On the Implementation of the Canonical Quantum Simplicity Constraint

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May 19, 2011

Abstract

In this paper, we are going to discuss several approaches to solve the quadratic and linear simplicity constraints in the context of the canonical formulations of higher dimensional General Relativity and Supergravity developed in [1, 2, 3, 4, 5, 6]. Since the canonical quadratic simplicity constraint operators have been shown to be anomalous in any dimension $D \geq 3$ in [3], non-standard methods have to be employed to avoid inconsistencies in the quantum theory. We show that one can choose a subset of quadratic simplicity constraint operators which are non-anomalous among themselves and allow for a natural unitary 1-1 map to the SU(2)-based Ashtekar-Lewandowski Hilbert space in $D = 3$. The linear constraint operators on the other hand are non-anomalous by themselves, however their solution space will be shown to differ in $D = 3$ from the expected Ashtekar-Lewandowski Hilbert space. We comment on possible strategies to make a connection to the quadratic theory. We emphasise that many ideas developed in this paper are certainly incomplete and should be considered as suggestions for possible starting points for more satisfactory treatments in the future.

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Contents

1 Introduction

In [1, 2], gravity in any dimension $D+1 \geq 3$ has been formulated as a gauge theory of SO$(1, D)$ or of the compact group SO$(D+1)$, irrespective of the spacetime signature. The resulting theory has been obtained on two different routes, a Hamiltonian analysis of the Palatini action making use of the procedure of gauge unfixing[1], and on the canonical side by an extension of the ADM phase space. The additional constraints appearing in this formulation, the simplicity constraints, are well known. They constrain bivectors to be simple, i.e. the antisymmetrised product of two vectors. Originally introduced in Plebanski's [10] formulation of General Relativity as a constrained BF theory in $3+1$ dimensions, they have been generalised to arbitrary dimension in [11]. Moreover, discrete versions of the simplicity constraints are a standard ingredient of the Spin Foam approaches to quantum gravity [12, 13, 14], see [15, 16] for reviews, and recently were also used in Group Field theory [17]. Two different versions of simplicity constraints are considered in the literature, which are either quadratic or linear in the bivector fields. The quantum operators corresponding to the quadratic simplicity constraints have been found to be anomalous both in the covariant [18] as well as in the canonical picture [19, 3]. On the covariant side, this lead to one of the major points of critique about the Barrett-Crane model [12]: The anomalous constraints are imposed strongly[2], which may imply erroneous elimination of physical degrees of freedom [20]. This triggered the development of the new Spin Foam models [21, 22, 13, 18, 14, 23], in which the quadratic simplicity constraints are replaced by linear simplicity constraints. The linear version of the constraints is slightly stronger than the quadratic constraints, since in $3+1$ dimensions the topological solution is absent. The

[1] See [7, 8, 9] for original literature on gauge unfixing.

[2] Strongly here means that the constraint operator annihilates physical states, $\hat{C}\,|\psi\rangle = 0\ \forall\, |\psi\rangle \in \mathcal{H}_{phys}$

corresponding quantum operators are still anomalous (unless the Immirzi parameter takes the values $\gamma = \pm\sqrt{\zeta}$, where ζ denotes the internal signature, or $\gamma = \infty$). Therefore, in the new models (parts of) the simplicity constraints are implemented weakly to account for the anomaly. Also, the newly developed U(N) tools [24, 25, 26] have been recently applied to solve the simplicity constraints [27, 28].

In this paper, we are not going to import techniques for solving the simplicity constraints which were developed in other contexts, but we are going to take an unbiased look at them from the canonical perspective in the hope of finding new clues for how to implement the constraints correctly. Of course, in the end an experiment will have to decide which implementation, if any, will be the correct one. Since such experiments are missing up to now, the general guidelines are of course mathematical consistency of the approach, as well as comparison with the classical implementation of the simplicity constraints in $D = 3$, where the usual SU(2) Ashtekar variables exist. If a satisfactory implementation in $D = 3$ can be constructed, the hope would then be that this procedure has a natural generalisation to higher dimensions.

The paper will be divided into two parts. We will begin with investigating the quadratic simplicity constraint operators which have been shown to be anomalous in [3]. It will be illustrated that choosing a recoupling scheme for the intertwiner naturally leads to a maximal closing subset of simplicity constraint operators. Next, the solution to this subset will be shown to allow for a natural unitary 1-1 map to the SU(2) based Ashtekar-Lewandowski Hilbert space in $D = 3$ and we will finish the first part with several remarks on this quantisation procedure. In the second part, we will analyse the strong implementation of the linear simplicity constraint operators since they are non-anomalous from start. The resulting intertwiner space will be shown to be one-dimensional, which is problematic because this forbids the construction of a natural 1-1 map to the SU(2) based Ashtekar-Lewandowski Hilbert space. In contrast to the quadratic case, the linear simplicity constraint operators will be shown to be problematic when acting on edges. We will discuss several possibilities of how to resolve these problem and finally introduce a mixed quantisation, in which the linear simplicity constraints will be substituted by the quadratic constraints plus a constraint ensuring the equality of the timelike normals N^I and $n^I(\pi)$.

2 The Quadratic Simplicity Constraint Operators

2.1 A Maximal Closing Subset of Vertex Constraints

It has been shown in [3] that the necessary and sufficient building blocks of the quadratic simplicity constraint operator acting on a vertex v are given by

$$R^e_{[IJ} R^{e'}_{KL]} f_\gamma = 0 \quad \forall e, e' \in \{e'' \in E(\gamma); v = b(e'')\}. \tag{2.1}$$

Since not all of these building blocks commute with each other, i.e. the ones sharing exactly one edge, we will have to resort to a non-standard procedure in order to avoid an anomaly in the quantum theory. The strong imposition of the above constraints, leading to the Barrett-Crane intertwiner [12], was discussed in [11]. A master constraint formulation of the vertex simplicity constraint operator was proposed in [3], however apart from providing a precise definition of the problem, this approach has not lead to concrete solution up to now.

In this paper, we are going to explore a different strategy for implementing the quadratic vertex simplicity constraint operators which is guided by two natural requirements:

1. The imposition of the constraints should be non-anomalous.

2. The imposition of the simplicity constraint operator should, at least on the kinematical level, lead to the same Hilbert space as the quantisation of the classical theory without a simplicity constraint. More precisely, there should exist a natural unitary 1-1 map from the solution space of the quadratic simplicity constraint operators $\mathcal{H}_{\text{simple}}$ to the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{\text{AL}}$ in $D = 3$.

The concept of gauge unfixing [7, 8, 9] which was successfully used in order to derive the classical connection formulation of General Relativity [1, 2] used in this paper was originally developed in the context of anomalous gauge theory, where it was observed that first class constraints can turn into second class constraints after quantisation [29, 30, 31, 32, 33]. This is however precisely what is happening in our case: The classically Abelian simplicity constraints become a set of non-commuting operators due to the regularisation procedure used for the fluxes. The natural question arising is thus: How does a set of maximally commuting vertex simplicity constraint operators look like?

Theorem 1. *Given a N-valent vertex $v \in \gamma$, the set*

$$\epsilon_{IJKL\overline{M}} R^{IJ}_{e_1} R^{KL}_{e_1} = \ldots = \epsilon_{IJKL\overline{M}} R^{IJ}_{e_N} R^{KL}_{e_N} = 0 \tag{2.2}$$

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$$\begin{aligned} \epsilon_{IJKL\overline{M}} \left(R^{IJ}_{e_1} + R^{IJ}_{e_2}\right)\left(R^{KL}_{e_1} + R^{KL}_{e_2}\right) &= 0 \\ \epsilon_{IJKL\overline{M}} \left(R^{IJ}_{e_1} + R^{IJ}_{e_2} + R^{IJ}_{e_3}\right)\left(R^{KL}_{e_1} + R^{KL}_{e_2} + R^{KL}_{e_3}\right) &= 0 \\ &\ldots \\ \epsilon_{IJKL\overline{M}} \left(R^{IJ}_{e_1} + \ldots + R^{IJ}_{e_{N-2}}\right)\left(R^{KL}_{e_1} + \ldots + R^{KL}_{e_{N-2}}\right) &= 0 \end{aligned} \tag{2.3}$$

generates a closed algebra of vertex simplicity constraint operators. Under the assumption that no linear combinations with different multi-indices are allowed[3]*, the set is maximal in the sense that adding new vertex constraint operators spoils closure.*

Proof. Closure can be checked by explicit calculation. In order to understand why the calculation works, recall that right invariant vector fields generate the Lie algebra so$(D+1)$ as [3]

$$\left[R^e_{IJ}, R^{e'}_{KL}\right] = \frac{1}{2}\delta_{e,e'}\left(\eta_{JK} R^e_{IL} + \eta_{IL} R^e_{JK} - \eta_{IK} R^e_{JL} - \eta_{JL} R^e_{IK}\right) \tag{2.4}$$

and thus infinitesimal rotations. The commutativity of (2.2) has been discussed in [3]. Further, we see that every element of (2.3) operates on (2.2) as an infinitesimal rotation. The same is also true for the elements in (2.3): Taking the ordering from above, every constraint operates as an infinitesimal rotation on all constraints prior in the list. Since the commutator is antisymmetric in the exchange of its arguments, closure, i.e. commutativity up to constraints, of (2.3) follows.

