Baez and Muniain* [55] is a simple and pleasant introduction to several ideas and techniques in the field.

- Introductions:

    - This "living review" may serve as a simple introduction to some basic ideas of quantum gravity in the loop formalism.

    - *Ashtekar and Lewandowski* [33] have a good introduction to loop quantum gravity that balances mathematical precision with physical ideas.

    - *Smolin's "Invitation to loop quantum gravity"* [277] is readable and complete, especially for possible physical consequences of the theory.

    - *Thiemann's lectures* [292] (not to be confused with the book) are a very good and readable straightforward introduction to loop quantum gravity.

    - *Ashtekar* has another nice introduction, especially to the recent advances in the field ("Four recent advances and a dozen frequently asked questions"), in [21].

    - *Perez's* lectures [230] give a very nice pedagogical introduction to loop gravity, including the spin-foam formalism.

    - *PhD Theses.* Of the numerous very good PhD theses that are online, I single out the one of Daniele Oriti [213], with a nice introduction to the spin-foam formalism.

- An old, but interesting, general introduction to the new variables, which includes several of the mathematical developments in the quantum theory, is given by *Ashtekar's Les Houches 1992 lectures* [19]. Another simple introduction is in [241].

- Others:

  - Several good listings of introductory literature to the subject can be found on the web. See for instance:
    * The "Basic curriculum for Quantum Gravity" in the "Christine's Background Independence" page of Christine C. Dantas [95] (scroll down).
    * The "Reading Guide to loop Quantum Gravity" by Seth Major [189].
    * Dan Christensen's webpage [87].
    * The webpage of the Loop Quantum Gravity Group of the National University of Singapore [208].

  - A resource for finding relevant literature is the comprehensive *"Bibliography of Publications Related to Classical and Quantum Gravity in terms of Connection and Loop Variables"*, organized chronologically. The original version was compiled by Peter Hübner in 1989. It has subsequently been updated by Gabriella Gonzales, Bernd Brügmann, Monica Pierri, Troy Shiling and Christopher Beetle. It is now updated by Alejandro Corichi and Alberto Hauser. The latest version can be found on the net at [91].

  - Lee Smolin's "Introduction to Quantum Gravity" course is online [272].

  - And so are the notes of John Baez's "Quantum gravity seminar" [40].

- The loop gravity community (often extended to several other non-string approaches) meets at regular conferences. The last meetings have been LOOPS 04 in Marseille, France [185], LOOPS 05 in Potsdam, Germany [186], and LOOPS 07 in Morelia, Mexico [187].

- Some of the institutions where loop quantum gravity is studied are

  - *Institute for Gravity and Geometry* [222], Penn State University, PA, U.S.A.
  - *Perimeter Institute* [232], Waterloo, ON, Canada.
  - *Albert Einstein Institute* [200], Potsdam, Germany.
  - *Louisiana State University*, Baton Rouge, LA, U.S.A.
  - *Center of Theoretical Physics of Luminy* [172], Marseille, France.
  - *The University of Nottingham*, Nottingham, U.K.
  - *Warsaw University*, Warsaw, Poland.
  - *CSIC*, Madrid, Spain.
  - *UNAM*, Mexico City, Mexico.
  - *Ecole National Superieur*, Lyon, France.
  - *Scuola Nazionale Superiore di Pisa*, Pisa, Italy.
  - *Beijing Normal University*, Beijing, China.
  - *Université de Tours*, Tours, France.
  - *Université de Montpellier*, Montpellier, France.
  - *Universitá di Bologna*, Bologna, Italy.
  - *Morelia University*, Morelia, Mexico.

- *Imperial College* [153], London, U.K.
- *Institute for Theoretical Physics*, Utrecht, The Netherlands.
- *Universidad de la República*, Montevideo, Uruguay.
- *University of Western Ontario*, London, ON, Canada.
- *University of New Brunswick*, Saint John, NB, Canada.
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# 5 Ideas and Physical Inputs

The main physics hypotheses on which loop quantum gravity relies are general relativity and quantum mechanics. In other words, loop quantum gravity is a rather conservative "quantization" of general relativity, with its traditional matter couplings, and it is based only on physical ideas well supported by the experiments. In this sense it is very different from string theory, which is based on a new strong physical hypothesis (the Universe is made up of strings).

Of course, "quantization" is far from being a straightforward algorithm, particularly for a nonlinear field theory. Rather, it is a poorly-understood inverse problem: to find a quantum theory with the given classical limit. Various choices are made in constructing the quantum theory. I discuss these choices below.

## 5.1 Quantum field theory on a differentiable manifold

The main idea behind loop quantum gravity is that one take general relativity seriously. General relativity is the discovery that the spacetime metric and the gravitational field are the same physical entity. A quantum theory of the gravitational field is therefore also a quantum theory of the spacetime metric. It follows that quantum gravity cannot be formulated as a quantum field theory over a *metric* manifold, because there is no (classical) metric manifold whatsoever in a regime in which gravity (and therefore the metric) is a quantum variable.

One can conventionally split the spacetime metric into two terms, consider one of the the two terms a background that gives a metric structure to spacetime and treat the other as the quantum field. This is the procedure on which perturbative quantum gravity, perturbative strings, as well as several current nonperturbative string theories, are based. In this framework one assumes that the causal structure of spacetime is determined by the underlying background metric alone, and not by the full metric. Contrary to this, loop quantum gravity assumes that the identification between the gravitational field and the metric-causal structure of spacetime holds, and must be taken into account even in the quantum regime. No split of the metric is made, and there is no background metric on spacetime.

One can still describe spacetime as a (differentiable) manifold (a space without metric structure), over which quantum fields live. A classical metric structure will then be defined only by expectation values of the gravitational field operator. Thus, the problem of quantum gravity is the problem of understanding what is *a quantum field theory on a manifold*, as opposed to quantum field theory on a metric space. This is what gives quantum gravity its distinctive flavor, so different from ordinary quantum field theory. In all versions of ordinary quantum field theory, the metric of spacetime plays an essential role in the construction of basic theoretical tools (creation and annihilation operators, canonical commutation relations, Gaussian measures, propagators . . . ); these tools cannot be used in quantum fields over a manifold.

Technically, the difficulty due to the absence of a background metric is circumvented in loop quantum gravity by defining the quantum theory as a representation of a Poisson algebra of classical observables, which can be defined without using the background metric. The idea that the quantum algebra at the basis of quantum gravity is not the canonical commutation-relation algebra, but the Poisson algebra of a different set of observables, has long been advocated by Chris Isham [154], whose ideas have been very influential in the birth of loop quantum gravity.[6] The algebra on which loop gravity is based, is the loop algebra [265]. Why this algebra?

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[6]Loop quantum gravity is an attempt to solve the last problem in Isham's lectures [154].

## 5.2   Why loops?

In choosing the loop algebra as the basis for the quantization, we are essentially assuming that Wilson loop operators are well defined in the Hilbert space of the theory; in other words, that certain states concentrated on one-dimensional structures (loops and graphs) have finite norm. This is a subtle nontrivial assumption. It is the key assumption that characterizes loop gravity, and is the one that looks most suspicious to scientists that have the habit of conventional background-dependent quantum field theory. If the approach turned out to be wrong, it will likely be because this assumption is wrong. Where does this assumption comes from and why is it dependable?

It comes from an old line of thinking in theoretical physics, according to which the natural variables describing gauge theories are loop-like. This idea has been variously defended by Wilson, Polyakov, Mandelstam, and many others, and, in a sense, can be traced to the very origin of gauge theory, in the intuition of Faraday. According to Faraday, the degrees of freedom of the electromagnetic field are best understood as lines in space: Faraday lines. Can we describe a quantum field theory in terms of its "Faraday lines"?

Consider first this question in a simplified context: on a lattice. The answer is then yes. In a lattice formulation of Yang–Mills theory, the physical Hilbert space of the theory is spanned by well-defined quantum states that are supported by loops on the lattice. These states can be written as traces of the holonomy operator around the loop. They are eigenstates of the electric-field operator, and they precisely represent quantized excitations of a single Faraday line. They are the exact analog of the loop quantum gravity spin-network states.

The attempt to take the continuum limit of this picture, however, fails in Yang–Mills theory. The reason is that when the lattice spacing converges to zero, the "physical width" of the individual loop states shrinks to zero, and the loop states become ill-defined infinite-norm states with one-dimensional support.

However, remarkably, this does not happen in a diffeomorphism-invariant theory. This is because, in the absence of a metric background, there is no sense in "shrinking down" the states. In fact, the size of the state is determined by the metric, which is determined by the gravitational field, which, in turn, is determined by the state itself. An explicit computation shows that refining the lattice space has no effect on the size of the loop states themselves: it only reflects a physically irrelevant change of coordinates.

Thus, in a diffeomorphism invariant theory, we can take the formal continuum limit of the lattice loop states. Once we factor away the gauge transformations defined by the diffeomorphisms, what remains are the abstract physical loop states, which are not localized in a space, but rather that define themselves by the physical excitations of the geometry, as will become clear in the following section.

## 5.3   Physical meaning of diffeomorphism invariance and its implementation in the quantum theory

Conventional field theories are *not* invariant under a diffeomorphism acting on the dynamical fields. (Every field theory, suitably formulated, is trivially invariant under a diffeomorphism acting on *everything*.) General relativity, on the contrary, is invariant under such transformations. More precisely, every general relativistic theory has this property. Thus, diffeomorphism invariance is not a feature of just the gravitational field: it is a feature of physics, once the existence of relativistic gravity is taken into account. One can say that the gravitational field is not particularly "special" in this regard: rather, diff-invariance is a property of the physical world that can be disregarded only in the approximation in which the dynamics of gravity are neglected. What is this property? What is the physical meaning of diffeomorphism invariance?

Diffeomorphism invariance is the technical implementation of a physical idea, due to Einstein.

The idea is a modification of the pre-general-relativistic (pre-GR) notions of space and time. In pre-GR physics, we assume that physical objects can be localized in space and time with respect to a fixed non-dynamical background structure. Operationally, this background spacetime can be defined by means of physical reference-system objects. These objects are considered as dynamically decoupled from the physical system that one studies. This conceptual structure works well in pre-GR physics, but it fails in a relativistic gravitational regime. In general relativistic physics, the physical objects are localized in space and time only with respect to one another. If we "displace" all dynamical objects in spacetime at once, we are not generating a different state, but an equivalent mathematical description of the same physical state. Hence, diffeomorphism invariance.

Accordingly, a physical state in GR is not "located" somewhere [256, 246, 244, 259] (unless an appropriate gauge fixing is made). Pictorially, GR is not physics over a stage, it is the dynamical theory of everything, including the stage itself.

