Towards Loop Quantum Supergravity (LQSG)

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Abstract

Should nature be supersymmetric, then it will be described by Quantum Supergravity at least in some energy regimes. The currently most advanced description of Quantum Supergravity and beyond is Superstring Theory/M-Theory in 10/11 dimensions. String Theory is a top to bottom approach to Quantum Supergravity in that it postulates a new object, the string, from which classical supergravity emerges as a low energy limit. On the other hand, one may try more traditional bottom to top routes and apply the techniques of Quantum Field Theory.

Loop Quantum Gravity (LQG) is a manifestly background independent and non perturbative approach to the quantisation of classical General Relativity, however, so far mostly without supersymmetry. The main obstacle to the extension of the techniques of LQG to the quantisation of higher dimensional Supergravity is that LQG rests on a specific connection formulation of General Relativity which exists only in $D + 1 = 4$ dimensions.

In this paper we introduce a new connection formulation of General Relativity which exists in all spacetime dimensions. We show that all LQG techniques developed in $D + 1 = 4$ can be transferred to the new variables in all dimensions and describe how they can be generalised to the new types of fields that appear in Supergravity Theories as compared to standard matter, specifically Rarita − Schwinger and p-form gauge fields.

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String/M Theory (ST) [1, 2] and Loop Quantum Gravity (LQG) [3, 4] are rather different programmes that aim for a consistent synthesis of the principles of General Relativity and Quantum Theory. String/M Theory is necessarily 10/11 dimensional, necessarily supersymmetric and is perturbatively defined on appropriate background spacetimes. Loop Quantum Gravity, on the other hand, is to date restricted to 4 spacetime dimensions, does not need supersymmetry and by design is background independently and non perturbatively defined. It is therefore very hard to compare these two approaches.

One possibility to make contact between them consists in the consideration of String/M Theory on spacetime manifolds for which the excess dimensions are compactified and in regimes where supersymmetry is broken, so that only an effective 4d theory of General Relativity and the Standard Model plus quantum corrections survives, which then can be compared to the sector of LQG with small geometry fluctuations around the chosen background. We refer to [5] for the state of the art of String Theory Phenomenology but it transpires that there are many possibilities for doing this and there appears to be no specific model that one can compare LQG to.

Another possibility would be to generalise LQG to higher dimensions including supersymmetric matter in order to compare the methods of the two theories directly in 10/11 dimensions. This idea appears to be easier to implement because in its fundamental dimension, String/M Theory is much simpler to describe and there are much less choices to be made. Concretely, String Theory can be considered as a specific proposal for quantising classical Supergravity [6] in 10/11 dimensions on a background defined by a solution to the classical Supergravity equations of motion. With the possible exception of $N = 8$ Supergravity in 4 spacetime dimensions [7, 8], perturbative approaches seem to fail due to non renormalisability[1] whence non perturbative methods appear to be more promising. Thus, the natural question arises, how to apply non perturbative methods, such as those of LQG, to the quantisation of classical Supergravity in 10/11 dimensions. This is a subject of obvious interest to Supergravity research, but, to the best of our knowledge, the literature on it is rather sparse [9].

When pursuing this question further in the context of LQG, one almost immediately gets stuck: LQG in 4 dimensions is fundamentally based on a connection formulation of the gravitational degrees of freedom ("Graviton") and the construction of this connection and the properties it has and which make it so adapted for purposes of quantisation, exists only in 4 spacetime dimensions [10, 11, 12]. Specifically, in 4 dimensions, General Relativity can be described, in its Hamiltonian form, by an SU(2) Yang – Mills phase space, subject to Gauß, spatial diffeomorphism and Hamiltonian constraints. The key properties of this formulation are:

I. The connection A_a^j and its canonically conjugate momentum E_j^a (Yang-Mills "electric field") are real valued, the $*-$relations of the associated Poisson algebra are trivial. Here $a, b, c, .. = 1, 2, 3$ are spatial, tensorial indices and $j, k, l, .. = 1, 2, 3$ are su(2) indices.

II. The Poisson algebra is of the usual, simple CCR form

$$\{A_a^j(x), A_b^k(y)\} = \{E_j^a(x), E_k^b(y)\} = 0; \quad \{E_j^a(x), A_b^k(y)\} = -G\gamma\delta_b^a\delta_j^k\delta(x, y), \qquad (1)$$

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[1]More precisely, to date no counterterm has been found for $N = 8$ SUGRA in 4 dimensions in any loop contribution carried out so far, some of them way beyond 3 loop order. This could mean that the theory is perturbatively finite like QED in 4 dimensions. It does not mean that it is a UV completion of General Relativity because the perturbation theories most likely diverges. Therefore strictly speaking, also here non perturbative methods need to be employed.

where G is Newton's constant and γ is a free, real valued parameter, called the Immrizi parameter.

III. The gauge group SU(2) is compact.

These three properties are essential for the whole LQG framework. Without them, LQG would not exist. Properties I. and II. imply that there is a sufficiently simple $*-$algebra $\mathfrak{A}$ of functions on phase space separating its points so that one has a chance to find non trivial Hilbert space representations thereof. Property III. implies that the holonomies of A are valued in a compact set. It is therefore possible to construct a probability measure on the space of (distributional) connections. The Hilbert space is then an L_2 space of functions of connections which makes sense due to (1) because on such a space the connection acts by multiplication which is only consistent with the algebra if the connection is Poisson self-commuting.

Put together, this enabled to develop a rigorous kinematical mathematical framework [13, 14] and to find a background independent representation of the holonomy – flux *-algebra $\mathfrak{A}$ which was later shown to be the unique one [15, 16] when one insists on a unitary representation of the spatial diffeomorphism group. Moreover, this representation is also well adapted to the quantum dynamics as one may rigorously implement the spatial diffeomorphism constraint [17] and the Hamiltonian constraint [18] including standard matter [19, 20] without anomalies.

