New Variables for Classical and Quantum Gravity in all Dimensions II. Lagrangian Analysis

N. Bodendorfer[1,*] T. Thiemann[1,2,†] A. Thurn[1,‡]

[1] Inst. for Theoretical Physics III, FAU Erlangen – Nürnberg,
Staudtstr. 7, 91058 Erlangen, Germany

and

[2] Perimeter Institute for Theoretical Physics,
31 Caroline Street N, Waterloo, ON N2L 2Y5, Canada

May 19, 2011

Abstract

We rederive the results of our companion paper, for matching spacetime and internal signature, by applying in detail the Dirac algorithm to the Palatini action. While the constraint set of the Palatini action contains second class constraints, by an appeal to the method of gauge unfixing, we map the second class system to an equivalent first class system which turns out to be identical to the first class constraint system obtained via the extension of the ADM phase space performed in our companion paper. Central to our analysis is again the appropriate treatment of the simplicity constraint. Remarkably, the simplicity constraint invariant extension of the Hamiltonian constraint, that is a necessary step in the gauge unfixing procedure, involves a correction term which is precisely the one found in the companion paper and which makes sure that the Hamiltonian constraint derived from the Palatini Lagrangian coincides with the ADM Hamiltonian constraint when Gauß and simplicity constraints are satisfied. We therefore have rederived our new connection formulation of General Relativity from an independent starting point, thus confirming the consistency of this framework.

---

[*]bodendorfer@theorie3.physik.uni-erlangen.de
[†]thiemann@theorie3.physik.uni-erlangen.de, tthiemann@perimeterinstitute.ca
[‡]thurn@theorie3.physik.uni-erlangen.de

Contents

1   Introduction

In our companion paper [1], we developed a higher dimensional connection formulation for General Relativity with only first class constraints and Poisson commuting connections in any spacetime dimension $D + 1 \geq 3$. The motivation was to make contact to approaches to quantum gravity other than Loop Quantum Gravity [2, 3] which require higher dimensions. In our companion paper we discovered this connection formulation by a judicious extension of the $D + 1$ ADM phase space supplemented by first class Gauß and simplicity constraints which by construction lead back to the ADM phase space upon symplectic reduction.

This approach has the advantage that it is rather simple, allows for $SO(1, D)$ or $SO(D + 1)$ as structure group irrespective of the spacetime signature and that in addition it admits a free parameter that is similar to but yet rather different from the Immirzi parameter in $3 + 1$ dimensions. However, one may ask whether this connection formulation can be obtained from an action principle, just as the LQG connection formulation can be obtained from the Holst action [4]. In this paper, we answer this question in the affirmative. The appropriate action to choose will be simply the $D + 1$ Palatini action. However, this action based approach has its limitations as compared to the Hamiltonian approach [1]: There is no Immirzi like freedom and the structure group is tied to the spacetime signature: It is necessarily $SO(1, D)$ for Lorentzian spacetime signature and $SO(D + 1)$ for Euclidean spacetime signature. This makes this approach less favourable with an eye towards quantisation of the Lorentzian theory which requires a compact structure group. Yet the efforts of the present paper are not in vain as our results confirm the achievements of [1] via an alternative route. Maybe the most astonishing outcome is that we obtain a pure first class theory while it is well known that the Palatini formulation is plagued by second class constraints. The resolution of the apparent contradiction is that we have to apply an additional step in order to arrive at the first class formulation which goes by the name *gauge unfixing*.

In more detail, we do the following: We start from the Palatini formulation of, say, Lorentzian GR in $D + 1$ spacetime dimensions with structure group $SO(1, D)$. Following strictly Dirac's canonical analysis, this formulation, naturally leads to an $SO(1, D)$ connection A and a $so(1, D)$ valued vector density π which is canonically conjugate to the connection. However, in addition to the $SO(1, D)$ Gauß constraint, the D-dimensional spatial diffeomorphism constraint and the Hamiltonian constraint, there is an additional primary constraint S which requires the momentum π to derive from (the pull back to the leaves of the foliation) of a co-$(D + 1)$-bein. We call it simplicity constraint because it is precisely the temporal spatial part of the simplicity constraint of a higher dimensional Plebanski formulation [5]. The stability of the constraint S with respect to the canonical Hamiltonian enforces a secondary constraint D and (S, D) form a second class pair. The situation is of course completely the same as in $D = 3$ dimensions. In $D = 3$ dimensions one can now either consider this $SO(1, 3)$ connection formulation and try to quantise the corresponding Dirac bracket [6] with non Dirac bracket commuting connections or one imposes the time gauge and reduces the (Holst modified) theory to a Dirac bracket commuting $SU(2)$ (or $SO(3)$) connection formulation. In higher dimensions also both possibilities exist except that imposing the time gauge does not lead to a $SO(D)$ connection formulation but rather the extended ADM formulation already derived in [7]. Thus the second strategy does not lead to the desired connection formulation with compact $SO(D)$ precisely due to the dimensional mismatch between D and $D(D - 1)/2$. Thus, in order to have a connection formulation only the first possibility remains but then the complication with the Dirac bracket arises. It is at this point where gauge unfixing comes into play: By a systematic, allowed modification of the Hamiltonian constraint which does not alter its first class character, the S constraints become Poisson commuting with all but the D constraints. One can therefore consider the D constraints

as gauge fixing conditions for the S constraints and impose only the first class constraints. This way one can map the second class constraint system to an equivalent first class constraint system and replace the complicated Dirac bracket by the simple ordinary Poisson bracket with Poisson commuting connections. In the end, this formulation is identical to the one of our companion paper for matching spacetime and internal signature as well as unit Immirzi like parameter.

This paper is is organised as follows:

In section 2, we will perform a canonical analysis of the higher dimensional Palatini theory in the spirit of [8] following strictly Dirac's procedure [9]. This approach uses vielbeins at every stage of the construction. In 3 we start from a different formulation of the simplicity constraints which avoids the introduction of vielbeins and is closer in spirit to the Plebanski formulation of General Relativity. As in the $3 + 1$-dimensional case, the Hamiltonian formulation of higher dimensional Palatini theory has second class constraints which can either be solved to obtain geometrodynamics or the Dirac bracket has to be implemented on a Hilbert space. To circumvent these problems, we apply the procedure of gauge unfixing which we review in in section 4 and apply to GR in section 5. The result is an $SO(1, D)$ or $SO(D)$ connection formulation for Lorentzian or Euclidean General Relativity respectively with first class constraints only and a connection variable which is Poisson self-commuting, the price to pay is one extra term in the Hamiltonian constraint. We finish with some concluding remarks in section 6.

In the appendix we complete the paper by specialising to $D + 1 = 4$ and supplying the strict Immirzi term that is peculiar to 4 spacetime dimensions.

2   The $(D + 1)$-dimensional Palatini Hamiltonian

In this chapter, we will derive the Hamiltonian formulation of $(D + 1)$-dimensional Palatini theory. To account for the mismatch of degrees of freedom between the connection and the vielbein, a new constraint, the simplicity constraint, will appear. The Dirac algorithm will be applied to check for further constraints. Finally, we will select a maximal first class set of constraints and solve the second class constraints.

2.1   Legendre Transformation

We start with the Palatini action in $D + 1$ dimensions and $\kappa = 1$:

$$S = \frac{s}{2} \int_{\mathcal{M}} d^{D+1}X \, e e^{\mu I} e^{\nu J} F_{\mu\nu IJ}. \tag{2.1}$$

$\mathcal{M}$ denotes the space-time manifold of topology $\mathbb{R} \times \sigma$. $\mathcal{M}$ foliates into hypersurfaces $\Sigma_t := X_t(\sigma)$, where $X_t : \sigma \to M$ is an embedding defined by $X_t(\sigma) = X(t, \sigma)$. $x^a, a, b, c = 1, \ldots, D$ are local coordinates on σ [3]. s denotes the space-time metric signature, i.e. $g_{\mu\nu} = \mathrm{diag}(s, 1, \ldots, 1)$ for a flat space-time ($\mu, \nu = 0, 1, ..., D$). Since we start from the Palatini action, Sylvesters theorem tells us that the signature of the internal metric has to coincide with the space-time metric signature. $e^{\mu I}$ denotes the vielbein and $F_{\mu\nu IJ} := \partial_\mu A_{\nu IJ} - \partial_\nu A_{\mu IJ} + [A_\mu, A_\nu]_{IJ}$ is the field strength of the $SO(1, D)$ ($SO(D + 1)$) connection $A_{\mu IJ}$, $I, J = 0, ..., D$.

The split in space and time is performed analogously to the $4d$ case. We split the time-evolution vector field into

$$T^\mu = N n^\mu + N^\mu, \quad n_\mu N^\mu = 0. \tag{2.2}$$

N is called the lapse function and N^μ the shift vector field. n^μ is the unit future pointing vector field normal to the spatial slices Σ_t, i.e. $n^\mu n_\mu = s$ and $n^\mu \partial_\mu X_t = 0$. We use

$$\delta^\mu{}_\nu = (g^\mu{}_\nu - s n^\mu n_\nu) + s n^\mu n_\nu =: q^\mu{}_\nu + s n^\mu n_\nu \tag{2.3}$$

to project the vielbein as

$$e_I^\mu = q^\mu{}_\nu \, e_I^\nu + s e_I^\nu n_\nu n^\mu =: {}^\|e_I^\mu + s n_I n^\mu. \tag{2.4}$$

With $e = N\sqrt{q}$, the action becomes

$$S = s \int_{\mathcal{M}} d^{D+1}X \, N \left(\frac{1}{2}\sqrt{q} \, {}^\|e^{\mu I \|}e^{\nu J} F_{\mu\nu IJ} + s\sqrt{q} n^I n^{\mu \|}e^{\nu J} F_{\mu\nu IJ} \right). \tag{2.5}$$

The next step is to rewrite the action using

$$
\begin{aligned}
N\sqrt{q} {}^\|e^{\mu I} n^J n^\nu F_{\mu\nu IJ} &= -\frac{1}{2}\pi'^{\mu IJ}(T^\nu - N^\nu)F_{\mu\nu IJ} \\
&= \frac{1}{2}\pi'^{\mu IJ}\mathcal{L}_T A_{\mu IJ} + \frac{1}{2}(T^\nu A_{\nu IJ})G'^{IJ} - N^\mu \mathcal{H}'_\mu
\end{aligned}
\tag{2.6}
$$

and

$$\frac{1}{2}N\sqrt{q} {}^\|e^{\mu I \|}e^{\nu J} F_{\mu\nu IJ} = -s\underaccent{\sim}{N}\mathcal{H}', \tag{2.7}$$

where

$$\pi'^{\mu IJ} := 2n^{[I \|}E^{\mu|J]} := 2\sqrt{q}n^{[I \|}e^{\mu|J]} \,, \quad \underaccent{\sim}{N} := -N/\sqrt{q} \,, \tag{2.8}$$

$$G'^{IJ} := D_\mu \pi'^{\mu IJ} := \partial_\mu \pi'^{\mu IJ} + [A_\mu, \pi'^\mu]^{IJ}, \tag{2.9}$$

$$\mathcal{H}' := \frac{1}{2}\pi'^{\mu IK}\pi'^{\nu J}{}_K F_{\mu\nu IJ} \quad \text{and} \quad \mathcal{H}'_\mu := \frac{1}{2}\pi'^{\nu IJ} F_{\mu\nu IJ}. \tag{2.10}$$

We use the primes to indicate the dependence on π'^{aIJ} and remark that later in the analysis, we will substitute π'^{aIJ} by π^{aIJ} which denotes the canonically conjugate momentum of A_{aIJ}, and neglect the primes. The split is completed by pulling back all spatially projected quantities to σ by X. The connection is pulled back as

$$A_{aIJ}(t,x) = ({}^\|A_{\mu IJ}\partial_a X^\mu)(X(x,t)) = {}^\|A_{\mu IJ}(X(t,x))\partial_a X^\mu(t,x). \tag{2.11}$$

Special care has to be taken for the co-D-bein, since it has a contravariant external index. We can use the spatially restricted metric $q_{\mu\nu}$ from the ADM formalism to pull the external index down. The pull-back is then possible as

$$e_a^I(t,x) = (q_{\mu\nu}{}^\|e^{\nu I}\partial_a X^\mu)(X(x,t)) = q_{\mu\nu}{}^\|e^{\nu I}(X(t,x))\partial_a X^\mu(t,x). \tag{2.12}$$

Furthermore, we can pull back the restricted metric similar to the ADM formalism, see [3] for details, and use it to raise the spatial index of the pulled back D-bein. A push-forward using X of the spatial co-D-bein yields again the projected co-D-bein we started with when carefully writing out all the terms.

We split $d^{D+1}X$ into $dt \, d^D x$ with $d^D x$ being the integration over the spatial coordinates x^a on σ. Note that $e = N\sqrt{q}$ and $\sqrt{q}d^D x$ is the induced measure on σ.

During the split, the lapse function underwent minor changes but we will still, as in the $(3+1)$-dimensional case, call $\underaccent{\sim}{N} = -N/\sqrt{q}$ the lapse function (of density weight -1) and denote it by N. As will be shown later, the constraints $\mathcal{H}$, $\mathcal{H}_a$ and G^{IJ} retain their interpretation also

in the higher-dimensional case. We will therefore refer to $\mathcal{H}$ as the Hamiltonian constraint, to $\mathcal{H}_a$ as the diffeomorphism constraint and G^{IJ} as the Gauß constraint.

We stress that we have to use the Lie derivative along the time evolution vector field as a time derivative and denote it in this chapter by the standard upper dot. In a special coordinate system where $T^\mu = \text{const.}$, the Lie derivative would reduce to a normal derivative, but in the general case, we have to take the deformation of the spatial slices into account.

After decomposition as well as the substitution $\lambda_{IJ} = -(T \cdot A)_{IJ}$ and the abuse of notation $\underset{\sim}{N} \to N$, the action reads

$$S = \int dt\, L = \int dt \int_\sigma d^D x \left(\frac{1}{2} \pi'^{aIJ} \mathcal{L}_T A_{aIJ} - N\mathcal{H}' - N^a \mathcal{H}'_a - \frac{1}{2} \lambda_{IJ} G'^{IJ} \right). \tag{2.13}$$

The Legendre transformation is performed according to Dirac [9]. The total Hamiltonian is defined by

$$\begin{aligned}
H_T &= \left(\int_\sigma d^D x \sum_i p_i(x) \dot{q}^i(x) \right) - L + \text{constraints} \\
&= \int_\sigma d^D x \left(N\mathcal{H}' + N^a \mathcal{H}'_a + \frac{1}{2} \lambda_{IJ} G'^{IJ} + \alpha P_N + \alpha^a P_{N^a} + \frac{1}{2} \alpha_{IJ} P_{\lambda_{IJ}} \right. \\
&\qquad\qquad\qquad \left. + \frac{1}{2} c_{aIJ} S^{aIJ} + \gamma_I^a P_{E_I^a} \right).
\end{aligned} \tag{2.14}$$

The Ps denote the canonically conjugate momenta to the indicated variables and π^{aIJ} is the momentum to A_{aIJ}. We remark that we put a factor of $1/2$ in front of the added terms which are summed over antisymmetric indices because all independent values are summed twice.

$$S^{aIJ} = \pi^{aIJ} - \pi'^{aIJ} \tag{2.15}$$

is called the simplicity constraint. We redefine[1] the Lagrange multiplier of the simplicity constraint to replace all π'^{aIJ} by π^{aIJ} in $\mathcal{H}'$, $\mathcal{H}'_a$ and G'^{IJ} and drop the primes to indicate the dependence on π^{aIJ} only in $\mathcal{H}$, $\mathcal{H}_a$ and G^{IJ}.

It is convenient to split the simplicity constraint into a boost and a non-boost part by decomposing its multiplier as

$$c_{aIJ} = 2n_{[I} \bar{c}_{a|J]} + \bar{c}_{aIJ}. \tag{2.16}$$

Using the projector $\bar{\eta}^I{}_J := \eta^I{}_J - sn^I n_J$, we can explicitly write $\bar{c}_{aIJ} = \bar{\eta}^K{}_I \bar{\eta}^L{}_J c_{aKL}$ and $\bar{c}_{aI} = -sc_{aIJ} n^J$. In the following, we will use this notation to split other objects with the above index structure in the same way. The simplicity constraint then splits into

$$\begin{aligned}
\frac{1}{2} c_{aIJ} S^{aIJ} = 0 \quad &\Leftrightarrow \quad \frac{1}{2} \bar{c}_{aIJ} \bar{S}^{aIJ} := \frac{1}{2} \bar{c}_{aIJ} \bar{\pi}^{aIJ} = 0, \\
&\qquad s\bar{c}_{aI} S^{aI} := -\bar{c}_{aI} \left(\pi^{aIJ} n_J + s\, E^{aI} \right) = 0, \tag{2.17}
\end{aligned}$$

where the first constraint ensures that $\bar{\pi}^{aIJ} = 0$ and hence $\pi^{aIJ} = 2n^{[I} B^{a|J]}$ for some B^{aJ} and the second constraint sets $B^{aJ} = E^{aJ}$. We will absorb the minus sign in the second line into $\bar{c}_{aI}$ for convenience.

Notice that the $D+1$ conditions $n^I E_I^a = 0$, $n^I n_I = s$ completely fix n_I as a function of E_I^a. It will turn out to be convenient to treat n^I is as an independent field and to constrain it in the

---

following such that it is given as a function of the E^{aI} on the corresponding constraint surface. We therefore add the constraints $n^I n_I - s \approx 0$ and $E^{aI} n_I \approx 0$ to the Hamiltonian with the Lagrange multiplier ρ and ρ_a as well as the primary constraint that its conjugate momentum vanishes. After these considerations, the total Hamiltonian reads

$$
\begin{aligned}
H_T \;=\; \int_\sigma d^D x \bigg(& N\mathcal{H} + N^a \mathcal{H}_a + \lambda_{IJ} G^{IJ} + \alpha P_N + \alpha^a P_{N^a} + \frac{1}{2}\alpha_{IJ} P_{\lambda_{IJ}} + \frac{1}{2}\bar{c}_{aIJ}\bar{S}^{aIJ} \\
& + \bar{c}_{aI} S^{aI} + \gamma_I^a P_{E_I^a} + \gamma_I P_{n_I} + \rho(n^I n_I - s) + \rho_a E^{aI} n_I \bigg)
\end{aligned}
\tag{2.18}
$$

and the symplectic potential is given by

$$
\frac{1}{2}\pi^{aIJ}\dot{A}_{aIJ} + P_{E_I^a}\dot{E}_I^a + P_{n_I}\dot{n}_I + P_N \dot{N} + P_{N^a}\dot{N}^a + \frac{1}{2}P_{\lambda_{IJ}}\dot{\lambda}_{IJ},
\tag{2.19}
$$

The non-vanishing Poisson-brackets are accordingly

$$
\left\{ A_{aIJ}(x), \pi^{bKL}(y) \right\} = 2\delta^{(D)}(x-y)\delta_a^b \eta_I^{[K}\eta_J^{L]}, \quad \left\{ E_I^a(x), P_{E_j^b}(y) \right\} = \delta^{(D)}(x-y)\delta_b^a \eta_I^J, \quad (2.20)
$$

$$
\{n_I(x), P_{n_J}(y)\} = \delta^{(D)}(x-y)\eta_I^J, \qquad\qquad \{N(x), P_N(y)\} = \delta^{(D)}(x-y),
$$

$$
\{N^a(x), P_{N^b}(y)\} = \delta^{(D)}(x-y)\delta_b^a, \qquad\qquad \{\lambda_{IJ}(x), P_{\lambda_{KL}}(y)\} = 2\delta^{(D)}(x-y)\eta_I^{[K}\eta_J^{L]}.
$$

2.2 Constraint Analysis

We start the constraint analysis with the total Hamiltonian H_T given in (2.18). We need to require preservation of the smeared constraints in time and therefore check upon the weak vanishing of their Poisson brackets with the Hamiltonian H_T. We will denote the smearing test functions by $f_{\cdot}$ with corresponding index structure in order to to distinguish them from the smearing Lagrange multipliers α, α^a, α_{IJ}, $\bar{c}_{aIJ}$, $\bar{c}_{aI}$, γ_I^a, γ_I, ρ, ρ_a and smearing phase space functions N, N^a, λ_{IJ} that appear within H_T and which, in contrast to the test functions, will become in part successively constrained during the stability analysis. The (weak) vanishing of the Poisson bracket of the smeared constraints with the Hamiltonian must hold for all test functions and their full index range.