Loop quantum gravity is an implementation of this relational notion of spacetime localization in quantum field theory. In particular, the basic quantum field theoretical excitations are not excitations over a space, but rather excitations of the "stage" itself. In greater detail, we define quantum states that correspond to loop-like and, more generally, graph-like excitations of the gravitational field on a differential manifold (spin networks); but then, when factoring away diffeomorphism invariance, the location of the states becomes irrelevant. The only remaining information contained in the graph is then its abstract graph structure and its knotting. Thus, diffeomorphism-invariant physical states are labeled by *s*-knots: equivalence classes of graphs under diffeomorphisms. An *s*-knot represents an elementary quantum excitation of space. It is not here or there, since it *is* the space with respect to which here and there can be defined. An *s*-knot state is an elementary quantum of space. See Figure 1, and the relative discussion in Section 7.
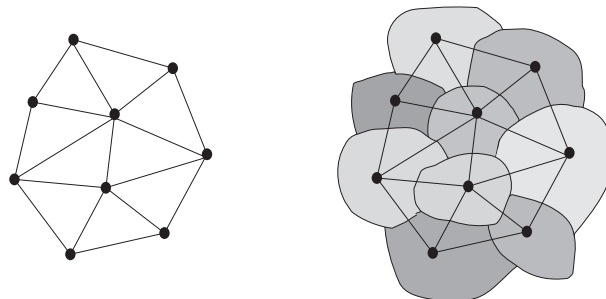


**Figure 1:** A spin network and the "chunks", or quanta, of space it describes.

In this manner, loop quantum gravity binds the new notion of space and time introduced by general relativity with quantum mechanics. As I illustrate later on, the existence of such elementary quanta of space is a consequence of the quantization of the spectra of geometrical quantities.

## 5.4   Problems *not* addressed

Quantum gravity is an open problem that has been investigated for over seventy years now. When one contemplates two deep problems, one is tempted to believe that they are related. In the history of physics, there are surprising examples of two deep problems solved by one stroke (the unification of electricity and magnetism and the nature of light, for instance); but there are also many examples in which a great hope to solve more than one problem at once was disappointed (finding the theory of strong interactions and getting rid of quantum-field-theory infinities, for instance). Quantum gravity has been asked, at one time or another, to address almost every

deep open problem in theoretical physics (and beyond). Here is a list of problems that have been connected to quantum gravity in the past, but about which loop quantum gravity has little to say:

### Interpretation of quantum mechanics.

Loop quantum gravity is a standard quantum (field) theory. Pick your favorite interpretation of quantum mechanics, and use it for interpreting the quantum aspects of the theory. I will refer to two such interpretations below. When discussing the quantization of area and volume, I will use the relation between eigenvalues and outcomes of measurements performed with classical physical apparatuses; when discussing evolution, I will refer to the histories interpretation. The peculiar way of describing time evolution in a general relativistic theory may require some appropriate variants of standard interpretations, such as Hartle's generalized quantum mechanics [140], or a suitable generalization of canonical quantum theory [261, 243, 245, 242]. But loop quantum gravity has no help to offer the scientists who have speculated that quantum gravity will solve the measurement problem. For a different point of view, see [278]. My own ideas on the interpretation of quantum mechanics are in [255] and [261]. On the other hand, I think that solving the problem of the interpretation of quantum mechanics might require relational ideas connected with the relational nature of spacetime revealed by general relativity. These issues are discussed in detail in my book [261].

### Quantum cosmology.

The expression "Quantum cosmology" is used with several different meanings. First, it is used to designate the quantum theory of the cosmological gravitational degrees of freedom of our universe. The application of loop gravity to this problem is substantial. Second, it is used to designate the theory of the entire universe as a quantum system without external observer [139], with or without gravity. The two meanings are unrelated, but confusion is common. Quantum gravity is the theory of one dynamical entity: the quantum gravitational field (or the spacetime metric), just one field among the many degrees of freedom of the universe. Precisely as for the theory of the quantum electromagnetic field, we can always assume that we have a classical observer with classical measuring apparatuses measuring gravitational phenomena, and therefore study quantum gravity under the assumption that there is an observer, which is not part of the quantum system studied.

### Unifications of all interactions or "Theory of Everything".

A common criticism of loop quantum gravity is that it does not unify all interactions. But the idea that quantum gravity can be understood *only* in conjunction with other fields is an interesting hypothesis, certainly not an established truth.

### Masses of the elementary particles.

As far as I see, nothing in loop quantum gravity suggests that one could compute masses from quantum gravity.

### Origin of the Universe.

A sound quantum theory of gravity is needed to understand the physics of the Big Bang. The converse is probably not true: we should be able to understand the small-scale structure of spacetime, even if we do not understand the origin of the Universe.

**Arrow of time.**

Roger Penrose has argued for some time that it should be possible to trace the time asymmetry in the observable Universe to quantum gravity.

**Physics of the mind.**

Penrose has also speculated that quantum gravity is responsible for the wave function collapse, and, indirectly, governs the physics of the mind [226].

A problem that has been repeatedly tied to quantum gravity, and which loop quantum gravity *is* able to address, is the problem of the ultraviolet infinities in quantum field theory. The very peculiar nonperturbative short-scale structure of loop quantum gravity introduces a physical cutoff. Since physical spacetime itself comes in quanta in the theory, there is literally no space in the theory for the very high momentum integrations that originate from the ultraviolet divergences.

# 6 Formalism

I give here a simple technical description of the formalism of loop quantum gravity. For a more detailed construction, see [261, 294, 292, 33].

## 6.1 Classical theory

The starting point of the construction of quantum theory is classical general relativity, formulated in terms of the Sen–Ashtekar–Barbero connection [271, 16, 61]. Detailed introductions to the (complex) Ashtekar formalism can be found in the book [18] and in the conference proceedings [103]. The real version of the theory is presently the most widely used.

Classical general relativity can be formulated in phase-space form as follows [18, 61]. Fix a three-dimensional manifold $M$ (compact and without boundaries) and consider a smooth real $SU(2)$ connection $A_a^i(x)$ and a vector density $\tilde{E}_i^a(x)$, transforming in the vector representation of $SU(2)$ on $M$. We use $a, b, \ldots = 1, 2, 3$ for spatial indices and $i, j, \ldots = 1, 2, 3$ for internal indices. The internal indices can be viewed as labeling a basis in the Lie algebra of $SU(2)$ or the three axis of a local triad. We indicate coordinates on $M$ as $x$. The relation between these fields and conventional metric gravitational variables is as follows: $\tilde{E}_i^a(x)$ is the (densitized) inverse triad, related to the three-dimensional metric $g_{ab}(x)$ of constant-time surfaces by

$$g \, g^{ab} = \tilde{E}_i^a \tilde{E}_i^b, \tag{1}$$

where $g$ is the determinant of $g_{ab}$; and

$$A_a^i(x) = \Gamma_a^i(x) + \gamma \, k_a^i(x); \tag{2}$$

$\Gamma_a^i(x)$ is the spin connection associated to the triad. (This is defined by $\partial_{[a} e_{b]}^i = \Gamma_{[a}^i e_{b]j}$, where $e_a^i$ is the triad). $k_a^i(x)$ is the extrinsic curvature of the constant-time three-surface.

In Equation (2), $\gamma$ is a constant, denoted the Immirzi parameter, that can be chosen arbitrarily (it will enter the Hamiltonian constraint) [152, 151, 150]. Different choices for $\gamma$ yield different versions of the formalism, all equivalent in the classical domain. If we choose $\gamma$ to be equal to the imaginary unit, $\gamma = \sqrt{-1}$, then $A$ is the standard Ashtekar connection, which can be shown to be the projection of the self-dual part of the four-dimensional spin connection on the constant-time surface. If we choose $\gamma = 1$, we obtain the real Barbero connection. The Hamiltonian constraint of Lorentzian general relativity has a particularly simple form in the $\gamma = \sqrt{-1}$ formalism; while the Hamiltonian constraint of Euclidean general relativity has a simple form when expressed in terms of the $\gamma = 1$ real connection. Other choices of $\gamma$ are viable as well. Different choices of $\gamma$ are genuinely physical physically? nonequivalent in the quantum theory, since they yield "geometrical quanta" of different magnitude [270]. It has been argued that there is a unique choice of $\gamma$ yielding the correct $1/4$ coefficient in the Bekenstein–Hawking formula [170, 171, 253, 22, 254, 92], but the matter is still under discussion; see for instance [160].

The spinorial version of the Ashtekar variables is given in terms of the Pauli matrices $\sigma_i, i = 1, 2, 3$, or the $SU(2)$ generators $\tau_i = -\frac{\mathrm{i}}{2} \sigma_i$, by

$$\tilde{E}^a(x) = -\mathrm{i} \, \tilde{E}_i^a(x) \, \sigma_i = 2 \tilde{E}_i^a(x) \, \tau_i, \tag{3}$$

$$A_a(x) = -\frac{\mathrm{i}}{2} \, A_a^i(x) \, \sigma_i = A_a^i(x) \, \tau_i. \tag{4}$$

Thus, $A_a(x)$ and $\tilde{E}^a(x)$ are $2 \times 2$ anti-Hermitian complex matrices.

The theory is invariant under local $SU(2)$ gauge transformations, three-dimensional diffeomorphisms of the manifold on which the fields are defined, as well as under (coordinate) time translations generated by the Hamiltonian constraint. The full dynamical content of general relativity is captured by the three constraints that generate these gauge invariances.

The Lorentzian Hamiltonian constraint does not have a simple polynomial form if we use the real connection (2). For a while, this fact was considered an obstacle to defining the quantum Hamiltonian constraint; therefore, the complex version of the connection was mostly used. However, Thiemann has succeeded in constructing a Lorentzian quantum-Hamiltonian constraint [285, 289, 291] in spite of the non-polynomiality of the classical expression. This is why the real connection is now widely used. This choice has the advantage of eliminating the old "reality conditions" problem, namely the problem of implementing nontrivial reality conditions in the quantum theory.

Alternative versions of the classical formalism used as a starting point for the quantization have been explored in the literature. Of particular interest is the approach followed by Alexandrov, who has argued for a formalism where the full local $SO(3,1)$ symmetry of the tetrad formalism is manifestly maintained [4, 5, 6]. One of the advantages of this approach is that it sheds light on the relationship with covariant spin-foam formalism (see below). Its main difficulty is to fully keep track of the complicated second-class constraints and the resulting nontrivial Dirac algebra.