It transpires that LQG heavily relies on a connection formulation with the properties listed above. This connection can be found by two independent methods. The first stays purely within the Hamiltonian framework [10] and uses an extension of the ADM phase space of General Relativity [21] to the afore mentioned Yang Mills phase space which is subjected to the additional su(2) Gauß constraint in order that its symplectic reduction recovers the ADM phase space. The core of the proof of this so called *symplectic reduction theorem* is the observation, that the spin connection of the triad $e_j^a = E_j^a / \sqrt{|\det(E)|}$ has a potential, that is, there exists a functional $F[E]$ such that $\Gamma_a^j(x) = \delta F / \delta E_j^a(x)$. The second method uses the Lagrangian framework and starts from the Holst generalisation [22, 23] of the Palatini action in order to accommodate the Immirzi parameter. This is actually an SO(1,3) Yang-Mills theory phase space in Lorentzian signature, however, it is subject to second class constraints which in particular imply that the connection is not self-commuting with respect to the corresponding Dirac bracket [24]. Therefore properties II. and III. listed above are not satisfied and no kinematical Hilbert space representation of the Dirac bracket algebra have been found so far. In order to obtain the SU(2) Yang-Mills phase space without second class constraints, one therefore imposes the so called time gauge which fixes the boosts of SO(1,3) and solves the second class constraints. What remains from the so(1,3) connection is the afore mentioned su(2) connection whose Dirac brackets are the Poisson brackets (1).

One would now guess that one can simply repeat either of these methods in dimensions $D + 1 > 4$. However, this is not the case. As has been shown in [25], the second route leads to a SO(D) gauge theory in the time gauge but it is not a theory with a connection. In order to obtain a connection formulation one would therefore need the analog of a topological Immirzi term, but such a term is not available in all dimensions. Without time gauge, one does have an SO(1, D) connection formulation but subject to a Dirac bracket which leads to the same complication as in $D + 1 = 4$. Moreover, the gauge group SO(1, D) is not compact so that the functional analytic and measure theoretic tools mentioned before do not apply. The first route also meets difficulties:

In D spatial dimensions, the metric of the ADM phase space has $D(D + 1)/2$ degrees of freedom while an so(D) connection has $D^2(D - 1)/2$ degrees of freedom of which the so(D) Gauß constraint fixes $D(D - 1)/2$. The symplectic reduction of the Yang-Mills phase space by the Gauß constraint therefore leaves $D^2(D - 1)/2 - D(D - 1)/2 = D(D - 1)^2/2$ degrees of freedom

which equals $D(D+1)/2$ precisely for $D = 3$. If one wants to go beyond $D = 3$ one therefore must add more constraints and/or change the gauge group in order to match the correct amount of ADM degrees of freedom and these constraints should better be first class in order to avoid complicated Dirac brackets and associated non commuting connections.

The general analysis of the first route has been started in [26, 27]. We take an unbiased viewpoint and consider a general Yang – Mills phase space with gauge group G, connection A_a^α and conjugate momentum π_α^a where $\alpha = 1, .., N = \dim(G)$ denotes the Lie algebra index. There are DN degrees of freedom in the connection of which the Gauß constraint removes N. If there are no other constraints then we must have $(D-1)N = D(D+1)/2$. The only positive integer solutions to this equation are $(D, N) = (2, 3), (3, 3)$ which again corresponds to 3d or 4d gravity respectively with gauge groups SO(1,2) and SO(3) respectively (with Lorentzian signature). Thus, necessarily more constraints are required.

As has been demonstrated in [26], one can obtain GUT theories by varying G. We are for the time being only interested in General Relativity and ask for the group G of minimal dimension that accomplishes all our requirements. To constrain the possible choices we try to follow as closely as possible the treatment of [10, 26] in D=3 and consider a "square root" e_a^I of the spatial metric $q_{ab} = \eta_{IJ} e_a^I e_b^J$ where $I, J, K, ... = 1, ..n \geq D$. Here η defines a G invariant metric of signature (p, q) with $q \geq D$ in order that q_{ab} is positive definite. This constrains the gauge group to be SO(p, q), $n = p + q$ which has dimension $N = n(n-1)/2$. In order to build the extrinsic curvature we need to construct from e_a^I the so called hybrid connection Γ_{aIJ} defined by

$$D_a e_b^I = \partial_a e_b^I - \Gamma_{ab}^c e_c^I + \Gamma_a{}^I{}_J e_b^J = 0 \tag{2}$$

and define the extrinsic curvature by

$$\sqrt{\det(q)}\, K_a{}^b := [A_{aIJ} - \Gamma_{aIJ}]\pi^{bIJ}, \tag{3}$$

where A is the so(p, q) connection and π its conjugate momentum. Notice that (3) is meaningful since $A - \Gamma$ transforms as a Lie algebra valued one form and not as a connection under SO(p, q). The question is of course whether Γ exists. The fact that $\Gamma_{a(IJ)} = 0$ leads to the consistency condition (all internal indices are moved with η)

$$e_{(c|I} \partial_{a|} e_{b)}^I - \Gamma_{(c|a|b)} = 0, \tag{4}$$

which can be shown to be identically satisfied. Therefore of the $D^2 n$ equations (2) for the $Dn(n-1)/2$ coefficients Γ_{aIJ} only $D^2 n - D^2(D+1)/2$ are independent. Requiring that Γ_{aIJ} can be uniquely solved for leads to a quadratic equation with the 2 roots $n = D$, $n = D + 1$, that is, either $p = 0, q = D$ or $p = \pm 1, q = D$.