To prove maximality of the set we will show that, having chosen a subset of simplicity constraints as given in (2.2) and (2.3), adding any other linear combination of the building blocks (2.1) spoils the closure of the algebra. To this end, we make the most general Ansatz

$$\sum_{1 \leq i < j < N} \alpha_{ij}\ \epsilon_{IJKL\overline{M}} R^{IJ}_i R^{KL}_j \tag{2.5}$$

for an N-valent vertex. Note that the diagonal terms ($i = j$) are proportional to (2.2) and therefore do not have to be taken into account in the above sum, and that $R_N = \sum_{i=1}^{N-1} R_i$ can

[3] A superposition of different multi-indices seems to be highly unnatural since an anomaly with the Gauß constraint has to be expected. We are however currently not aware of a proof which excludes this possibility from the viewpoint of a maximal closing set.

be dropped due to gauge invariance. Moreover, α_{ij} can be chosen such that for fixed j' not all $\alpha_{ij'}$ $(i < j')$ are equal. Otherwise, with $\alpha_{ij'} := \alpha_{j'}$ we find the term $\alpha_{j'}\epsilon_{IJKL\overline{M}}R^{IJ}_{1\ldots(j'-1)}R^{KL}_{j'}$ in the sum, which can be expressed as a linear combination of (2.2) and (2.3) and therefore can be dropped. Consider

$$
\begin{aligned}
& \left[\epsilon_{IJKL\overline{M}}R^{IJ}_{12}R^{KL}_{12}, \epsilon_{ABCD\overline{E}}\left(\alpha_{13}R^{AB}_{1}R^{CD}_{3} + \alpha_{23}R^{AB}_{2}R^{CD}_{3} + \ldots\right)\right] \\
\approx \; & \sum_{j=3}^{N-1} 2\alpha_{1j}\; \epsilon_{IJKL\overline{M}}R^{IJ}_{2}f^{KL\;AB}{}_{MN}R^{MN}_{1}\epsilon_{ABCD\overline{E}}R^{CD}_{j} \\
+ \; & \sum_{j=3}^{N-1} 2\alpha_{2j}\; \epsilon_{IJKL\overline{M}}R^{IJ}_{1}f^{KL\;AB}{}_{MN}R^{MN}_{2}\epsilon_{ABCD\overline{E}}R^{CD}_{j} \\
\approx \; & \sum_{j=3}^{N-1} 2(\alpha_{1j} - \alpha_{2j})\; \epsilon_{IJKL\overline{M}}R^{IJ}_{2}f^{KL\;AB}{}_{MN}R^{MN}_{1}\epsilon_{ABCD\overline{E}}R^{CD}_{j},
\end{aligned}
\tag{2.6}
$$

where we dropped terms proportional to (2.2) in the first and in the second step. For a closing algebra, the right hand side of (2.6) necessarily has to be proportional to (a linear combination of) simplicity building blocks (2.1). Terms containing R_j $(j \geq 3)$ have to vanish separately (In general, one could make use of gauge invariance to "mix" the contributions of different R_j. However, in the case at hand this will produce terms containing R_N, which do not vanish if the contributions of different R_js did not already vanish separately).

We start with the case $D = 3$. The summands on the right hand sides of (2.6) are proportional to

$$\delta^{ABC}_{IJK}(R_j)_{AB}(R_2)^{IJ}(R_1)^{K}{}_{C}, \tag{2.7}$$

where we used the notation $\delta^{I_1\ldots I_n}_{J_1\ldots J_n} := n!\;\delta^{I_1}_{[J_1}\delta^{I_2}_{J_2}\ldots\delta^{I_n}_{J_n]}$. To show that this expression can not be rewritten as a linear combination of the of building blocks (2.1) we antisymmetrise the indices $[ABIJ]$, $[ABKC]$ and $[IJKC]$ and find in each case that the result is zero.

For $D > 3$, the summands are proportional to

$$\delta^{ABC\overline{E}}_{IJK\overline{M}}(R_j)_{AB}(R_2)^{IJ}(R_1)^{K}{}_{C}. \tag{2.8}$$

Whatever multi-index $\overline{E}$ we might have chosen in the Ansatz (2.5), we can always restrict attention to those simplicity constraints in the maximal set which have the same multi-index $\overline{M} = \overline{E}$. Then, the same calculation as in the case of $D = 3$ shows that the antisymmetrisations of the indices $[ABIJ]$, $[ABKC]$ and $[IJKC]$ vanish.

Therefore, the only possibilities are (a) the trivial solution $\alpha_{1j} = \alpha_{2j} = 0$ or (b) $\alpha_{1j} = \alpha_{2j}(\neq 0)$, which implies that the terms on the right hand side of (2.6) are a rotated version of $\epsilon_{IJKL\overline{M}}R^{IJ}_{1}R^{KL}_{2}$. The second option (b) is, for $j = 3$, excluded by our choice of α_{ij} and we must have $\alpha_{13} = \alpha_{23} = 0$. Next, consider $j = 4$ and suppose we have $\alpha_{14} = \alpha_{24} := \alpha' \neq 0$. Then, we can define $\alpha'_{34} := \alpha_{34} - \alpha'$ and find the terms $\alpha'\epsilon_{IJKL\overline{M}}R^{IJ}_{123}R^{KL}_{4} + \alpha'_{34}\epsilon_{IJKL\overline{M}}R^{IJ}_{3}R^{KL}_{4}$ in (2.5). The first term again is already in the chosen set, which implies we can set $\alpha_{14} = \alpha_{24} = 0$ w.l.o.g. by changing $\alpha_{34} \to \alpha'_{34}$ (We will drop the prime in the following). This immediately generalises to $j > 4$, and we have w.l.o.g. $\alpha_{1j} = \alpha_{2j} = 0$ $(3 \leq j < N)$.

Suppose we have calculated the commutators of $\epsilon_{IJKL\overline{M}}R^{IJ}_{1\ldots i}R^{KL}_{1\ldots i}$ $(i = 2, \ldots, n)$ with (2.5) and found that for closure, we need $\alpha_{ij} = 0$ for $1 \leq i \leq n$ and $i < j < N$. Then,

$$\left[\epsilon_{IJKL\overline{M}} R^{IJ}_{1...(n+1)} R^{KL}_{1...(n+1)}, \epsilon_{ABCD\overline{E}} \left(\sum_{j=n+2}^{N-1} \alpha_{(n+1)j} R^{AB}_{(n+1)} R^{CD}_{j} + ... \right) \right] \approx$$

$$\approx \sum_{j=(n+2)}^{N-1} 2\alpha_{(n+1)j}\ \epsilon_{IJKL\overline{M}} R^{IJ}_{1...n} f^{KL\ AB}{}_{MN} R^{MN}_{(n+1)} \epsilon_{ABCD\overline{E}} R^{CD}_{j}, \tag{2.9}$$

which, by the reasoning above, again is not a linear combination of any simplicity building blocks for any choice of $\alpha_{(n+1)j}$, and therefore only the trivial solution $\alpha_{(n+1)j} = 0$ $(n+1 < j < N)$ leads to closure of the algebra. □

2.2 The Solution Space of the Maximal Closing Subset

In order to interpret this set of constraints recall from [3] that the constraints in (2.2) are the same as the diagonal simplicity constraints acting on edges of γ and can be solved by demanding the edge representations to be simple. The remaining constraints (2.3) can be interpreted as specifying a recoupling scheme for the intertwiner ι at v: Couple the representations on e_1 and e_2, then couple this representation to e_3, and so forth, see fig. 1. We call the intermediate virtual edges e_{12}, e_{123}, ... and denote the highest weights of the representations thereon by $\vec{\Lambda}_{12}, \vec{\Lambda}_{123}, \ldots$ Since we can use gauge invariance at all the intermediate intertwiners in the recoupling scheme, e.g., $R_{e_1} + R_{e_2} = R_{e_{12}}$, we have

$$\epsilon_{IJKL\overline{M}} \left(R^{IJ}_{e_1} + R^{IJ}_{e_2}\right) \left(R^{KL}_{e_1} + R^{KL}_{e_2}\right) = \epsilon_{IJKL\overline{M}} R^{IJ}_{e_{12}} R^{KL}_{e_{12}} = 0 \tag{2.10}$$

and thus that the representation on e_{12} has to be simple, i.e.

$$\vec{\Lambda}_{12} = (\lambda_{12}, 0, ..., 0) \qquad \lambda_{12} = 0, 1, 2, ... \tag{2.11}$$

Using the same procedure, all intermediate representations are required to be simple and the intertwiner is labeled by $N-3$ "spins" $\lambda_i \in \mathbb{N}_0$. We call an intertwiner where all internal lines are labeled with simple representations simple.