Since N, N^a and λ_{IJ} only appear linearly and their momenta are constrained to vanish, stability of these momenta yields immediately the constraints

$$
\mathcal{H} \approx 0 \;,\quad \mathcal{H}_a \approx 0 \;,\quad \text{and}\quad G^{IJ} \approx 0,
\tag{2.21}
$$

called the Hamiltonian, diffeomorphism, and Gauß constraints. Stability of S^{aI} requires

$$
\begin{aligned}
0 \;\approx\; & \int_\sigma d^D x\, \bar{f}_{aI}(x) \left\{ \pi^{aIJ}(x) n_J(x) + s E^{aI}(x), H_T \right\} \\
=\; & \int_\sigma d^D x\, \bar{f}_{aI} \left(*^{aIJ} n_J + \pi^{aIJ}\gamma_J + s\gamma^{aI} \right) \\
=\; & \int_\sigma d^D x\, \bar{f}_{aI} \left(*^{aIJ} n_J + \bar{\pi}^{aIJ}\gamma_J + s\bar{\gamma}^{aI} \right).
\end{aligned}
\tag{2.22}
$$

In the last line we noticed that only the piece $\bar{\gamma}^{aI}$ orthogonal to n^I survives since $\bar{f}_{aI}$ is projected. This equation can therefore be solved for the Lagrange multiplier $\bar{\gamma}^{aI}$. $*^{aIJ}$ denotes the terms which come from Poisson commuting π^{aIJ}. We do not compute them explicitly as they will be of no importance after solving the second class constraints. Likewise

$$
\begin{aligned}
0 \;\approx\; & \int_\sigma d^D x\, f_a(x) \left\{ E_I^a(x) n^I(x), H_T \right\} \\
=\; & \int_\sigma d^D x\, f_a \left(\gamma_I^a n^I + \gamma^I E_I^a \right),
\end{aligned}
\tag{2.23}
$$

which can be solved for the piece $\gamma_I^a n^I$ of γ_I^a.

Next, we calculate

$$0 \approx \int_\sigma d^D x \, f(x) \left\{ n^I(x) n_I(x) - s, H_T \right\} = 2 \int_\sigma d^D x \, f \, n^I \gamma_I. \tag{2.24}$$

This requires that $\gamma^I = \bar{\gamma}^I$, i.e. the Hamiltonian flow does not change the length of n^I. In the same way,

$$0 \approx \int_\sigma d^D x f_I^a(x) \left\{ P_{E_I^a}(x), H_T \right\} = \int_\sigma d^D x f^{aI} (\bar{c}_{aI} - \rho_a n_I). \tag{2.25}$$

It follows that $\bar{c}_{aI} = 0 = \rho_a$. As a consequence, both $\bar{c}_{aI} S^{aI}$ and $\rho_a E^{aI} n_I$ drop from the Hamiltonian. Further,

$$0 \approx \int_\sigma d^D x \, f_I(x) \left\{ P_{n_I}(x), H_T \right\} \approx \int_\sigma d^D x \, f^I \left(-2n_I \rho + \bar{c}_{aIJ} E^{aJ} \right). \tag{2.26}$$

After projecting f^I orthogonally and perpendicularly to n^I we conclude that $\rho = 0$ and $\bar{c}_{aIJ} E^{aJ} = 0$. This means that $\bar{c}_{aIJ}$ is trace free and we can parametrise its unconstrained part by a matrix $\bar{c}'_{aIJ}$ orthogonal to n^I with the same symmetries according to

$$\bar{c}_{aIJ} \equiv \bar{c}_{aIJ}^T = \bar{c}'_{aIJ} - \frac{2}{D-1} E_{a[I} \bar{c}'_{cK|J]} E^{cK}. \tag{2.27}$$

Here E_{aI} denotes the inverse of E^{aI} defined by $E^{aI} E_{bI} = \delta_b^a$ and $E_{bI} n^I = 0$, i.e. $E^{aI} E_{aJ} = \bar{\eta}_J^I$.

The remaining constraint $\bar{S}^{aIJ}$ yields

$$\begin{aligned}
0 &\approx \int_\sigma d^D x \, \bar{f}_{aIJ}(x) \left\{ \bar{\pi}^{aIJ}(x), H_T \right\} \\
&\approx \int_\sigma d^D x \, \bar{f}_{aIJ} \left(-2N D_b \left(\pi^{b[I|K} \pi^{a|J]K} \right) \right. \\
&\qquad \left. + 2 E^{a[J} \left(-\bar{\gamma}^{I]} + n_K \lambda^{K|I]} + s(\partial_b N) E^{b|I]} - s N^b \, D_b n^{I]} \right) \right) \\
&\approx \int_\sigma d^D x \, (-2s) N \bar{f}_{aIJ} E^{b[I} D_b E^{a|J]} \\
&\qquad + 2 E^{a[J} \left(-\bar{\gamma}^{I]} + n_K \lambda^{K|I]} + s(\partial_b N) E^{b|I]} - s N^b \, D_b n^{I]} \right) \right), \quad (2.28)
\end{aligned}$$

where D is the covariant differential associated with the connection A (it acts only on internal indices and neglects tensorial indices and density weights). We have used the simplicity constraint several times in order to arrive at this weak identity. We can decompose the test function as $\bar{f}_{aIJ} = \bar{f}'_{aIJ} + \bar{f}_{[I} E_{|a|J]}$ and conclude that the trace and trace free part of (2.28) have to vanish separately. Consider the quantity

$$D^{aIJ} := E^{b[I} D_b E^{aJ]}. \tag{2.29}$$

It is neither transversal nor trace free. Since the test function $\bar{f}_{aIJ}$ is transversal actually only the transversal part

$$\bar{D}^{aIJ} = \bar{\eta}_K^I \bar{\eta}_L^J D^{aKL} \tag{2.30}$$

enters (2.28). We split off the its trace part

$$\bar{D}^{aIJ} = \bar{D}_T^{aIJ} + \frac{2}{D-1} E^{a[I} E_{bK} \bar{D}^{bK|J]} \tag{2.31}$$

8

and rewrite (2.28) as

$$
\begin{aligned}
0 \quad &\approx \quad \int_\sigma d^D x \,(-2s) N \bar{f}_{aIJ} \\
&\qquad \times \left[\bar{D}_T^{aIJ} + 2E^{a[J} \left(-\bar{\gamma}^{I]} + n_K \lambda^{K|I]} + s(\partial_b N) E^{b|I]} - sN^b \, D_b n^{I]} - \frac{1}{D-1} E_{bK} \bar{D}^{bK|I]} \right) \right] \\
&=: \quad \int_\sigma d^D x \, [\bar{f}_{aIJ}^T (-2sN \, \bar{D}_T^{aIJ}) + 2\bar{f}_I \,(-\bar{\gamma}_{J]} + \bar{D}_{J]})].
\end{aligned}
\tag{2.32}
$$

Vanishing of the trace part now requires that $\bar{\gamma}_I = \bar{D}_I$ and vanishing of the trace free part yields the secondary constraint $\bar{D}_T^{aIJ}$ which is manifestly transversal and trace free.

To summarise, the outcome of the analysis so far is that $\bar{c}_{aI} = \rho = \rho_a = \gamma^I n_I = \bar{c}_{aIJ} E^{aJ} = 0$ and that $\gamma_I^a = \gamma_{I0}^a$, $\bar{\gamma}^I = \bar{\gamma}_0^I$ are fixed functions on phase space. The Hamiltonian that stabilises the primary constraints therefore reduces to

$$
\begin{aligned}
H_T \quad = \quad &\int_\sigma d^D x \left(N\mathcal{H} + N^a \mathcal{H}_a + \frac{1}{2}\lambda_{IJ} G^{IJ} + \alpha P_N + \alpha^a P_{N^a} + \frac{1}{2}\alpha_{IJ} P_{\lambda_{IJ}} + \frac{1}{2}\bar{c}_{aIJ}^T \bar{S}^{aIJ} \right. \\
&\qquad \left. + \gamma_{I0}^a P_{E_I^a} + \bar{\gamma}_{I0} P_{n_I} \right)
\end{aligned}
\tag{2.33}
$$

and the secondary constraints are G^{IJ}, $\mathcal{H}_a$, $\mathcal{H}$, $\bar{D}_T^{aIJ}$. Here $\bar{c}_{aIJ}^T$ denotes the trace free part of $\bar{c}_{aIJ}$ etc.

In the next step of the canonical analysis, the secondary constraints have to be checked for consistency and then we must subdivide (linear combinations of) the constraints into those of first and second class respectively. Anticipating the result, we will show that S, D form a second class pair while linear combinations of the remaining constraints are first class. We begin with the constraints $\mathcal{H}$, $\mathcal{H}_a$, and G^{IJ} which only depend on the variables π^{aIJ} and A_{aIJ} but not on N, N^a λ_{IJ}, n^I, E_I^a whence they have vanishing brackets with the primary constraints P, P_a, $P_{\lambda_{IJ}}$, P_{n^I}, $P_{E_I^a}$. Thus, to ensure their stability, only their brackets with themselves and the trace free part of the simplicity constraint $\bar{S}^{aIJ}$ appearing in the Hamiltonian above have to be checked.

We begin with the Gauß constraint. The action of the smeared Gauß constraint on the phase space variables is

$$
\left\{ A_{aIJ}, \frac{1}{2} G^{KL}[f_{KL}] \right\} \quad = \quad -D_a f_{IJ},
\tag{2.34}
$$

$$
\left\{ \pi^{aIJ}, \frac{1}{2} G^{KL}[f_{KL}] \right\} \quad = \quad [f, \pi^a]^{IJ},
\tag{2.35}
$$

meaning that A_{aIJ} transforms as an internal connection and π^{aIJ} as an internal second rank contravariant tensor. The Gauß constraint closes with itself,

$$
\left\{ \frac{1}{2} G^{IJ}[f_{IJ}], \frac{1}{2} G^{KL}[\gamma_{KL}] \right\} = \frac{1}{2} G^{IJ} \left[\lambda_{IK}\gamma^K{}_J - \gamma_{IK}\lambda^K{}_J \right],
\tag{2.36}
$$

which was expected as the difference between two rotations is again a rotation. The factors $1/2$ are included because of the antisymmetry of the multiplier. Since the Hamiltonian constraint

and the diffeomorphism constraint do not have free internal indices, the calculations yield as expected

$$\left\{ \frac{1}{2} G^{IJ}[f_{IJ}], \mathcal{H}_a[N^a] \right\} = 0, \tag{2.37}$$

$$\left\{ \frac{1}{2} G^{IJ}[f_{IJ}], \mathcal{H}[N] \right\} = 0. \tag{2.38}$$

It follows that

$$\left\{ \frac{1}{2} G^{IJ}[f_{IJ}], H_T \right\} \approx -\bar{c}_{aIJ}^T E^{aI} n^K \lambda_K{}^J = 0, \tag{2.39}$$

since $\bar{c}_{aIJ}^T$ is trace free whence the Gauß constraint is already stable.

The diffeomorphism constraint as given generates spatial diffeomorphisms mixed with internal rotations. In order to see this, we define

$$\tilde{\mathcal{H}}_a := \mathcal{H}_a - \frac{1}{2} A_{aIJ} G^{IJ} = \frac{1}{2} \pi^{bIJ} \partial_a A_{bIJ} - \frac{1}{2} \partial_b \left(\pi^{bIJ} A_{aIJ} \right). \tag{2.40}$$

The action of $\tilde{\mathcal{H}}_a$ on the phase space variables is

$$\left\{ A_{aIJ}, \tilde{\mathcal{H}}_b[f^b] \right\} = f^b \partial_b A_{aIJ} + (\partial_a f^b) A_{bIJ} = \mathcal{L}_f A_{aIJ}, \tag{2.41}$$

$$\left\{ \pi^{aIJ}, \tilde{\mathcal{H}}_b[f^b] \right\} = f^b \partial_b \pi^{aIJ} - (\partial_b f^a) \pi^{bIJ} + (\partial_b f^b) \pi^{aIJ} = \mathcal{L}_f \pi^{aIJ}, \tag{2.42}$$

meaning that A_{aIJ} transforms like the components of a one-form and π^{aIJ} like the components of a vector density under infinitesimal diffeomorphisms generated by the Lie derivative. Straight forward calculations yield

$$\left\{ \mathcal{H}_a[f^a], \mathcal{H}_b[N^b] \right\} = \mathcal{H}_a[(\mathcal{L}_f N)^a] - \frac{1}{2} G^{IJ}[f^a N^b F_{abIJ}], \tag{2.43}$$

$$\{ \mathcal{H}_a[f^a], \mathcal{H}[N] \} = \mathcal{H}[\mathcal{L}_f N] + G^{IJ}[N f^a \pi^b{}_I{}^K F_{abJK}]. \tag{2.44}$$

The Lie derivatives are given by $(\mathcal{L}_f N)^a = f^b \partial_b N^a - N^b \partial_b f^a$ when N^a is considered as a vector field and $(\mathcal{L}_f N) = f^b \partial_b N - N \partial_b f^b$ where N is considered as a scalar density of weight minus one. It follows

$$\left\{ \mathcal{H}_b[f^b], H_T \right\} \approx \left\{ \tilde{\mathcal{H}}_b[f^b], H_T \right\} \approx N^b \bar{c}_{aIJ}^T E^{aJ} (\partial_b n^I) = 0, \tag{2.45}$$

again due to trace freeness of $\bar{c}_{aIJ}^T$. Thus also the spatial diffeomorphism constraint is already stabilised.

Left with the Hamiltonian constraint, we calculate

$$\begin{aligned} \{ \mathcal{H}[f], \mathcal{H}[N] \} &= -\frac{1}{2} \mathcal{H}_a \left[(f \partial_b N - N \partial_b f) \pi^{aIJ} \pi^b_{IJ} \right] \\ &\quad + \int_\sigma d^D x \frac{3s}{2} (D-3)! (f \partial_a N - N \partial_a f) \pi^a_{[IJ} \pi^b_{KL]} \pi^{cIJ} F^{KL}_{cb} \\ &\approx -\frac{1}{2} \mathcal{H}_a \left[(f \partial_b N - N \partial_b f) \pi^{aIJ} \pi^b_{IJ} \right], \end{aligned} \tag{2.46}$$

which is more involved than the other calculations but it is helpful to make use of the fact that due to the antisymmetry of the Poisson bracket, only terms proportional to $M \partial_b N - N \partial_b M$ can survive. The second term vanishes because of the total antisymmetrisation in the πs when using the simplicity constraint. We used an important relation concerning the contraction of $so(1, D)$

structure constants which is discussed in appendix A. It follows by the same calculation as in (2.28) that

$$\{\mathcal{H}[f], H_T\} \approx \{\mathcal{H}[f], \bar{S}^{aIJ}[\bar{c}^T_{aIJ}]\} \approx -\bar{D}^{aIJ}_T[f\bar{c}^T_{aIJ}] \approx 0. \tag{2.47}$$

In conclusion, the Gauß, spatial diffeomorphism and Hamiltonian constraint are already stabilised.

Therefore, among the secondary constraints, the only constraint left to be checked for consistency is $\bar{D}^{aIJ}_T$. For this, it will be important to analyse the Poisson bracket

$$\left\{\bar{S}^{aIJ}[\bar{f}_{aIJ}], D^{bKL}[\bar{f}'^T_{bKL}]\right\} = \int d^D x \; \bar{f}_{aIJ} F^{aIJ,bKL} \bar{f}'^T_{bKL} \tag{2.48}$$

with

$$F^{aIJ,bKL} = 4s E^{a[K} \bar{\eta}^{L][J} E^{b|I]} \tag{2.49}$$

and we have absorbed the transverse trace free projections in $\bar{D}^{aIJ}_T$ into transverse trace free smearing functions. Using this definition, the stability requirement for D^{aIJ} reads

$$\{D^{bKL}[\bar{f}^T_{bKL}], H_T\} = \int_\sigma d^D x \; \bar{f}^T_{aIJ} \left(-\frac{1}{2} F^{aIJ,bKL} \bar{c}^T_{bKL} + \bar{\Sigma}^{aIJ}_T\right) \approx 0, \tag{2.50}$$

where $\bar{\Sigma}^{aIJ}_T$ denotes all the contributions from H_T different from $\bar{S}^{aIJ}[\bar{c}^T_{aIJ}]$. We remark that all partial derivatives were integrated away from $\bar{f}^T_{aIJ}$ onto $\bar{\Sigma}^{bKL}_T$ and that a possible longitudinal part of $\bar{\Sigma}^{aIJ}$ of the form $n^{[I}\bar{\Sigma}^{aJ]}$ and a possible trace part of the form $E^{a[I}\bar{v}^{J]}$ would vanish because of the contraction with $\bar{f}^T_{aIJ}$.