Finally the number of constraints additional to the Gauß constraint needed is given by $Dn(n-1)/2 - n(n-1)/2 - D(D+1)/2$ which equals $D^2[D-3]/2$ for $n = D$ and $D(D+1)[D-2]/2$ for $n = D + 1$. The question is of course what these constraints should be. A natural choice is that these constraints should somehow impose that π^{aIJ} is entirely determined by e_a^I. The excess number of degrees of freedom in π^{aIJ} as compared to e_a^I is given by $Dn(n-1)/2 - Dn$ which equals precisely the number of additional constraints needed for both $n = D$ and $n = D + 1$. In order to write π^{aIJ} purely in terms of e_a^I we require an internal vector n^I built from e_a^I such that

$$\pi^{aIJ} = 2\sqrt{\det(q)} q^{ab} n^{[I} e_b^{J]}. \tag{5}$$

There is no way to construct n^I out of e_a^I algebraically for $n = D$ so that in this case we must resort to $D = 3$. For $n = D + 1$, however, we may build the unit normal

$$n_I := \frac{1}{D!\,\sqrt{\det(q)}}\, \epsilon_{IJ_1..J_D}\, \epsilon^{a_1..a_D}\, e_{a_1}^{J_1}..e_{a_D}^{J_D} \tag{6}$$

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satisfying $n_I n^I = \pm 1$ for SO$(\pm 1, D)$.

We conclude that if we want to obtain a connection representation with compact gauge group for *Lorentzian General Relativity*, a natural choice is to consider the phase space of an SO$(D+1)$ Yang-Mills theory subject to an so$(D+1)$ Gauß constraint and additional *simplicity constraints* which impose that π^{aIJ} is determined by a generalised D-bein e_a^I via (5) where $n_I e_a^I = n_I n^I - 1 = 0$ is the unit normal determined by e_a^I. In fact, one might have almost guessed that:

If one performs the Hamiltonian analysis of the Lorentzian Palatini action in $D+1$ dimensions (see e.g. [28] and references therein) then one obtains a primary constraint precisely of the form (5). However, as secondary constraints one obtains an so$(1, D)$ Gauß constraint next to spatial diffeomorphism and Hamiltonian constraints plus one more constraint. This last constraint, let us call it D-constraint, is a second class constraint partner to the simplicity constraint S. Thus, the Hamiltonian analysis of the $D+1$ Palatini action for *Lorentzian gravity* fails to deliver a canonical theory of connections with the properties I., II. and III. listed above for two reasons:
1. The gauge group is SO$(1, D)$ rather than SO$(D+1)$ and thus non compact.
2. The theory suffers from second class constraints and thus leads to Dirac bracket non commuting connections.

As it turns out [28], the second problem can be circumvented by employing the machinery of *gauge unfixing* [29, 30]. Under certain conditions, which are satisfied for our second class pair (S, D), it is possible to trade the second class system under consideration for an equivalent first class system equipped with the original Poisson bracket rather than the Dirac bracket so that the connection remains Poisson commuting. However, the first problem cannot be overcome starting from the Palatini action for Lorentzian General Relativity. Therefore, the canonical theory that we are about to describe does not have an obvious Lagrangian origin (other than by backwards Legendre transform).

The first step [31] consists in writing both the hybrid connection and the simplicity constraint purely in terms of π^{aIJ}. The guideline for doing this consists in replacing the solution $\Gamma_{aIJ}[e]$ of (2), which can be computed explicitly, by a function $\Gamma_{aIJ}[\pi]$ of π alone such that it reduces to $\Gamma_{aIJ}[e]$ when $\pi^{aIJ} = 2n^{[I}E^{a|J]}$ where $E^{aI} = \sqrt{\det(q)}q^{ab}e_b^I$. This has been done explicitly in [31] which leads to a rather complicated expression which can be displayed as a rational homogeneous function of degree zero in terms of π and its first partial derivatives and which transforms as an so$(D+1)$ connection.

Next one shows [31] that the condition that π^{aIJ} is of the form $2n^{[I}E^{a|J]}[E]$ is equivalent to the condition

$$S^{aIJ;bKL} := \pi^{a[IJ}\ \pi^{|b|KL]} = 0 \tag{7}$$

provided that for any non zero vector n the object $Q^{ab} = \pi^{aIK}\pi^{bJL}\delta_{IJ}n_K n_L$ is non degenerate. The proof follows closely the seminal paper [32] in which the possibility of a higher dimensional, canonical version of LQG is also contemplated.

We are now in the position to establish the relation with the ADM phase space. We postulate the following non trivial Poisson brackets

$$\{A_{aIJ}(x), \pi^{bKL}(y)\} := 2\beta\, G\, \delta_a^b\, \delta_{[I}^K\, \delta_{J]}^L\, \delta(x, y), \tag{8}$$

where β is a free real parameter and consider the following quantities

$$\det(q)\, q^{ab} := -\text{Tr}(\pi^a \pi^b), \quad \sqrt{\det(q)}\, K_a{}^b := -\frac{1}{\beta}\text{Tr}([A_a - \Gamma_a]\pi^b) \tag{9}$$

which play the role of the intrinsic D metric and the extrinsic curvature. Then one can show [31] that with the usual formula $P^{ab} := \sqrt{\det(q)}q^{ac}[K_c{}^b - \delta_c^b\, K_d{}^d]$ one obtains the following