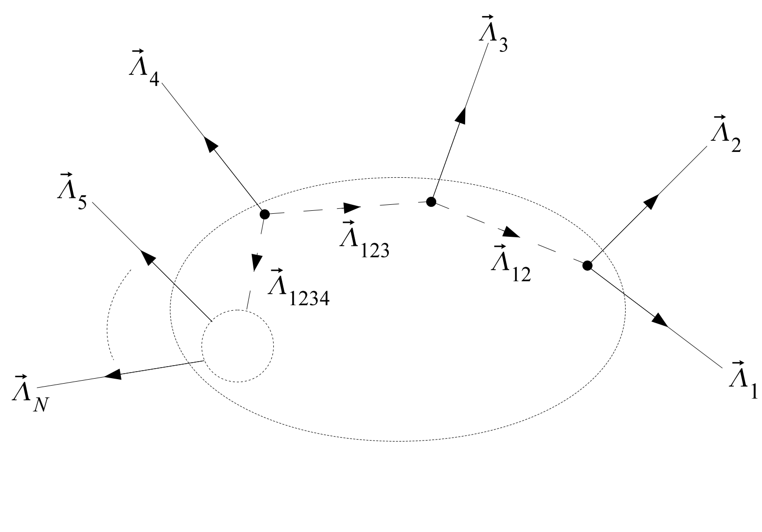

Figure 1: Recoupling scheme corresponding to the subset of quadratic vertex simplicity constraint operators (2.3).

Denote by $\mathcal{I}_N^{\mathrm{SU}(2)}$ the set of SU(2) intertwiners and by $\mathcal{I}_{s,N}^{\mathrm{Spin}(D+1)}$ the set of simple Spin$(D+1)$ intertwiners. Recalling that an N-valent SU(2) intertwiner can be expressed in the same recoupling basis and calling the intermediate spins j_i, we see that the map

$$\begin{aligned} F : \mathcal{I}_{s,N}^{\mathrm{Spin}(D+1)} &\rightarrow \mathcal{I}_N^{\mathrm{SU}(2)} \\ \frac{1}{2}\lambda_i &\mapsto j_i \end{aligned} \tag{2.12}$$

is unitary (with respect the scalar products induced by the respective Ashtekar-Lewandowski measures, see [3]) and $1-1$. The motivation for the factor 1/2 comes from the fact that $\vec{\Lambda} = (1, 0)$ in $D = 3$ corresponds to the familiar $j_+ = j_- = 1/2$ and the area spacings of the SO(4) and the SU(2) based theories agree using this identification, cf [3].

2.3 Remarks

- Since the choice of the maximal closing subset of the simplicity constraint operators is arbitrary, no recoupling basis is preferred a priori. On the SU(2) level, a change in the recoupling scheme amounts to a change of basis in the intertwiner space and therefore poses no problems. On the level of simple Spin$(D+1)$ representations however, a choice in the recoupling scheme affects the property "simple", since the non-commutativity of constraint operators belonging to different recoupling schemes means that kinematical states cannot have the property simple in both schemes.

- There exist recoupling schemes which are not included in the above procedure, e.g., take $N = 6$ and the constraints $\epsilon R_{12} R_{12} = \epsilon R_{34} R_{34} = \epsilon R_{56} R_{56} = 0$ and couple the three resulting simple representations. The theorem should however generalise to those additional recoupling schemes.

- It is doubtful if the action of the Hamiltonian constraint leaves the space of simple intertwiners in a certain recoupling scheme invariant. To avoid this problem, one could use a projector on the space of simple intertwiners in a certain recoupling scheme to restrict the Hamiltonian constraint on this subspace and average later on over the different recoupling schemes if they turn out to yield different results. The possible drawbacks of such a procedure are however presently unclear to the authors and we refer to further research.

- It would be interesting to check whether the dropped constraints are automatically solved in the weak operator topology (matrix elements with respect to solutions to the maximal subset).

- The imposition of the constraints can be stated as the search for the joint kernel of a maximal set of commuting generalised area operators

  $$\mathrm{Ar}_{\overline{M}}[S] := \sum_{U \in \mathcal{U}} \sqrt{\frac{1}{4} \epsilon_{IJKL\overline{M}} \pi^{IJ}(S_U) \pi^{KL}(S_U)|}. \tag{2.13}$$

  Notice, however, that for $D > 3$ these generalised area operators, just as the simplicity constraints, are not gauge invariant while in $D = 3$ they are

- In $D = 3$ we have the following special situation:
  We have two classically equivalent extensions of the ADM phase at our disposal whose respective symplectic reduction reproduces the ADM phase space. One of them is the Ashtekar-Barbero-Immirzi connection formulation in terms of the gauge group SU(2) with

additional SU(2) Gauß constraint next to spatial diffeomorphism and Hamiltonian constraint, and the other is our connection formulation in terms of SO(4) with additional SO(4) Gauß constraint and simplicity constraint. Both formulations are classically completely equivalent and thus one should expect that also the quantum theories are equivalent in the sense that they have the same semiclassical limit. Let us ask a stronger condition, namely that the joint kernel of SO(4) Gauß and simplicity constraint of the SO(4) theory is unitarily equivalent to the kernel of the SU(2) Gauß constraint of the SU(2) theory. To investigate this first from the classical perspective, we split the SO(4) connection and its conjugate momentum (A^{IJ}, π_{IJ}) into self-dual and anti-selfdual parts $A^j_\pm, \pi^\pm_j)$ which then turn out to be conjugate pairs again. It is easy to see that the SO(4) Gauß constraint G_{IJ} splits into two SU(2) Gauß constraints $G^\pm_j$, one involving only self-dual variables and the other only anti-selfdual ones which therefore mutually commute as one would expect. The SO(4) Gauß constraint now asks for separate SU(2) gauge invariance for these two sectors. Thus a quantisation in the Ashtekar-Isham-Lewandowski representation would yield a kinematical Hilbert space with an orthonormal basis $T^+_{s_+} \otimes T^-_{s_-}$ where $S_\pm$ are usual SU(2) invariant spin networks. The simplicity constraint, which in $D = 3$ is Gauß invariant and can be imposed after solving the Gauß constraint, from classical perspective asks that the double density inverse metrics $q^{ab}_\pm = \pi^a_{j\pm}\pi^b_{k\pm}\delta^{jk}$ are identical. This is classically equivalent to the statement that corresponding area functions $\mathrm{Ar}_\pm(S)$ are identical for every S. The corresponding statement in the quantum theory is, however, again anomalous because it is well known that area operators do not commute with each other. On the other hand, neglecting this complication for a moment, it is clear that the quantum constraint can only be satisfied on vectors of the form $T^+_{s_+} \otimes T^-_{s_-}$ for all S if s_+, s_- share the same graph and SU(2) representations on the edges because if S cuts a single edge transversally then the area operator is diagonal with an eigenvalue $\propto \sqrt{j(j+1)}$ and we can always arrange such an intersection situation by choosing suitable S. By a similar argument one can show that the intertwiners at the edges have to be the same. But this is only a sufficient condition because in a sense there are too many quantum simplicity constraints due to the anomaly. However, the discussion suggests that the joint kernel of both SO(4) and simplicity constraint is the closed linear span of vectors of the form $T^+_s \otimes T^-_s$ for the *same* spin network $s = s_+ = s_-$. The desired unitary map between the Hilbert spaces would therefore simply be $T_s \mapsto T^+_s \otimes T^-_s$.

This can be justified abstractly as follows: From all possible area operators pick a maximal commuting subset $\mathrm{Ar}^\pm_\alpha$ using the axiom of choice (i.e. pick a corresponding maximal set of surfaces S_α). We may construct an adapted orthonormal basis $T^\pm_\lambda$ diagonalising all of them[4] such that $\mathrm{Ar}^\pm_\alpha T^\pm_\lambda = \lambda_\alpha T^\pm_\lambda$. Now the constraint

$$\mathrm{Ar}^+_\alpha \otimes \mathbb{1} = \mathbb{1} \otimes \mathrm{Ar}^-_\alpha$$

can be solved on vectors $T^+_{\lambda_+} \otimes T^-_{\lambda_-}$ by demanding $\lambda_+ = \lambda_-$. The desired unitary map would then be $T_\lambda \mapsto T^+_\lambda \otimes T^-_\lambda$. Thus the question boils down to asking whether a maximal closing subset can be chosen such that the eigenvalues λ are just the spin networks s. We leave this to future research.