## 6.2   Quantum kinematics

Certain classical quantities play a very important role in the quantum theory. These are: traces of the holonomy of the connection, which are labeled by loops on the 3-manifold; and surface integrals of the triad. Given a loop $\alpha$ in $M$ define:

$$\mathcal{T}[\alpha] = -\mathrm{Tr}[U_\alpha], \tag{5}$$

where $U_\alpha \sim \mathcal{P}exp\{\oint_\alpha A\}$ is the parallel propagator of $A_a$ along $\alpha$, which is defined by $U_\alpha(0) = \mathbb{1}$,

$$\frac{d}{ds}U_\alpha(s) = \frac{d\alpha^a(s)}{ds}\ A_a(\alpha(s))\ U_\alpha(s) \tag{6}$$

and $U_\alpha = U_\alpha(1)$. Given a two-dimensional surface $S$, define

$$E[S,f] = \int_S dS_a\ \tilde{E}_i^a\ f^i, \tag{7}$$

where $f$ is a function on the surface $S$, taking values in the Lie algebra of $SU(2)$. These two quantities are naturally represented as quantum operators on the Hilbert space of the quantum theory, which I now define.

Consider a "Schrödinger-like" representation formed by quantum states that are functionals $\psi(A)$ of the connection. On these states, the two quantities $\mathcal{T}[\alpha]$ and $E[S,f]$ act naturally: the first as a multiplicative operator, the second as the functional derivative operator

$$E[S,f] = G\hbar \int_S dS_a\ f^i\ \frac{\delta}{\delta A_a^i}. \tag{8}$$

This defines an heuristic quantization, à la Wheeler–DeWitt. What is needed to make it mathematically precise, and to make it usable for concrete calculations is a precise characterization of the class of functionals $\psi(A)$ to be considered, and, especially, a definition of the scalar product among these. Let us do so.

The class of functionals that we will use is formed by (the closure in the Hilbert-space norm of the linear span of) functionals of a particular class, denoted "cylindrical states". These are defined as follows. Pick a graph $\Gamma$, say with $n$ links, denoted $\gamma_1 \ldots \gamma_n$, immersed in the manifold $M$. Let $U_i(A) = U_{\gamma_i}$, $i = 1, \ldots, n$ be the parallel transport operator of the connection $A$ along $\gamma_i$. $U_i(A)$ is an element of $SU(2)$. Pick a function $f(g_1 \ldots g_n)$ on $[SU(2)]^n$. The graph $\Gamma$ and the function $f$ determine a functional of the connection as follows

$$\psi_{\Gamma,f}(A) = f(U_1(A), \ldots, U_n(A)). \tag{9}$$

(These states are called cylindrical states because they were introduced in [28, 29, 30] as cylindrical functions for the definition of a cylindrical measure.) Notice that we can always "enlarge the graph", in the sense that if $\Gamma$ is a subgraph of $\Gamma'$, we can always write

$$\psi_{\Gamma,f}(A) = \psi_{\Gamma',f'}(A) \tag{10}$$

by simply choosing $f'$ independent from the $U_i$'s of the links, which are in $\Gamma'$ but not in $\Gamma$. Thus, given any two cylindrical functions, we can always view them as having the same graph (formed by the union of the two graphs). Given this observation, we define the scalar product between any two cylindrical functions [173, 28, 29, 30] by

$$(\psi_{\Gamma,f}, \psi_{\Gamma,h}) = \int_{SU(2)^n} dg_1 \ldots dg_n \, \overline{f(g_1 \ldots g_n)} \, h(g_1 \ldots g_n), \tag{11}$$

where $dg$ is the Haar measure on $SU(2)$. This scalar product extends by linearity to finite linear combinations of cylindrical functions. It is not difficult to show that Equation (11) defines a well-defined scalar product on the space of these linear combinations. Completing the space of these linear combinations in the Hilbert norm, we obtain a Hilbert space $\mathcal{H}$. This is the (unconstrained) quantum-state space of loop gravity.

The main property of $\mathcal{H}$ is that it carries a natural unitary representation of the diffeomorphism group and of the group of the local $SU(2)$ transformations, obtained transforming the argument of the functionals. In fact, the essential property of the scalar product (11) is that it is invariant under both these transformations. The operators $\mathcal{T}[\alpha]$ and $E[S, f]$ are well-defined self-adjoint operators in this Hilbert space.

A number of observations are in order.

- The construction of this quantum representation may seem arbitrary, but a powerful theorem [176, 111, 112], called the LOST theorem (from the initials of the authors of one of the two versions of the theorem), states that, under rather general assumptions, this representation is *unique*, up to unitary equivalence. This is in the same sense in which the usual Schrödinger representation of nonrelativistic quantum mechanics is unique. The main hypothesis of the theorem is the diffeomorphism invariance of the theory. This shows that diffeomorphism invariance is a powerful constraint on the form of the quantum field theory.

- $\mathcal{H}$ is nonseparable. After factoring away diffeomorphism invariance, we obtain a separable Hilbert space (see Section 6.4).

- From the point of view of the physical intuition, cylindrical functions can be seen first of all as a convenient way to span the space of the functional of the connection. (In a suitable topology, any functional of the connection can be approximated by a linear combination of such functions.) On the other hand, the choice reflects the physics. In Yang–Mills theory, this choice would lead to an inconsistent theory based on a nonseparable Hilbert space. Here, on the other hand, diffeomorphism invariance cures the nonseparability. This is the mathematical implementation of the physical argument concerning the existence of the continuous limit of loop states, which was given in Section 5.2.

- Standard spectral theory holds on $\mathcal{H}$, and it turns out that using spin networks (discussed below) one can express $\mathcal{H}$ as a direct sum over finite-dimensional subspaces, which have the structure of Hilbert spaces of spin systems; this makes practical calculations very manageable.

Using Dirac notation, we write

$$\Psi(A) = \langle A|\Psi\rangle, \tag{12}$$

in the same manner in which one may write $\psi(x) = \langle x|\Psi\rangle$ in ordinary quantum mechanics. As in that case, $|A\rangle$ is not a normalizable state.

## 6.3   Loop and spin network states

A subspace $\mathcal{H}_0$ of $\mathcal{H}$ is formed by states invariant under $SU(2)$ gauge transformations. We now define an orthonormal basis in $\mathcal{H}_0$. This basis represents a very important tool for using the theory. It was introduced in [269] and developed in [47, 48]; it is denoted 'spin network basis'.

First, given a loop $\alpha$ in $M$, there is a normalized state $\psi_\alpha(A)$ in $\mathcal{H}$, which is obtained by taking $\Gamma = \alpha$ and $f(g) = -Tr(g)$. Namely

$$\psi_\alpha(A) = -\text{Tr}U_\alpha(A). \tag{13}$$

We introduce Dirac notation for the abstract states, and denote this state as $|\alpha\rangle$. These states are called loop states. Using Dirac notation, we can write

$$\psi_\alpha(A) = \langle A|\alpha\rangle. \tag{14}$$

It is easy to show that loop states are normalizable. Products of loop states are normalizable as well. Following tradition, we also denote with $\alpha$ a multiloop, namely a collection of (possibly overlapping) loops $\{\alpha_1, \ldots, \alpha_n, \}$, and we call

$$\psi_\alpha(A) = \psi_{\alpha_1}(A) \times \ldots \times \psi_{\alpha_n}(A) \tag{15}$$

a multiloop state. (Multi-)loop states represented the main tool for loop quantum gravity before the discovery of the spin network basis. Linear combinations of multiloop states (over-)span $\mathcal{H}$, and, therefore, a generic state $\psi(A)$ is fully characterized by its projections on the multiloop states, namely by

$$\psi(\alpha) = (\psi_\alpha, \psi). \tag{16}$$

The "old" loop representation was based on representing quantum states in this manner, namely by means of the functionals $\psi(\alpha)$ over loop space defined in Equation (16). Equation (16) can be explicitly written as an integral transform, as we will see in Section 6.5.

Next, consider a graph $\Gamma$. A "coloring" of $\Gamma$ is given by the following.

1. Associate an irreducible representation of $SU(2)$ to each link of $\Gamma$. Equivalently, associate to each link $\gamma_i$ a half-integer number $j_i$, the spin of the irreducible representation.

2. Associate an invariant tensor $v$ in the tensor product of the representations $j_1 \ldots j_n$ to each node of $\Gamma$ in which links with spins $j_1 \ldots j_n$ meet. An invariant tensor is an object with $n$ indices in the representations $j_1 \ldots j_n$ that transform covariantly. If $n = 3$, there is only one invariant tensor (up to a multiplicative factor), given by the Clebsch–Gordan coefficient. An invariant tensor is also called an *intertwiner*. All intertwiners are given by the standard Clebsch–Gordan theory. More precisely, for fixed $j_1 \ldots j_n$, the invariant tensors form a finite-dimensional linear space. Pick a basis $v_i$ is this space, and associate one of these basis elements to the node. Notice that invariant tensors exist only if the tensor product of the representations $j_1 \ldots j_n$ contains the trivial representation. This yields a condition on the coloring of the links. For $n = 3$, this is given by the well -known Clebsch-Gordan condition: each color is not larger than the sum of the other two, and the sum of the three colors is even.

Indicate a colored graph by $\{\Gamma, \vec{j}, \vec{v}\}$, or simply $S = \{\Gamma, \vec{j}, \vec{v}\}$, and denote it a "spin network". (It was Penrose who first had the intuition that this mathematics could be relevant for describing the quantum properties of the geometry, and who gave the first version of spin-network theory [223, 224].)

Given a spin network $S$, we can construct a state $\Psi_S(A)$ as follows. Take the propagator of the connection along each link of the graph in the representation associated to that link, and then,

at each node, contract the matrices of the representation with the invariant tensor. We obtain a state $\Psi_S(A)$, which we also write as

$$\psi_S(A) = \langle A|S\rangle. \tag{17}$$

One can then show the following.

- The spin network states are normalizable.

- They are $SU(2)$ gauge invariant.

- Each spin network state can be decomposed into a finite linear combination of products of loop states.

- The (normalized) spin network states form an orthonormal basis for the gauge $SU(2)$ invariant states in $\mathcal{H}$ (choosing the basis of invariant tensors appropriately).

The spin network states provide a very convenient basis for the quantum theory, with a direct physical interpretation. This follows from the fact that the spin network states are eigenstates of area and volume operators, therefore they are states in which the three-dimensional geometry is well defined. See [261] for details.

Consider the relations between the loop states

$$\psi(\alpha) = \langle \alpha|\psi\rangle \tag{18}$$

and the states $\psi(A)$ giving the amplitude for a connection field configuration $A$, and defined by

$$\psi(A) = \langle A|\psi\rangle. \tag{19}$$

Here $|A\rangle$ are "eigenstates of the connection operator", or, more precisely (since the operator corresponding to the connection is ill defined in the theory), the generalized states that satisfy

$$T[\alpha]\,|A\rangle = -Tr[\mathcal{P}e^{\int_\alpha A}]\,|A\rangle. \tag{20}$$

We can write, for every spin network $S$, and every state $\psi(A)$

$$\psi(S) = \langle S|\psi\rangle = (\psi_S, \psi). \tag{21}$$

This equation defines a unitary mapping between the two presentations of $\mathcal{H}$: the "loop representation", in which one works in terms of the basis $|S\rangle$; and the "connection representation", in which one uses wave functionals $\psi(A)$.