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non trivial Poisson brackets

$$\{P^{ab}(x), q_{cd}(y)\} = -G\, \delta_c^{(a}\, \delta_d^{b)}\, \delta(x,y) \tag{10}$$

modulo terms that vanish on the joint constraint surface defined by the simplicity constraint (7) and the so$(D+1)$ Gauß constraint

$$G^{IJ} = \partial_a \pi^{aIJ} + [A_a, \pi]^{IJ}. \tag{11}$$

This is non trivial and heavily relies on the fact that the hybrid connection $\Gamma_{aIJ}[\pi]$ has a *weak potential*, that is, there is a functional $F[\pi]$ such that $\Gamma_{aIJ}(x) = \delta F/\delta \pi^{aIJ}(x)$ modulo terms that vanish on the simplicity constraint surface. Without this property, the bracket $\{P^{ab}(x), P^{cd}(y)\}$ would not vanish on the constraint surface. It is also not difficult to show that the algebra of the simplicity and Gauß constraints is first class and that the variables (9) are weak Dirac observables with respect to both constraints.

Thus, as with the usual LQG variables [10], the (weak) integrability of the spin (hybrid) connection is central to show that the symplectic reduction of the SO$(D+1)$ Yang-Mills phase space by Gauß and simplicity constraints results in the ADM phase space. That the ADM spatial diffeomorphism constraints C_a and Hamiltonian constraint C, when expressed in terms of (9), weakly Poisson commute with $S^{aIJ;bKL}$, G^{IJ} and that their algebra among themselves, as compared to the ADM phase space, is unchanged modulo terms vanishing when $S^{aIJ;bKL} = G^{IJ} = 0$ is a simple corollary. We conclude that we have found a connection formulation for Lorentzian General Relativity in $D+1$ dimensions with $D \geq 2$ with all the desired properties, in particular, all four types of constraints form a first class algebra.

Remarkably, all we have said so far can be performed for all four combinations of the spacetime and internal signatures (s,ζ) respectively where $s = \pm 1$ for Euclidian and Lorentzian General Relativity respectively and $\zeta = \pm 1$ for SO$(D+1)$ and SO$(1,D)$ respectively.

Similar to the situation with the usual variables [10], the constraints take a simple form when written in terms of the curvature F_{abIJ} of A_{aIJ}. One finds modulo Gauß and simplicity constraints [31]

$$C_a = -\text{Tr}(F_{ab}\pi^b), \tag{12}$$

$$\sqrt{\det(q)}C = -\zeta \text{Tr}(F_{ab}\pi^a \pi^b) + \frac{1}{(D-1)^2}\, [\zeta - \frac{s}{\beta^2}]\, [K_a{}^b K_b{}^a - (K_c{}^c)^2] - K_{aIJ}^{TT}\, T^{aIJ;bKL}\, K_{bKL}^{TT}.$$

The spatial diffeomorphism constraint takes unsurprisingly a form analogous to the formulation in $D = 3$. Also the first two terms in C look familiar. However, the third term in the expression for C is new. Here $K_{aIJ} = (A_{aIJ} - \Gamma_{aIJ})/\beta$ and K_{aIJ}^{TT} is transversal $n^I K_{aIJ}^{TT} = 0$ and trace free $E^{aI} K_{aIJ}^{TT} = 0$ on the solution $\pi^{aIJ} = 2n^{[I}E^{a|J]}$ of the simplicity constraint, it can be written as $P_{aIJ}{}^{bKL}[\pi]\, K_{bKL}$ where $P[\pi]$ is a transverse tracefree projector and just depends on π. Also $T^{aIJ;bKL}[\pi]$ is a tensor constructed entirely from π. The significance of this third term is that it removes the dependence of the curvature on the transverse tracefree components of K_{aIJ} which are pure gauge with respect to the simplicity constraints and on which the ADM variables do not depend. Notice that for matching internal and spacetime signature $\zeta = s$ and $\beta = 1$ the second term in C vanishes and the Hamiltonian constraint simplifies, a feature that also is familiar from the usual formulation. In this case one can arrive at (12) also starting from the Palatini action by the method of gauge unfixing [28] where now the third term is generated when making the Hamiltonian constraint invariant under the gauge transformations generated by the simplicity constraint so that the afore mentioned second class partner can be dropped.

In order to quantise the Hamiltonian constraint, we observe that the terms quadratic in K_{aIJ} can be written, as in the $D = 3$ situation [18] as [33]

$$K_{aIJ}(x) \propto \{A_{aIJ}(x), \{V, C_E\}\}, \tag{13}$$

where $V = \int d^D x \sqrt{\det(q)}$ is the total volume and

$$\sqrt{\det(q)}\, C_E := -\mathrm{Tr}(F_{ab}\pi^a\pi^b) \tag{14}$$

is the "Euclidian piece" of the Hamiltonian constraint. Similar remarks apply to the tensors T, P which can be treated by analogous Poisson bracket identities as displayed in [18, 19]. Notice that the kinematical Hilbert space techniques [13, 14, 15, 16] as well as the treatment of the spatial diffeomorphism constraint [17] have been formulated for canonical theories of connections for compact gauge groups in any dimension and thus can be applied to our situation without further effort. In particular, the holonomy flux algebra now consists of $SO(D+1)$ valued holonomies of A along piecewise analytic (or semianalytic) one dimensional paths and $so(D+1)$ valued fluxes of π through semianalytic $D-1$ surfaces.