- In $D \neq 3$ the afore mentioned split into selfdual and anti-selfdual sector is meaningless and we must stick with the dimension independent scheme outlined above. An astonishing feature of this scheme is that after the proposed implementation of the simplicity constraints,

[4] If the maximal set still separates the points of the classical configurations space, this should leave no room for degeneracies, that is the λ_α completely specify the eigenvector. We will assume this to be the case for the following argument.

the size of the kinematical Hilbert space *is the same for all dimensions $D \geq 3$!* By "size", we mean that the spin networks are labelled by the same sets of quantum numbers on the graphs. Of course, before imposing the spatial diffeomorphism constraint these graphs are embedded into spatial slices of different dimension and thus provide different amounts of degrees of freedom. However, after implementation of the diffeomorphism constraint, most of the embedding information will be lost and the graphs can be treated almost as abstract combinatorial objects. Let us neglect here, for the sake of the argument, the possibility of certain remaining moduli, depending on the amount of diffeomorphism invariance that one imposes, which could a priori be different in different dimensions. In the case that the proposed quantisation would turn out to be correct, that is, allow for the correct semiclassical limit, this would mean that *the dimensionality of space would be an emergent concept* dictated by the choice of semiclassical states which provide the necessary embedding information. A possible caveat to this argument is the remaining Hamiltonian constraint and the algebra of Dirac observables which critically depend on the dimension (for instance through the volume operator or dimension dependent coefficients, see [1, 2]) and which could require to delete different amounts of degrees of freedom depending on the dimension.

This idea of dimension emergence is not new in the field of quantum gravity, however, it is interesting to possibly see here a concrete technical realisation which appears to be forced on us by demanding anomaly freedom of the simplicity constraint operators. Of course, these speculations should be taken with great care: The number of degrees of freedom of the classical theory certainly *does* strongly depend on the dimension and therefore the speculation of dimension emergence could fail exactly when we try to construct the semiclassical sector with the solutions to the simplicity constraints advertised above. This would mean that our scheme is wrong. On the other hand, there are indications [34] that the semiclassical sector of the LQG Hilbert space already in $D = 3$ is entirely described in terms of 6-valent vertices. Therefore, the higher valent graphs which in $D = 3$ could correspond to pure quantum degrees of freedom, could account for the semiclassical degrees of freedom of higher dimensional General Relativity. Since there is no upper limit to the valence of a graph, this would mean that already the $D = 3$ theory contains all higher dimensional theories!

Obviously, this puzzle asks for thorough investigation in future research.

- The discussion reveals that we should compare the amount of degrees of freedom that the classical and the quantum simplicity constraint removes. This is a difficult subject, because there is no well defined scheme that attributes quantum to classical degrees of freedom unless the Hilbert space takes the form of a tensor product, where each factor corresponds to precisely one of the classical configuration degrees of freedom. The following "counting" therefore is highly heuristic and speculative:

  In the case $D = 3$, the classical simplicity constraints remove 6 degrees of freedom from the constraint surface per point on the spatial slice. In order to count the quantum degrees of freedom that are removed by the quantum simplicity constraint when acting on a spin network function, we make the following, admittedly naive analogy:
  We attribute to a point on the spatial slice an N-valent vertex v of the underlying graph γ which is attributed to the spatial slice. This point is equipped with degrees of freedom labelled by edge representations and the intertwiner. Every edge incident at v is shared by exactly one other vertex (or returns to v which however does not change the result).

Therefore, only half of the degrees of freedom of an edge can be attributed to one vertex. We take as edge degrees of freedom the $\lfloor \frac{D+1}{2} \rfloor$ Casimir eigenvalues of SO$(D+1)$ labelling the irreducible representation. The edge simplicity constraint removes all but one of these Casimir eigenvalues, thus per edge $\lfloor \frac{D-1}{2} \rfloor$ edge degrees of freedom are removed. Further, a gauge invariant intertwiner is labelled by a recoupling scheme involving $N-3$ irreducible representations not fixed by the irreducible representations carried by the edges adjacent to the vertex in question, which are fully attributed to the vertex (there are $N-2$ virtual edges coming from coupling 1,2 then 3 etc. until N but the last one is fixed due to gauge invariance). We take as vertex degrees of freedom these $N-3$ irreducible representations each of which is labelled again by $\lfloor \frac{D+1}{2} \rfloor$ Casimir eigenvalues. The vertex simplicity constraint again deletes all but one of these eigenvalues, thus it removes $(N-3)\lfloor \frac{D-1}{2} \rfloor$ quantum degrees of freedom. We conclude that the quantum simplicity constraint removes

$$(N-3+\frac{N}{2})\lfloor \frac{D+1}{2} \rfloor$$

quantum degrees of freedom per point (N valent vertex) where $N-3$ accounts for the vertex and $N/2$ for the N edges counted with half weight as argued above. This is to be compared with the classical simplicity constraint which removes $D^2(D-1)/2-D$ degrees of freedom per point. Requiring equality we see that vertices of a definitive valence N_D are preferred in D spatial dimensions which for large D grows quadratically with D. Specifically for $D=3$ we find $N_3=6$. Thus, our naive counting astonishingly yields the same preference for 6-valent graphs in $D=3$ as has been obtained in [34] by completely different methods. From the analysis of [34], it transpires that $N_3=6$ has an entirely geometric origin and one thus would rather expect $N_D=2D$ (hypercubulations) and this may indicate that our counting is incorrect.

3 The Linear Simplicity Constraint Operators

3.1 Regularisation and Anomaly Freedom

Since the linear simplicity constraint is a vector of density weight one, it is most naturally smeared over $(D-1)$-dimensional surfaces. The regularisation of the objects

$$S^b(S) := \int_S b_{L\overline{M}}(x)\epsilon^{IJKL\overline{M}}N^I(x)\pi^{aJK}(x)\epsilon_{ab_1...b_{D-1}}dx^{b_1}\wedge ... \wedge dx^{b_{D-1}} \tag{3.1}$$

therefore is completely analogous to the case of flux vector fields. The corresponding quantum operator

$$\hat{S}^b(S)f = \hat{Y}^{\epsilon b\hat{N}}(S)f = p^*_{\gamma_S}\hat{Y}^{\epsilon b\hat{N}}_{\gamma_S}(S)f_{\gamma_S} = p^*_{\gamma_S}\sum_{e\in\gamma_S}\epsilon(e,S)\epsilon^{IJKL\overline{M}}b_{L\overline{M}}(b(e))\hat{N}^I(b(e))R^{JK}_e f_{\gamma_S} \tag{3.2}$$

has to annihilate physical states for all surfaces $S\subset\sigma$ and all semianalytic functions $b^{I\overline{M}}$ of compact support, where p_γ denotes the cylindrical projection and γ_S is a graph adapted to the surface S. Since we can always choose surfaces which intersect a given graph only in one point, this implies that the constraint has to vanish when acting on single points of a given graph. In [3], it has been shown that the right invariant vector fields actually are in the linear span of the flux vector fields. Therefore, it is necessary and sufficient to demand that

$$\epsilon^{IJKL\overline{M}}\; b_{L\overline{M}}(b(e))\; \hat{N}^I(b(e))\; R^{JK}_e \cdot f_\gamma = 0 \tag{3.3}$$

for all points of γ (which can be be seen as the beginning point of edges by suitably subdividing and inverting edges). Since $\hat{N}^I$ acts by multiplication and commutes with the right invariant vector fields, see [5] for details, the condition is equivalent to[5]

$$\bar{R}_e^{IJ} \cdot f_\gamma = 0, \tag{3.4}$$

i.e. the generators of rotations stabilising N^I have to annihilate physical states. Before imposing this conditions on the quantum states, we have to consider the possibility of an anomaly. Classically, both, the linear and the quadratic simplicity constraint are Poisson self-commuting. The quadratic constraint is known to be anomalous both in the Spin Foam [18] as well as in the canonical picture [19, 3] and thus should not be imposed strongly. Also the linear simplicity constraint is anomalous (at least if $\gamma \neq 1$ in the Euclidean theory. But $\gamma = 1$ is ill-defined for SO(4), see e.g., [35]). Surprisingly, in the case at hand, we do not find an anomaly. But that is just because the generators of rotations stabilising N^I form a closed subalgebra! Direct calculation yields, choosing (without loss of generality) $\gamma_{SS'}$ to be a graph adapted to both surfaces S, S',