Thus we have to solve

$$\frac{1}{2}[F^{aIJ,bKL}\bar{c}^T_{bKL}]_{TT} = \bar{\Sigma}^{aIJ}_T, \tag{2.51}$$

where $(.)_{TT}$ denotes the transverse trace free part of a tensor. Notice that the tensor F in (2.49) is symmetric under the exchange of the index triples aIJ and bIJ, antisymmetric in IJ and KL but not trace free (contraction with $E_{aI}, E_{aJ}, E_{bK}, E_{bL}$ does not vanish) while manifestly transverse (contraction with n_I, n_J, n_K, n_L vanishes). Notice also that the index positions of a, b cannot be interchanged as otherwise contraction of F with a trace free tensor would vanish. The strategy to solve (2.51) will be to compute the inverse of F on the space of just transverse tensors and then to determine the trace free projection of its action on trace free and transverse tensors. A lengthy but straight forward calculation yields

$$\left(F^{-1}\right)_{aIJ,bKL} = \frac{s}{4} E_{aA} E_{bB} \left(\bar{\eta}^{AB} \bar{\eta}_{K[I}\bar{\eta}_{J]L} - 2\bar{\eta}^B_{[I}\bar{\eta}_{J][K}\bar{\eta}^A_{L]}\right). \tag{2.52}$$

This formula can be discovered by the observation that the two tensors on the right hand side are the only traverse tensors with the correct index structure and the same symmetries as F. One then just has to calculate the corresponding contractions in order to determine the coefficients displayed. Thus

$$F^{aIJ,cMN} \left(F^{-1}\right)_{cMN,bKL} = \delta^a_b \; \bar{\eta}^I_{[K} \bar{\eta}^J_{L]} \tag{2.53}$$

is the unit operator on transverse tensors of type $\bar{c}^{bKL}$ which are antisymmetric in K, L. Since F is invertible, we can set

$$\bar{c}^{T0}_{aIJ} = 2P^{bKL}_{aIJ} \left(F^{-1}\right)_{bKL,cMN}\bar{\Sigma}^{cMN}_T, \tag{2.54}$$

with the transversal trace freeness projector

$$P^{aIJ}_{bKL} = \delta^a_b \; \bar{\eta}^I_{[K} \bar{\eta}^J_{L]} + \frac{2}{D-1} E^{a[I} \; \bar{\eta}^{J]}_{[K} E_{bL]}. \tag{2.55}$$

11

All constraints are stable at this point and we are left with the Hamiltonian

$$
\begin{aligned}
H_T \;=\; & \int_\sigma d^D x \bigg(N\mathcal{H} + N^a \mathcal{H}_a + \frac{1}{2}\lambda_{IJ}G^{IJ} + \alpha P_N + \alpha^a P_{N^a} + \frac{1}{2}\alpha_{IJ}P_{\lambda_{IJ}} \\
& + \frac{1}{2}\bar{c}^{T0}_{aIJ}\bar{S}^{aIJ} + \gamma^a_{I0}P_{E^a_I} + \bar{\gamma}_{I0}P_{n_I} \bigg),
\end{aligned}
\tag{2.56}
$$

where the "0" remind us of the fact that those Lagrange multipliers have been replaced by phase space dependent functions.

2.3 Degrees of Freedom

In this section, we count the degrees of freedom of the derived Hamiltonian system to see if they match those of General Relativity. Before we can count the constraints, we have to identify a maximal subset of first class constraints. Comparing with the other available canonical formulations, we suspect the first class constraints to be $\mathcal{H}$, $\mathcal{H}_a$, G^{IJ}, P_N, P_{N^a}, and $P_{\lambda_{IJ}}$. The first three constraints from this list are not of the first class by themselves. Since the following works for all three constraints, we will denote by C either $\mathcal{H}[N]$, $\mathcal{H}_a[N^a]$, or $G^{IJ}[\lambda_{IJ}]$. We can make all of them first class with respect to the D constraint by the substitution

$$
C \to C - \int \bar{S}^{aIJ}_T \left(\{\bar{D}_T,\bar{S}_T\}^{-1}\right)_{aIJ,bKL}\left\{\bar{D}^{bKL}_T,C\right\}. \tag{2.57}
$$

This notation is symbolic (notice that the pointwise Poisson brackets are distributional): The matrix $\{\bar{D}_T,\bar{S}_T\}$ is ultralocal and what is meant is its non distributional factor. As shown earlier in the constraint analysis, C is first class with respect to the traceless part of $\bar{S}^{aIJ}$. In fact, only the Hamiltonian constraint contributes and yields the constraint $\bar{D}_T$. The trace and the traceless parts of $\bar{S}^{aIJ}$ and its multiplier are accessible via the projector defined in (2.27). Next, we need to check the Poisson bracket with S^{aI}. It turns out that the substitution

$$
C \to C + s\int \left\{C,\pi^{aIJ}\right\}n_J P_{E^a_I} \tag{2.58}
$$

forces C to Poisson commute with S^{aI} and all the other constraints. The only constraint which does not Poisson commute with C at this point is the trace part of $\bar{S}^{aIJ}$. Again, we perform a substitution

$$
C \to C - \frac{s}{D-1}E_{bK}\left\{C,\pi^{bKJ}\right\}E^a_J n_I P_{E^a_I} + \frac{1}{D-1}E_{aI}\left\{C,\pi^{aIJ}\right\}\bar{\eta}_{JK}P_{n_K} \tag{2.59}
$$

and see that C is now of the first class. For this, we have to recheck that C still Poisson commutes with S^{aI}, which it does.

The rest of the constraints are of the second class. An easy proof for this statement is to refer to the next section, where the constraints are solved. Since the solution of the proposed second class constraints yields a non-degenerate Poisson bracket, we are sure that we did not solve any first class constraints. It is however possible to explicitly decompose the second class constraints into second class pairs.

First of all, we note that except for $\bar{D}^{aIJ}_T$, all remaining constraints Poisson commute with the trace free part of $\bar{S}^{aIJ}$. We can thus perform the substitution

$$
C \to C - \int \bar{S}^{aIJ}_T \left(\{\bar{D}_T,\bar{S}_T\}^{-1}\right)_{aIJ,bKL}\left\{D^{bKL}_T,C\right\} \tag{2.60}
$$

for all the remaining constraints and the first set of second class pairs is

$$\left\{ \bar{S}_T^{aIJ}[\bar{f}_{aIJ}^T], \bar{D}_T^{bKL}[\bar{g}_{bKL}^T] \right\} = \int_\sigma d^D x \, \bar{f}_{aIJ}^T F^{aIJ,bKL} \bar{g}_{bKL}^T. \tag{2.61}$$

It is easy to see that another set of pairs is given by

$$\left\{ P_{E_I^a}[f_I^a], S^{aI}[\bar{g}_{aI}] + E^{aI} n_I[g_a] \right\} = \int_\sigma d^D x \, f^{aI} \left(-s\bar{g}_{aI} - n_I g_a \right). \tag{2.62}$$

To obtain yet another set of second class pairs which Poisson commute with the first two pairs, we realise that the linear combination

$$P'_{n_I}[\gamma_I] := P_{n_I}[\gamma_I] + s P_{E_J^a}[\gamma_I \pi^{aI}{}_J] \tag{2.63}$$

Poisson commutes with all the above constraints. The third set of second class pairs is given by

$$\left\{ P'_{n_I}[f_I], (n^I n_I - s)[g] + \bar{S}^{aIJ}[\bar{g}_{[I} E_{a|J]}] \right\} \approx - \int_\sigma d^D x \, f_I \left(2g n^I - s(1-D)\bar{g}_I \right). \tag{2.64}$$

We emphasise that the three sets of pairs yield three invertible Dirac matrices. Since constraints from different sets of pairs Poisson commute with each other, the whole Dirac matrix is invertible because its determinant is the product of the three subdeterminants coming from the three sets.

The transition to the extended Hamiltonian is not necessary since all first class constraints are already contained with arbitrary multipliers in the total Hamiltonian.

The counting of the degrees of freedom goes as follows:

Variable	DoF		Constraint	Number
A_{aIJ}	$\frac{D^2(D+1)}{2}$		First class	(count twice!)
π^{aIJ}	$\frac{D^2(D+1)}{2}$		$\mathcal{H}$	1
λ_{IJ}	$\frac{D(D+1)}{2}$		$\mathcal{H}_a$	D
N	1		G^{IJ}	$\frac{D(D+1)}{2}$
N^a	D		P_N	1
$P_{\lambda_{IJ}}$	$\frac{D(D+1)}{2}$		P_{N^a}	D
P_N	1		$P_{\lambda_{IJ}}$	$\frac{D(D+1)}{2}$
P_{N^a}	D		Second class	
n^I	$D+1$		S^{aIJ}	$\frac{D^2(D+1)}{2}$
E^{aI}	$D(D+1)$		$\bar{D}_T^{aIJ}$	$\frac{D^2(D-1)}{2} - D$
P_{n_I}	$D+1$		$E^{aI} n_I$	D
$P_{E^{aI}}$	$D(D+1)$		$n^I n_I - s$	1
			$P_{E^{aI}}$	$D(D+1)$
			P_{n_I}	$D+1$
Sum:	$D^3 + 4D^2 + 7D + 4$		Sum:	$D^3 + 3D^2 + 8D + 6$

The difference between the degrees of freedom and the effective number of constraints is $(D-2)(D+1)$ and matches the degrees of freedom of gravity in the Hamiltonian formulation.

2.4 Solution of the Second Class Constraints

The solution of the second class constraints is done analogously to the treatment by Peldan [10]. We start by solving the first class constraints P_N, P_{N^a}, and $P_{\lambda_{AB}}$ by treating N, N^a, and λ_{AB} as Lagrange multipliers.

To solve the second class constraints, we use the Ansatz

$$A_{aIJ} = \Gamma_{aIJ} + K_{aIJ}. \tag{2.65}$$

Γ_{aIJ} is the hybrid spin connection defined by

$$\nabla_a E^{bI} = \partial_a E^{bI} + \Gamma^b_{ac} E^{cI} + \Gamma_a{}^I{}_J E^{bJ} - \Gamma^c_{ca} E^{bI} = 0 \tag{2.66}$$

and is explicitly given by

$$\Gamma_{aIJ} = 2s E_{b[I} n_{J]} n^K \partial_a E^b_K + E_{b[I} \bar{\eta}_{J]K} \partial_a E^{bK} + \Gamma^c_{ab} E^b_{[J} E_{c|I]}, \tag{2.67}$$

where

$$\Gamma^c_{ab} := \frac{1}{2} q^{cd} \left(\partial_a q_{bd} + \partial_b q_{ad} - \partial_d q_{ab} \right) \tag{2.68}$$

is the Levi-Civita connection. Notice that

$$\nabla_a E^{aI} = \partial_a E^{aI} + \Gamma_a{}^I{}_J E^{aJ} = 0. \tag{2.69}$$

Further, we decompose K_{aIJ} into $\bar{K}_{aIJ}$ and $2n_{[I} K_{a|J]}$ similar as for π^{aIJ} where $K_{aI} \propto K_{aIJ} n^J$ is automatically transversal. At the same time, we solve the simplicity constraints to $\pi^{aIJ} = 2n^{[I} E^{aJ]}$. The requirement that n^I is orthogonal to E^{aI} and has unit length leads to

$$n^I = \frac{\epsilon^{IJ_1...J_D} E^{a_1}_{J_1} \cdots E^{a_1}_{J_D} \epsilon_{a_1...a_D}}{D! \sqrt{\det E^{aI} E^b_I}}. \tag{2.70}$$

In the following, we will always mean $n_I = n_I(E^{aJ})$ and thus have solved the constraints $n^I n_I - s \approx 0$, $E^{aI} n_I \approx 0$, and $P_{n_I} \approx 0$.

The boost (longitudinal) part of the Gauß constraint becomes

$$n_{[I} \lambda_{J]} D_a \pi^{aIJ} = s \lambda_I \bar{K}_a{}^{KI} E^a_K \tag{2.71}$$

and coincides with the trace part of $\bar{K}_{aIJ}$. Next, we insert this Ansatz into D^{aIJ} and calculate

$$\begin{aligned} f^T_{aIJ} \bar{D}^{aIJ}_T &= \bar{f}^T_{aIJ} D^{aIJ} = -2s \bar{f}^T_{aIJ} E^{b[I} D_b E^{a|J]} \\ &= -2s \bar{f}^T_{aIJ} E^{b[I} \left(D_b E^{a|J]} + \Gamma^a_{bc} E^{c|J]} - \Gamma^c_{bc} E^{a|J]} \right) \\ &= \frac{1}{2} \bar{f}^T_{aIJ} F^{aIJ,bKL} \bar{K}^T_{bKL}. \end{aligned} \tag{2.72}$$

In the second line, we have added terms to construct a covariant derivative compatible with E^{aI}. Both added terms are zero, the first because of the antisymmetry in $[IJ]$ and the second one because of the trace freeness of $\bar{f}^T_{aIJ}$. We see that $\bar{D}^{aIJ}_T = 0$ implies the vanishing of the trace free part of $\bar{K}_{aIJ}$. Thus Gauß and D constraint together imply that $\bar{K}_{aIJ}$ vanishes whence $K_{aIJ} = 2n_{[I} K_{aJ]}$.

Since we solved second class constraints, we have to perform a symplectic reduction and determine the new symplectic structure. In addition to the above considerations, we set $P_{E_I^a} = 0$. The symplectic potential now reads

$$
\begin{aligned}
\frac{1}{2}\pi^{aIJ}\dot{A}_{aIJ} &= n^{[I}E^{a|J]}\left(\dot{\Gamma}_{aIJ} + \dot{K}_{aIJ}\right) \\
&= n^I\left((\nabla_a E_I^a)\dot{} - \nabla_a \dot{E}_I^a + E^{aJ}\dot{K}_{aIJ}\right) \\
&= -\partial_a(n^I \dot{E}_I^a) + n^I E^{aJ}\dot{K}_{aIJ} \\
&= -\dot{E}^{aJ}n^I K_{aIJ} - \dot{n}^I E^{aJ}K_{aIJ} \\
&= \dot{E}^{aJ}(n_J E_a^I \bar{K}_{bIK}E^{bK} - sK_{aJ}) \\
&=: \dot{E}^{aJ}K'_{aJ},
\end{aligned}
\tag{2.73}
$$

where we have dropped total time derivatives and divergences, in the second before the last step we used that $\dot{n}^I$ is transversal. [2] Notice also that we keep the trace part of $\bar{K}_{aIJ}$ since we do not solve first class constraints at this point.

In the last step, we have to express the remaining constraints $\mathcal{H}$, $\mathcal{H}_a$, and G^{IJ} in terms of the new canonical variables. The calculation yields

$$
\frac{1}{2}f_{IJ}G^{IJ} = -f^{IJ}E_{[I}^a K'_{a|J]},
\tag{2.74}
$$

$$
N^a \mathcal{H}_a \approx 2sN^a \nabla_{[a}E^{bJ}K'_{b]J},
\tag{2.75}
$$

$$
N\mathcal{H} \approx N\left(s\frac{1}{2}E^{aI}E^{bJ}R_{abIJ} - E^{a[I}E^{b|J]}K'_{aI}K'_{bJ}\right).
\tag{2.76}
$$

We have neglected terms proportional to the Gauß constraint in the expressions for $N^a\mathcal{H}_a$ and $N\mathcal{H}$ as well as total derivatives. $R_{abIJ} := 2\partial_{[a}\Gamma_{b]IJ} + [\Gamma_a, \Gamma_b]_{IJ}$ denotes the field strength of the hybrid spin connection. Up to a global factor, the time gauge $n^I = (1, 0, \ldots, 0)$ and the solution of the boost part of the Gauß constraint lead to the ADM formalism with internal $SO(D)$ gauge group as derived in [3].

3 Equivalent Formulation

3.1 The BF-Simplicity Constraint

Freidel, Krasnov and Puzio [5] have shown that higher-dimensional Einstein gravity can be written as a constrained BF theory. Since their starting point is a BF theory where there are a priori no vielbeins, they have to use a different simplicity constraint to avoid introducing vielbeins. In the following, we will review their simplicity constraint and discuss its relation to our formulation. To distinguish the two versions of the simplicity constraint, we will call the simplicity constraint dealt with in this section the BF-simplicity.

We will sketch the main proof given in [5] and adapt it to our canonical treatment. This step is necessary because the original paper considers path integral quantisation for which only the action is needed and the whole vielbein is encoded in an object $B_{IJ}^{\mu\nu}$. In our case, the components $\underset{\sim}{N}$ and N^a of the vielbein are not considered because they function as Lagrange multipliers. In the following, we will denote by $\overline{M}$ a $(D-3)$ multi-index.

Let us begin by citing the main result of [5]:

---

[2] We also used $\nabla_a n^I = 0$ which follows from $E_I^a n^I = n^I n_I - s = \nabla_a E_I^b = 0$: We have for the longitudinal part $n_I \nabla_a n^I = \nabla_a(n_I n^I/2) = 0$ and for the transversal part $E_I^b \nabla_a n^I = \nabla_a(E_I^b n^I) = 0$.

Theorem (Freidel, Krasnov, Puzio)**.**
In dimension $D > 3$ a generic B field satisfies the constraints

$$\epsilon^{\overline{M}IJKL}\tilde{B}_{IJ}^{\mu\nu}\tilde{B}_{KL}^{\rho\sigma} = \tilde{\epsilon}^{[\alpha]\mu\nu\rho\sigma}\tilde{c}_{[\alpha]}^{\overline{M}} \tag{3.1}$$

for some coefficients c with $[\alpha]$, $\overline{M}$ totally skew tensorial and Lie algebra combinations respectively of length $D-3$, if and only if it comes from a frame field. In other words, a non-degenerate B satisfies the constraints (3.1) if and only if there exist e_I^μ such that

$$\tilde{B}_{IJ}^{\mu\nu} = \pm|e|e_I^{[\mu}e_j^{\nu]}, \tag{3.2}$$

where $|e|$ is the absolute value of the determinant of the inverse matrix $e_{\mu I}$.

The theorem also holds for $D = 3$ with the additional appearance of a topological sector which we will neglect in the following. Clearly, our desired simplicity constraint guaranteeing $\pi^{aAB} = 2n^{[A}E^{a|B]}$ seems to be a special case of this theorem.

The constraints are divided into the categories

$$\begin{array}{llll}
\text{simplicity:} & \tilde{B}_{[IJ}^{\mu\nu}\tilde{B}_{KL]}^{\mu\nu} = 0 & \mu, \nu & \text{distinct,} \\
\text{intersection:} & \tilde{B}_{[IJ}^{\mu\nu}\tilde{B}_{KL]}^{\nu\rho} = 0 & \mu, \nu, \rho & \text{distinct,} \\
\text{normalisation:} & \tilde{B}_{[IJ}^{\mu\nu}\tilde{B}_{KL]}^{\rho\sigma} = \tilde{B}_{[IJ}^{\mu\rho}\tilde{B}_{KL]}^{\sigma\nu} & \mu, \nu, \rho, \sigma & \text{distinct,}
\end{array}$$

from which only the first two are relevant for us since the last one does not appear in our case where $B_{IJ}^{\mu\nu} \to B_{IJ}^{t\nu} \to \pi_{IJ}^a$.

We find it convenient for the following considerations to look at π_{IJ}^a as a two form $\pi_{IJ}^a dx^I \wedge dx^J$. It can be shown that for a two-form B_{IJ}

$$B_{[IJ}B_{KL]} = 0 \;\Leftrightarrow\; B_{IJ} = u_{[I}v_{J]}, \tag{3.3}$$

which corresponds to the simplicity constraint. Therefore, all π_{IJ}^a factor into $u_{[I}^a v_{J]}^a$ (no summation). To complete the proof, we have to relate the different u_I^a to each other. For this purpose, it is proved in [5] that for two two-forms B_{IJ} and B_{IJ}',

$$B_{[IJ}B_{KL]}' = 0 \;\Leftrightarrow\; B_{IJ} = u_{[I}v_{J]} \text{ and } B_{IJ}' = u_{[I}w_{J]}, \tag{3.4}$$

meaning that the two two-forms share a common factor which is unique up to scaling. In our case, this relation is ensured by the intersection constraint where $\nu = t$.