## 6.4 Diffeomorphism invariance

The next step in the construction of the theory is to factor away diffeomorphism invariance. This is a key step for two reasons. First of all, $\mathcal{H}$ is a "huge" nonseparable space. It is far "too large" for a quantum field theory. However, most of this redundancy is gauge, and disappears when one solves the diffeomorphism constraint, defining the diff-invariant Hilbert space $\mathcal{H}_{Diff}$. This is the reason for which the loop representation, as defined here, is only of value in diffeomorphism invariant theories.

The second reason is that $\mathcal{H}_{Diff}$ turns out to have a natural basis labeled by knots. More precisely by "s-knots". An s-knot $s$ is an equivalence class of spin networks $S$ under diffeomorphisms. An s-knot is characterized by its "abstract" graph (defined only by the adjacency relations between links and nodes), by the coloring, and by its knotting and linking properties, as in knot theory.

Thus, the physical quantum states of the gravitational field turn out to be essentially classified by knot theory.

There are various equivalent ways of obtaining $\mathcal{H}_{Diff}$ from $\mathcal{H}$. One can use regularization techniques for defining the quantum operator corresponding to the classical diffeomorphism constraint in terms of elementary loop operators, and then find the kernel of such operator. Equivalently, one can factor $\mathcal{H}$ by the natural action of the diffeomorphism group that it carries. Namely

$$\mathcal{H}_{Diff} = \frac{\mathcal{H}}{Diff(M)}. \tag{22}$$

There are several rigorous ways for defining the quotient of a Hilbert space by the unitary action of a group. See in particular the construction in [34], which follows the ideas of Marolf and Higuchi [194, 196, 197, 145].

In the quantum gravity literature, a big deal has been made of the problem that a scalar product is not defined on the space of solutions of a constraint $\hat{C}$, defined on a Hilbert space $\mathcal{H}$. This, however, is a false problem. It is true that if zero is in the continuum spectrum of $\hat{C}$, then the corresponding eigenstates are generalized states and the $\mathcal{H}$ scalar product is not defined between them. But the generalized eigenspaces of $\hat{C}$, including the kernel, *nevertheless* inherit a scalar product from $\hat{H}$. This can be seen in a variety of equivalent ways. For instance, it can be seen from the following theorem. If $\hat{C}$ is self-adjoint, then there exists a measure $\mu(\lambda)$ on its spectrum and a family of *Hilbert* spaces $\mathcal{H}(\lambda)$ such that

$$\mathcal{H} = \int d\mu(\lambda)\mathcal{H}(\lambda), \tag{23}$$

where the integral is the continuous sum of Hilbert spaces described, for instance, in [138]. Clearly $\mathcal{H}(0)$ is the kernel of $\hat{C}$ *equipped with a scalar product.* This is discussed, for instance, in [239].

When factoring away the diffeomorphisms in the quantum-theory finite-dimensional moduli spaces associated with high valence, nodes appear [137]. Because of these, the resulting Hilbert space is still nonseparable. These moduli parameters, however, have no physical significance and do not play any role in the quantum theory. They can be discarded by judicious choice of the functional space in which the fields are defined [110, 299, 300], or in other ways [294].

## 6.5   Other structures in $\mathcal{H}$

The mathematical foundations of loop quantum gravity have been developed to the level of rigor of mathematical physics. This has introduced some heavy mathematical tools, sometimes unfamiliar to the average physicist, at the price of widening the language gap between scientists who study quantum gravity and other parts of the community. There is good reason for seeking a mathematical-physics level of precision in quantum gravity. In the development of conventional quantum field theory mathematical rigor could be low because extremely accurate empirical verifications assured physicists that "the theory may be mathematically meaningless, but it is nevertheless physically correct, and therefore the theory must make sense, even if we do not understand well how." In quantum gravity this indirect experimental reassurance is lacking and the claim that the theory is well founded can be based only on a solid mathematical control. Given the unlikelihood of finding direct experimental corroboration, the research can only aim for the moment at the goal of finding a *consistent* theory, with correct limits in the regimes that we control experimentally. High mathematical rigor is the only assurance of the consistency of the theory. Quantum field theory on manifolds is an unfamiliar terrain in which the experience accumulated in conventional quantum field theory is often useless and sometimes misleading.

One may object that a rigorous definition of quantum gravity is a vain hope, given that we do not even have a rigorous definition of QED, presumably a much simpler theory. The objection

is particularly valid from the point of view of a physicist who views gravity "just as any other field theory; like the ones we already understand". But the (serious) difficulties of QED and of other conventional field theories are ultraviolet. The physical hope supporting the quantum gravity research program is that the ultraviolet structure of a diffeomorphism-invariant quantum field theory is profoundly different from the one of conventional theories. Indeed, recall that in a very precise sense there is no short distance limit in the theory; the theory naturally cuts itself off at the Planck scale, due to the very quantum discreteness of spacetime. Thus, the hope that quantum gravity could be defined rigorously may be optimistic, but it is not ill founded.

After these comments, let me briefly mention some of the structures that have been explored in $\mathcal{H}$. First of all, the spin-network states satisfy the Kauffman axioms of the tangle theoretical version of recoupling theory [162] (in the "classical" case $A = -1$) at all the points (in 3D space) where they meet. For instance, consider a 4-valent node of four links colored $a, b, c, d$. The color of the node is determined by expanding the 4-valent node into a trivalent tree; in this case, we have a single internal link. The expansion can be done in different ways (by pairing links differently). These are related to each other by the recoupling theorem of pg. 60 in Ref. [162]

$$\begin{array}{c} b \\ \diagdown \end{array} \underset{a}{\overset{j}{\diagup}} \begin{array}{c} c \\ \diagdown \end{array} = \sum_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} \begin{array}{c} b \qquad c \\ \diagdown i \diagup \\ \diagup \quad \diagdown \\ a \qquad d, \end{array} \tag{24}$$

where the quantities $\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}$ are $SU(2)$ six-j symbols (normalized as in [162]). Equation (24) follows just from the definitions given above. Recoupling theory provides a powerful computational tool in this context (see [298, 77, 88]).

Since spin network states satisfy recoupling theory, they form a Temperley–Lieb algebra [162]. The scalar product (11) in $\mathcal{H}$ is also given by the Temperley–Lieb trace of the spin networks, or, equivalently by the Kauffman brackets, or, equivalently, by the chromatic evaluation of the spin network.

Next, $\mathcal{H}$ admits a rigorous representation as an $L_2$ space, namely a space of square-integrable functions. To obtain this representation, however, we have to extend the notion of connection, to a notion of "distributional connection". The space of the distributional connections is the closure of the space of smooth connection in a certain topology. Thus, distributional connections can be seen as limits of sequences of connections, in the same manner in which distributions can be seen as limits of sequences of functions. Usual distributions are defined as elements of the topological dual of certain spaces of functions. Here, there is no natural linear structure in the space of the connections, but there is a natural duality between connections and curves in $M$: a smooth connection $A$ assigns a group element $U_\gamma(A)$ to every segment $\gamma$. The group elements satisfy certain properties. For instance if $\gamma$ is the composition of the two segments $\gamma_1$ and $\gamma_2$, then $U_\gamma(A) = U_{\gamma_1}(A)U_{\gamma_2}(A)$.

A generalized connection $\bar{A}$ is defined as a map that assigns an element of $SU(2)$, which we denote as $U_\gamma(\bar{A})$ or $\bar{A}(\gamma)$, to each (oriented) curve $\gamma$ in $M$, satisfying the following requirements: i) $\bar{A}(\gamma^{-1}) = (\bar{A}(\gamma))^{-1}$; and ii) $\bar{A}(\gamma_2 \circ \gamma_1) = \bar{A}(\gamma_2) \cdot \bar{A}(\gamma_1)$, where $\gamma^{-1}$ is obtained from $\gamma$ by reversing its orientation, $\gamma_2 \circ \gamma_1$ denotes the composition of the two curves (obtained by connecting the end of $\gamma_1$ with the beginning of $\gamma_2$) and $\bar{A}(\gamma_2) \cdot \bar{A}(\gamma_1)$ is the composition in $SU(2)$. The space of such generalized connections is denoted $\overline{\mathcal{A}}$. The cylindrical functions $\Psi_{\Gamma,f}(A)$, defined in Section 6.3 as functions on the space of smooth connections, extend immediately to generalized connections

$$\Psi_{\Gamma,f}(\bar{A}) = f(\bar{A}(\gamma_1), \ldots, \bar{A}(\gamma_n)). \tag{25}$$

We can define a measure $d\mu_0$ on the space of generalized connections $\overline{\mathcal{A}}$ by

$$\int d\mu_0[\bar{A}]\,\Psi_{\Gamma,f}(\bar{A}) \equiv \int_{SU(2)^n} dg_1 \ldots dg_n\, f(g_1 \ldots g_n). \tag{26}$$

In fact, one may show that Equation (26) defines (by linearity and continuity) a well-defined absolutely-continuous measure on $\overline{\mathcal{A}}$. This is the Ashtekar–Lewandowski (or Ashtekar–Lewandowski–Baez) measure [28, 29, 30, 45]. Then, one can prove that $\mathcal{H} = L_2[\overline{\mathcal{A}}, d\mu_0]$ under the natural isomorphism given by identifying cylindrical functions. It follows immediately that the transformation (16) between the connection representation and the "old" loop representation is given by

$$\psi(\alpha) = \int d\mu_0[\bar{A}]\,\overline{\mathrm{Tr}\mathcal{P}e^{\int_\alpha A}}\,\Psi(\bar{A}). \tag{27}$$

This is the loop transform formula [265]; here it becomes rigorously defined.

Furthermore, $\mathcal{H}$ can be seen as the projective limit of the projective family of the Hilbert spaces $\mathcal{H}_\Gamma$, associated to each graph $\Gamma$ immersed in $M$. $\mathcal{H}_\Gamma$ is defined as the space $L_2[SU(2)^n, dg_1 \ldots dg_n]$, where $n$ is the number of links in $\Gamma$. The cylindrical function $\Psi_{\Gamma,f}(A)$ is naturally associated to the function $f$ in $\mathcal{H}_\Gamma$, and the projective structure is given by the natural map (10) [34, 198].