$$\begin{aligned}
\left[\hat{S}^b_{\gamma_{SS'}}(S), \hat{S}^{b'}_{\gamma_{SS'}}(S')\right] f_{\gamma_{SS'}} &= \left[\sum_{e\in\gamma_{SS'}} \ldots \bar{R}_e^{IJ}, \sum_{e'\in\gamma_{SS'}} \ldots \bar{R}_{e'}^{AB}\right] f_{\gamma_{SS'}} = \sum_{e\in\gamma_{SS'}} \ldots \left[\bar{R}_e^{IJ}, \bar{R}_e^{AB}\right] f_{\gamma_{SS'}} \\
&= \sum_{e\in\gamma_{SS'}} \ldots \bar{\eta}^I{}_K \bar{\eta}^J{}_L \bar{\eta}^A{}_C \bar{\eta}^B{}_D \; f^{KL\;CD}{}_{MN} R_e^{MN} f_{\gamma_{SS'}} \\
&= \sum_{e\in\gamma_{SS'}} \ldots \bar{\eta}^I{}_K \bar{\eta}^J{}_L \bar{\eta}^A{}_C \bar{\eta}^B{}_D \left(\eta^{L][C} \delta^{D]}{}_{[M} \delta_{N]}{}^{[K}\right) R_e^{MN} f_{\gamma_{SS'}} \\
&= \sum_{e\in\gamma_{SS'}} \ldots \bar{R}_e^{MN} f_{\gamma_{SS'}},
\end{aligned} \tag{3.5}$$

where the operator in the last line is in the linear span of the vector fields $\hat{S}^b(S)$. The classical constraint algebra is not reproduced exactly (the commutator does not vanish identically), but the algebra of quantum simplicity constraints closes, they are of the first class. Therefore, strong imposition of the quantum constraints does make mathematical sense.

Note that up to now, we did not solve the Gauß constraint. The quantum constraint algebra of the simplicity and the Gauß constraint can easily be calculated and reproduces the classical result

$$\begin{aligned}
&\left[\hat{S}^b(S), \hat{G}^{AB}[\Lambda_{AB}]\right] p^*_{\gamma_S} f_{\gamma_S} \\
=& p^*_{\gamma_S} \left[\sum_{e'\in E(\gamma_S), v=b(e')} \epsilon(e',S) \epsilon^{IJKL\overline{M}} b_{L\overline{M}}(v) \hat{N}^I(v) R_{e'}^{JK}, \Lambda_{AB}(v) \left(\sum_{e\in E(\gamma_S), v=b(e)} R_e^{AB} + R_N^{AB}\right)\right] f_{\gamma_S} \\
=& p^*_{\gamma_S} \Lambda_{AB}(v) \epsilon^{IJKL\overline{M}} b_{L\overline{M}}(v) \sum_{e\in E(\gamma_S), v=b(e)} \epsilon(e,S) \left(\hat{N}^I(v) \left[R_e^{JK}, R_e^{AB}\right] + \eta^{I[A} N^{B]}(v) R_e^{JK}\right) f_{\gamma_S} \\
=& p^*_{\gamma_S} \Lambda_{AB}(v) \epsilon^{IJKL\overline{M}} b_{L\overline{M}}(v) \sum_{e\in E(\gamma_S), v=b(e)} \epsilon(e,S) \left(\hat{N}^I(v) 2\eta^{KA} R_e^{JB} + \eta^{I[A} N^{B]}(v) R_e^{JK}\right) f_{\gamma_S} \\
=& \hat{S}^{(-\Lambda\cdot b)}(S) p^*_{\gamma_S} f_{\gamma_S}.
\end{aligned} \tag{3.6}$$

where we used $R_N^{AB} := \frac{1}{2}\left(N^A \frac{\partial}{\partial N_B} - N^B \frac{\partial}{\partial N_A}\right)$. It follows that the simplicity constraint operator does not preserve the Gauß invariant subspace (in other words, as in the classical theory, the

[5]Use the decomposition of X_{IJ} into its rotational ($\bar{X}_{IJ} := \bar{\eta}_I^K \bar{\eta}_J^L X_{KL}$) and "boost" parts ($\bar{X}_I := -\zeta N^J X_{IJ}$) with respect to N^I in (3.3).

Gauß constraint does not generate an ideal in the constraint algebra). This implies that the joint kernel of both Gauß and simplicity constraint must be a proper subspace of the Gauß invariant subspace. It is therefore most convenient to look for the joint kernel on in the kinematical (non Gauß invariant) Hilbert space.

3.2 Solution on the Vertices

Consider a slight modification of the usual gauge-variant spin network functions, where the intertwiners $i_v = i_v(N)$ are square integrable functions of N^I. Let v be a vertex of γ and $e_1, \ldots, e_n$ the edges of γ incident at v, where all orientations are chosen such that the edges are all outgoing at v. Then we can write the modified spin network functions

$$\begin{aligned} T_{\gamma,\vec{l},\vec{i}}(A,N) &:= (i_v(N))_{\vec{K}_1\ldots\vec{K}_n} \prod_{i=1}^{n} \Big(\pi_{l_{e_i}}(h_{e_i}(A))\Big)_{\vec{K}_i\vec{K}'_i} (M_v)_{\vec{K}'_1\ldots\vec{K}'_n} \\ &= \operatorname{tr}\Big(i_v(N)\cdot\otimes_{i=1}^{n}\pi_{l_{e_i}}(h_{e_i}(A))\cdot M_v\Big), \end{aligned} \tag{3.7}$$

where M_v contracts the indices corresponding to the endpoints of the edges e_i and represents the rest of the graph γ. These states span the combined Hilbert space for the normal field and the connection $\mathcal{H}_T = \mathcal{H}_{\text{grav}} \otimes \mathcal{H}_N$ (cf. [5]) and they will prove convenient for solving the simplicity constraints. Choose the surface S' such that it intersects a given graph γ' only in the vertex $v' \in \gamma'$. The action of $\hat{S}^b(S')$ on the vertex v' of a spin network $T_{\gamma',\vec{l},\vec{i}}(A,N)$ implies with (3.4) that

$$\begin{aligned} \hat{S}^b(S')_{\gamma'} T_{\gamma',\vec{l},\vec{i}}(A,N) &= 0 \\ \Longleftrightarrow\ \operatorname{tr}\Big(\Big(i_v(N)\bar{\tau}^{IJ}_{\pi_{l_e}}\Big)\cdot\otimes_{i=1}^{n}\pi_{l_{e_i}}\left(h_{e_i}(A)\right)\cdot M_v\Big) &= 0\ \forall e \text{ at } v', \end{aligned} \tag{3.8}$$

where $\tau^{IJ}_{\pi_{l_e}}$ here denote the generators of $\mathrm{SO}(D+1)$ in the representation π_{l_e} of the edge e and the bar again denotes the restriction to rotational components (w.r.t. N^I). The above equation implies that the intertwiner i_v, seen as a vector transforming in the representation $\bar{\pi}_{l_e}$ dual to π_{l_e} of the edge e, has to be invariant under the $\mathrm{SO}(D)_N$ subgroup which stabilises the N^I. By definition [36], the only representations of $\mathrm{SO}(D+1)$ which have in their space nonzero vectors which are invariant under a $\mathrm{SO}(D)$ subgroup are of the representations of class one (cf. also appendix A), and they exactly coincide with the simple representations used in Spin Foams [11]. It is easy to see that the dual representations of simple representations are simple representations. Therefore, all edges must be labelled by simple representations of $\mathrm{SO}(D+1)$. Moreover, $\mathrm{SO}(D)$ is a massive subgroup of $\mathrm{SO}(D+1)$ [36], so that the (unit) invariant vector $\xi_{l_e}(N)$ in the representation $\bar{\pi}_{l_e}$ is unique, which implies that the allowed intertwiners $i_v(N)$ are given by the tensor product of the invariant vectors of all n edges and potentially an additional square integrable function $F_v(N)$, $i_v(N) = \xi_{l_{e_1}}(N) \otimes \ldots \otimes \xi_{l_{e_n}}(N) \otimes F_v(N)$. Going over to normalised gauge invariant spin network functions implies that $F_v(N) = 1$, and the resulting intertwiner space solving the simplicity and Gauß constraint becomes one-dimensional, spanned by $I_v(N) := \xi_{l_{e_1}}(N) \otimes \ldots \otimes \xi_{l_{e_n}}(N)$. We will call these intertwiners and vertices coloured by them linear-simple. For an instructive example of the linear-simple intertwiners, consider the defining representation (which is simple since the highest weight vector is $\Lambda = (1, 0, \ldots, 0)$, cf. appendix A). The unit vector invariant under rotations (w.r.t. N^I) is given by N^I and for edges in the defining representation incoming at v we simply contract $h_e^{IJ} N_J$. If the constraint is acting on an interior point of an analytic edge, this point can be considered as a trivial two-valent vertex and the above result applies. Since this has to be true for all surfaces, a spin network function solving the constraint would need to have linear-simple intertwiners at every point of its graph γ, i.e. at infinitely many points, which is in conflict with the definition of cylindrical functions (cf. [37]). In the next section, we comment on a possibility of how to implement this idea.