Combining these two arguments, we realise that π_{IJ}^a factors into one-forms with a common factor. Introducing the correct density weight and a suitable normalisation, we obtain

$$\pi_{IJ}^a = \pm 2\sqrt{q}n_{[I}^{\parallel}e_{J]}^a. \tag{3.5}$$

The property of n^I being time-like in the Lorentzian case will be enforced by another constraint.

The sign can be absorbed into n^I for $D + 1$ even, the otherwise appearing signs can be absorbed into the Lagrange multipliers in the Hamiltonian. We remark that in the general case discussed in [5], the normalisation constraints are necessary and the proof becomes considerably longer.

Without the normalisation constraints and smeared over space, (3.1) reduces to

$$S_{\overline{M}}^{ab}\left[c_{ab}^{\overline{M}}\right] = \int_\sigma d^D x\, \frac{1}{4}c_{ab}^{\overline{M}}\epsilon_{IJKL\overline{M}}\pi^{aIJ}\pi^{bKL}. \tag{3.6}$$

The Lagrange multiplier $c_{ab}^{\overline{M}}$ can be chosen to be symmetric in the index pair ab.

16

3.2 Constraint Analysis

We start with the action

$$S = \int_{\mathbb{R}} dt \int_{\sigma} d^D x \left(\frac{1}{2} \pi^{aIJ} \dot{A}_{aIJ} - N\mathcal{H} - N^a \mathcal{H}_a - \frac{1}{2} \lambda_{IJ} G^{IJ} - c_{ab}^{\overline{M}} S_{\overline{M}}^{ab} \right), \tag{3.7}$$

where we used the notation from the previous section. The action is motivated by [5], but we can also arrive at it by taking the action from the previous chapter, dropping the variables E^{aI}, n_I, and all constraints containing them, and introducing the BF-simplicity constraint. Since we want the metric to be positive definite, we impose the constraint

$$s\pi^{aIJ} \pi_{IJ}^b \approx 2qq^{ab} > 0, \tag{3.8}$$

where the greater sign means positive definiteness of matrices. In the Lorentzian case, the relation is only satisfied if n^I is time-like, because $E^{aI} E_I^b$ would be indefinite otherwise. Such a constraint is called non-holonomic and does not reduce the degrees of freedom.

From the above action "we read off" the non-vanishing Poisson brackets as $\{A_{aIJ}, \pi^{bKL}\} = 2\delta_a^b \delta_{[I}^K \delta_{J]}^L$. To see that this is correct, i.e. to read off the symplectic structure rather than going through the constraint analysis, consider the following generic situation. All variables appearing in the action have to be considered as configuration variables and corresponding velocities to begin with. Since

$$P_{A_{aIJ}} = \frac{\delta L}{\delta \dot{A}_{aIJ}} = \pi^{aIJ} \tag{3.9}$$

cannot be solved for $\dot{A}_{aIJ}$, $P_{A_{aIJ}} - \pi^{aIJ} \approx 0$ becomes a constraint and has to be added to the Hamiltonian and smeared with a Lagrange multiplier μ_{aIJ}. The same is true for

$$P_{\pi^{aIJ}} = \frac{\delta L}{\delta \dot{\pi}^{aIJ}} = 0. \tag{3.10}$$

We will call the corresponding multiplier ν^{aIJ}. Before proceeding with the canonical analysis, we can use $P_{A_{aIJ}} - \pi^{aIJ} \approx 0$ to substitute all π^{aIJ} for $P_{A_{aIJ}}$ by redefining μ_{aIJ}. Stability of $P_{A_{aIJ}} - \pi^{aIJ} \approx 0$ requires to adjust ν^{aIJ}. Stability of $P_{\pi^{aIJ}} \approx 0$ requires us to set $\mu_{aIJ} = 0$. The stability of the other constraints is equivalent to the case where we "read off" the symplectic structure and we assume it to be satisfied. We can now solve the constraints $P_{A_{aIJ}} - \pi^{aIJ} \approx 0$ and $P_{\pi^{aIJ}} \approx 0$ and perform a symplectic reduction which gives us the new Poisson bracket

$$\{A_{aIJ}, \pi^{bKL}\} = 2\delta_a^b \delta_{[I}^K \delta_{J]}^L. \tag{3.11}$$

Most of the canonical analysis is the same as in the previous chapter and we will only describe the differences. The Poisson bracket

$$\begin{aligned}
\{\mathcal{H}[M], \mathcal{H}[N]\} &= -\frac{1}{2} \mathcal{H}_a \left[(M\partial_b N - N\partial_b M)\pi^{aIJ}\pi_{IJ}^b \right] \\
&\quad + s\frac{1}{4} S_{\overline{M}}^{ab} \left[(M\partial_a N - N\partial_a M)\epsilon_{IJKL}{}^{\overline{M}} \pi^{cIJ} F_{cb}^{KL} \right]
\end{aligned} \tag{3.12}$$

of two Hamiltonian constraints reproduces exactly the BF-simplicity constraint and shows that the theory would be inconsistent without this constraint. The BF-simplicity constraint is stable under spatial diffeomorphisms and internal rotations as reflected by the Poisson brackets

$$\left\{ \tilde{\mathcal{H}}_a[N^a], S_{\overline{M}}^{ab}[c_{ab}^{\overline{M}}] \right\} = -S_{\overline{M}}^{ab} \left[(\mathcal{L}_N c)_{ab}^{\overline{M}} \right] \tag{3.13}$$

17

and

$$\left\{\frac{1}{2}G^{IJ}[\lambda_{IJ}], S^{ab}_{\overline{M}}[c^{\overline{M}}_{ab}]\right\} = S^{ab}_{\overline{M}}\left[\sum_{i=1}^{D-3} \lambda^{M_i}{}_{M'_i} c_{ab}^{M_1...M_{i-1}M'_i M_{i+1}...M_{D-3}}\right], \tag{3.14}$$

and trivially commutes with itself. As in the previous chapter, the Poisson bracket with the Hamiltonian constraint

$$\left\{S^{ab}_{\overline{M}}[c^{\overline{M}}_{ab}], \mathcal{H}[N]\right\} = D^{ab}_{\overline{M}}\left[Nc^{\overline{M}}_{ab}\right] + S^{ab}_{\overline{M}}[\ldots] \tag{3.15}$$

imposes a new constraint

$$D^{ab}_{\overline{M}} = -\epsilon_{IJKL\overline{M}}\pi^{cIJ}\left(\pi^{(a|KN}D_c\pi^{b)L}{}_N\right). \tag{3.16}$$

To show its stability, we have to show that its Poisson bracket with the BF-simplicity is invertible. Irrespective of this, we emphasise that $D^{ab}_{\overline{M}}$ is stable under internal rotations, reflected by

$$\left\{\frac{1}{2}G^{IJ}[\lambda_{IJ}], D^{ab}_{\overline{M}}[d^{\overline{M}}_{ab}]\right\} = D^{ab}_{\overline{M}}\left[\sum_{i=1}^{D-3} \lambda^{M_i}{}_{M'_i} d_{ab}^{M_1...M_{i-1}M'_i M_{i+1}...M_{D-3}}\right] + S^{ab}_{\overline{M}}[\ldots]. \tag{3.17}$$

Concerning the diffeomorphism constraint, it is easy to see that we can extend the covariant derivative in $D^{ab}_{\overline{M}}$ to act on spatial indices via the Christoffel symbols. Namely, adding the corresponding terms to the constraint, we see that, due to the symmetry of the Christoffel symbols in their lower indices, the added terms are proportional to simplicity constraints. $D^{ab}_{\overline{M}}$ therefore transforms like a scalar density of weight $+3$ under spatial diffeomorphisms and the Poisson bracket with the diffeomorphism constraint has to be proportional to the $D^{ab}_{\overline{M}}$. Another easy way to do this calculation is to use the Jacobi identity after expressing $D^{ab}_{\overline{M}}$ as a Poisson bracket. We do not know of any nice way to express the Poisson bracket of $D^{ab}_{\overline{M}}$ with the Hamiltonian constraint and will leave the discussion of this bracket open, as its value is not important in the following.

Counting of the degrees of freedom which the BF-simplicity constraint reduces, i.e. $\pi^{aIJ} \rightarrow E^{aI}$, yields $D^2(D-1)/2 - D$ which is for $D > 3$ less than the number of Lagrange multipliers $\frac{1}{2}D(D+1)\binom{D+1}{4}$. The BF-simplicity constraints are therefore not independent and the matrix formed by calculating the Poisson bracket with $D^{ab}_{\overline{M}}$ cannot be invertible. The solution to this problem is to find an independent set of BF-simplicity and D constraints which still enforce the same constraint surface. The constraints of the previous chapter do have this property and lead us to the following Ansatz. We choose some internal time-like vector n^I with $n^I n_I = s$ which may vary as a function of the spatial coordinates and decompose π^{aIJ} as

$$\pi^{aIJ} = \bar{\pi}^{aIJ} + 2n^{[I}E^{a|J]} \tag{3.18}$$

as in the previous chapter. We also define E_{aI} by $E_{aI}E^{bI} = \delta^b_a$ and $n^I E_{aI} = 0$. Together with the normalisation condition $n^I n_I = s$ this means that $n^I = n^I[E]$ can be considered as a function of E^a_I only and thus does not count as independent degree of freedom. The BF-simplicity constraints plus the non-holonomic constraint are equivalent with $\bar{\pi}^{aIJ} = 0$ and n_I being time-like in the Lorentzian case. However, $\bar{\pi}^{aIJ}$ has $D^2(D-1)/2$ degrees of freedom and E^a_I has $D(D+1)$ which together yields $D^2(D+1)/2 + D$ degrees of freedom while π^{aIJ} has only $D^2(D+1)/2$ degrees of freedom. It follows that $\bar{\pi}^{aIJ}$ and E^a_I cannot be considered as independent degrees of freedom, there must be D additional relations among them. Indeed, in the companion paper [1] we will argue[3] that it is always possible to arrange that $\bar{\pi}^{aIJ} = \bar{\pi}^{aIJ}_T$ is

---

[3]This is not trivial: For $D \geq 3$ one cannot use closed formulas for a proof. It is apparently necessary to make use of fixed point theorems.

automatically trace free with respect to E_{aI}. These would be the missing D relations and now the BF-Simplicity constraints are equivalent with the $D^2(D-1)/2 - D$ constraints $\bar{\pi}_T^{aIJ} = 0$ which in number match with the constraints $\bar{K}_{aIJ}^T = 0$ to which the constraints $D_{\overline{M}}^{ab} = 0$ reduce as we will now show below. At the moment we have no proof of this for $D \geq 3$ thus we will make the assumption that π^{aIJ} can always be decomposed in this way. In other words, we will only allow π^{aIJ} of the following form: There is a tensor E^{aI} with $Q^{ab} = \eta_{IJ} E^{aI} E^{bJ}$ positive definite. Let $n^I[E]$ be the unique normal vector satisfying $E^{aI} n_I = 0$, $n^I n_I = s$. Take any tensor t^{aIJ} and construct from it $\bar{t}_T^{aIJ}[t, E]$ using $E, n[E]$. Then $\pi^{aIJ} := \bar{t}_T^{aIJ} + 2n^{[I} E^{aJ]}$ and automatically $E^{aI} = -s\pi^{aIJ} n_J$. For π^{aIJ} constructed in this way, the fixed point equation derived in [1] has obviously non trivial solutions and the question is whether such π^{aIJ} are generic.

Concerning the $D_{ab}^{\overline{M}}$ constraint, we make the same Ansatz as in the previous chapter and set

$$A_{aIJ} = \Gamma_{aIJ} + \bar{K}_{aIJ} + 2n_{[I} K_{a|J]}. \tag{3.19}$$

A short calculation yields

$$\bar{f}_{(a|IJ}\pi_{|b)KL}\epsilon^{IJKL\overline{M}} D_{\overline{M}}^{ab} = -\bar{f}_{(a|IJ}\pi_{|b)KL}\epsilon^{IJKL\overline{M}} \epsilon_{ABCD\overline{M}}\pi^{cAB}\pi^{(a|C}{}_E D_c \pi^{b)DE}$$
$$\approx -(D-3)!(D-1)\bar{K}_{aIJ} F^{aIJ,bKL}\bar{f}_{bKL}, \tag{3.20}$$

where $F^{aIJ,bKL}$ denotes the same matrix as in (2.49). We defined $\pi_{bKL} := q^{-1}q_{ab}\pi^a{}_{KL}$, where $q^{-1}q_{ab}$ is the inverse matrix of $\frac{s}{2}\pi^{aIJ}\pi^b{}_{IJ}$, such that $\pi^{aIJ}\pi_{bIJ} = 2s\delta_b^a$. We notice that $\bar{f}_{aIJ}$ can be chosen traceless with respect to E^{aI}, since any trace part would drop out in the combination $\bar{d}_{(a|IJ}\pi_{b)KL}\epsilon^{IJKL\overline{M}}$ modulo the BF-Simplicity constraint. The subset of D constraints parametrised by $\bar{f}_{aIJ} = \bar{f}_{aIJ}^T$ as above thus sets the trace free part of $\bar{K}_{aIJ}$ to zero. When inserting the solution of the BF-simplicity constraint into the full $D_{\overline{M}}^{ab}$ constraint, we get

$$D_{\overline{M}}^{ab} \approx -2s\epsilon_{ABC}{}^D{}_{\overline{M}} n^A E^{cB} E^{(a|C} E^{b)E} \bar{K}_{cED} \tag{3.21}$$

and immediately verify that the solution $\bar{K}_{aIJ}^T = 0$ solves all the D constraints because the trace part of $\bar{K}_{cED}$ drops out in the above combination.

From these considerations, we realise that it is legitimate to use the Lagrange multipliers displayed in (3.20) and therefore only a subset of the D constraints. It follows that we only have to check the stability of this subset of constraints. To form the Dirac matrix, we choose similarly a subset of BF-simplicity constraints equivalent to $\bar{\pi}_T^{aIJ} = 0$ and calculate

$$\int d^D x \int d^D y \, [\bar{f}_{(a|IJ}^T \pi_{b)KL}\epsilon^{IJKL\overline{M}}](x) \left\{ S_{\overline{M}}^{ab}(x), D_{\overline{N}}^{cd}(y) \right\} [\bar{g}_{(c|MN}^T \pi_{d)OP}\epsilon^{MNOP\overline{N}}](y)$$
$$\approx 4(D-1)^2((D-3)!)^2 \int d^D x \, \bar{f}_{aIJ}^T F^{aIJ,bKL} \bar{g}_{bKL}^T. \tag{3.22}$$

We can therefore adjust the multiplier of the BF-simplicity such that the independent subset of D constraints is stable under time evolution and finish the canonical analysis. Since the Dirac matrix is invertible, the chosen subset of BF-simplicity constraints has to be independent. The number of BF-simplicities in this subset is equivalent to the number of degrees of freedom in a transverse trace free matrix, i.e. $D^2(D-1)/2 - D = D(D+1)(D-2)/2$ and matches the degrees of freedom which are to be taken out of the system by the full BF-simplicity constraints and all BF-simplicity constraints can thus be derived by taking the linear span of this subset.

The solution of the constraints proceeds analogously to the previous chapter, the only difference being that we do not need to solve the momenta associated with E^{aI} and n^I. The two formulations presented are therefore equivalent.

3.3 Degrees of Freedom

As in the previous chapter, we check the degrees of freedom of the Hamiltonian system derived using the BF-simplicity constraint. For $\mathcal{H}$ to become a first class constraint, we construct the linear combination (using the same abuse of notation as before)

$$\tilde{\mathcal{H}} := \mathcal{H} - \int \bar{S}_T^{aIJ} \left(\{\bar{D}_T, \bar{S}_T\}^{-1} \right)_{aIJ,bKL} \left\{ \bar{D}_T^{bKL}, \mathcal{H} \right\}. \tag{3.23}$$

Since the Dirac matrix between the independent BF-simplicity and D constraints is invertible, they are of the second class. The rest of the constraints is of the first class.

Variable	DoF	Constraint	DoF
A_{aIJ}	$\frac{D^2(D+1)}{2}$	First class	(count twice!)
π^{aIJ}	$\frac{D^2(D+1)}{2}$	$\tilde{\mathcal{H}}$	1
λ_{IJ}	$\frac{D(D+1)}{2}$	$\mathcal{H}_a$	D
N	1	G^{IJ}	$\frac{D(D+1)}{2}$
N^a	D	P_N	1
$P_{\lambda_{IJ}}$	$\frac{D(D+1)}{2}$	P_{N^a}	D
P_N	1	$P_{\lambda_{IJ}}$	$\frac{D(D+1)}{2}$
P_{N^a}	D	Second class	
		S_M^{ab}	$\frac{D^2(D-1)}{2} - D$
		D_M^{ab}	$\frac{D^2(D-1)}{2} - D$
Sum:	$D^3 + 2D^2 + 3D + 2$	Sum:	$D^3 + D^2 + 4D + 4$

The difference between the degrees of freedom and the weighted sum of the constraints is again $(D+1)(D-2)$ and matches those of General Relativity.

4 Review of Gauge Unfixing

The name "gauge unfixing" suggests that this is a procedure in some sense inverse to "gauge fixing". To see to what extent this is indeed the case it is useful to recall some facts about gauge fixing first. After that we focus on the gauge unfixing case. This review section can be skipped by readers familiar with gauge (un)fixing although we add a few extra twists to it. We have combined material from several sources: To the best of our knowledge, the pioneering paper on gauge unfixing of second class theories is [11] and the general theory was developed in [12, 13]. Parts of this theory were independently rediscovered from the point of view of a first class theory in [14, 15], see also [16, 17, 18].

4.1 Gauge Fixing

Recall that gauge fixing of a *first class system* with first class constraints S_I (where I takes values in some index set) on a phase space $\mathcal{M}$ consists in imposing an equal number of gauge fixing conditions D_I such that the matrix M with entries $M_{IJ} := \{S_I, D_J\}$ is regular. The gauge fixing conditions, modulo the problem of Gribov copies, select a unique point on each

gauge orbit of the S_I. Here the gauge orbit of a point $m \in \mathcal{M}$ is the set[4]

$$[m] := \{\alpha_\beta(m), \ \beta^I \in \mathbb{R}\}, \quad \alpha_\beta(f) := \exp(\beta^I \{S_I, .\}) \cdot f, \tag{4.1}$$

where $\alpha_\beta(f)$ is the gauge flow with parameter β applied to the (smooth) function f on phase space. To qualify as an admissible gauge fixing condition, at least on the constraint surface

$$\overline{\mathcal{M}} := \{m \in \mathcal{M}; \ S_I(m) = 0 \ \forall I\}, \tag{4.2}$$

it must be possible to reach the selected section

$$\sigma_D(\overline{\mathcal{M}}) := \{m \in \overline{\mathcal{M}}; \ D_I(m) = 0 \ \forall I\} \tag{4.3}$$

from any other section of $\overline{\mathcal{M}}$.