Finally, Ashtekar and Isham [27] have recovered the representation of the loop algebra by using C*-algebra representation theory: the space $\overline{\mathcal{A}}/\mathcal{G}$, where $\mathcal{G}$ is the group of local $SU(2)$ transformations (which acts in the obvious way on generalized connections), is precisely the Gelfand spectrum of the Abelian part of the loop algebra. One can show that this is a suitable norm closure of the space of smooth $SU(2)$ connections over physical space, modulo gauge transformations.

Thus, a number of powerful mathematical tools are at hand for dealing with nonperturbative quantum gravity. These include Penrose's spin network theory, $SU(2)$ representation theory, Kauffman tangle theoretical recoupling theory, Temperley–Lieb algebras, Gelfand's $C^*$algebra, spectral-representation theory, infinite-dimensional measure theory and differential geometry over infinite-dimensional spaces.

## 6.6  Dynamics: I. Hamiltonian

The definition of the theory is completed by giving the Hamiltonian constraint. A number of approaches to the definition of a Hamiltonian constraint have been attempted in the past, with various degrees of success. Thiemann has succeeded in providing a regularization of the Hamiltonian constraint that yields a well-defined, finite operator. Thiemann's construction [285, 289, 291] is based on several clever ideas. I will not describe it here. Rather, I will sketch below in a simple manner the final form of the constraint (for the Lapse = 1 case), following [252]. For a complete treatment, see [294].

I begin with the Euclidean Hamiltonian constraint. We have

$$\hat{H}|s\rangle = \sum_i \sum_{(IJ)} \sum_{\epsilon=\pm 1} \sum_{\epsilon'=\pm\frac{1}{2}} A_{\epsilon\epsilon'}(j_i...j_n)\,\hat{D}_{i;(IJ),\epsilon\epsilon'}\,|s\rangle. \tag{28}$$

Here $i$ labels the nodes of the s-knot $s$; $(IJ)$ labels couples of (distinct) links emerging from $i$. $j_1...j_n$ are the spins of the links emerging from $i$. $\hat{D}_{i;(IJ)\epsilon\epsilon'}$ is the operator that acts on an s-knot by: (i) creating two additional nodes, one along each of the two links $I$ and $J$, (ii) creating a novel link, colored $\frac{1}{2}$, joining these two nodes, (iii) assigning the coloring $j_I + \epsilon$ and, respectively, $j_J + \epsilon'$ to the links that join the new formed nodes with the node $i$. This is illustrated in Figure 2.

The coefficients $A_{\epsilon\epsilon'}(j_i...j_n)$, which are finite, can be expressed explicitly (but in a rather laborious way) in terms of products of linear combinations of Wigner $6j$ symbols of $SU(2)$. The
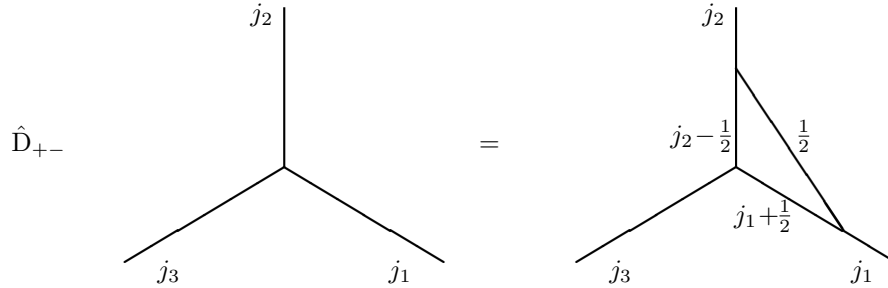
**Figure 2:** Action of $\hat{D}_{i;(IJ)\epsilon\epsilon'}$.

Lorentzian Hamiltonian constraint is given by a similar expression, but quadratic in the $\hat{D}$ operators.

The operator defined above is obtained by introducing a regularized expression for the classical Hamiltonian constraint, written in terms of elementary loop observables, turning these observables into the corresponding operators and taking the limit. The construction works rather magically, relying on the fact [267] that certain operator limits $\hat{O}_\delta \to_{\delta \to 0} \hat{O}$ turn out to be finite on diff-invariant states, thanks to the fact that, for $\delta$ and $\delta'$ sufficiently small, $\hat{O}_\delta|\Psi\rangle$ and $\hat{O}_{\delta'}|\Psi\rangle$ are diffeomorphic equivalent. Thus, here diff invariance plays again the crucial role in the theory.

During the last years, Thomas Thiemann has introduced the idea of replacing the full set of quantum constraints with a single ("master") constraint [293]. The development of this formulation of the dynamics is in progress.

## 6.7   Dynamics: II. Spin foams

Alternatively, the dynamics of the spin-network states can be defined via the spin-foam formalism. This is a covariant, rather than canonical, language, which can be used to define the dynamics of loop quantum gravity, in the same sense in which giving the covariant *vertex amplitude* defines the dynamics of the photon and electron states.

In three dimensions, the spin-foam formalism gives the well-known Ponzano–Regge model. Here the vertex amplitude turns out to be given by an $SU(2)$ Wigner $6j$ symbol. The relation between this model and loop quantum gravity has been pointed out long ago [248], and the complete equivalence has been proven by Alejandro Perez and Karim Noui [210].

In four dimensions, an important role has been been played by the Barrett–Crane model, a much-studied spin-foam theory constructed [97, 221] from the definition of the vertex as an $SU(2)$ Wigner $10j$ symbol, given by Louis Crane and John Barrett in [66]. There are several difficulties in using the Barrett–Crane model for defining the dynamics of 4D loop quantum gravity [53, 52]. First, the Barrett–Crane model is formulated keeping local Lorentz invariance manifest, while loop quantum gravity is not. This problem has been investigated by Sergei Alexandrov, see [4, 5, 7] and references therein. Second, the Barrett–Crane model appears to have fewer degrees of freedom than general relativity on a spacelike surface, because it fixes the values of the intertwiners. More importantly, the low-energy limit of the propagator defined by the Barrett–Crane model does not seem to be correct [2, 3]. An important recent development, however, has been the introduction of a new vertex amplitude, defined by the square of the $SU(2)$ Wigner $15j$ symbol, which may correct all these problems [106]. This has given rise to a rapid development [105, 181, 182, 227] and formulation of a class of spin-foam models that may provide a viable definition of the LQG dynamics [104, 114, 6], and are currently under intense investigation. Some preliminary results appear to be encouraging [188].

What characterizes the spin-foam formalism is the fact that it can be derived in a surprising variety of different ways, which all converge to essentially the same structure. The formalism can be obtained (i) from loop gravity, (ii) as a quantization of a discretization of general relativity on a simplicial triangulation, (iii) as a generalization of matrix models to higher dimensions, (iv) from a quantization of the elementary geometry of tetrahedra and 4-simplices, and (v) by quantizing general relativity from its formulation as a constrained $BF$ theory, by imposing constraints on quantum topological theory. Each of these derivations sheds some light on the formalism. For detailed introductions to the spin-foam formalism see [50, 229, 230, 212, 64, 177]. A recent derivation as the quantization of a discretization of general relativity is in [105, 104], which can also be seen as an independent derivation of the loop-gravity canonical formalism itself. The PhD thesis of Daniele Oriti [213] is also a very good introduction. Here I give only a simple heuristic description of the way spin foams appear from loop gravity.

In his PhD thesis, Feynman introduced a path-integral formulation of quantum mechanics, deriving it from the canonical formalism. He considered a perturbation expansion for the matrix elements of the evolution operator

$$U(T) \equiv e^{i \int_0^T dt \ \int d^3x \ \hat{H}(x)}. \tag{29}$$

One can search for a covariant formulation of loop quantum gravity following the same idea [257, 236]. The matrix elements of the operator $U(T)$ obtained exponentiating the (Euclidean) Hamiltonian constraint in the proper-time gauge (the operator that generates evolution in proper time) can be expanded in a Feynman sum over paths. In conventional QFT each term of a Feynman sum corresponds naturally to a certain Feynman diagram, namely a set of lines in spacetime meeting at vertices (branching points). A similar structure of the terms appears in quantum gravity, if one uses the spin-network basis, but the diagrams are now given by *surfaces* is spacetime that branch along edges, which in turn meet at vertices.

This is a consequence of the fact that the Hamiltonian operators acts only on nodes. The histories of $s$-knots (abstract spin networks), evolving under such an action, are branched surfaces, which carry spins on the faces (swept by the links of the spin network) and intertwiners on the edges (swept by the nodes of the spin network). These have been called "spin foams" by John Baez [48], who has studied the general structure of theories defined in this manner [49, 50].

Thus, the time evolution of a spin network, which generates spacetime, is given by a spin foam. The spin foam describes the evolving gravitational degrees of freedom. The formulation is "topological" in the sense that one must sum over topologically-nonequivalent surfaces only, and the contribution of each surface depends on its topology only. This contribution is given by the product of the amplitude of the elementary "vertices", namely points where the edges branch.

The physical transition amplitude between two s-knot states $|s_i\rangle$ and $|s_f\rangle$ turns out to be given by summing over all (branching, colored) surfaces $\sigma$ that are bounded by the two s-knots $s_i$ and $s_f$

$$\langle s_f | s_i \rangle = \sum_{\substack{\sigma \\ \partial\sigma = s_i \cup s_f}} \mathcal{A}[\sigma]. \tag{30}$$

The weight $\mathcal{A}[\sigma](T)$ of the spin foam $\sigma$ is given by a product over the $n$ vertices $v$ of $\sigma$:

$$\mathcal{A}[\sigma] \sim \prod_v A_v(\sigma). \tag{31}$$

The contribution $A_v(\sigma)$ of each vertex is given by the matrix elements of the Hamiltonian constraint operator between the two s-knots obtained by slicing $\sigma$ immediately below and immediately above the vertex. They turn out to depend only on the colors of the surface components
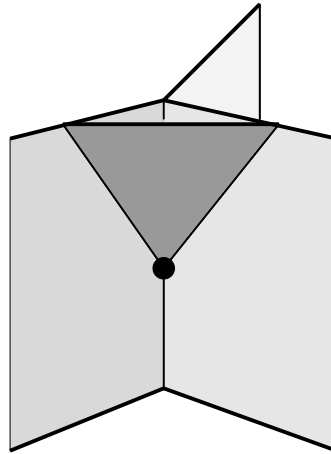
**Figure 3:** The elementary vertex.

immediately adjacent the vertex $v$. The sum turns out to be finite and explicitly computable order by order.

As in the usual Feynman diagrams, the vertices describe the elementary interactions of the theory. Here, in particular, one sees that the complicated structure of the Thiemann Hamiltonian, which makes a node split into three nodes, corresponds to a geometrically very simple vertex. Figure 3 is a picture of the elementary vertex. Notice that it represents nothing but the spacetime evolution of the elementary action of the Hamiltonian constraint, given in Figure 2. An example of a surface in the sum is given in Figure 4.