3.3 Edge Constraints

As noted above, the imposition of the linear simplicity constraint operators acting on edges is problematic, because it does not, as one might have expected, single out simple representations, but demand that at every point where it acts, there should be a linear-simple intertwiner. The problem with this type of solution is that all intertwiners, even trivial intertwiners at all interior points of edges, have to be linear-simple, which is however in conflict with the definition of a cylindrical function, in other words, there would be no holonomies left in a spin network because every point would be a N-dependent vertex.

It could be possible to resolve this issue using a rigging map construction [38, 39, 40] of the type

$$\eta(T_{\gamma,\vec{l},\vec{l}_N,\vec{i}})[T_{\gamma',\vec{l}',\vec{l}'_N,\vec{i}'}] := \lim_{\mathfrak{P}_\gamma \ni p_\gamma \to \infty} C\left(p_\gamma, T_\gamma, T_{\gamma'}\right) \left\langle T^{p_\gamma}_{\gamma,\vec{l},\vec{l}_N,\vec{i}}\ ,\ T_{\gamma',\vec{l}',\vec{l}'_N,\vec{i}'} \right\rangle_{\text{kin}}, \tag{3.9}$$

where $\mathfrak{P}_\gamma$ is the set of finite point sets p of a graph γ, $p = \{\{x_i\}_{i=1}^N | x_i \in \gamma \,\forall\, i, N < \infty\}$. $\mathfrak{P}_\gamma$ is partially ordered by inclusion, $q \succeq p$ if p is a subset of q, so that the limit is meant in the sense of net convergence with respect to $\mathfrak{P}_\gamma$. By the prescription $T^{p_\gamma}_{\gamma,\vec{l},\vec{l}_N,\vec{i}}$ we mean the projection of $T_{\gamma,\vec{l},\vec{l}_N,\vec{i}}$ onto linear-simple intertwiners at every point in p and $C\left(p_\gamma, T_\gamma, T_{\gamma'}\right)$ is a numerical factor. Assuming this to work, consider any surface S intersecting γ'. We (heuristically) find

$$\begin{aligned}
\eta(T_\gamma)[\hat{S}^b(S)T_{\gamma'}] &= \lim_{\mathfrak{P}_\gamma \ni p_\gamma \to \infty} C\left(p_\gamma, T_\gamma, T_{\gamma'}\right) \left\langle T^p_\gamma, \hat{S}^b(S)T_{\gamma'} \right\rangle_{\text{kin}} \\
&= \lim_{\mathfrak{P}_\gamma \ni p_\gamma \to \infty} C\left(p_\gamma, T_\gamma, T_{\gamma'}\right) \left\langle [\hat{S}^b(S)]^\dagger T^p_\gamma, T_{\gamma'} \right\rangle_{\text{kin}} \\
&= \lim_{\mathfrak{P}_\gamma \ni p_\gamma \to \infty} C\left(p_\gamma, T_\gamma, T_{\gamma'}\right) \left\langle \hat{S}^b(S)T^p_\gamma, T_{\gamma'} \right\rangle_{\text{kin}} = 0,
\end{aligned} \tag{3.10}$$

since the intersection points of S with γ will eventually be in p_γ and $\hat{S}^b(S)$ is self-adjoint.

We were however not able to find such a rigging map with satisfactory properties. It is especially difficult to handle observables with respect to the linear simplicity constraint and to implement the requirement, that the rigging map has to commute with observables. It therefore seems plausible to look for non-standard quantisation schemes for the linear simplicity constraint operators, at least when acting on edges. Comparison with the quadratic simplicity constraint suggests that also the linear constraint should enforce simple representations on the edges, see the following remarks as well as section 3.5 for ideas on how to reach this goal.

3.4 Remarks

The intertwiner space at each vertex is one-dimensional and thus the strong solution of the unaltered linear simplicity constraint operator contrasts the quantisation of the classically imposed simplicity constraint at first sight. A few remarks are appropriate:

- One could argue that the intertwiner space at a vertex v is infinite-dimensional by taking into account holonomies along edges e' originating at v and ending in a 1-valent vertex v'. Since e' and v' are assigned in a unique fashion to v if the valence of v is at least 2, we can consider the set $\{v, e', v'\}$ as a new "non-local" intertwiner. Since we can label e' with an arbitrary simple representation, we get an infinite set of intertwiners which are orthogonal in the above scalar product. This interpretation however does not mimic the classical imposition of the simplicity constraints or the above imposition of the quadratic simplicity constraint operators.

- The main difference between the formulation of the theory with quadratic and linear simplicity constraint respectively is the appearance of the additional normal field sector in the linear case. Thus one could expect that one would recover the quadratic simplicity constraint formulation by ad hoc averaging the solutions of the linear constraint over the normal field dependence with the probability measure ν_N defined in [5]. Indeed, if one does so, then one recovers the solutions to the quadratic simplicity constraints in terms of the Barrett-Crane intertwiners in $D = 3$ and higher dimensional analogs thereof as has been shown long ago by Freidel, Krasnov, and Puzio [11]. Such an average also deletes the solutions with "open ends" of the previous item by an appeal to Schur's lemma. Since after such an average the N dependence of all solutions disappears, we can drop the μ_N integral in the kinematical inner product since μ_N is a probability measure. The resulting effective physical scalar product would then be the Ashtekar-Lewandowski scalar product of the theory between the solutions to the quadratic simplicity constraints. Such an averaging would also help with the solution of the edge constraints, since a 2-valent linear-simple intertwiner is averaged as

$$\int_{S^D} d\nu(N)\, \bar{\xi}_l^{\alpha}(N)\xi_l^{\beta}(N) = \frac{1}{d_{\pi_l}}\delta^{\alpha\beta}, \tag{3.11}$$

  thus yielding a projector on simple representations.

- It can be easily checked that the volume operator as defined in [3], and therefore also more general operators like the Hamiltonian constraint, do not leave the solution space to the linear (vertex) simplicity constraints invariant. A possible cure would be to introduce a projector $\mathcal{P}_S$ on the solution space and redefine the volume operator as $\hat{\mathcal{V}} := \mathcal{P}_S \hat{V} \mathcal{P}_S$. Such procedures are however questionable on the general ground that anomalies can always be removed by projectors.

- If one accepts the usage of the projector $\mathcal{P}_S$, calculations involving the volume operator simplify tremendously since the intertwiner space is one-dimensional. We will give a few examples which can be calculated by hand in a few lines, restricting ourselves to the defining representation of SO$(D+1)$, where the SO$(D)_N$ invariant unit vector is given by N^I.

  Having direct access to N^I, one can base the quantisation of the volume operator on the classical expression

$$\det q = \left| \frac{1}{D!} \epsilon_{IJ_1 \ldots J_D} N^I \left(\pi^{a_1 K_1 J_1} N_{K_1}\right) \ldots \left(\pi^{a_D K_D J_D} N_{K_D}\right) \epsilon_{a_1 \ldots a_D} \right|^{\frac{1}{D-1}}. \tag{3.12}$$

  In the case $D+1$ uneven, this choice is much easier than the expression quantised in [3]. In the case $D+1$ even, the above choice is of the same complexity[6] as the one in [3], but leads to a formula applicable in any dimension and therefore, for us, is favoured. Proceeding as

[6] Up to $(N)^{D+1}$, but in the chosen representation $\hat{N}$ acts by multiplication and therefore is less problematic than additional powers of right invariant vector fields.

in [3], we obtain for the volume operator

$$\hat{V}(R) = \int_R d^D p\, |\widehat{\det(q)(p)}|_\gamma = \int_R d^D p\, \hat{V}(p)_\gamma, \tag{3.13}$$

$$\hat{V}(p) = \left(\frac{\hbar}{2}\right)^{\frac{D}{D-1}} \sum_{v\in V(\gamma)} \delta^D(p,v)\hat{V}_{v,\gamma}, \tag{3.14}$$

$$\hat{V}_{v,\gamma} = \left| \frac{i^D}{D!} \sum_{e_1,\ldots,e_D\in E(\gamma),\, e_1\cap\ldots\cap e_D=v} s(e_1,\ldots,e_D)\hat{q}_{e_q,\ldots,e_D} \right|^{\frac{1}{(D-1)}}, \tag{3.15}$$

$$\hat{q}_{e_1,\ldots,e_D} = \epsilon_{IJ_1\ldots J_D}\hat{N}^I \left(R_{e_1}^{K_1J_1}\hat{N}_{K_1}\right)\ldots\left(R_{e_D}^{K_DJ_D}\hat{N}_{K_D}\right). \tag{3.16}$$

Note that the operator $\hat{q}_{e_1,\ldots,e_D}$ is built from D right invariant vector fields. Since these are antisymmetric, $\hat{q}^T_{e_1,\ldots,e_D} = (-1)^D\hat{q}_{e_1,\ldots,e_D}$. In the case at hand, we have to use the projectors $\mathcal{P}_S$ to project on the allowed one-dimensional intertwiner space, the operator $\mathcal{P}_S\hat{q}\mathcal{P}_S$ therefore has to vanish for the case $D+1$ even (an antisymmetric matrix on a one-dimensional space is equal to 0). However, the volume operator depends on $\hat{q}^2$, and $\mathcal{P}_S\hat{q}^2\mathcal{P}_S$ actually is a non-zero operator in any dimension, though trivially diagonal. Therefore, also $\hat{\mathcal{V}}$ is diagonal.