At least locally, the constraint surface acquires the structure of a fibre bundle where the fibres are given by the gauge orbits (considered as subsets of $\overline{\mathcal{M}}$) and the base space is the set of equivalence classes

$$\widehat{\mathcal{M}} := \{[m]; \ m \in \overline{M}\} \tag{4.4}$$

called the reduced phase space. Under the above conditions there is a bijection between $\sigma_D(\overline{\mathcal{M}})$ and $\widehat{M}$: Given $m \in \sigma_D(\overline{\mathcal{M}})$ one obtains $[m] \in \widehat{\mathcal{M}}$ via (4.1) and given $[m]$ (considered as a subset of $\overline{\mathcal{M}}$) one computes the unique point $m' \in [m]$ such that $D_I(m') = 0$ for all I, that is $m' = [m] \cap \sigma_D(\overline{\mathcal{M}})$. However, while the construction of $\widehat{\mathcal{M}}$ is *canonical*, i.e. does not use any structure other than S_I which canonically follow from the Dirac algorithm applied to the singular Lagrangian in question, the cross section $\sigma_D(\overline{\mathcal{M}})$ uses the additional input of D which, except for the regularity condition on M, is rather arbitrary.

The observables of the first class system are the gauge invariant functions evaluated on the constraint surface. These therefore only depend on the equivalence classes $[m]$. It appears that the construction of such gauge invariant functions is generically impossible for sufficiently complicated constraints S_I. This turns out to be correct if one is interested in these observables as functions on $\overline{\mathcal{M}}$. However, given a set of gauge fixing conditions D_I, not only can one write an explicit formula for these observables but one can also compute their Poisson algebra. This also then displays the relation between the spaces $\sigma_D(\overline{\mathcal{M}})$ and $\overline{\mathcal{M}}$ in explicit form. Given a function f on $\mathcal{M}$ one can define a weak Dirac observable by the formula

$$O^{(D)}(f) := [\alpha_\beta(f)]_{\alpha_\beta(D)=0}, \tag{4.5}$$

where the superscript (D) is to make it explicit that this formula is not canonical but depends on the chosen gauge fixing. This formula has to be understood in the following way: First one computes the gauge flow of f at $m \in \overline{\mathcal{M}}$ with real valued (phase space independent) constants β^I, that is

$$\alpha_\beta(f) := f + \sum_{n=1}^{\infty} \frac{1}{n!} \beta^{I_1}..\beta^{I_n} \{S_{I_1}, \{.., \{S_{I_n}, f\}..\}\} \tag{4.6}$$

and then one solves the condition $\alpha_\beta(D_I) = 0$ for all I for $\beta^I = \gamma^I(m)$ and inserts the corresponding phase space dependent function into (4.6). The value $\gamma(m)$ is thus the parameter needed in order to map m to that point on its orbit $[m]$ at which the D_I vanish. It is not

---

[4]In case that the first class constraints close with non trivial structure functions only, it maybe necessary to apply several of the Poisson automorphisms α_β with different β because the α_β do not form a group under concatenation in this case.

difficult to check that indeed $\{S_I, O_f\} \approx 0$, and that $O^{(D)}$ preserves the pointwise addition and multiplication of functions

$$O^{(D)}(f+g) = O^{(D)}(f) + O^{(D)}(g), \quad O^{(D)}(fg) = O^{(D)}(f)\, O^{(D)}(g). \tag{4.7}$$

Moreover, the following remarkable formula holds[5]

$$\{O^{(D)}(f), O^{(D)}(g)\} \approx \{O^{(D)}(f), O^{(D)}(g)\}^*_{S,D} \approx O^{(D)}(\{f,g\}^*_{S,D}), \tag{4.8}$$

where $\{.,.\}^*_{S,D}$ is the Dirac bracket of the *second class system* of constraints S_I, D_I. Since a sufficient number of the $O^{(D)}(f)$ serves as coordinates of $\widehat{\mathcal{M}}$ we see that the Poisson bracket on the reduced phase space $\widehat{\mathcal{M}}$ is given by the Dirac bracket and $O^{(D)}$ is a Dirac bracket homomorphism from the algebra of smooth functions on $\mathcal{M}$ to the one on $\widehat{\mathcal{M}}$.

It should be stressed, however, that this algebra of observables is not canonical, it depends on the choice of admissible gauge fixing D which is an extra input necessary for their very construction. Nevertheless, once we have made such a choice, we see that a first class system S together with a gauge fixing condition D is completely equivalent to the second class system S, D. Namely, for a second class system the reduced phase space consists simply in the constraint manifold

$$\overline{\overline{\mathcal{M}}} := \{m \in \mathcal{M}; \ \ S_I(m) = D_I(m) = 0 \ \forall \ I\} \equiv \sigma_D(\overline{\mathcal{M}}), \tag{4.9}$$

which precisely coincides with the gauge section (4.3), and the symplectic structure on $\overline{\overline{\mathcal{M}}}$ is given by the Dirac bracket

$$\{f,g\}^*_{S,D} = \{f,g\} - \{f, C_\alpha\}\, [M^{-1}]^{\alpha\beta}\, \{C_\beta, g\}, \tag{4.10}$$

where $\{C_\alpha\} = \{S_I, D_I\}$ and $M_{\alpha\beta} = \{C_\alpha, C_\beta\}$ is non degenerate by construction. When restricting $O^{(D)}$ to $\overline{\overline{\mathcal{M}}}$ which is in bijection with $\widehat{\mathcal{M}}$ as we have seen, it becomes a Dirac bracket isomorphism.

4.2 Gauge Unfixing

We now consider a second class system with constraints S_I, D_I with the special structure that S_I is a first class subalgebra of constraints, that is

$$\{S_I, S_J\} = f_{IJ}{}^K\, S_K \tag{4.11}$$

for certain structure functions $f_{IJ}{}^K$ and $M_{IJ} := \{S_I, D_J\}$ is supposed to be non degenerate on the constraint surface

$$\overline{\overline{\mathcal{M}}} := \{m \in \mathcal{M}; \ \ S_I(m) = D_I(m) = 0 \ \forall \ I\}, \tag{4.12}$$

which is equipped with the Dirac bracket (4.10). In [13] we find conditions under which linear combinations of a given set of second class constraints can be subdivided into sets S_I and D_I subject to (4.11). Here we will simply assume that this has been achieved.

We have seen in the previous section that a first class system S_I together with additional gauge fixing conditions D_I is equivalent with the second class system S_I, D_I. The idea of gauge

---

[5]The first identity holds because the S constraints form a subalgebra. The Dirac matrix $M_{\alpha\beta} = \{C_\alpha, C_\beta\}$, $\{C_\alpha\} = \{S_I, D_I\}$ on the constraint surface therefore has the symbolic structure $M = \begin{pmatrix} 0 & B \\ -B & A \end{pmatrix}$ and its inverse is given by $M^{-1} = \begin{pmatrix} B^{-1}AB^{-1} & -B^{-1} \\ B^{-1} & 0 \end{pmatrix}$ so that the Dirac bracket $\{F,G\}^*$ contains no terms $\propto \{F,D\}\{G,D\}$.

unfixing is now simply to interpret the given second class system of constraints S_I, D_I as just a first class system S_I to which the particular gauge fixing conditions D_I have been added.

This point of view has the following advantage towards quantisation:

For a first class system of constraints we have two possible quantisation strategies, namely A. Operator Constraint Quantisation and B. Reduced Phase Space Quantisation. The advantage of option A. is that one can work with the simple Poisson bracket algebra on the kinematical phase space $\mathcal{M}$ for which Hilbert space representations are typically easy to find and the task is to find those which support the S_I as densely defined, closable and non anomalous operators. The disadvantage is that one has to equip the joint kernel of the constraints with a new (physical) inner product which carries a representation of the observables of the theory and while there are general tools available such as group averaging, it is generically not possible to determine the physical Hilbert space in closed form. The disadvantage of option B. is that the Dirac bracket algebra on the reduced phase space is typically so complicated that no Hilbert space representations can be found. On the other hand, if one manages to do so, then one has direct access to the physical Hilbert space and the algebra of observables. Now in case that option B. is inhibited due to the complexity of the Dirac bracket algebra which is typically the case for second class systems, option A. appears to be the only possible approach to quantisation. As we will see, one can do even better than that, but let us assume for the moment that we take a second class system S_I, D_I with complicated Dirac bracket algebra and therefore drop D_I and just perform an operator constraint quantisation of S_I.

At first sight, this strategy seems to be false for at least two reasons:

1. From the point of view of the first class system, the gauge fixing conditions D_I are just one of an infinite number of possible choices, the first class system does not know about the D_I and therefore one can drop the D_I. However, we are not given a first class system, we are given a second class system and from the point of view of the second class system, the D_I are *canonical*, they follow canonically from Dirac's stabiliser algorithm applied to the given singular Lagrangian. It seems therefore to be wrong to forget about the special role of the D_I within the first class system as we would drop information that is forced on us by Dirac's algorithm. However, imposing the D_I as operators as well in the quantum theory is not possible, that is, the joint kernel of the D_I, S_I is just the zero vector.

2. The canonical Hamiltonian H of the second class system as derived via Dirac's stabiliser algorithm is typically not gauge invariant with respect to the S_I which would not be the case for a true first class system with just the constraints S_I. In fact, in many applications the second class structure S_I, D_I arises from primary constraints S_I and a canonical Hamiltonian of the form

$$H' = H_0 + \lambda^I \, S_I, \tag{4.13}$$

with nontrivial H_0 independent of the S_I (that is $[H_0]_{S=0} \neq 0$) and the D_I arise as secondary constraints from the stability requirement

$$0 \overset{!}{=} \{H', S_I\} \approx \{H_0, S_I\} =: D_I, \tag{4.14}$$

where $\{S_I, S_J\} \propto S_K \approx 0$ was used. The stability of the D_I fixes the Lagrange multipliers λ^I

$$0 \overset{!}{=} \{H', D_I\} = \{H_0, S_I\} + \lambda^J \{S_J, D_I\} \quad \Rightarrow \quad \lambda^I = -[M^{-1}]^{JI} \, \{H_0, S_J\} =: \lambda_0^I, \tag{4.15}$$

so that the stabilised, first class Hamiltonian (it weakly commutes with all the constraints S_I, D_I) reads

$$H = H_0 + \lambda_0^I \, S_I. \tag{4.16}$$

23

It is not gauge invariant with respect to just the constraints S_I since $\{H, S_I\} \approx D_I$ so that the constraints D_I appear again as a consistency condition.

We now explain how both obstacles can be overcome. We deal first with the second issue: We simply make the canonical Hamiltonian H gauge invariant with respect to the S_I by using the map $O^{(D)}$ displayed in (4.5) that is, we replace H by

$$\tilde{H} := O^{(D)}(H). \tag{4.17}$$

To see that this is an allowed Hamiltonian within the second class system we need to compute $\tilde{H}$ in some detail. As one can show [15, 18] one has explicitly

$$O^{(D)}(H) = H + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} [-D_{I_k}] \{S'^{I_1}, ..\{S'^{I_n}, H\}..\}, \tag{4.18}$$

where $S'^I = [M^{-1}]^{IJ} S_J$ so that $\{S'^I, D_J\} = \delta^I_J$ modulo S. We have

$$
\begin{aligned}
\tilde{H} - H &= -D_I \{S'^I, H\} + \mathcal{O}(D^2) = -D_I \left([M^{-1}]^{IJ} \{S_J, H\} + \{M^{IJ}, H\} S_J\right) + \mathcal{O}(D^2) \\
&= -D_I \left([M^{-1}]^{IJ}[D_J + N_J^K S_K] + \{M^{IJ}, H\} S_J\right) + \mathcal{O}(D^2) = \mathcal{O}(D^2, DS) \tag{4.19}
\end{aligned}
$$

for some N_J^K. Therefore $\tilde{H}$ and H differ by terms at least quadratic in the constraints and thus do not spoil the first class structure of H. Therefore $\tilde{H}$ is an admissible Hamiltonian for the second class system which is simultaneously weakly invariant with respect to the S_I. This is also the reason why one did not choose $\tilde{H}' = O^{(G)}(H)$ for some gauge fixing conditions $G_I \neq D_I$ because by a similar calculation as in (4.19) one would now compute $\tilde{H}' - H = O(DG, SG, G^2)$ but G_I is no constraint and thus $\tilde{H}'$ is not an admissible Hamiltonian for the second class system. Notice also that H and $\tilde{H}$ generate the same equations of motion on $\overline{\overline{\mathcal{M}}}$.

We now come to the second issue. The question is: How can it be that the first class constrained Hamiltonian system $(\tilde{H}, S_I)$ be equivalent to the second class system $(\tilde{H}, S_I, D_I)$? The first class system does not know about the D_I. It is true that if we choose the special gauge fixing conditions $G_I := D_I = 0$ for the first class system, then the reduced phase spaces of the two systems are indeed isomorphic as we have shown above. However, the choice of G_I is arbitrary from the point of view of the first class system as long as the matrix with entries $\{S_I, G_J\}$ is non degenerate and therefore it appears that one has to still somehow feed the additional information about the special role of the gauge fixing condition $G_I = D_I$ into the first class system. However, this is not the case:
The point is simply that an arbitrary gauge condition $G_I = 0$ is related by a gauge transformation generated by the S_I to the gauge condition $D_I = 0$. Therefore the observables of the form $O^{(G)}(f)$ are in fact linear combinations, with phase space independent coefficients, of the observables of the form $O^{(D)}(f)$. This follows simply from the fact that for gauge invariant functions F (with respect to the S_I) we have $F \approx O^{(D)}(F)$. Applied to $F = O^{(G)}(f)$ it follows

$$O^{(G)}(f) \approx O^{(D)}\left(O^{(G)}(f)\right). \tag{4.20}$$

Hence any observable of the form $O^{(G)}(f)$ can be written as $O^{(D)}(f')$ for some other function $f' = O^{(G)}(f)$. Since the roles of G_I, D_I can be interchanged we see that the range of the maps $O^{(D)}, O^{(G)}$ is the same. Since the algebra of the $O^{(G)}(f)$ and of the $O^{(D)}(f)$ can be computed using the original Poisson bracket on the unreduced phase space we see that the algebra of the $O^{(D)}(f)$ and $O^{(G)}(f)$ are isomorphic, i.e. it does not matter whether we display one and the same observable F in the form $F = O^{(D)}(f)$ or in the form $F = O^{(G)}(f')$.

What is different are of course the maps $O^{(D)}$, $O^{(G)}$ which provide different gauge invariant extensions of a given function f. Only the map $O^{(D)}$ yields an isomorphism with the Dirac bracket algebra of the second class system. However, this does not mean that one cannot use $O^{(G)}$ to construct gauge invariant observables. It just means that the identification between the Dirac bracket algebra of functions on $\overline{\overline{\mathcal{M}}}$ with the Poisson bracket algebra on $\widehat{\mathcal{M}}$ is rather complicated to write down because the correct gauge invariant function is $O^{(D)}(f) \approx O^{(G)}(O^{(D)}(f))$ and not just $O^{(G)}(f)$.

Remarks:
1.
This last observation now is also the key to a reduced phase space quantisation approach to second class systems (H, S_I, D_I): After having replaced it by an equivalent first class system $(\tilde{H}, S_I)$ one can now make use of the local Abelianisation theorem (see e.g. [19] and references therein) and replace the constraints S_I by an equivalent, strongly Abelian set $S'_I = \pi_I + h_I(\phi^I; q^a, p_a)$ at least locally in phase space where the system of first class constraints S_I has been solved for some of the momenta π_I in terms of its conjugate variables ϕ^I and the remaining canonical pairs (q^a, p_a). Using the natural gauge fixing condition $G_I = \phi_I$ the algebra of the $Q^a := O^{(G)}(q^a)$, $P_a := O^{(G)}(p_a)$ coincides with the algebra of the q^a, p_a because the corresponding Dirac bracket does not affect the subalgebra of functions of q^a, p_a. Since the algebra of the Q^a, P_a is simple it can be quantised. This is surprising because we could have chosen to solve the constraints $S_I = D_I = 0$ for $S'_I = \pi_I - \Pi_I(q^a, p_a)$, $D'_I = \phi^I - \Phi^I(q^a, p_a)$ from the outset so that the reduced phase space is parametrised by the q^a, p_a but the corresponding Dirac bracket $\{p_a, q^b\}^* \neq \delta_a^b$ is not simple. The reason is of course that the functions Q^a, P_a are genuinely different from q^a, p_a, in fact they are nontrivial functions of ϕ^I, q^a, p_a built in such a way that they have a simple Dirac bracket with respect to S, D. Moreover $\{Q^a, P_b\} = \{Q^a, P_b\}^*_{S,D}$ due to gauge invariance. This holds for any two pairs of gauge invariant functions, in particular for the Hamiltonian $\tilde{H}$.
2.
For generally covariant systems H_0 is not a true Hamiltonian but rather a linear combination of different constraints $H_0 = \mu^A C'_A$, typically a closed subalgebra of the form $\{C'_A, C'_B\} = f_{AB}{}^C C'_C$ such that $\{C'_A, S_I\} = f_{AI}{}^J S_J$ for $A \neq 0$ and $\{C'_0, S_I\} = D_I$ thus $\{H_0, S_I\} \approx \mu^0 D_I$. In our applications it will turn out that $\{C'_A, D_I\} = \tilde{f}_{AI}{}^K D_K$, $A \neq 0$ and $\{C'_0, D_I\}$ is not weakly zero. The Dirac stabiliser algorithm then replaces C'_0 by $C_0 = C'_0 - M^{JI}\{C'_0, D_J\}S_I$ so that $\{C_0, D_I\} = 0$ while $C'_A = C_A$ for $A \neq 0$ and H' is replaced by $H = \mu^A C_A$. The C_A now close among themselves modulo S_I. Application of $O^{(D)}$ replaces H by $\tilde{H} = \mu^A \tilde{C}_A$, $\tilde{C}_A = O^{(D)}(C_A)$. Now modulo S_I constraints

$$\{\tilde{C}_A, \tilde{C}_B\} \approx O^{(D)}(\{C_A, C_B\}^*_{S,D}) \tag{4.21}$$

and

$$\{C_A, C_B\}^*_{S,D} \propto \{C_A, C_B\}, \quad \{C_A, S_I\}\{C_B, S_J\}, \quad \{C_A, S_I\}\{C_B, D_J\}, \quad \{C_A, D_I\}\{C_B, D_J\},$$
$$\propto C_A, S_I, D_I. \tag{4.22}$$

Since $O^{(D)}(D_I) \approx 0$ it follows that the $\tilde{C}_A$ and the S_I form a first class algebra.
3.
Whether gauge unfixing is feasible depends largely on the question whether the series that determines $\tilde{H}$ terminates. Fortunately, in our application this will be the case.
4.
The formula $O^{(G)}(f) \approx O^{(D)}(O^{(G)}(f))$ does not display the fact that G can be reached from D

via a gauge transformation. However, using the fact that $O^{(D)}(D_I) \approx 0$ and that $O^{(G)}(f)$ is a power series in G we also have

$$O^{(G)}(f) \approx O^{(D)}\left(\left[\exp(\beta_I\{(\tilde{M}^{-1})^{IJ}S_J,.\})\cdot f]_{\beta=-(G-D)}\right]\right),\qquad (4.23)$$

with $\tilde{M}_{IJ} = \{S, G_J\}$. Notice that the argument of $O^{(D)}$ on the right hand side of (4.23) is not gauge invariant and that it is the gauge transform of f with respect to the weakly Abelian constraints $\tilde{S}_I = [\tilde{M}^{-1}]^{IJ}S_J$ from the gauge $G = 0$ to the gauge $D = 0$ as desired.