The resulting form of the covariant formulation of quantum gravity as a sum over spin-foam amplitudes is quite different from that of background-dependent quantum field theory.

Spin foams have a very nice geometric interpretation as the dual to a simplicial decomposition of spacetime. Vertices corresponding to four-volumes, edges to three-volumes, etc. This last point is a very elegant feature, and has an immediately intuitive explanation in terms of loop quantum gravity operators. The area operator counts lines in a spin network, corresponding to faces in a spin foam; the volume operator counts vertices in a spin network, corresponding to lines in a spin foam.

Finiteness of some spin-foam models at all orders of perturbation theory has been proven [228, 94, 93].

## 6.8  Dynamics: III. Group field theory

Very strictly related to the spin-foam language is an intriguing formalism that has been developing in recent years: group field theory. This has emerged from dual formulations of topological theories [76], and can be seen as a higher-dimensional version of the duality between matrix models and fluctuating geometries in 2D. Group field theories are standard quantum field theories defined over a group manifold, characterized by a peculiar nonlocal interaction term. They have a remarkable property: the Feynman expansion generates a sum over Feynman graphs that have a direct interpretation as a spin-foam model. In other words, the "discrete geometries" summed over can be seen as Feynman graphs of the group field theory.

Intuitively, the individual quanta of the group field theory can be seen as the quanta of space predicted by loop quantum gravity (see Section 7), and their Feynman histories make up spacetime.

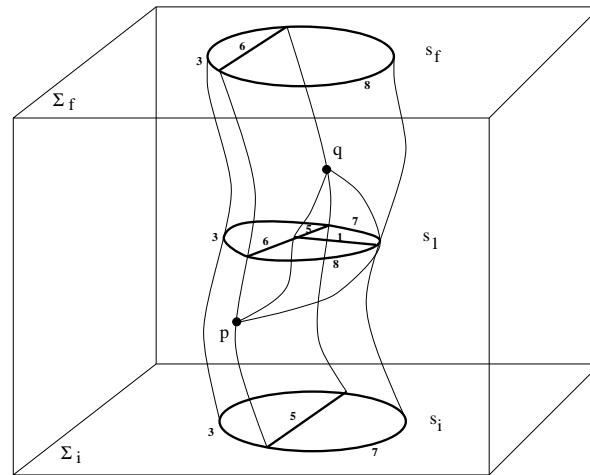The advantage of this formulation of quantum gravity is that it precisely fixes the sum over

**Figure 4:** A term of second order.

spin foams, and that it allows a number of theoretical tools from standard quantum field theory to be imported directly into the background independent formalism. In this sense, this approach has similarities with the philosophy of the Maldacena duality in string theory: a nonperturbative theory is dual to a more quantum field theory. But here there is no conjecture involved: the duality between certain spin-foam models and certain group field theories is a theorem.

The application of group field theory to quantum gravity, begun in the first attempts to frame the Barrett–Crane vertex into a complete theory [97]. It was later shown that any spin-foam model can be written as a group field theory [237, 238]. The subject of group field theory has greatly developed recently, and I refer to the reviews [113, 217, 216] for an update and complete references.

## 6.9 Matter couplings

The coupling of fermions to the theory [206, 207, 54, 287] works easily. All the important results of the pure GR case survive in the GR + fermions theory. Not surprisingly, fermions can be described as open ends of "open-spin networks".

The extension of the theory to the Maxwell field [169, 126] and Yang–Mills [290] also works smoothly. Remarkably, the Yang–Mills term in the quantum Hamiltonian constraint can be defined in a rigorous manner, extending the pure gravity methods, and *ultraviolet divergences do not appear*, strongly supporting the expectation that the natural cutoff introduced by quantum gravity might cure the ultraviolet difficulties of conventional quantum field theory. For an up-to-date account and complete references, see Thiemann's book [294].

The coupling of matter in the spin-foam and group field theory formalism is not yet clear, in spite of considerable work in this direction. In the spin-foam formalism, an interesting line of investigation has studied the coupling of particles in 3D, treated as topological defects [118, 119, 209, 117, 116, 220, 108]. Extensions of this formalism to 4D are considered in [59, 56]. On the coupling of gauge fields to spin foams, see [219, 201, 281].

Matter couplings in the group field theory formalism have been studied especially in 3D. See [120, 215, 167, 109].

## 6.10   Variants: fundamental discreteness

What I have describe above is the conservative and most widely used formulation of loop quantum gravity. A number of variants have appeared over the years. I sketch here some of those that are currently under investigation.

The common theme of these variants is to take seriously spacetime discreteness, which in loop quantum gravity is derived as a *result* of conservative quantization of continuum classical general relativity, and to use it, instead, as a *starting point* for the formulation of the fundamental theory.

The common point of view is then that the fundamental quantum theory will be discrete. There is no continuum limit in the sense in which lattice QCD has (presumably) a continuum quantum field-theory limit. Rather, states can approximate classical continuum GR.

- **Causal histories**

  A point of view on the dynamics somewhat intermediate between the canonical one and the spin-foam one has been developed over the years by Fotini Markopoulou and Lee Smolin [191, 192, 190]. See Smolin's introduction [277] and complete references therein. The idea is that, after having understood that quantum space can be described by a basis of spin-network states, and that evolution happens discretely at the nodes, then the problem of determining the dynamics is reduced to the study of the possible elementary "moves", such as the one that takes the graph in the left-hand side of Figure 1 into the one in the right-hand side, and their amplitudes. Ideally, a space of background-independent theories is given by the space of these possible moves and their amplitudes.

  Markopoulou, Smolin and their collaborators have developed this point of view in a number of intriguing directions. See for instance [193]. I refer to Smolin's review for an overview, and I mention here only one recent idea that I have found particularly intriguing. Together with T. Konopka, they have introduced in [165] a model where the degrees of freedom live on a complete graph (all nodes connected to one another) and the physics is invariant under the permutations of all the points. This opens up the possibility of the model having a "low-energy" phase in which physics on a low-dimensional lattice emerges, the permutation symmetry is broken to the translation group of that lattice, and a "high-temperature", as well as a disordered, phase, where the permutation symmetry is respected and the average distance between degrees of freedom is small. This may serve as a paradigm for the emergence of classical geometry in background-independent models of spacetime. The appeal of the idea is its possible application in a cosmological scenario, in relation to the horizon problem. The horizon problem is generally presented as the puzzle raised by the fact that different parts of the universe appear to have emerged thermalized from an initial phase of the life of the universe in which the causal structure determined by *classical* general relativity has them causally disconnected. It seems to me natural to suspect that the problem is in the application of classical ideas to the very early universe, where, instead, quantum gravity dominates and there is no classical causal structure in place. The approach of Konopka, Markopoulou and Smolin gives a means to model a possibility of avoiding the horizon problem with a transition from the high-temperature phase, in which the points of the universe are all in direct causal connection, to the low-temperature phase, in which the classical causal structure gets established.

  A different approach to the introduction of causality at the microscopic level in the spin-foam formalism is due to Daniele Oriti and Etera Livine [178, 179]; see also [214]. This includes the construction of new causal models, as well as the extraction of causal amplitudes from existing models.

- **Algebraic Quantum Gravity**

  Thomas Thiemann and Kristina Giesel have introduced a top down approach [136, 135] called Algebraic Quantum Gravity (AQG). The quantum kinematics of AQG is determined by an abstract $*-$algebra generated by a countable set of elementary operators labelled by a given algebraic graph. The quantum dynamics of AQG is governed by a single Master Constraint operator. AQG is inspired by loop quantum gravity; the difference is the absence of any fundamental topology or differential structure in the setting up of the theory. This is quite appealing, since in loop quantum gravity one has the strong impression that the topology and the differential structure used in the setting up of the theory are residual of the classical limit, namely of the "inverse-problem" way in which the theory is deduced from its classical limit, and should play no real role in the fundamental theory. In AQG, the information about the topology and differential structure of the spacetime manifold, as well as about the background metric to be approximated, comes from the quantum state itself, when this state is a coherent state that approximates a classical state.

- **Uniform discretization**

  Motivated by the same problems as those considered by Giesel and Thiemann in [134], Campiglia, Di Bartolo, Gambini and Pullin have studied a lattice formulation of the fundamental theory, denoted "uniform discretization" [82]. Though starting from different points, the end prescription of this approach and AQG can, in some cases, be quite close in practical implementation.

  The uniform discretization is an evolution of the "consistent discretization" [131] approach, studied by Rodolfo Gambini and Jorge Pullin. The key idea is that, when one uses connection variables and holonomies, a natural regularization arena for theories is the lattice. The introduction of a lattice in a theory like general relativity is a highly nontrivial affair. It destroys at the most basic level the fundamental symmetry of the theory, the invariance under diffeomorphisms. The consequences of this can be far reaching. If one just discretizes the equations of general relativity, the resulting set of discrete equations is not even consistent and the equations cannot be solved simultaneously (this is well known, for instance, in numerical relativity, where "free evolution" schemes violate the constraints). The uniform (and the earlier consistent) discretization approaches attempt to take these discrete theories seriously. In particular, to recognize that they do not have the same symmetries as continuum GR, but have, at best, "approximate symmetries". Quantities that were constraints in the continuum theory become evolution equations, and quantities that were Lagrange multipliers become dynamical variables determined by evolution. In the simplest view, called consistent discretization, these theories were analyzed directly as they were constructed by the discretization procedure [101]. In the "uniform discretization" [82] one constructs theories in which one controls precisely via the initial data "how much they depart from the continuum" in the sense of how large the constraints are.

  The resulting theories are straightforward to quantize. Since the constraints are not identically zero, all the conceptual problems of having to enforce the constraints go away. One can construct relational descriptions of reality [124] and probe properties like how much does the use of real clocks and rulers destroy unitarity and entanglement in quantum theories [123, 125].

  This approach has been explored in several finite-dimensional models [82], and is being studied in midi-superspace implementations, where it could yield a viable numerical quantum gravity approach to spherical gravitational collapse [83].

# 7  Physical Results

In Section 6, I have sketched the basic structure of loop quantum gravity. This structure has been applied to derive a number of physical results. I list below a few of these. For a far more detailed list of results, see for instance [277] where *forty-two*[7] results are listed!

- **Planck-scale discreteness of space**

  The central physical result obtained from loop quantum gravity is the evidence for a physical quantum discreteness of space at the Planck scale. This is manifested in the fact that certain operators corresponding to the measurement of geometrical quantities, in particular area and volume, have discrete spectra. According to the standard interpretation of quantum mechanics (which we adopt), this means that the theory predicts that a physical measurement of an area or a volume will yield quantized results. In particular, since the smallest eigenvalues are of Planck scale, this implies that there is no way of observing areas smaller than the Planck scale. Space comes, therefore, in "quanta" in the same manner as the energy of an oscillator. The spectra of the area and volume operators have been computed in detail in loop quantum gravity. These spectra have a complicated structure, and they constitute detailed quantitative physical predictions of loop quantum gravity on Planck-scale physics. If we had experimental access to Planck-scale physics, they would allow the theory to be empirically tested in detail.