The following remark is due at this point for the case of $D = 3$:
In the usual formulation the connection A^{LQG} is an $SU(2)$ connection and is related to the extrinsic curvature by

$$A^{\mathrm{LQG}}_{ajk} - \Gamma^{\mathrm{SPIN}}_{ajk}[E] = \gamma\epsilon_{jkl}K^l_a; \quad A^{\mathrm{LQG}}_{a0j} = \Gamma^{\mathrm{SPIN}}_{a0j}[E] \equiv 0, \tag{15}$$

where $K_{ab} = K^j_a e^j_b$ and where $\Gamma^{\mathrm{SPIN}}_{ajk}[E]$ is the spin connection of the co-triad e^j_a built from E^a_j. Neither A^{LQG} nor Γ^{SPIN} have a "boost" part, the information about the extrinsic curvature is encoded in the rotational part of A^{LQG} and depends on the Immirzi parameter γ. On the other hand, in the "time gauge" $n^I = \delta^I_0$

$$A^{\mathrm{NEW}}_{ajk} - \Gamma^{\mathrm{HYB}}_{ajk}[\pi] \approx \mathrm{S} - \text{gauge}, \quad A^{\mathrm{NEW}}_{a0j} - \Gamma^{\mathrm{HYB}}_{a0j}[\pi] = \beta K^j_a, \tag{16}$$

where $\Gamma^{\mathrm{HYB}}[\pi]$ is the hybrid connection built from π. The tracefree rotational components of the new connection are pure gauge and the trace piece of the rotational components vanishes by the boost part of the Gauß constraint. Therefore the information about the extrinsic curvature is encoded in the boost part of the new connection and depends on the parameter β. It transpires that the Immirzi parameter γ and β have nothing to do with each other and that the new formulation is a rather different extension of the ADM phase space as compared to the LQG formulation even in $D = 3$ which nevertheless have the same symplectic reduction, namely the ADM phase space. In [28] we show that for $D = 3$ one can generalise the exposition given so far by considering a 2 parameter family of connection formulations depending on both γ, β. The essential features remain the same, the connection remains Poisson commuting.

The price to pay for the higher dimensional connection formulation are the additional simplicity constraints. On the other hand, this makes this canonical formulation appear much closer to the spin foam models [34] which aim to be a path integral formulation of LQG in $D + 1 = 4$ dimensions. For instance, we expect that the $\beta = 1, \zeta = s$ theory with arbitrary γ corresponds to the "new spin foam models" [35, 36] because both can be obtained from the Holst modification of the Palatini action without imposing the time gauge. Notice that our formulation is in the continuum rather than on a fixed triangulation. In particular, we must define the simplicity constraint operators on spin network functions for all graphs not only simplicial ones. The necessity of doing that in order to make contact with the canonical formulation has been recently emphasised also in [37].

The quadratic, canonical quantum simplicity constraints are naturally quantised by smearing the two factors of π in (7) over two independent $D - 1$ surfaces in order to obtain flux operators which are then shrunk to a point x [33]. Then two types of simplicity constraints arise: Either x is an interior point of an edge (edge simplicity constraints) or a vertex (vertex simplicity constraints). Taking advantage of the flexibility in the choice of the surfaces available in the

continuum formulation as well as taking linear combinations one can show that one can reduce the simplicity constraints to the following building blocks

$$S_{e,e'}^{IJKL} := X_e^{[IJ} X_{e'}^{KL]} = 0 \tag{17}$$

for all pairs of edges e, e' of a the given graph which share a vertex. Here X_e^{IJ} is right invariant vector field on the copy of $SO(D+1)$ associated with the edge e. The edge simplicity constraints arise for $e = e'$, the vertex simplicity constraints for $e \neq e'$. These are the analogs of the diagonal and cross simplicity constraints appearing in the spin foam literature which involve either one face or two faces sharing a "temporal edge". These faces arise from our edges by "time evolving" them into faces.

The edge simplicity constraints enforce the irreducible $SO(D+1)$ representations on the edges of the $SO(D + 1)$ spin network functions to be simple representations which have already been described in great detail in [32] for $SO(D+1)$. The vertex simplicity constraints are anomalous and strong imposition leads to the higher dimensional analog of the Barrett-Crane intertwiner also described already in [32]. To avoid the anomaly one can invent several strategies none of which are entirely satisfactory. Either one can try to construct a vertex master constraint [33] for each vertex whose solutions, however, are beyond any analytic control at the moment. Another possibility is to consider for each vertex a preferred recoupling scheme which then selects a maximal commuting subset of vertex simplicity constraints which in turn select a so called simple intertwiner [38] which appears to establish a unitary map between the old and new formulation in $D + 1 = 4$. The unnatural feature here is that the notion of simple intertwiner depends on the chosen recoupling scheme. Yet another option in the spirit of the "improved spin foam models" [35, 36] is to consider linear simplicity constraints [38] instead. These arise in the canonical framework by considering the time normal n^I as an independent quantum field subject to the normalisation constraint $\mathcal{N} = n^I n_I - 1$ and to replace the quadratic constraints (7) by the linear constraints

$$S^{aIJK} = \pi^{a[IJ} n^{K]}. \tag{18}$$

The framework described so far can be adapted to this formulation of the simplicity constraint and all essential features are preserved. This formulation automatically avoids the topological sector for $D = 3$. The then needed Hilbert space representation for the time normal field is supplied in [39]. Again, it is natural to smear the π appearing in (18) over arbitrary surfaces. Notice that now the wave functions depend on both A and n where A is located along the edges and n along the vertices. The building blocks of the simplicity constraint are now the operators

$$X_{e,x}^{[IJ} n^{K]}(x) \tag{19}$$

for each point x on the edge e and each edge e of the graph. Here $X_{e,x}$ denotes the right invariant vector field on the copy of $SO(D + 1)$ corresponding to segment of e which starts at x and ends at the end point of e. The nice thing is that these constraints are non anomalous because they generate the Lie algebra of the $SO(D)$ subgroup of $SO(D+1)$ stabilising n, however, they enforce that at each point of an edge there should be a specific n dependence which maps out of the space of spin network functions described above. While some strategies for matching linear and quadratic simplicity constraints are spelled out in [38] based on ideas from group averaging, none of them can be considered as complete so far and they deserve further research.