5.

An important final comment concerns the dynamics of the theory (we consider for simplicity only one pair of second class constraints but the same discussion applies, with more notational load, to the general case):

Suppose first that $H' = H_0 + \lambda^I S_I$, that is H_0 is not constrained to vanish. From the point of view of the second class system the Hamiltonian that drives the dynamics of the system is H or equivalently $\tilde{H}$ via the Dirac bracket evaluated on the constraint surface of the second class system $\overline{\overline{\mathcal{M}}}$, that is

$$\dot{f}_{|S=D=0} = [\{\tilde{H}, f\}^*_{S,D}]_{S=D=0} = \{H_{|S=D=0}, f_{|S=D=0}\}^*_{S,D}. \qquad (4.24)$$

On the other hand, from the point of view of the first class system, the Hamiltonian is $\tilde{H}$ which acts on gauge (S-) invariant functions which we write in the form $F = O^{(D)}(f)$ on the constraint surface of the first class system $\overline{\mathcal{M}}$, that is

$$\dot{F}_{|S=0} = \{\tilde{H}, F\}_{|S=0} = \{O^{(D)}(H), O^{(D)}(f)\}_{|S=0} = O^{(D)}(\{H, f\}^*_{S,D})_{|S=0}. \qquad (4.25)$$

Comparing (4.24) and (4.25) we see that the time evolutions are isomorphic when mapping $f_{|S=D=0}$ to $O^{(D)}(f)_{|S=0}$.

Now we consider the case that $H_0 = C$ itself is constrained to vanish. Then also the Hamiltonian $\tilde{H}$ is constrained to vanish from the point of view of the second class system since is a linear combination of the three constraints C, S, D. Now the following subtlety arises: From the point of view of the first class system, the Hamiltonian $\tilde{H}$ is *not* constrained to vanish because the first class system is only subject to the constraints C, S. But this would clearly be wrong: The first class system would only have the constraint S and this would lead to a different dimensionality of the reduced phase space than in the second class system. The correct point of view is the following: The second class system is equivalently described by the three types of constraints $\tilde{H}, S, D$ of which $\tilde{H}$ constitutes a first class set of constraints while (S, D) constitute a second class system of constraints. From the point of view of the first class system we just forget about the D constraints and instead consider the first class constraint system $\tilde{H}, S$. The counting of physical number of degrees of freedom is now correct again because both first class constraints $\tilde{H}, S$ count twice in the first class system while in the second class system $\tilde{H}, S, D$ only $\tilde{H}$ counts twice and S, D only count once. This also makes sure that there is no true Hamiltonian in both schemes. To compare the observables from both points of view, let $S_1 := S$, $S_2 := \tilde{H}$, $D_1 := D$, $D_2 := G$ where the gauge fixing condition G is chosen in such a way that the matrix with entries $M_{IJ} = \{S_I, D_J\}$ is non singular. It is easy to see that the second class system (S_I, D_I) is of the type to which gauge unfixing applies and the discussion proceeds from here just as in the general case.

5 Application of Gauge Unfixing to Gravity

We now want to apply the ideas of gauge unfixing to higher dimensional General Relativity and start with the Hamiltonian system derived in section 3. The second class constraints are given

by $S^{ab}_{\overline{M}} \approx D^{ab}_{\overline{M}} \approx 0$. As we pointed out in section 3.2, the constraints are not independent and the Dirac matrix is not invertible. We will neglect this fact for the moment and remark that we can deal with it using the independent sets of constraints discussed above. We remark that gauge unfixing has been applied previously to $2 + 1$-dimensional linearised massive gravity [20].

The general discussion of the previous section suggests that the simplicity invariant extension of the Hamiltonian constraint involves an infinite series which is beyond any analytical control already at the classical level. Luckily, the Dirac matrix depends only on π^{aIJ} and therefore commutes with the BF-simplicity constraint. Hence repeated commutators acting on functions that depend polynomially on A vanish beyond the order of the polynomial. We calculate explicitly

$$\tilde{H} = H - \frac{1}{2} D^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\ cd} D^{cd}_{\overline{N}}, \tag{5.1}$$

where terms up to the second order contributed, since H is quadratic in the connection. The effect of the extra term in the Hamiltonian can be seen when solving the simplicity constraint and reducing the theory to the ADM variables. When doing the calculation (2.76), we have to use $D \sim (F\bar{K}^T)^{aIJ} = 0$ to eliminate a term proportional to $\bar{K}^T_{aIJ} F^{aIJ,bKL} \bar{K}^T_{bKL}$. This is not necessary any more because the additional $-1/2DF^{-1}D$ precisely counters this term.

The Gauß and diffeomorphism constraints only obtain extra terms proportional to the BF-simplicity constraints which can be neglected in the first class theory. We can use the projector identities to calculate the new constraint algebra

$$\left\{\tilde{G}, \tilde{G}\right\} = \tilde{G} + S, \tag{5.2}$$

$$\left\{\tilde{G}, \tilde{\mathcal{H}}^a\right\} = S, \tag{5.3}$$

$$\left\{\tilde{G}, \tilde{\mathcal{H}}\right\} = S, \tag{5.4}$$

$$\left\{\tilde{\mathcal{H}}_a, \tilde{\mathcal{H}}_b\right\} = \tilde{\mathcal{H}}_a + \tilde{G} + S, \tag{5.5}$$

$$\left\{\tilde{\mathcal{H}}_a, \tilde{\mathcal{H}}\right\} = \tilde{\mathcal{H}} + \tilde{G} + S, \tag{5.6}$$

$$\left\{\tilde{G}, S\right\} = S, \tag{5.7}$$

$$\left\{\tilde{\mathcal{H}}_a, S\right\} = S, \tag{5.8}$$

$$\left\{\tilde{\mathcal{H}}, S\right\} = 0. \tag{5.9}$$

By construction it closes without the D constraint and displays a first class structure.

Concerning gauge invariant phase space functions, it is clear that a vanishing commutator with the BF-simplicity constraint does not constrain the dependence on π^{aIJ}. Additionally, these functions may only depend on the simplicity invariant extension of A_{aIJ} which is given explicitly by

$$\tilde{A}_{aIJ} = A_{aIJ} + D^{cd}_{\overline{N}} \left(F^{-1}\right)^{\overline{N}\,\overline{M}}_{cd,ab} \epsilon_{\overline{M}IJKL} \pi^{bKL}, \tag{5.10}$$

since A_{aIJ} changes under simplicity gauge transformations as

$$\delta^S A_{aIJ} := \left\{A_{aIJ}, S^{bc}_{\overline{M}}[c^{\overline{M}}_{bc}]\right\} = c^{\overline{M}}_{ab} \epsilon_{IJKL\overline{M}} \pi^{bKL}. \tag{5.11}$$

We still have to give a sense to $\left(F^{-1}\right)^{\overline{N}\,\overline{M}}_{cd,ab}$. As we have shown in section 3, it is enough to consider only a subspace of Lagrange multipliers for the BF-simplicity and $D^{ab}_{\overline{M}}$ constraints parametrised by the projected test functions

$$d^{\overline{M}}_{ab} = \bar{d}_{(a|IJ}\pi_{b)KL}\epsilon^{IJKL\overline{M}}. \tag{5.12}$$

On this subspace, $F^{aIJ,bKL}$ was shown to be invertible. We therefore make the Ansatz

$$\left(F^{-1}\right)^{\overline{N}\,\overline{M}}_{cd,ab} = \gamma \epsilon^{EFGH\overline{N}} \pi_{(c|EF} \left(F^{-1}\right)_{d)GH,(a|AB} \pi_{b)CD} \epsilon^{ABCD\overline{M}} \tag{5.13}$$

for some constant γ, where

$$\left(F^{-1}\right)_{aIJ,bKL} := \frac{s}{4(D-1)} \pi_{aAC} \pi_{bBD} \left(\pi^{cEC} \pi_{cE}{}^{D} - s\eta^{CD}\right) \left(\eta^{AB} \eta^{K[I} \eta^{J]L} - 2\eta^{LA} \eta^{B[I} \eta^{J]K}\right) \tag{5.14}$$

only depends on π^{aIJ} and reduces to the correct expression on the simplicity constraint surface when contracted in the above equation. Insertion into $\tilde{A}$ yields

$$\gamma = \frac{1}{4(D-1)^2((D-3)!)^2} \tag{5.15}$$

when demanding that $\tilde{A}$ is independent of D, i.e. that the $\bar{K}^T_{aIJ}$ term is cancelled. Since all simplicity invariant phase space functions are arbitrary functions of $\tilde{A}_{aIJ}$ and π^{aIJ}, we have shown that the proposed expression for $\left(F^{-1}\right)^{\overline{N}\,\overline{M}}_{cd,ab}$ yields the desired results. This can of course also be obtained by direct inversion of the projected version of the matrix F. This way we obtain a connection formulation for gravity in $D + 1 > 3$ without second class constraints. Notice however that the observables (with respect to the simplicity constraint) $(\tilde{A}, \pi)$ have complicated Poisson brackets, only the brackets of the canonical pair (A, π) are simple, therefore suggesting a Dirac quantisation approach (quantisation at the kinematical level).

Let us summarise and compare with the connection formulation in $D + 1 = 4$:

1.
On the surface where the simplicity constraint vanishes, $\pi^{aIJ} = 2n^{[I} E^{a|J]}$, we can describe the situation more explicitly. From the above formula it is obvious that both $n^I \delta^S A_{aIJ} = 0$ and $E^{aI} \delta^S A_{aIJ} = 0$, since we always may choose $c^{\overline{M}}_{[ab]} = 0$. Thus, when decomposing the connection $A_{aIJ} = \Gamma_{aIJ} + \bar{K}_{aIJ} + 2n_{[I} K_{a|J]}$ into hybrid spin connection and rotational (i.e. transversal) and boost (i.e longitudinal) components of hybrid contorsion K_{aIJ}, we find that the simplicity constraint generates on-shell gauge transformations of the trace free part of the rotational (transversal) components of the $SO(1, D)$ hybrid contorsion $\bar{K}^T_{aIJ}$. As we have seen in equation (2.71), the remaining trace component of the rotational part $E^{aI} \bar{K}_{aIJ}$ is proportional to the boost part of the Gauß constraint and vanishes if $n_I G^{IJ} = 0$ holds. In total, we find that observables in this connection theory may not depend on the value of the rotational components of the $SO(1, D)$ hybrid contorsion at all. The whole physical information contained in the connection is encoded in the boost components of the contorsion, which becomes conjugate to the vielbein after solving the simplicity constraint. Therefore, when removing the boost gauge freedom by choosing the time gauge, there is no physical information left in the $SO(D)$ connection.
2.
In $D + 1 = 4$, this formulation therefore differs from the formulation in terms of real Ashtekar variables considered in [21], which remains a connection formulation also after imposing the time gauge. This is achieved by mixing boost and rotational components of the connection using the total antisymmetric tensor, i.e. $^{(\beta)}A_{ajk} = A_{ajk} - \beta\epsilon_{0ijk} A_{a0i}$, to "rotate" physical degrees of freedom into the rotational components of the connection. Thus, this procedure exploits a peculiarity of dimension $D + 1 = 4$, and therefore is not possible in any other dimension. In $D + 1 = 4$ it is possible to arrive at this connection formulation in terms of $^{(\beta)}A$ also by enlarging the ADM phase space instead of starting from the Holst action [21]. This turns out to be true also for the new connection formulation derived above as we have shown in our companion paper

[1]. Following this route allows for the introduction of a free parameter similar to the Barbero-Immirzi parameter, but the transformation made to obtain the connection is very different in nature since there is no mixing of boost and rotational parts.

3.

The internal gauge group in the case of Lorentzian external signature is $SO(1, D)$, for which the techniques developed in [22, 23] are not available. It is therefore desirable to obtain a formulation with a compact internal gauge group as is the case in $D + 1 = 4$. The idea, similar to the situation in $D + 1 = 4$, is to add a correction term to the Euclidean Hamiltonian constraint which changes the external signature after reduction to ADM variables. This has been achieved in [1] where we derived the formulation presented in this section and solved the task to obtain a compact gauge group formulation even for Lorentzian signature. In contrast to $D+1 = 4$ this is not achieved by choosing a gauge in a formulation coming from a Lorentzian gauge theory action but by simply choosing a compact gauge group extension of the Lorentzian ADM formulation. This formulation therefore cannot be obtained from a Lorentzian connection formulation.

6   Conclusions

We have shown that General Relativity in dimensions $D+1 \geq 3$ can be written as a gauge theory with either $SO(D + 1)$ or $SO(1, D)$ as the internal gauge group. We derived this result both from a canonical analysis of the Palatini Lagrangian in this paper and from an extension of the ADM phase space using Hamiltonian methods in [1]. In both cases, the simplicity constraints in addition to the Gauß constraints played a key role in order to match the correct number of degrees of freedom. However, there are two differences:

First, the Lagrangian methods are restricted to a matching of spacetime and internal signature, that is, we get necessarily $SO(1, D)$ for Lorentzian General Relativity. There is no room for an Immirzi like parameter (without a topological term in the Lagrangian which we did not consider here since in higher dimensions such a term is not known in explicit form depending on D). For the Hamiltonian methods there is no such restriction, in particular we can have $SO(D + 1)$ for Lorentzian General Relativity and arbitrary Immirzi like Parameter. The price to pay is that this connection formulation does not have a Lagrangian origin, on the other hand, Loop Quantum Gravity methods are available for quantisation because the gauge group is compact which is a huge advantage.

Secondly, as is well known, the Palatini action gives rise to second class constraints and an associated non trivial Dirac bracket with respect to which the connection is not commuting (see however [6, 24]) thus forbidding a connection formulation (the connection cannot act as a multiplication operator in the quantum theory) [25]. On the other hand the Hamiltonian extension of the ADM phase space does not know about any Dirac bracket and starts with a connection that is Poisson commuting and with only first class constraints.

This second observation raises the questions how these two descriptions can possibly be simultaneously valid? The answer to this puzzle is that (for matching spacetime and internal signature) the Hamiltonian description can be obtained from the Lagrangian description by gauge unfixing the second class theory into the first class theory. The secondary second class constraint partner to the primary second class simplicity constraint thereby gets removed and gets built into the Hamiltonian constraint in order to achieve the first class property. From the Hamiltonian perspective this secondary second class constraint precisely arises as a counterterm that removes a piece in the "natural", primary Hamiltonian constraint (expressed in terms of curvature) absent in the correct first class Hamiltonian constraint.

The description at which we arrived in this paper is therefore very different from the platform from which Lorentzian Loop Quantum Gravity in $D + 1 = 4$ starts. There is no time gauge

and therefore the gauge group is SO(4) rather than SU(2). The price to pay are the additional simplicity constraints and a more complicated Hamiltonian constraint. For $D + 1 = 4$ this description is therefore more complicated than the standard description and thus less useful. However, for $D + 1 > 4$ we do not know of any other gauge theory formulation of General Relativity with 1. compact gauge group and 2. only first class constraints and 3. standard symplectic structure with Poisson commuting connections. We will address the differences in the quantum theories between the present formulation and standard LQG in a future publication. We expect that there will be close relations with the way the simplicity constraint is treated in the EPRL Spin Foam model [26, 27, 28, 29, 30] as well as with projected spin networks [31]. In fact, it is very likely that the present formulation will present the bridge between the kinematical Hilbert space of standard LQG and the boundary Hilbert space of current Spin Foam models. The reason is that the simplicity constraints will require the SO(4) representations and intertwiners to be "simple", i.e. there will be some relation between the left and right handed spin labeling an irreducible SO(4) representation precisely as in current Spin Foam models so that effectively one deals with an SU(2) theory as in standard LQG. What was missing so far is the interpretation of the connection in terms of which that effective SU(2) theory is formulated as compared to the Ashtekar-Barbero connection. Since Spin Foam models start from the Palatini (or equivalently the Plebanski) action as in our case, it is very suggestive to assume that the Spin Foam boundary Hilbert space is formulated in terms of our formulation with the twist of the gauge unfixing procedure as otherwise one could not have a connection representation.

In our companion papers, we will extend the methods developed for 3+1 Loop Quantum Gravity to our $D + 1$ theory and show that it can be quantised analogously [32], extend the treatment to standard matter [33], discuss possible solution strategies for the simplicity constraint [34], and finally develop tools allowing the loop quantisation of a large class of Supergravity theories [35, 36].

A The Lie algebra so$(1, D)$

In this appendix, we generalise a so$(1, 3)$ structure constant identity given in [8]. In our notation,

$$(T_{AB})^I{}_J = \eta^I{}_{[A}\eta_{B]J} \tag{A.1}$$

denotes the generators of so$(1, D)$ in the fundamental representation. The *antisymmetric* index pair AB labels the $D(D+1)/2$ generators, I and J are matrix indices, also antisymmetric. In the following, a generator T_{AB} will always have a label, but the matrix indices will be mostly suppressed. Insertion of the definitions shows that the generators satisfy the usual Lorentz algebra

$$[T_{AB}, T_{CD}]^I{}_J = 2\eta_{A][C}\left(T_{D][B}\right)^I{}_J =: f_{AB,CD,}{}^{EF}\left(T_{EF}\right)^I{}_J \tag{A.2}$$

with

$$f_{AB,CD,EF} = -2\eta_{B][C}\eta_{D][E}\eta_{F][A} = -2\mathrm{Tr}\left(T_{AB}T_{CD}T_{EF}\right). \tag{A.3}$$

We further define the Cartan-Killing metric

$$q_{IJ,KL} = \eta_{I[K}\eta_{L]J} \ \Leftrightarrow\ -\mathrm{Tr}\left(T_{AB}T_{CD}\right) = (T_{AB})^{IJ} q_{IJ,KL} (T_{CD})^{KL} \tag{A.4}$$

and the object

$$(q^{*\overline{M}})_{IJ,KL} = \frac{1}{2}\epsilon_{IJKL}{}^{\overline{M}} \tag{A.5}$$

defining the dual

$$T_{AB}^{*\overline{M}} = (q^{*\overline{M}})_{AB,}{}^{CD}T_{CD} \tag{A.6}$$

generators. We note that self-duality is a concept reserved for $3 + 1$ dimensions.

These definitions lead us to the main result of this appendix:

$$f_{AB,CD,IJ}f_{EF,GH,}{}^{IJ} = \frac{1}{2}q_{AB,EF}q_{GH,CD} - \frac{\eta_{\overline{MN}}}{2(D-3)!}(q^{*\overline{M}})_{AB,EF}(q^{*\overline{N}})_{GH,CD} - EF \leftrightarrow GH. \tag{A.7}$$

It can be proven by carefully inserting the definitions and writing out explicitly each term.