  The discreteness of area and volume is derived as follows. Consider a surface $\Sigma$. The physical area $A$ of $\Sigma$ depends on the metric, namely on the gravitational field. In a quantum theory of gravity, the gravitational field is a quantum field operator, and therefore the area of $\Sigma$ is described by a quantum operator $\hat{A}$. What is the quantum operator $\hat{A}$ in nonperturbative quantum gravity? It can easily be worked out by writing the standard expression for the area of a surface and replacing the metric with the appropriate function of the loop variables. Promoting these loop variables to operators, we obtain the area operator $\hat{A}$. The precise construction of this operator requires regularizing the classical expression and then taking the limit of a sequence of operators, in a suitable operator topology [268, 98, 121, 75, 31]. For a complete presentation of the details of this construction, see [261, 294]. The resulting area operator $\hat{A}$ acts as follows on a spin-network state $|S\rangle$ (assuming here for simplicity that $S$ is a spin network without nodes on $\Sigma$):

  $$\hat{A}[\Sigma]\,|S\rangle = \left( \frac{l_0^2}{2} \sum_{i \in \{S \cap \Sigma\}} \sqrt{j_i(j_i+1)} \right)\,|S\rangle \tag{32}$$

  where $i$ labels the intersections between the spin network $S$ and the surface $\Sigma$, and $j_i$ is the spin of the link of $S$ crossing the $i-th$ intersection. This result shows that the spin-network states (with a finite number of intersection points with the surface and no nodes on the surface) are eigenstates of the area operator. The corresponding spectrum is labeled by multiplets $\vec{j} = (j_1, ..., j_n)$ of positive half integers, with arbitrary $n$, and given by

  $$A_{\vec{j}}[\Sigma] = l_0^2 \sum_i \sqrt{j_i(j_i+1)}. \tag{33}$$

  A similar result can be obtained for the volume [268, 183, 184, 32, 98, 175, 286]. The eigenvalues of the volume of a region $R$ turn out to be determined by the intertwiners of the *nodes* of the spin network contained in $R$. The two results on area and volume offer a

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[7]Forty-two, of course, is The Answer...

compelling physical interpretation of the spin-network states. These are quantum states in which space is made by a set of "chunks", or quanta of space, which are represented by the nodes of the spin network, connected by surfaces, which are represented by the links of the spin networks. The intertwiners on the nodes are the quantum numbers of the volume of the chunks, while the spins on the links are the quantum numbers of the area of the surfaces that separate the chunks. See Figure 5.
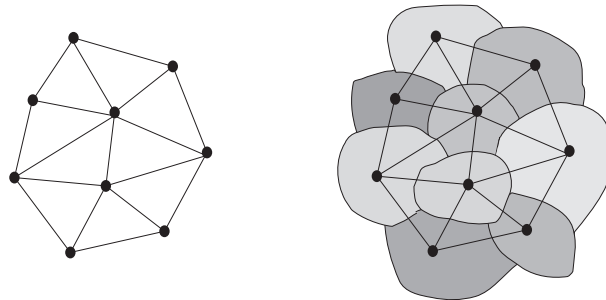


**Figure 5:** A spin network and the "chunks", or quanta, of space it describes.

Two comments are in order.

*(i)* The reader will wonder why area and volume seem here to play a role more central than length, when classical geometry is usually described in terms of lengths. The reason is that the length operator is difficult to define and has a difficult physical interpretation, see [288]. Whether this is simply a technical difficulty, or it reflects some deep fact, is not clear to me. The basic field of the theory is not the tetrad $e$, which assigns length to line elements, but rather the 2-form $E = e \wedge e$, which assigns areas to surface elements. Another way to say it is that the loop representation is based on the theory of quantized angular momentum. Angular momentum is not a vector but a bivector, so it corresponds not to an arrow but to an oriented area element. On this, see Baez's [49]. On the relation between the $E$ field and area, see [247].

*(ii)* Area and volume are not gauge-invariant operators. Therefore, we cannot directly interpret them as representing physical measurements, according to the conventional interpretation of quantum gauge systems. There are three reason, however, to take the discreteness of their spectra as an indication of the physical discreteness of spacetime.

   *(a)* A realistic measurement of an area or a volume refers to a surface or a region determined physically, for instance by some physical object. For example, I can measure the area of the surface of a certain table at a certain time. In the dynamical theory that describes the gravitational field, as well as the table (and the clock), the area of the surface of the table is a diffeomorphism-invariant quantity $A$, which depends on gravitational as well as matter variables. In the quantum theory, $A$ will be represented by a diffeomorphism-invariant operator. It is completely plausible to assume that the operator $A$ is the same mathematical operator as the pure gravity area operator. This is because we can gauge fix the matter variables, and use matter location as coordinates, so that non-diff-invariant observables in the pure gravity theory correspond precisely to diff-invariant observables in the matter + gravity theory, as they do in the classical theory [249]. In other words, the fact that these geometrical operators have discrete spectra is true *in any gauge*.

   *(b)* The discreteness depends on the commutation structure of the relevant geometrical quantities. This does not change according to the specific versions of the quantities

to which it is applied. Compare this with the angular momentum in nonrelativistic quantum theory: the angular momentum is always quantized, and it always has the same eigenvalues, irrespective of whether it is the angular momentum of an atom, a proton or a molecule. This is because the angular momentum observables may be different in the different cases, but their commutation structure remains the same. Similarly, the commutation structure of the components of the area of any physical object can be uniquely dictated by the geometry of the gravitational field, not by specific features of the object.

*(c)* Quantum mechanical discreteness is a *kinematical* property of a system, independent of the system's *dynamics*. For instance, the momentum $p$ of a particle in a box is quantized, independently from the form of the Hamiltonian $H(q,p)$ of the particle. To make precise sense of this crucial observation in the context of gravity, where dynamics and gauges are mixed up, requires a careful analysis of the way the formalism of quantum theory may be extended in the case of generally covariant systems. This is done in [261]. There discreteness is recognized as always associated to the spectra of the *partial* observables [260]. This analysis provides a foundation for the claim that area and volume are predicted to be discrete in loop quantum gravity.

- **Quantum cosmology**

  The application of loop quantum gravity to cosmology is one of it most spectacular achievements. The main result is that the initial singularity is controlled by quantum effects. The reason is not difficult to grasp. In the classical theory, the volume of the universe goes continuously to zero at the Big Bang singularity. In the quantum theory, one has transition amplitudes between finite-volume eigenvalues. The singularity is controlled by a mechanics very similar to quantum mechanism that stabilizes the orbit of an electron around the nucleus. This opens up the possibility of studying the physics of the very initial universe and also the physical evolution *across* the Big Bang. The region around the Big Bang is a region where spacetime enters a genuine quantum regime, which cannot be described in terms of a conventional spacetime manifold, but that can still be described by the quantum theory. For a detailed description of techniques and results of loop cosmology, see the comprehensive *Living Review* article by Martin Bojowald [73]. For a nice introduction see Abhay Ashtekar [20].

- **Black hole singularity**

  The same techniques applied in quantum cosmology can be utilized to study quantum spacetime in the neighborhood of the classical singularity at the center of a black hole. Again, the singularity is controlled by quantum effects. Again, the region around the classical singularity is a region in which spacetime enters a genuine quantum regime, which cannot be described in terms of a conventional spacetime manifold, but that can still be described by quantum theory [24, 204]. This opens up a new possible "paradigm" [23] for describing the final evolution of a black hole.

- **Black hole entropy**

  Indirect arguments strongly support the idea that a Schwarzschild black hole of (macroscopic) area $A$ behaves as a thermodynamical system governed by the Bekenstein–Hawking entropy [142, 143, 68, 296]

$$S = \frac{k}{4\hbar G_{\text{Newton}}}\, A \tag{34}$$

($k$ is the Boltzmann constant; here I put the speed of light equal to one, but write the Planck and Newton constants explicitly). A physical understanding and a first principles derivation of this relation require quantum gravity, and therefore represent a challenge for every candidate theory of quantum theory. The Bekenstein–Hawking expression (34) for the entropy of a Schwarzschild black hole of surface area $A$ can be derived from loop quantum gravity via a statistical mechanical computation [170, 171, 253, 22]. The derivation has been extended to various classes of black holes; see [25] and references therein.

This derivation is based on the idea that the entropy of the black hole originates from the microstates of the horizon that correspond to a given macroscopic configuration [297, 85, 84, 58, 57]. Physical arguments indicate that the entropy of such a system is determined by an ensemble of configurations of the horizon with fixed area [253]. In quantum theory these states are finite in number and can be counted [170, 171]. Counting these microstates using loop quantum gravity yields

$$S = \frac{c}{\gamma} \, \frac{k}{4\hbar G_{\mathrm{Newton}}} \, A. \tag{35}$$

$\gamma$ is defined in Section 6, and $c$ is a real number of the order of unity that emerges from the combinatorial calculation (roughly, $c \sim 1/4\pi$). If we choose $\gamma = c$, we get Equation (34) [270, 92]. Thus, the theory is compatible with the numerical constant in the Bekenstein–Hawking formula, but does not lead to it univocally. The precise significance of this fact is under discussion. In particular, the meaning of $\gamma$ is unclear. Jacobson has suggested [159, 160] that finite renormalization effects may affect the relation between the bare and the effective Newton constant, and this may be reflected in $\gamma$. For discussion of the role of $\gamma$ in the theory, see [270]. On the issue of entropy in loop gravity, see also [275].

- **Low-energy limit: $n$-point functions**

  Loop quantum gravity is formulated in a background-independent language. Spacetime is not assumed a priori, but rather it is built up by the states of theory themselves. The relation between this formalism and the conventional formalism of quantum field theory on a given spacetime is far from obvious, and it is far from obvious how to recover low-energy quantities from the full background-independent theory. One would like, in particular, to derive the $n$-point functions of the theory from the background-independent formalism, in order to compare them with the standard perturbative expansion of quantum general relativity and therefore check that loop quantum gravity yields the correct low-energy limit.

  The search for a way to describe the low-energy degrees of freedom, namely "the graviton" in the background-independent formulation of loop quantum gravity, has a long history [157, 158]. To appreciate the difficulty, observe that the $n$-point functions are intrinsically defined on a background. In fact, they express correlations among the fluctuations of the quantum field *around a given background solution*. How can one extract this information from the background independent theory?