The simplest non-trivial calculation involves a D-valent non-degenerate (i.e. no three tangents to edges at v lie in the same plane) vertex v where all edges are labelled by the defining representation of $\mathrm{SO}(D+1)$ and thus the unique intertwiner which we will denote by $\left|N^{A_1}\ldots N^{A_D}\right\rangle$. We find

$$\begin{aligned}
\hat{q}_{e_1,\ldots,e_D}\left|N^{A_1}\ldots N^{A_D}\right\rangle &= s(e_1,...,e_D)\left(-\frac{1}{2}\right)^D\left|N_I\epsilon^{IA_1\ldots A_D}\right\rangle, \\
\hat{q}_{e_1,\ldots,e_D}\left|N_I\epsilon^{IA_1\ldots A_D}\right\rangle &= s(e_1,...,e_D)\left(\frac{1}{2}\right)^D D!\left|N^{A_1}\ldots N^{A_D}\right\rangle, \\
\hat{V}_v\left|N^{A_1}\ldots N^{A_D}\right\rangle &= \left(\left(\frac{1}{4}\right)^D D!\right)^{\frac{1}{2(D-1)}}\left|N^{A_1}\ldots N^{A_D}\right\rangle,
\end{aligned} \tag{3.17}$$

i.e. for those special vertices, the volume operator preserves the simple vertices. For vertices of higher valence and/or other representations, we need to use the projectors. Of special interest are the vertices of valence $D+1$ (triangulation) and $2D$, where every edge has exactly one partner which is its analytic continuation through v (cubulation). We find

$$\begin{aligned}
\hat{\mathcal{V}}_v\left|N^{A_1}\ldots N^{A_{D+1}}\right\rangle &= \left(\left(\frac{1}{4}\right)^D (D+1)!\right)^{\frac{1}{2(D-1)}}\left|N^{A_1}\ldots N^{A_{D+1}}\right\rangle, \\
\hat{\mathcal{V}}_v\left|N^{A_1}\ldots N^{A_{2D}},\mathrm{cubic}\right\rangle &= \left(\left(\frac{1}{2}\right)^D (D)!\right)^{\frac{1}{2(D-1)}}\left|N^{A_1}\ldots N^{A_{2D}},\mathrm{cubic}\right\rangle.
\end{aligned} \tag{3.18}$$

The dimensionality of the spatial slice now appears as a quantum number like the spins labelling the representations on the edges and it could be interesting to consider a large dimension limit in the spirit of the large N limit in QCD.

- When introducing an Immirzi parameter in $D = 3$ [2], i.e. using the linear constraint $\epsilon_{IJKL}N^J\pi^{aKL} \approx 0$ while having $\{A_{aIJ}(x), {}^{(\gamma)}\pi^{bKL}(y)\} = 2\delta_a^b\delta_{[I}^K\delta_{J]}^L\delta^{(D)}(x-y)$ with ${}^{(\gamma)}\pi^{aIJ} =$

$\pi^{aIJ} + 1/(2\gamma)\epsilon^{IJKL}\pi^a{}_{KL}$, the linear simplicity constraint operators become anomalous unless $\gamma = \pm\sqrt{\zeta}$, the (anti)self-dual case, which however results in non-invertibility of the prescription $^{(\gamma)}$. Repeating the steps in section 3.1, we find that these anomalous constraints require $\epsilon_{IJKL}N^I(R_e^{KL} - 1/(2\gamma)\epsilon^{KLMN}R_e^{MN}) \cdot f_\gamma = 0$. Since $\epsilon_{IJKL}N^I(R_e^{KL} - 1/(2\gamma)\epsilon^{KLMN}R_e^{MN})$ do not generate a subgroup, the constraint can not be satisfied strongly if the edge e transforms in an irreducible representation of $\mathrm{SO}(D+1)$ (by definition, the representation space does not contain an invariant vector).

In order to figure out the "correct" quantisation, one can try, in analogy to the strategy for the quadratic simplicity constraints, to weaken the imposition of the constraints at the quantum level. The basic difference between the linear and the quadratic simplicity constraints is that the time normal N^I is left arbitrary in the quadratic case and fixed in the linear case. In order to loose this dependence in the linear case, one could average over all N^I at each point in σ, which however leads to the Barrett-Crane intertwiners as described above. In analogy to the quadratic constraints, we could choose the subset

$$\begin{aligned}
\epsilon_{IJKL\overline{M}}N^J\left(R_{e_1}^{KL} + R_{e_2}^{KL}\right) &= 0 \\
\epsilon_{IJKL\overline{M}}N^J\left(R_{e_1}^{KL} + R_{e_2}^{KL} + R_{e_3}^{KL}\right) &= 0 \\
&\dots \\
\epsilon_{IJKL\overline{M}}N^J\left(R_{e_1}^{KL} + \ldots + R_{e_{N-2}}^{KL}\right) &= 0
\end{aligned} \tag{3.19}$$

for each N-valent vertex plus the edge constraints. As above, the choice of the subset specifies a recoupling scheme and the imposition of the constraints leads to the contraction of the virtual edges and virtual intertwiners of the recoupling scheme with the $\mathrm{SO}(D)_N$-invariant vectors $\xi_{l_{e_i}}(N)$ and their complex conjugates $\bar{\xi}_{l_{e_i}}(N)$, see fig. 2. Gauge invariance can still be used at each (virtual) vertex in this calculation in the form $\sum_i \bar{R}_{e_i} = 0$, which is sufficient since only $\bar{R}_{e_i}$ appears in the linear simplicity constraints. If we now integrate over each pair of $\xi_{l_{e_i}}(N)$ "generated" by the elements of the proposed subset of the simplicity constraint operators separately, we obtain projectors on simple representations for each of the virtual edges in the recoupling scheme. The integration over N^I for the edge constraints yields projectors on simple representations in the same manner. Finally, we obtain the simple intertwiners of the quadratic operators in addition to solutions where incoming edges are contracted with $\mathrm{SO}(D)_N$-invariant vectors $\xi_{l_{e_i}}(N)$. A few remarks are appropriate:

- Although this procedure yields a promising result, it contains several non-standard and ah-hoc steps which have to be justified. One could argue that the "correct" quantisation of the linear and quadratic simplicity constraints should give the same quantum theory, however, as is well known, classically equivalent theories result in general in non-equivalent quantum theories, which nevertheless can have the same classical limit.

- It is unclear how to proceed with "integrating out" N^I in the general case. For the vacuum theory, integration over every point in σ gives the Barrett-Crane intertwiner for the edges contracted with $\mathrm{SO}(D)_N$-invariant vectors. This type of integration would also get rid of the 1-valent vertices and thus allow for a natural unitary 1-1 map to the quadratic solutions as already mentioned above.

- When introducing fermions, there is the possibility for non-trivial gauge-invariant functions of N^I at the vertices which immediately results in the question of how to integrate out this N^I-dependence. Next to including those N^I in the above integration or to integrate out the remaining N^I separately, one could transfer this integration back into the scalar

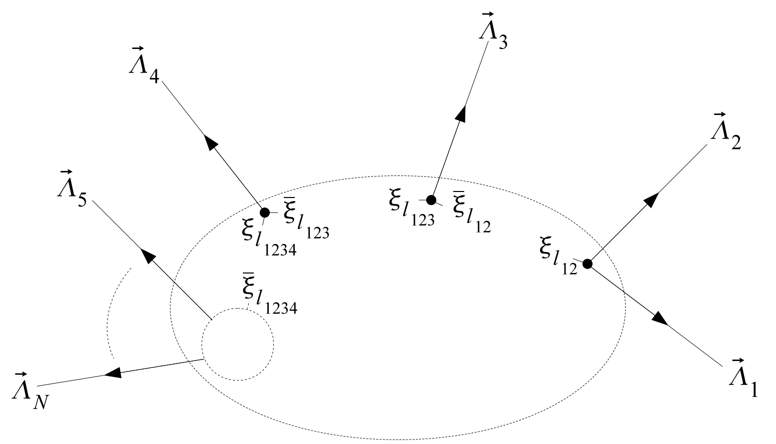

Figure 2: Recoupling scheme corresponding to the subset of linear vertex simplicity constraint operators (3.19).

product. Since the authors are presently not aware of an obvious way to decide about these issues, we will leave them for further research.