B Introduction of the Barbero-Immirzi Parameter in $3 + 1$ Dimensions

In the special case of $3 + 1$ dimensions, it is straight forward to introduce the Barbero-Immirzi parameter γ: Perform the canonical analysis of the Holst action without time gauge (given in, e.g., [21]) and use the method of gauge unfixing in complete analogy to the treatment in this paper. The matrix F, which has to be inverted for gauge unfixing, is very simple in this case, given by $F^{ab,cd} = 2q^2 q^{a[b}q^{c]d}$ [21]. Following these lines, we obtain a connection formulation with (first class) quadratic simplicity constraints and gauge group SO$(3, 1)$, which reduces to the Ashtekar-Barbero formulation after solving the Simplicity and boost Gauß constraints. (Another straight-forward calculations shows that the same procedure gives a possible solution to the open issue (i) in [37]). For quantisation purposes, it again would be nice to be able to work with the compact gauge group SO(4) instead of the Lorentz group. Moreover, the linear simplicity constraint, which is introduced in any dimension $D + 1$ in [35], is favoured in $3 + 1$ dimensions, since the quadratic simplicity constraint allows for unphysical solutions, usually called the topological sector. Last but not least, a formulation with Barbero-Immirzi parameter and linear simplicity constraints maximally mimics Spin Foams. In this appendix, we will show by extending ADM phase space that both, the formulations with flipped signature and with linear simplicity constraints, exist.

B.1 Flipped Signature and Quadratic Simplicity Constraints

We start with the formulation given in [35] with variables $\{K_{aIJ}, \pi^{bKL}\}$. In $3 + 1$ dimensions, we can introduce for real Barbero-Immirzi parameter γ, $\gamma^2 \neq \zeta$,

$$(\overset{(\gamma)}{\mathcal{M}})^{IJ}{}_{KL} := \eta_K^{[I}\eta_L^{J]} + \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL} \quad \text{and} \quad (\overset{(\gamma)}{\mathcal{M}}{}^{-1})^{IJ}{}_{KL} := \frac{\gamma^2}{\gamma^2 - \zeta}\left(\eta_K^{[I}\eta_L^{J]} - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}\right). \tag{B.1}$$

It is obvious that the transformation to the canonical pair of variables defined by

$$\underset{(\gamma,\beta)}{K}{}_{aIJ} := \beta(\overset{(\gamma)}{\mathcal{M}}{}^{-1})_{IJ}{}^{KL}K_{aKL} := \beta\overset{(\gamma)}{\mathcal{M}}{}^{-1}K_{aIJ}, \tag{B.2}$$

$$\overset{(\gamma,\beta)}{\pi}{}_{aIJ} := \frac{1}{\beta}(\overset{(\gamma)}{\mathcal{M}})^{IJ}{}_{KL}\pi^{aKL} := \frac{1}{\beta}\overset{(\gamma)}{\mathcal{M}}\pi^{aIJ}, \tag{B.3}$$

is canonical. Note that we introduced a second free parameter β coming from a constant rescaling, which already appeared in [1]. To obtain a connection formulation, we the would like to use the canonical pair of variables given by $\left\{A_{aIJ} := (\Gamma + \underset{(\gamma,\beta)}{K})_{aIJ}, \overset{(\gamma,\beta)}{\pi}{}^{aIJ}\right\}$. We will prove in the following that these variables are indeed a valid extension of the ADM phase space.

For later convenience, we introduce the notations

$$\overset{(\gamma,\beta)}{\pi}{}_{aIJ} := \frac{1}{q}q_{ab}\overset{(\gamma,\beta)}{\pi}{}^b{}_{IJ},$$

$$\pi^{aIJ} := \beta \cdot (\overset{(\gamma)}{\mathcal{M}}{}^{-1})^{IJ}{}_{KL}\overset{(\gamma,\beta)}{\pi}{}^{aKL},$$

$$\underset{(\gamma,\beta)}{\pi}{}^{aIJ} := \beta^2 \cdot (\overset{(\gamma)}{\mathcal{M}}{}^{-1})^{IJ}{}_{KL}(\overset{(\gamma)}{\mathcal{M}}{}^{-1})^{KL}{}_{MN}\overset{(\gamma,\beta)}{\pi}{}^{aMN},$$

where in the first line $\frac{1}{q}q_{ab}$ has to be understood as a function of $\overset{(\gamma,\beta)}{\pi}{}^{aIJ}$ as given in (B.5). Moreover, note that in $3 + 1$ dimensions the expression for the hybrid spin connection (cf. [1]) can be simplified to

$$\Gamma_{aIJ} := \zeta\pi_{b[I|K}\nabla_a\pi^b{}_{J]}{}^K := \zeta\left(\pi_{b[I|K}\partial_a\pi^b{}_{J]}{}^K + \pi_{b[I|K}\Gamma^b_{ac}\pi^c{}_{J]}{}^K\right), \tag{B.4}$$

where Γ^b_{ac} denotes the Christoffel symbols and therefore, ∇_a is the covariant derivative annihilating q_{ab}. The ADM variables, expressed in terms of A_{aIJ} and $\overset{(\gamma,\beta)}{\pi}{}^{aIJ}$, are given by

$$2\zeta q q^{ab} := \pi^{aIJ}\pi^b{}_{IJ} = \overset{(\gamma,\beta)}{\pi}{}^{aIJ}\underset{(\gamma,\beta)}{\pi}{}^b{}_{IJ}, \tag{B.5}$$

$$K^{ab} := -\frac{s}{4\sqrt{q}}\overset{(\gamma,\beta)}{\pi}{}^{(b|IJ}q^{a)c}(A-\Gamma)_{cIJ}, \tag{B.6}$$

$$P^{ab} = -s\sqrt{q}\left(K^{ab} - q^{ab}K_c{}^c\right) = \frac{1}{4}\left(q^{c(a}\overset{(\gamma,\beta)}{\pi}{}^{b)IJ} - q^{ab}\overset{(\gamma,\beta)}{\pi}{}^{cIJ}\right)(A-\Gamma)_{cIJ}. \tag{B.7}$$

Using the above equations (B.5, B.7), we find for the ADM constraints

$$\mathcal{H}_a = -2q_{ac}\nabla_b P^{bc}$$
$$= -\frac{1}{2}\nabla_b\left((A-\Gamma)_{aIJ}\overset{(\gamma,\beta)}{\pi}{}^{bIJ} - \delta_a^b(A-\Gamma)_{cIJ}\overset{(\gamma,\beta)}{\pi}{}^{cIJ}\right), \tag{B.8}$$

$$\mathcal{H} = -\left[\frac{s}{\sqrt{q}}\left(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd}\right)P^{ab}P^{cd} + \sqrt{q}R\right]$$
$$= -\frac{s}{8\sqrt{q}}\left(\overset{(\gamma,\beta)}{\pi}{}^{[a|IJ}\overset{(\gamma,\beta)}{\pi}{}^{b]KL}(A-\Gamma)_{bIJ}(A-\Gamma)_{aKL}\right) - \sqrt{q}R(\overset{(\gamma,\beta)}{\pi}). \tag{B.9}$$

In order to have the right number of physical degrees of freedom, the Gauß and quadratic simplicity constraints

$$G^{IJ} \quad := \quad D_a \overset{(\gamma,\beta)}{\pi}{}^{aIJ} \approx 2\,(A-\Gamma)_a{}^{[I}{}_K \overset{(\gamma,\beta)}{\pi}{}^{aK|J]}, \tag{B.10}$$

$$S^{ab} \quad := \quad \frac{1}{4}\epsilon^{IJKL}\pi^a_{IJ}\pi^b_{KL}, \tag{B.11}$$

are introduced. In $3+1$ dimensions, the quadratic simplicity constraint has additional solutions which lead to a theory not corresponding to General Relativity. We will exclude this sector by hand. In section B.2, we will introduce the linear version of the constraint like in [35], which does not have this additional solution sector. Using the (non-vanishing) Poisson brackets

$$\left\{A_{aIJ}(x), \overset{(\gamma,\beta)}{\pi}{}^{bKL}(y)\right\} := 2\delta^b_a\delta^K_{[I}\delta^L_{J]}\delta^3(x-y), \tag{B.12}$$

one can check that Γ_{aIJ} given in (B.4) transforms as a connection under the action of the Gauß constraint, and therefore (B.6, B.7, B.8, B.9) are invariant under gauge transformations. Since the matrix $\overset{(\gamma)}{\mathcal{M}}$ is built from intertwiners, (B.5) and (B.11) are gauge invariant by inspection. Simplicity invariance of (B.7, B.8, B.9) follows from

$$\left\{K^{ab}(x), S^{cd}(y)\right\} \quad = \quad -\frac{s}{4\sqrt{q}}\overset{(\gamma,\beta)}{\pi}{}^{(b|IJ}q^{a)e}(x)\left\{A_{eIJ}(x), \frac{1}{4}\epsilon^{KLMN}\pi^c_{KL}(y)\pi^d_{MN}(y)\right\}$$

$$= \quad -\frac{s}{4\sqrt{q}}\epsilon^{IJKL}\beta\overset{(\gamma)}{\mathcal{M}}{}^{-1}\overset{(\gamma,\beta)}{\pi}{}^{(b|IJ}q^{a)(c}\pi^{d)}_{KL}\,\delta^3(x-y)$$

$$= \quad -\frac{s}{2\sqrt{q}}\left(S^{b(d}q^{c)a} + S^{a(d}q^{c)b}\right)\delta^3(x-y) \approx 0. \tag{B.13}$$

What remains to be checked is if the ADM Poisson brackets are reproduced on the new phase space, which will by construction imply that the constraint algebra closes. The following Poisson brackets will be helpful in the sequel:

$$\left\{A_{aIJ}(x), q^{bc}(y)\right\} \quad = \quad \zeta\delta^3(x-y)\left(\frac{2}{q}\delta^{(b}_a\underset{(\gamma,\beta)}{\pi}{}^{c)}{}_{IJ} - q^{bc}\underset{(\gamma,\beta)}{\pi}{}_{aIJ}\right), \tag{B.14}$$

$$\left\{A_{aIJ}(x), q_{bc}(y)\right\} \quad = \quad -\zeta\delta^3(x-y)\left(2q_{a(b}\underset{(\gamma,\beta)}{\pi}{}_{c)IJ} - q_{bc}\underset{(\gamma,\beta)}{\pi}{}_{aIJ}\right), \tag{B.15}$$

$$\left\{A_{aIJ}(x), q(y)\right\} \quad = \quad \zeta\delta^3(x-y)q\cdot\underset{(\gamma,\beta)}{\pi}{}_{aIJ}, \tag{B.16}$$

$$\overset{(\gamma,\beta)}{\pi}{}^{cIJ}(x)\left\{A_{aIJ}(x), \pi_b{}^{KL}(y)\right\} \quad \approx \quad \delta^3(x-y)\left(\frac{2}{q}q_{ab}\bar{\eta}^K_{[M}\bar{\eta}^L_{N]} - \zeta\pi_a{}^{KL}\pi_{bMN}\right)\beta(\overset{(\gamma)}{\mathcal{M}}{}^{-1})^{MN}{}_{IJ}\overset{(\gamma,\beta)}{\pi}{}^{cIJ}$$

$$\approx \quad -2\delta^3(x-y)\pi_a{}^{KL}\delta^c_b. \tag{B.17}$$

The brackets

$$\left\{q_{ab}, q_{cd}\right\}_{(A, \overset{(\gamma,\beta)}{\pi})} = 0 \quad \text{and} \quad \left\{q_{ab}, P^{cd}\right\}_{(A, \overset{(\gamma,\beta)}{\pi})} = \delta^c_{(a}\delta^d_{b)} \tag{B.18}$$

are easily verified. The remaining Poisson bracket

$$\left\{ P^{ab}[A_{ab}], P^{cd}[B_{cd}] \right\}_{(A,\,\overset{(\gamma,\beta)}{\pi})}$$

$$= \int_\sigma d^3x \int_\sigma d^3y \left[\left(\frac{1}{2} A_{ab} q^{a[e} \overset{(\gamma,\beta)}{\pi}{}^{b]IJ} \right)(x) \left\{ A_{eIJ}(x), \left(q^{c[f} \overset{(\gamma,\beta)}{\pi}{}^{d]KL} \right)(y) \right\} \left(\frac{1}{2} B_{cd}(A - \Gamma)_{fKL} \right)(y) \right]$$

$$- [A \leftrightarrow B] \tag{B.19}$$

$$+ \int_\sigma d^3x \int_\sigma d^3y \left[\left(\frac{1}{2} A_{ab} q^{a[e} \overset{(\gamma,\beta)}{\pi}{}^{b]IJ} \right)(x) \left\{ A_{eIJ}(x), (-\Gamma_{fKL})(y) \right\} \left(\frac{1}{2} B_{cd} q^{c[f} \overset{(\gamma,\beta)}{\pi}{}^{d]KL} \right)(y) \right]$$

$$- [A \leftrightarrow B] \tag{B.20}$$

is much harder and therefore will be discussed in more detail. Here, A_{ab} and B_{cd} are test fields of compact support, which we can choose symmetric w.l.o.g., since P^{ab} is symmetric by definition. The second line (B.19) and third line (B.20) of the above equation vanish independently. For (B.19), we find using (B.14)

$$(B.19) = \ldots = \frac{1}{4} \int_\sigma d^3x \; A_{ab} B_{cd} q^{ac} \overset{(\gamma,\beta)}{\pi}{}^{[d|IJ} q^{b]e} (A - \Gamma)_{eIJ} \propto \bar{G}^{IJ}[\ldots],$$

which vanishes if the (rotational part of the) Gauß constraint holds. Before we proceed, we define $\alpha_f{}^e := \frac{1}{4} A_{ab} \left(q^{e(a} \delta_f^{b)} - q^{ab} \delta_f^e \right)$ and $\beta_h{}^g := \frac{1}{4} B_{cd} \left(q^{g(c} \delta_h^{d)} - q^{cd} \delta_h^g \right)$ and check that $\alpha_{[ef]} = 0 = \beta_{[gh]}$. Then, we find for the third line (skipping "$-[A \leftrightarrow B]$" for a moment)

$$(B.20) = \int_\sigma d^3x \int_\sigma d^3y \; \alpha_f{}^e \overset{(\gamma,\beta)}{\pi}{}^{fIJ}(x) \left\{ A_{eIJ}(x), (-\zeta) \pi_{bKM} \left(\nabla_g \pi^b{}_L{}^M \right)(y) \right\} \beta_h{}^g \overset{(\gamma,\beta)}{\pi}{}^{hKL}(y)$$

$$= -\zeta \int_\sigma d^3x \int_\sigma d^3y \; \alpha_f{}^e \overset{(\gamma,\beta)}{\pi}{}^{fIJ}(x) \Big[\left\{ A_{eIJ}(x), \pi_{bKM}(y) \right\} \left(\nabla_g \pi^b{}_L{}^M \right) \beta_h{}^g \overset{(\gamma,\beta)}{\pi}{}^{hKL}(y)$$

$$- \left\{ A_{eIJ}(x), \pi^b{}_L{}^M(y) \right\} \nabla_g \left(\pi_{bKM} \beta_h{}^g \overset{(\gamma,\beta)}{\pi}{}^{hKL} \right)(y) \Big] \tag{B.21}$$

$$- \zeta \int_\sigma d^3x \int_\sigma d^3y \; \alpha_f{}^e \overset{(\gamma,\beta)}{\pi}{}^{fIJ}(x) \left\{ A_{eIJ}(x), \Gamma^b{}_{ga}(y) \right\} \pi_{bKM} \pi^a{}_L{}^M \beta_h{}^g \overset{(\gamma,\beta)}{\pi}{}^{hKL}(y). \tag{B.22}$$

Again, (B.21) and (B.22) vanish separately. For (B.21), we find after a few steps using (B.17)

$$(B.21) = \ldots = 2\zeta \int_\sigma d^3x \; \alpha_f{}^e \nabla_g \left(\beta_h{}^g \cdot \mathrm{Tr} \left(\pi_e \; \pi^f \overset{(\gamma,\beta)}{\pi}{}^h \right) \right)$$

$$= 2\zeta \int_\sigma d^3x \; \alpha_{fe} \left[\nabla_g \left(\frac{1}{q} \beta_h{}^g \cdot \mathrm{Tr} \left(\pi^{[e} \; \pi^{f]} \overset{(\gamma,\beta)}{\pi}{}^h \right) \right) - (\nabla_g q^{ea}) \frac{1}{q} \beta_h{}^g \cdot \mathrm{Tr} \left(\pi_a \; \pi^f \overset{(\gamma,\beta)}{\pi}{}^h \right) \right] \approx 0,$$

which vanishes since the trace $\mathrm{Tr}(abc) := a^I{}_J b^J{}_K c^K{}_I$ of antisymmetric matrices a, b, c is antisymmetric when exchanging two matrices while α_{ab} is symmetric and $\nabla_a q_{bc} \approx 0$ by construction.