Notice that especially in $D = 3$ we are now faced with a real challenge and opportunity to make progress with the simplicity constraint:
We have two classically equivalent connection formulations, the one based on the Ashtekar-Barbero-Immirzi SU(2) connection with SU(2) Gauß constraint and the new SO(4) connection

formulation with SO(4) Gauß and simplicity constraint. One would therefore expect the corresponding quantum theories to be unitarily or at least semiclassically equivalent. Thus, the usual LQG formulation may help us to find a consistent and non anomalous definition of the simplicity constraint as argued in [38].

So far we have only considered the purely gravitational degrees of freedom (graviton). We now turn to matter. As far as standard matter is concerned, all the machinery developed for $D = 3$ straightforwardly generalises to arbitrary D as far as gauge bosons and (Higgs) scalars are concerned [19]. Slightly more attention is required when considering fermionic matter because the Lorentzian theory uses Dirac matrices for the $SO(1, D)$ Clifford algebra which we would like to recast in terms of the Dirac matrices for the $SO(D + 1)$ Clifford algebra in order that matter and gravitational degrees of freedom transform under the same gauge group $SO(D + 1)$. As far as Dirac fermions are concerned, one can proceed similar [40] as in the case $D = 3$ in [19]: Consider the Lorentzian theory first in time gauge $n^I = \delta_0^I$ which breaks the $SO(1, D)$ internal group down to the $SO(D)$ subgroup preserving the time gauge. In particular, the theory is now formulated in terms of E_j^a, K_a^j as in [25]. Now extend all constraints and variables in an $SO(D + 1)$ covariant way such that they reduce to the previous expressions in the time gauge. This can be done using that $E^{ai} = \pi^{ai0}$ in time gauge when the simplicity constraint holds, by subtracting terms proportional to K_{aIJ}^{TT} as we did for the gravitational degrees of freedom and by replacing the Lorentzian γ_L^0 Dirac matrix by the Euclidian one $\gamma_0 := \gamma_E^0 := i\gamma_L^0$ wherever it appears. For instance, the term $E_j^a \Psi^\dagger \gamma^0 \gamma_j \nabla_a \Psi$ where Ψ are the Dirac fermions and ∇_a is the covariant derivative defined by the $SO(D)$ spin connection of E_i^a, can be written as $\pi^{aIJ} \Psi^\dagger \gamma_{[I} \gamma_{J]} \mathcal{D}_a \Psi$ plus terms involving K_{aIJ} where $\mathcal{D}_a$ is the covariant derivative defined by A_{aIJ}. Here we used the fact that the $SO(D)$ spin connection is the $SO(D+1)$ hybrid connection in time gauge. The K_{aIJ} terms then need to be written as above in double Poisson brackets involving volume and Euclidian piece of the gravitational contribution to the Hamiltonian constraint. This produces 4-fermion terms, see [40] for details but apart from this one can copy the Hilbert space representation from the case $D = 3$ for Dirac fermions given in [20]. We conclude that standard matter can be treated just as in $D = 3$.

The reason why this worked so well is that both $SO(D + 1)$ and $SO(1, D)$ act on the same *complex* representation spaces. When we turn to Supergravity theories, this is no longer helps because Supergravity multiplets contain the Rarita – Schwinger field (gravitino) as a *Majorana fermion* and these belong to *real* representation spaces. In particular, the Lorentzian Dirac matrices (for our mostly plus signature choice in $D + 1 = 4, 10, 11$) are real valued which is crucial in order that the real vector space V formed by Majorana Fermions is preserved under $SO(1, D)$. Since $\gamma_E^0 = i\gamma_L^0$ is then purely imaginary this is no longer the case for $SO(D + 1)$ so that $SO(D + 1)$ does not act on the vector space of Majorana fermions. To see this explicitly, notice that $[\gamma^I, \gamma^J]/8$ are the generators of $so(1, D)$ or $so(D + 1)$ respectively in the spinor representation. We resolve the arising tension by noticing that while there is no action of $SO(D + 1)$ on V there is an action on the set of pairs (n, θ) where θ is a Majorana fermion and n is the time normal field. To see this, notice that the action of $g \in SO(D + 1)$ on n is simply by $n \mapsto g \cdot n$. Let us now build an $SO(D+1)$ matrix $A(n)$ from the components of n, for instance the one that is given in [39] and which maps n^I to the time gauge normal vector δ_0^I. Then one can show that $gA(n)^{-1} = A(g \cdot n)^{-1} R(g, n)$, where $R(g, n)$ is a rotation preserving the time gauge. Such a rotation is generated in the spinor representation by $[\gamma^j, \gamma^k]/8$, $j, k = 1, .., D$ and thus only involves the *real* Dirac matrices γ^j. We now consider the extension $\theta \mapsto A(n)^{-1} \cdot \theta$ off the time gauge and observe that the $SO(D + 1)$ orbit of these vectors is preserved as $g \cdot A(n)^{-1} \cdot \theta = A(g \cdot n)^{-1} \cdot R(g, n) \cdot \theta$. In other words, we have the action on pairs $g \cdot (n, \theta) = (g \cdot n, R(g, n) \cdot \theta)$. Starting from time gauge, one then rewrites all constraints in terms of these variables, replaces Lorentzian γ^0 by the Euclidian one and finally extends off the time gauge similar as for Dirac

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fermions. There is still a remaining complication which arises from the fact that there is a reality condition on the Majorana spinors which effectively gives rise to a non trivial Dirac antibracket and which prohibits to use the formalism of [20]. We supply the missing details and provide a proper Hilbert space representation of the Dirac antibracket [39].