3.5 Mixed Quantisation

Since the implementation of the quadratic simplicity constraints described above yields a more promising result than the implementation of the linear constraints, we can try to perform a mixed quantisation by noting that we can classically express the linear constraints for even D in the form

$$\frac{1}{4}\epsilon_{IJKL\overline{M}}\pi^{aIJ}\pi^{bKL} \approx 0, \quad N^I - n^I(\pi) \approx 0. \tag{3.20}$$

The phase space extension derived in [5] remains valid when interchanging the linear simplicity constraint for the above constraints. The reason for restricting D to be even is that we have an explicit expression for $n^I(\pi)$, see [1, 2]. Since a quantisation of $n^I(\pi)$ will most likely not commute with the Hamiltonian constraint operator, we resort to a master constraint. Note that the expression

$$M'_N := \frac{\left((N^I - n^I(\pi))\sqrt{q}^{D-1}\right)\delta_{IJ}\left((N^J - n^J(\pi))\sqrt{q}^{D-1}\right)}{\sqrt{q}^{2D-3}}, \tag{3.21}$$

which is the densitised square of $N^I - n^I(\pi)$, can be quantised as

$$\hat{M'}_N = 2\widehat{\sqrt{q}^{3-2D}}(\sqrt{|\hat{V}^I\hat{V}_I|} - N_I\hat{V}^I), \tag{3.22}$$

when using a suitable factor ordering, where a quantisation of $\sqrt{q}^{3-2D}$ is described in [3]. The solution space is not empty since the intertwiner

$$s(e_1, ..., e_D)\sqrt{D!}|N^{A_1}N^{A_2}\dots N^{A_D} > + |N_B\epsilon^{BA_1\dots A_D} > \tag{3.23}$$

is annihilated by $\hat{M}_N$, which can be easily checked when using the results of the volume operator acting on the solution space of the full set of linear simplicity constraint operators. In order to turn the expression into a well defined master constraint operator, we have to square it again and to adjust the density weight, leading to

$$\hat{\mathbf{M}}_N = 4\left(\widehat{\sqrt{q}^{5/2-2D}}(\sqrt{|\hat{V}^I\hat{V}_I|} - N_I\hat{V}^I)\right)^\dagger \widehat{\sqrt{q}^{5/2-2D}}(\sqrt{|\hat{V}^I\hat{V}_I|} - N_I\hat{V}^I), \tag{3.24}$$

which is by construction a self-adjoint operator with non-negative spectrum. We remark that it was necessary to use the fourth power of the classical constraint for quantisation, because the second power, having the desired property that its solution space is not empty, does not qualify as a well defined master constraint operator in the ordering we have chosen. There exists however no a priori reason why one should not take into account master constraint operators constructed from higher powers of classical constraints [41]. Curiously, the quadratic simplicity constraint operators as given above do not annihilate the solution displayed. Clearly, the calculations will become much harder as soon as vertices with a valence higher than D are used, since the building blocks of the volume operator will not be diagonal on the intertwiner space.

4 Conclusions

In this paper we have reported on several new ideas of how to treat the simplicity constraints which appear in our connection formulation of General Relativity in any dimension $D \geq 3$ [1, 2, 3, 4, 5, 6]. None of them is entirely satisfactory at this point and must be further developed. We hope that the discussion presented in this paper will be useful for an eventually consistent formulation.

Acknowledgements
NB and AT thank Emanuele Alesci, Jonathan Engle, Alexander Stottmeister, and Antonia Zipfel for numerous discussions as well as Karl-Hermann Neeb for email exchange about representation theory. NB and AT thank the German National Merit Foundation for financial support. The part of the research performed at the Perimeter Institute for Theoretical Physics was supported in part by funds from the Government of Canada through NSERC and from the Province of Ontario through MEDT.

A Simple Irreps of SO$(D+1)$ and Square Integrable Functions on the Sphere S^D

There is a natural action of SO$(D+1)$ on $F \in \mathcal{H} := L_2(S^D, d\mu)$ given by $\pi(g)F(N) := F(g^{-1}N)$. The $\pi(g)$ are called quasi-regular representations of SO$(D+1)$. The generators in this representation are of the form $\tau_{IJ} = \frac{1}{2}(\frac{\partial}{\partial N^I}N_J - \frac{\partial}{\partial N^J}N_I)$ and are known to satisfy the quadratic simplicity constraint $\tau_{[IJ}\tau_{KL]} = 0$ [11]. These representations are reducible. The representation space can be decomposed into spaces of harmonic homogeneous polynomials $\mathfrak{H}^{D+1,l}$ of degree l in $D+1$ variables, $L_2(S^D) = \sum_{l=0}^{\infty} \mathfrak{H}^{D+1,l}$. The restriction of $\pi(g)$ to these subspaces gives irreducible representations of SO$(D+1)$ with highest weight $\vec{\Lambda} = (l, 0, ..., 0)$, $l \in \mathbb{N}$. These are (up to equivalence) the only irreducible representations of SO$(D+1)$ satisfying the quadratic simplicity constraint [11] and therefore are mostly called simple representations in the Spin Foam community, which we will adopt in this work. Note that these representations have been studied quite extensively in the mathematical literature, where they are called most degenerate representations [42, 43, 44], (completely) symmetric representations [43, 45, 46, 47] or representations of class one (with respect to a SO(D) subgroup) [36]. The latter is due to the fact that these representations of SO$(D+1)$ are the only ones which have in their representations space a non-zero vector invariant under a SO(D) subgroup, which is exactly the definition of being of class one w.r.t. a subgroup given in [36]. An orthonormal basis in $\mathfrak{H}^{D+1,l}$ is given by generalisations of spherical harmonics to higher dimensions [36] which we denote $\Xi_l^{\vec{K}}(N)$,

$$\int_{S^D} \Xi_l^{\vec{K}}(N)\, \overline{\Xi_{l'}^{\vec{M}}(N)}\, dN = \delta_{l'}^{l} \delta_{\vec{M}}^{\vec{K}}, \tag{A.1}$$

where $\vec{K}$ denotes an integer sequence $\vec{K} := (K_1, \ldots, K_{D-2}, \pm K_{D-1})$ satisfying $l \geq K_1 \geq \ldots \geq K_{D-1} \geq 0$ and analogously defined $\vec{M}$. $F_l(N) \in \mathfrak{H}^{D+1,l}$ can be decomposed as $F_l(N) = \sum_{\vec{K}} a_{\vec{K}} \Xi_l^{\vec{K}}(N)$ where the sum runs over those integer sequences $\vec{K}$ allowed by the above inequality. Since $L_2(S^D) = \sum_{l=0}^{\infty} \mathfrak{H}^{D+1,l}$, any square integrable function $F(N)$ on the sphere can be expanded in a mean-convergent series of the form [36]

$$F(N) = \sum_{l=0}^{\infty} \sum_{\vec{K}_l} a_{\vec{K}_l}^{l} \Xi_l^{\vec{K}_l}(N). \tag{A.2}$$

Consider a recoupling basis [48] for the ONB of the tensor product of N irreps: Choose a labelling of the irreps $\vec{\Lambda}_1, ..., \vec{\Lambda}_N$. Then, consider the ONB

$$\left| \vec{\Lambda}_1, ..., \vec{\Lambda}_N; \vec{\Lambda}_{12}, \vec{\Lambda}_{123}, ..., \vec{\Lambda}_{1...N-1}; \vec{\Lambda}, \vec{M} \right\rangle, \tag{A.3}$$

(with certain restrictions on the values of the intermediate and final highest weights). A basis in the intertwiner space is given by

$$\left| \vec{\Lambda}_1, ..., \vec{\Lambda}_N; \vec{\Lambda}_{12}, \vec{\Lambda}_{123}, ..., \vec{\Lambda}_{1...N-1}; 0, 0 \right\rangle, \tag{A.4}$$

(with certain restrictions). A change of recoupling scheme corresponds to a change of basis in the intertwiner space. A basis in the intertwiner space of N simple irreps is given by

$$\left| \Lambda_1, ..., \Lambda_N; \vec{\Lambda}_{12}, \vec{\Lambda}_{123}, ..., \vec{\Lambda}_{1...N-1}; 0, 0 \right\rangle, \tag{A.5}$$

(with certain restrictions), since in the tensor product of two simple irreps, non-simple irreps appear in general [47, 46],

$$(\lambda_1, 0, ..., 0) \otimes (\lambda_2, 0, ..., 0) = \sum_{k=0}^{\lambda_2} \sum_{l=0}^{\lambda_2 - k} (\lambda_1 + \lambda_2 - 2k - l, l, 0, ..., 0) \quad (\lambda_2 \leq \lambda_1). \tag{A.6}$$

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