  A strategy for doing so has been introduced in [205]. It is based on the idea of considering a *finite* region of spacetime and studying the amplitude for having *given* boundary states around this region. By choosing the boundary states appropriately, one can study the physical configurations that fluctuate around a chosen *average* internal geometry. In particular, one can recover quantities that converge to the conventional $n$-point functions in the large-distance limit, by appropriately "adding quanta" to the boundary state that corresponds to an average internal flat geometry. Using this technique, a calculation of the graviton propagator has been completed in [262] to first order and in [69] to second order. Similar calculations have been completed in three dimensions [280]. An improved boundary state has

been studied in [180]. See also [102, 202]. The calculation of the complete tensorial structure of the propagator has been recently completed [2].

- **Observable effects**

  The best possibility for testing the theory seems to be via cosmology. However, the investigation of the possibility that quantum gravity effects are observable is constantly under investigation. Various possibilities have been considered, including quantum gravitational effects on light and particle propagation at very long distances [130, 8], which could perhaps be relevant for observations in progress such as AUGER and GLAST, and others. For an overview, see for instance [277, 199].

  The MAGIC telescope collaboration has recently reported the measurement of an energy-dependent time delay in the arrival of signals from the active galaxy Markarian 501. The measured phenomenological parameter governing this dependence is on the Planck scale [1]. Energy-dependent time delays in the arrival of signals from far away sources have long been suggested as possible quantum gravity effects [14, 15]. A quantum-gravity interpretation of the MAGIC observation does not appear to be likely at present (see for instance [67]), but the measurement shows that quantum-gravity effects *are* within the reach of current technology.

- **Noncommutative geometry from loop quantum gravity**

  Laurent Freidel and Etera Livine have shown that the low-energy limit of quantum gravity coupled with matter in three dimensions is equivalent to a field theory on a noncommutative spacetime [116]. This is a remarkable result because it directly connects the study of noncommutative spacetimes with quantum gravity. Work is in progress to understand the extent to which the result is also meaningful in four dimensions.

# 8   Open Problems and Current Lines of Investigation

- **Low-energy limit**

  The key open problem in the theory is to prove that loop gravity gives general relativity correctly in the low-energy, or classical, limit. Naively, this is the case, because replacing the quantum operators in the dynamical equations with their classical counterpart yields the equations of classical general relativity. But the background independent structure of the quantum theory is peculiar and unconventional, and a more convincing proof is certainly needed. Furthermore, we need to understand how the low-energy limit emerges from the background-independent theory in order to correct the low-order quantum corrections to classical general relativity.

- **Hamiltonian constraint**

  The kinematics of the theory is well understood both physically (quanta of area and volume, discrete geometry) and from the mathematical point of view. The part of the theory that is not yet fully under control is the dynamics, which is determined by the Hamiltonian constraint. A plausible candidate for the quantum Hamiltonian constraint is available, and provides a well-defined theory, but a number of variants are under investigation, both as far as the operator and the general formalism are concerned. The question is whether the theory defined has the correct classical limit (see $n$-point functions, below).

- **Spin foams and group field theory**

  Finding a version of the loop quantum gravity Hamiltonian and a spin-foam theory, in four dimensions, that could be proven to be equivalent, would be a major step ahead in understanding the theory. Substantial progress in this direction has been recently obtained with the introduction of the new vertex that corrects the difficulties of the Barrett–Crane one (see Section 6.7) and appears to provide a direct relation between the spin foam and the loop-gravity language. The viability of these new models now needs to be investigated in depth.

  In four dimensions, the coupling of matter in the spin foams and in the group field theory formalisms needs to be better understood.

- **Lorentzian theory**

  In the Hamiltonian formulation, the Lorentzian version of the Hamiltonian constraint is well defined (see [294]), but it has not been much studied yet. A number of Lorentzian spin-foam models have been investigated [65, 231, 51, 4, 5, 7], including with the presence of a cosmological constant [211]. Some of these have been proven to be finite [93, 86]. But it is not yet clear what is the best approach for defining the physical theory.

  There is another possibility. In conventional quantum field theory, the procedure of defining the physical theory as an analytic continuation of an Euclidean theory is extremely effective. The conventional Wick rotation does not make sense in general relativity, therefore this procedure cannot be generalized to gravity naively. But this does not imply that an appropriate generalization of this procedure could not be found, perhaps in terms of an analytic continuation in some physical boundary time variable. For instance, if the calculations of the $n$-point functions from Euclidean loop quantum gravity converge to the $n$-point functions of Euclidean perturbative quantum gravity, then it would make sense to analytically continue in the $n$-point functions temporal argument, and this is precisely a physical boundary time in the nonperturbative theory [69]. In other words, Euclidean loop quantum gravity might still be a tool for defining the physical theory, and not simply a useful example of background independent quantum field theory.

- **Quantum cosmology**

  Quantum cosmology is developing rapidly. It faces two major challenges: to prove that the models used truly derive from full loop quantum gravity, and to develop the models to the point at which they could lead to predictions that can be compared with cosmological observations. This is probably the best hope for finding a window to the empirical verification of loop quantum gravity.

- **Black holes**

  The derivation of the Bekenstein–Hawking entropy formula is a major success of loop gravity, but much remains to be understood. A clean derivation from the full quantum theory is not yet available. Such a derivation would require us to understand what, precisely, the event horizon in the quantum theory is. In other words, given a quantum state of the geometry, we should be able to define and "locate" its horizon (or the structure that replaces it in the quantum theory). To do so, we should understand how to effectively deal with the quantum dynamics, how to describe the classical limit (in order to find the quantum states corresponding to classical black-hole solutions), as well as how to describe asymptotically-flat quantum states.

  Besides these formal issues, at the root of the black-hole entropy puzzle is a basic physical problem, which, to my understanding, is still open. The problem is to understand how we can use basic thermodynamical and statistical ideas and techniques in a general covariant context.[8] To appreciate the difficulty, notice that statistical mechanics makes heavy use of the notion of energy (for example, in the definition of the canonical or microcanonical ensembles); but there is no natural local notion of energy associated to a black hole (or there are too many of such notions). Energy is an extremely slippery notion in gravity. Thus, how do we define the statistical ensemble? In other words, to compute the entropy (for example, in the microcanonical ensemble) of a normal system, we count the states with a given energy. In GR we should count the states with a given *what*? One may say black-hole states with a given area. But why? We understand why the number of states with given *energy* governs the thermodynamical behavior of normal systems. But why should the number of states with given *area* govern the thermodynamical behavior of the system, namely govern its heat exchanges with the exterior? For a discussion of this last point, see [254].

- **$n$-point functions**

  Computing scattering amplitudes from loop gravity is of interest for a number of reasons. First, it allows a connection with the conventional language of particle physics to be established, and therefore also to compare the loop theory with other approaches to quantum gravity. Second, it provides a direct test of the low-energy limit of the theory. Third, it opens the way to a systematical computation of the quantum corrections to general relativity. The calculation of these $n$-point functions has begun, but it is laborious, and much remains to be done to set up a consistent general formalism. One of the results of these calculations has been to rule out the Barrett–Crane vertex as a way to define the dynamics of loop gravity [2, 3], opening the way to the development of the new vertex.

- **Matter from braiding?**

  Sundance Bilson-Thompson, Fotini Markopoulou and Lee Smolin, have recently introduced the intriguing idea of the possibility of describing fermion degrees of freedom in terms of the braiding of the spin networks [70]. The idea is old; it can be traced to Lord Kelvin,

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[8]A general approach to this problem, and an idea about its connection with the "problem of time" in quantum gravity, have been developed in [250, 251, 90].

who suggested that the stability of atoms could be understood if atoms are different knots of vortex lines in the ether. It is soon to be understood if this idea can work in loop gravity, but the possibility is obviously very interesting.

# 9 Short Summary and Conclusion

The mathematics of loop quantum gravity is solidly defined, and is understood from several alternative points of view. Long-standing problems, such as the lack of a scalar product, the difficulty of controlling the overcompleteness of the loop basis, and the problem of implementing the reality condition in the quantum theory, have been successfully solved or sidestepped. The kinematics is given by a well-defined Hilbert space $\mathcal{H}$ that carries a representation of the basic operators. A convenient orthonormal basis in $\mathcal{H}$ is provided by the spin-network states, defined in Section 6.3. The diffeomorphism-invariant states are given by the $s$-knot states, and the structure and properties of the (diff-invariant) quantum states of the geometry are quite well understood (Section 6.4). These states give a description of quantum spacetime in terms of discrete excitations of the geometry. More precisely, in terms of elementary excitations carrying discretized quanta of area (Section 7).

The dynamics can be coded into the Hamiltonian constraint. A well-defined version of this constraint exists, and thus a complete and consistent theory exists, but it is not easy to extract physics from this theory and proof that the classical limit of this theory is correct classical general relativity is still lacking. Alternative versions of the Hamiltonian constraint have been proposed and are under investigation. In all these cases, the Hamiltonian has the crucial properties of acting on nodes only. This implies that its action is naturally discrete and combinatorial. This fact is at the root of the finiteness of the theory. The theory can be extended to include matter, and there are strong indications that ultraviolet divergences do not appear.

A spacetime-covariant version of the theory is given by the spin-foam formalism (Section 6.7). The group field theory formalisms (Section 6.8) provides a dual formulation of the theory that generates the spin-foam sum. In 4D, the precise relation between the different formalisms is under investigation.

A key physical result is given by the explicit computation of the eigenvalues of area and volume (see Equation (33)). This result is at the basis of the physical picture of a discrete spacetime. To be sure, discreteness is in the quantum mechanical sense: generically physical space is not an ensemble of quanta, but a continuous probabilistic superposition of ensembles of quanta.

The theory has numerous physical applications, including quantum cosmology, black hole physics and others.

The two main (related) open problems are to understand the description of the low energy regime within the theory and to find the correct version of the dynamics, either via the Hamiltonian constraint or via a spin-foam vertex.

The history of quantum gravity is a sequence of moments of great excitement followed by bitter disappointment. I distinctively remember, as a young student, listening to a very famous physicist announcing at a major conference that quantum gravity was solved (I think it was the turn of supergravity). The list of theories that claimed to be final and have then ended up forgotten or superseded is a reason for embarrassment for the theoretical physics community, in my opinion.

In my view, loop quantum gravity is the best we can do so far in trying to understand quantum spacetime, from a nonperturbative, background-independent point of view. Theoretically, we have reasons to suspect that this approach could represent a consistent quantum theory with the correct classical limit. The theory yields a definite physical picture of quantum spacetime and definite quantitative predictions, but a systematic way of extracting physical information is still lacking. Experimentally, there is no support for the theory, neither direct nor indirect. The spectra of area and volume computed within the theory could, or could not, be physically correct. I hope we may find a way to know in the not too distant future.

# 10    Acknowledgments

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