Supergravity theories not only contain the Rarita Schwinger field but often additional tensor fields such as the 3-index photon field A in 11d Supergravity. This field is self-interacting due to a Chern-Simons term in the action (the Hamiltonian is a polynomial of fourth order in A and the momentum π conjugate to it) and it is constrained by a twisted Gauß Law constraint which is of the form $G = d * \pi - c/2 F \wedge F$. Here, $F = dA$ is the curvature of A and c is related to the level of the Chern Simons theory. The appearance of the $F \wedge F$ term implies that one cannot simply solve the constraint by considering the theory as the 3-form equivalent of Maxwell theory in higher dimensions and for which gauge invariant states would simply be gauge invariant "Spin network states", i.e. functions of A integrated over closed 3-surfaces. In particular the analog of the LQG representation is inappropriate to solve the constraint. We resolve the tension [41] by considering the Weyl algebra generated by the exponentials of the Dirac observables F, $P = *\pi + cA \wedge F$ with respect to G which can be computed in closed form and show that the resulting $*-$algebra admits a state (positive linear functional) which is discontinuous in both F, P and is thus of the Narnhofer-Thirring type[2] [42]. The resulting background independent Hilbert space representation then follows from the GNS construction [44] and since the contribution of the 3-index Photon to the Hamiltonian and spatial diffeomorphism constraint can be written just in terms of the observables, they can be quantised in terms of Weyl elements [41].

Thus we believe to have provided the tools for an LQG type of quantisation of at least a subset of Supergravity theories in higher dimensions, notably 11d Supergravity and 4d $N = 8$ Supergravity which can be obtained from the 11d theory by dimensional reduction as well as some Supergravity theories in 10d and 4d. This by far does not exhaust all of them. Some Supergravity theories contain non Abelian p-form fields whose quantisation most probably requires techniques from higher gauge theory [43]. Other Supergravity theories only have an on shell formulation, that is, the algebra of local supersymmetry constraints closes only modulo the equations of motion making them second class off shell. We do not have yet any concrete proposal for how to deal with these kinds of Supergravity Theories, possibly ideas from gauge unfixing may be applied in order to turn such a system first class.

Many questions are not answered by our work, for instance:
The precise relation with String Theory/M Theory is not provided. Our formulation works but is most certainly not the most elegant one. The treatment of the simplicity constraint remains unsatisfactory. A formulation in terms of superfields is maybe more desirable. We hope to be able to improve on this in future publications and have many other applications in mind such as revisiting the AdS/CFT correspondence [45] and its integrability structure [46] between (classical?) Supergravity Theories in the bulk of 11d manifolds with asymptotic AdS topology and $N = 4$ Super Yang-Mills Theories on its conformal boundary which is a 10d conformal field theory.

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[2]This state is different from the Fock state and has the attractive feature of being ghost free while manifestly Poincaré invariant.

References

[1] M.B. Green, J. Schwarz and E. Witten. *Superstring Theory*, vol. 1,2, (Cambridge University Press, Cambridge, 1987).

[2] J. Polchinski. *String Theory*, vol.1: An introduction to the bosonic string, vol. 2: Superstring theory and beyond, (Cambridge University Press, Cambridge, 1998).

[3] C. Rovelli. *Quantum Gravity*, (Cambridge University Press, Cambridge, 2004).

[4] T. Thiemann. *Modern Canonical Quantum General Relativity*, (Cambridge University Press, Cambridge, 2007).

[5] D. Lüst. Seeing through the String Landscape - a String Hunter's Companion in Particle Physics and Cosmology. *JHEP* **0903** (2009), 149. [arXiv:0904.4601 [hep-th]]

[6] P. Van Nieuwenhuizen. Supergravity. *Phys. Rep.* **68** (1981), 189.

[7] N. Arkani-Hamed, F. Cachazo and J. Kaplan. What is the Simplest Quantum Field Theory? *JHEP* **1009** (2010), 016. [arXiv:0808.1446 [hep-th]]

[8] R. Kallosh. On UV Finiteness of the Four Loop N=8 Supergravity. *JHEP* **0909** (2009), 116. [arXiv:0906.3495 [hep-th]]

[9] S. Melosch and H. Nicolai. New canonical variables for $D = 11$ supergravity. *Phys. Lett.* **B416** (1998), 91-100. [hep-th/9709227]

[10] A. Ashtekar. New Hamiltonian formulation of general relativity. *Phys. Rev.* **D36** (1987), 1587-1602.

[11] F. Barbero. Real Ashtekar variables for Lorentzian signature space times. *Phys. Rev.* **D51** (1995), 5507-5510. [gr-qc/9410014]

[12] G. Immirzi. Quantum gravity and Regge calculus. *Nucl. Phys. Proc. Suppl.* **57** (1997), 65. [gr-qc/9701052]

[13] A. Ashtekar and C.J. Isham. Representations of the holonomy algebras of gravity and non-Abelian gauge theories. *Class. Quant. Grav.* **9** (1992), 1433. [hep-th/9202053]

[14] A. Ashtekar and J. Lewandowski. Representation theory of analytic holonomy $C^\star$ algebras. In *Knots and Quantum Gravity*, J. Baez (ed.), (Oxford University Press, Oxford 1994). [gr-qc/9311010]

[15] J. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann. Uniqueness of diffeomorphism invariant states on holonomy – flux algebras. *Commun. Math. Phys.* **267** (2006), 703-733. [gr-qc/0504147]

[16] C. Fleischhack. Representations of the Weyl algebra in quantum geometry. *Commun. Math. Phys.* **285** (2009), 67-140. [math-ph/0407006]

[17] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann. Quantisation of diffeomorphism invariant theories of connections with local degrees of freedom. *Journ. Math. Phys.* **36** (1995), 6456-6493. [gr-qc/9504018]

[18] T. Thiemann. Quantum spin dynamics (QSD). *Class. Quant. Grav.* **15** (1998), 839-873. [gr-qc/9606089]

[19] T. Thiemann. Quantum spin dynamics (QSD): V. Quantum gravity as the natural regulator of the Hamiltonian constraint of matter quantum field theories. *Class. Quant. Grav.* **15** (1998), 1281-1314. [gr-qc/9705019]

[20] T. Thiemann. Kinematical Hilbert spaces for fermionic and Higgs quantum field theories. *Class. Quant. Grav.* **15** (1998), 1487-1512. [gr-qc/9705021]

[21] R. M. Wald. *General Relativity*, (The University of Chicago Press, Chicago, 1989).

[22] S. Holst. Barbero's Hamiltonian derived from a generalised Hilbert-Palatini action. *Phys. Rev.* **D53** (1996), 5966. [gr-qc/9511026]

[23] N. Barros e Sá. Hamiltonian analysis of general relativity with the Immirzi parameter. *Int. J. Mod. Phys.* **D10** (2001), 261-272. [gr-qc/0006013]

[24] S. Alexandrov. SO(4,C) covariant Ashtekar-Barbero gravity and the Immirzi parameter. *Class. Quant. Grav.* **17** (2000), 4255-4268. [gr-qc/0005085]

[25] M. Han, Y. Ma, Y. Ding and L. Qin. Hamiltonian analysis of n-dimensional Palatini gravity with matter. *Mod. Phys. Lett.* **A20** (2005), 725-732. [gr-qc/0503024]

[26] P. Peldan. Actions for gravity, with generalisations: a review. *Class. Quant. Grav.* **11** (1994), 1087-1132. [gr-qc/9305011]

[27] P. Peldan. Ashtekar's variables for arbitrary gauge group. *Phys. Rev.* **D46** (1992), 2279-2282. [hep-th/9204069]

[28] N. Bodendorfer, T. Thiemann and A. Thurn. New Variables for Classical and Quantum Gravity in all Dimensions II. Lagrangian Analysis. [arXiv:1105.3704 [gr-qc]]

[29] P. Mitra and R. Rajaraman. Gauge Invariant Reformulation Of An Anomalous Gauge Theory. *Phys. Lett.* **B225** (1989), 267-271.

[30] A.S. Vytheeswaran. Gauge unfixing in second class constrained systems. *Annals Phys.* **236** (1994), 297-324.

[31] N. Bodendorfer, T. Thiemann and A. Thurn. New Variables for Classical and Quantum Gravity in all Dimensions I. Hamiltonian Analysis. [arXiv:1105.3703 [gr-qc]]

[32] L. Freidel, K. Krasnov and R. Puzio. BF description of higher dimensional gravity theories. *Adv. Theor. Math. Phys.* **3** (1999), 1289-1324. [hep-th/9901069]

[33] N. Bodendorfer, T. Thiemann and A. Thurn. New Variables for Classical and Quantum Gravity in all Dimensions III. Quantum Theory. [arXiv:1105.3705 [gr-qc]]

[34] A. Perez. Spin foam models for quantum gravity. *Class. Quant. Grav.* **20** (2003), R43. [gr-qc/0301113]

[35] J. Engle, R. Pereira and C. Rovelli The Loop-quantum-gravity vertex-amplitude. *Phys. Rev. Lett.* **99** (2007), 161301. [arXiv:0705.2388 [gr-qc]]

[36] L. Freidel and K. Krasnov. A New Spin Foam Model for 4d Gravity. *Class. Quant. Grav.* **25** (2008), 125018. [arXiv:0708.1595 [gr-qc]]

[37] W. Kaminski, M. Kisielowski and J. Lewandowski. Spin-Foams for All Loop Quantum Gravity. *Class. Quant. Grav.* **27** (2010), 095006. [arXiv:0909.0939 [gr-qc]]

[38] N. Bodendorfer, T. Thiemann and A. Thurn. On the Implementation of the Canonical Quantum Simplicity Constraint. [arXiv:1105.3708 [gr-qc]]

[39] N. Bodendorfer, T. Thiemann and A. Thurn. Towards Loop Quantum Supergravity (LQSG) I. Rarita-Schwinger Sector. [arXiv:1105.3709 [gr-qc]]

[40] N. Bodendorfer, T. Thiemann and A. Thurn. New Variables for Classical and Quantum Gravity in all Dimensions IV. Matter Coupling. [arXiv:1105.3706 [gr-qc]]

[41] N. Bodendorfer, T. Thiemann and A. Thurn. Towards Loop Quantum Supergravity (LQSG) II. p-Form Sector. [arXiv:1105.3710 [gr-qc]]

[42] H. Narnhofer and W. Thirring. Covariant QED without indefinite metric. *Rev. Math. Phys.* **SI1** (1992), 197 – 211.

[43] J. C. Baez and J. Huerta. An Invitation to Higher Gauge Theory. [arXiv:1003.4485 [hep-th]]

[44] R. Haag. *Local Quantum Physics*, (Springer Verlag, Berlin 1992).

[45] J. L. Petersen. Introduction to the Maldacena conjecture on AdS / CFT. *Int. J. Mod. Phys.* **A14** (1999), 3597-3672. [hep-th/9902131]

[46] Niklas Beisert et al. Review of AdS/CFT Integrability: An Overview. [arXiv:1012.3982 [hep-th]]