The remaining part (B.22) can be rewritten as

$$
\begin{aligned}
(B.22) =& - \zeta \int_\sigma d^3y \ \left\{ P^{ab}[A_{ab}], \Gamma^d_{gc}(y) \right\} \beta_h{}^g \ \mathrm{Tr}\left(\pi_d \ \pi^c {}^{(\gamma,\beta)}_{\pi}{}^h \right)(y) \\
=& - \zeta \int_\sigma d^3y \ \left[\left\{ P^{ab}[A_{ab}], q^{de}(y) \right\} q_{di}\Gamma_{gc\ e} + \left\{ P^{ab}[A_{ab}], \Gamma_{gc\ i}(y) \right\} \right] \frac{1}{q}\beta_h{}^g \ \mathrm{Tr}\left(\pi^i \ \pi^c {}^{(\gamma,\beta)}_{\pi}{}^h \right)(y) \\
=& - \zeta \int_\sigma d^3y \ \left[-\left\{ P^{ab}[A_{ab}], q_{jk}(y) \right\} \delta^{(j}_i \Gamma^{k)}_{gc} + \left\{ P^{ab}[A_{ab}], \partial_c q_{ig}(y) \right\} \right] \frac{1}{q}\beta_h{}^g \ \mathrm{Tr}\left(\pi^i \ \pi^c {}^{(\gamma,\beta)}_{\pi}{}^h \right)(y) \\
\approx& - \zeta \int_\sigma d^3y \ \left[A_{ab}\Gamma^b_{gc} + A_{ag}\vec{\partial}_c \right] \frac{1}{q}\beta_h{}^g \ \mathrm{Tr}\left(\pi^a \ \pi^c {}^{(\gamma,\beta)}_{\pi}{}^h \right) \\
=& - \zeta \int_\sigma d^3y \ A_{ab}\nabla_c \left(\frac{1}{q}\beta_h{}^b \ \mathrm{Tr}\left(\pi^a \ \pi^c {}^{(\gamma,\beta)}_{\pi}{}_h \right) \right) \\
=& \zeta \int_\sigma d^3y \ (\nabla_c A_{ab}) \frac{1}{4q}B_{ed}\left(q^{b(e}\delta^{d)}_h - q^{cd}\delta^b_h \right) \ \mathrm{Tr}\left(\pi^a {}^{(\gamma,\beta)}_{\pi}{}^c \ \pi^h \right) \\
=& \zeta \int_\sigma d^3y \ (\nabla_c A_{ab}) \frac{1}{4q}B^b{}_d \ \mathrm{Tr}\left(\pi^a {}^{(\gamma,\beta)}_{\pi}{}^c \ \pi^d \right).
\end{aligned}
$$

In the first step, we just reassembled the terms on the left hand side of the Poisson bracket, in the second we used the definition of the Christoffel symbol, in the third the formula for the derivative of the inverse matrix and antisymmetry of the trace in $(i \leftrightarrow c)$. In the fourth line we used the already known brackets of the metric q_{ab} and its conjugate momentum P^{cd}. Note that the density weight and index structure is such that the terms in the fourth line can be reassembled in a covariant derivate. In the sixth line the definition of $\beta_h{}^g$ is inserted, we integrated by parts and we used that $\mathrm{Tr}(ab{}^{(\gamma,\beta)}_{c}) = \mathrm{Tr}(a \ {}^{(\gamma,\beta)}_{b} c)$ (this trace property can be shown using the definition of the matrices ${}^{(\gamma)}_{\mathcal{M}}$). Thus we find that the second summand appearing in the definition of $\beta_h{}^g$ vanishes due to antisymmetry of the trace in the indices $(a \leftrightarrow b)$. If we now restore the antisymmetry in the test fields $(A \leftrightarrow B)$, we obtain

$$
\begin{aligned}
(B.22) \ =& \ \frac{\zeta}{4} \int_\sigma d^3y \ \left[(\nabla_c A_{ab}) B^b{}_d - (\nabla_c B_{ab}) A^b{}_d \right] \frac{1}{q}\mathrm{Tr}\left(\pi^a {}^{(\gamma,\beta)}_{\pi}{}^c \ \pi^d \right) \\
\approx& \ \frac{1}{4\beta\gamma} \int_\sigma d^3y \ \epsilon^{cda} \ \nabla_c \left(A_{ab}B^b{}_d \right) \\
=& \ \frac{1}{4\beta\gamma} \int_\sigma d^3y \ \partial_c \left(\epsilon^{cda} A_{ab}B^b{}_d \right) = 0,
\end{aligned}
$$

where we used the simplicity constraint in the second line and then dropped a surface term. We leave the case where σ has a boundary for further research. This furnishes the proof of the validity of the formulation.

B.2 Linear Simplicity Constraints

As will be demonstrated in [35], we can extend the result to the case of a linear simplicity (and normalisation) constraints when introducing additional phase space degrees of freedom $\{N^I, P_J\}$. The theory with linear simplicity constraint has the non-vanishing Poisson brackets (B.12) as well as

$$
\left\{ N^I(x), P_J(y) \right\} = \delta^I_J \delta^3(x - y), \tag{B.23}
$$

and the constraints are given by

$$\mathcal{H}_a \;\; := \;\; \frac{1}{2}\overset{(\gamma,\beta)}{\pi}{}^{bIJ}\partial_a A_{bIJ} - \frac{1}{2}\partial_b\left(\overset{(\gamma,\beta)}{\pi}{}^{bIJ}A_{aIJ}\right) + P_I\partial_a N^I, \tag{B.24}$$

$$\mathcal{H} \;\; := \;\; -\frac{s}{8\sqrt{q}}\left(\overset{(\gamma,\beta)}{\pi}{}^{[a|IJ}\overset{(\gamma,\beta)}{\pi}{}^{b]KL}\left(A-\Gamma\right)_{bIJ}\left(A-\Gamma\right)_{aKL}\right) - \sqrt{q}R(\overset{(\gamma,\beta)}{\pi}), \tag{B.25}$$

$$G^{IJ} \;\; := \;\; D_a\overset{(\gamma,\beta)}{\pi}{}^{aIJ} + 2P^{[I}N^{J]}, \tag{B.26}$$

$$S^{aI} \;\; := \;\; \epsilon^{IJKL}N_J\pi^a_{KL}, \tag{B.27}$$

$$\mathcal{N} \;\; := \;\; N^I N_I - \zeta. \tag{B.28}$$

Note that for the proof that the ADM Poisson brackets are reproduced on the extended phase space in section B.1 we just needed $\bar{G}^{IJ} \approx 0$ and $S^{ab} \approx 0$. Since the solutions to the quadratic and linear simplicity (and normalisation) constraints coincide[6] and on shell, we did not change the rotational parts of the Gauß constraint, this implies that the ADM brackets are also reproduced in the case at hand if we express q_{ab} and P^{cd} again as given in equations (B.5, B.7). Therefore, we just need to check if the constraint algebra closes. The change in the definition of G^{IJ} and $\mathcal{H}_a$ is needed since the newly introduced constraints S^{aI} and $\mathcal{N}$ have to transform nicely under spatial diffeomorphisms and gauge transformations for the constraint algebra to close. Since the hybrid spin connection (B.4) transforms covariantly under spatial diffeomorphisms by inspection and we already noted that it transforms as a connection under gauge transformations, also the Hamilton constraint transforms covariantly under the action of G^{IJ} and $\mathcal{H}_a$. The only non-trivial Poisson bracket is the one of two Hamilton constraints. But since the ADM brackets are reproduced, we know that the result of this Poisson bracket on-shell is the ADM spatial diffeomorphism constraint. Therefore, we just need to show that $\mathcal{H}_a$ reduces correctly. This will be shown in the following section and completes the proof of the validity of this formulation.

B.3   Solving the Linear Simplicity and Normalisation Constraints

It is instructive to solve first the simplicity and then the boost Gauß constraint ("time gauge"), which will in the first step lead to a formulation similar to the one given in [21] and then to the formulation in Ashtekar-Barbero variables. We will treat the case with linear simplicity constraints, since in $3+1$ dimensions, the linear constraint has the additional advantage that its only solution is General Relativity, while the quadratic simplicity constraint also allows for the topological solution.

   The solution to the linear simplicity and normalisation constraint is given by $\pi^{aIJ} = 2n^{[I}E^{a|J]}$ (cf. [35]) and therefore

$$\overset{(\gamma,\beta)}{\pi}{}^{aIJ} = \frac{1}{\beta}\left(2n^{[I}E^{a|J]} + \frac{1}{\gamma}\epsilon^{IJKL}n_K E_{aL}\right). \tag{B.29}$$

For the connection, we make the Ansatz $A_{aIJ} = \Gamma_{aIJ} + \underset{(\gamma,\beta)}{K}{}_{aIJ}$. Using the results of [8, 1, 35],

---

[6]Note that in the considerations in section B.1, we neglected the topological sector when solving the quadratic simplicity constraints.

the symplectic potential becomes

$$
\begin{aligned}
\frac{1}{2}\overset{(\gamma,\beta)aIJ}{\pi}\dot{A}_{aIJ} + P^I \dot{N}_I
&= \frac{1}{2}\overset{(\gamma,\beta)aIJ}{\pi}\dot{\Gamma}_{aIJ} + \frac{1}{2}\overset{(\gamma,\beta)aIJ}{\pi}\underset{(\gamma,\beta)}{\dot{K}}_{aIJ} + P^I \dot{N}_I \\
&= \frac{1}{2\beta\gamma}\epsilon^{IJKL}n_K E_{aL}\dot{\Gamma}_{aIJ} + n^{[I}E^{a|J]}\dot{K}_{aIJ} + P^I \dot{N}_I \\
&= -(n^{[I}\dot{E}^{a|J]})\left(K_{aIJ} + \frac{1}{2\beta\gamma}\epsilon_{IJ}{}^{KL}\Gamma_{aKL}\right) + P^I \dot{N}_I \\
&\approx -\dot{E}^{aJ}\left(n^I K_{aIJ} + \frac{1}{2\beta\gamma}\epsilon_{IJ}{}^{KL}n^I \Gamma_{aKL} + n_J P^I E_{aI}\right. \\
&\qquad\qquad \left. - n_J E_a^I E^{bK}\left(\bar{K}_{bIK} + \frac{1}{2\beta\gamma}\epsilon_{IK}{}^{LM}\Gamma_{bLM}\right)\right) \\
&=: \ E^{aJ}\dot{A}_{aJ}. \tag{B.30}
\end{aligned}
$$

In the next step we express the remaining constraints in terms of the new canonical variables. The reduction of the Gauß constraint yields

$$
\begin{aligned}
\frac{1}{2}\Lambda_{IJ}G^{IJ}
&= \frac{1}{2}\Lambda_{IJ}D_a \overset{(\gamma,\beta)aIJ}{\pi} + \Lambda_{IJ}P^I N^J \\
&\approx \Lambda_{IJ}\underset{(\gamma,\beta)}{K}{}^I{}_{aK}\overset{(\gamma,\beta)aKJ}{\pi} + \Lambda_{IJ}P^I N^J \\
&= \Lambda_{IJ}K^I_{aK}\pi^{aKJ} + \Lambda_{IJ}P^I N^J \\
&= \Lambda_{IJ}K^I_{aK}\left(n^K E^{aJ} - n^J E^{aK}\right) + \Lambda_{IJ}P^I N^J \\
&\approx -\Lambda_{IJ}E^{aJ}\left(n_K K_a^{KI} - n^I E_{aL}E^{bK}K_b^L{}_K + n^I P^K E_{aK}\right) \\
&= -\Lambda_{IJ}E^{aJ}\left(A_a^I - \frac{1}{2\beta\gamma}\left(\epsilon_M{}^{IKL}n^M\Gamma_{aKL} - n^I E_a^N E^{bK}\epsilon_{NK}{}^{LM}\Gamma_{bLM}\right)\right) \\
&= \Lambda_{IJ}\left(E^{aI}A_a^J + \frac{1}{2\beta\gamma}\epsilon^{IJKL}\partial_a\left(n_K E_L^a\right)\right), \tag{B.31}
\end{aligned}
$$

which after time gauge $n^I = \delta_0^I$ and solution of the boost part of the Gauß constraint obviously reproduces the SU(2) Gauß constraint of the Ashtekar-Barbero formulation. The diffeomorphism constraint becomes

$$
\begin{aligned}
\mathcal{H}_a &= \frac{1}{2}\overset{(\gamma,\beta)bIJ}{\pi}\partial_a A_{bIJ} - \frac{1}{2}\partial_b\left(\overset{(\gamma,\beta)bIJ}{\pi}A_{aIJ}\right) + P_I \partial_a N^I \approx \dots \\
&\approx E^{bI}\partial_a A_{bI} - \partial_b\left(E^{bI}A_{aI}\right). \tag{B.32}
\end{aligned}
$$

In order to complete the proof of the validity of the formulation with linear simplicity constraints of section B.2, we have to show that this constraint becomes the ADM spatial diffeomorphism constraint after solving the Gauß constraint. Alternatively, we can show that it coincides on-shell with the spatial diffeomorphism constraint of section B.1,

$$
\begin{aligned}
\mathcal{H}_a &= -\frac{1}{2}\nabla_b\left((A-\Gamma)_{aIJ}\overset{(\gamma,\beta)bIJ}{\pi} - \delta_a^b(A-\Gamma)_{cIJ}\overset{(\gamma,\beta)cIJ}{\pi}\right) \approx \dots \\
&\approx E^{bI}\partial_a A_{bI} - \partial_b\left(E^{bI}A_{aI}\right) + \frac{1}{2}\Gamma_{aKL}G^{KL}, \tag{B.33}
\end{aligned}
$$

which it does. For the above calculations, the following results may be helpful:

$$
\begin{aligned}
\bar{R}_{abIJ} &= E_I^c E_{dJ}R_{abc}{}^d, \tag{B.34} \\
R_{abIJ}n^I E^{bJ} &= 0, \tag{B.35} \\
\epsilon^{IJKL}R_{abIJ}n_K E_L^b &= 0. \tag{B.36}
\end{aligned}
$$

For the first line, expand an internal vector λ_I with $\lambda_I n^I = 0$ into $\lambda_I = e_I^a \lambda_a$, $\lambda_a = e_a^I \lambda_I$, use $D_a e_b^J = 0$ and compare $[D_a, D_b]\lambda_J$ with $[D_a, D_b]\lambda_c$. The second line follows from the fact that Γ_{aIJ} annihilates $n^I = n^I(E)$ and the third line is a consequence of the first line and the algebraic Bianci identity. Finally, the Hamilton constraint gives

$$\mathcal{H} = -\frac{s}{8\sqrt{q}} \left({}^{(\gamma,\beta)}\pi^{[a|IJ}{}^{(\gamma,\beta)}\pi^{b]KL} (A-\Gamma)_{bIJ} (A-\Gamma)_{aKL} \right) - \sqrt{q} R({}^{(\gamma,\beta)}\pi) \tag{B.37}$$

$$\approx -\frac{s}{2\sqrt{q}} E^{[a|I} E^{b]J} \left(A_{bI} - \frac{1}{2\beta\gamma}\epsilon_{MI}{}^{KL} n^M \Gamma_{bKL} \right) \left(A_{aJ} - \frac{1}{2\beta\gamma}\epsilon_{NJ}{}^{AB} n^N \Gamma_{aAB} \right) - \sqrt{q} R(E).$$

We leave the task of exactly relating this formulation to the similar one given in [21] for future research.

B.4   Time Gauge

Choosing time gauge $n^I = \delta_0^I \Leftrightarrow E^{a0} = 0$ always implies solving its second class partner, the boost part $G^{0i} = -E^{ai} A_a^0$ of the Gauß constraint $(i,j,\ldots \in \{1,2,3\})$. In order to obtain expressions which can be easily compared to results in the literature (e.g. [3]), we introduce the rescaled[7] variables $A_{ai} \to A'_{ai} := \frac{\tilde\gamma}{2} A_{ai}$ and $E^{bj} \to E'^{bj} := \frac{2}{\tilde\gamma} E^{bj}$, where $\tilde\gamma := 2\zeta\beta\gamma$. The result is

$$E^{aI} \dot{A}_{aI} \to 2 E'^{ai} \dot{A}'_{ai}, \tag{B.38}$$

$$G^{IJ} \to \frac{1}{4}\epsilon^{kij} G_{ij} = \partial_a E'^{ak} - \epsilon^{kij} A'_{ai} E'^a{}_j, \tag{B.39}$$

$$\mathcal{H}_a \to \frac{1}{2}\mathcal{H}_a = E'^{bi} \partial_a A'_{bi} - \partial_b \left(E'^{bi} A'_{ai} \right), \tag{B.40}$$

$$\mathcal{H} \to \mathcal{H} = -\frac{s}{2\sqrt{q}} E^{[a|i} E^{b]j} \left(A_{bi} - \frac{1}{\tilde\gamma}\epsilon_i{}^{kl}\Gamma_{bkl} \right) \left(A_{aj} - \frac{1}{\tilde\gamma}\epsilon_j{}^{mn}\Gamma_{amn} \right) - \sqrt{q} R(E) \tag{B.41}$$

$$\approx -\frac{s}{\sqrt{q}} F'_{abij} E'^{ai} E'^{bj} + \left(\frac{s}{\tilde\gamma^2} - 1 \right) \sqrt{q} R(\tilde\gamma E'), \tag{B.42}$$

where terms proportional to the Gauß constraint have been dropped in the expression for the Hamilton constraint (cf. [3]). At this stage, only the combination of the parameters $\gamma \cdot \beta$ is left and one could ask if one should have worked with one parameter from the beginning. Note that the (quadratic) simplicity constraint implies $\frac{1}{2}\epsilon_{IJKL} {}^{(\gamma,\beta)}\pi^{aIJ}{}^{(\gamma,\beta)}\pi^{bKL} = \frac{2\zeta\gamma}{\gamma^2+\zeta} {}^{(\gamma,\beta)}\pi^{aIJ}{}^{(\gamma,\beta)}\pi^b{}_{IJ}$ and therefore

$$2\zeta q q^{ab} = \pi^{aIJ}\pi^b{}_{IJ} = \left(\frac{\gamma^2}{\gamma^2-\zeta} \right)^2 \left[\left(1 + \frac{\zeta}{\gamma^2} \right) {}^{(\gamma,\beta)}\pi^{aIJ}{}^{(\gamma,\beta)}\pi^b{}_{IJ} - \frac{1}{\gamma}\epsilon_{IJKL} {}^{(\gamma,\beta)}\pi^{aIJ}{}^{(\gamma,\beta)}\pi^{bKL} \right]$$

$$\approx \frac{\gamma^2\beta^2}{\gamma^2+\zeta} {}^{(\gamma,\beta)}\pi^{aIJ}{}^{(\gamma,\beta)}\pi^b{}_{IJ}. \tag{B.43}$$

We expect that the square root of this factor, $\frac{\gamma\beta}{\sqrt{\gamma^2+\zeta}}$, will appear in the spectrum of the area operator. It seems improbable that the two parameters γ, β appear just in this peculiar combination in the spectra of operators and therefore, at the quantum level one should be able to distinguish between γ and β.

---

[7]The factor of two appears since we want to obtain the Poisson brackets $\{A'_{ai}(x), E'^{bj}(y)\} = \frac{1}{2}\delta_a^b \delta_i^j \delta^3(x-y)$, which brings the formulation closer to the one given in [3].

B.5 Formulation with two commuting SU(2) Connections

Note that we could have chosen time gauge before solving the simplicity and normalisation constraints by setting $N^I = \delta_0^I$ and solving the boost part of the Gauß constraint $G^{0i} = D_a \, {}^{(\gamma,\beta)}\pi^{ai} - P^i$, where we used the notation ${}^{(\gamma,\beta)}\pi^{ai} := {}^{(\gamma,\beta)}\pi^{a0i}$. Furthermore, we define $A_{ai} := A_{a0i}$ and $D_a B^i := \partial_a B^i + A_{aij} B^j$. Then we find

$$\frac{1}{2} {}^{(\gamma,\beta)}\pi^{aIJ} \dot{A}_{aIJ} + P^I \dot{N}_I \quad \rightarrow \quad \frac{1}{2} {}^{(\gamma,\beta)}\pi^{aij} \dot{A}_{aij} + {}^{(\gamma,\beta)}\pi^{ai} \dot{A}_{ai} \tag{B.44}$$

and

$$G^{IJ} \quad \rightarrow \quad G^{ij} = D_a \, {}^{(\gamma,\beta)}\pi^{aij} + 2 \, {}^{(\gamma,\beta)}\pi^{a[i} A_a^{j]}, \tag{B.45}$$

$$S^{aI} \quad \rightarrow \quad S^{ai} = \epsilon^{ijk} {}^{(\gamma,\beta)}\pi^a_{\ jk} - \frac{2\zeta}{\beta} {}^{(\gamma,\beta)}\pi^{ai}, \tag{B.46}$$

$$\mathcal{H}_a \quad \rightarrow \quad \mathcal{H}_a = \frac{1}{2} {}^{(\gamma,\beta)}\pi^{bij} \partial_a A_{bij} - \frac{1}{2} \partial_b \left({}^{(\gamma,\beta)}\pi^{bij} A_{aij} \right) + {}^{(\gamma,\beta)}\pi^{bi} \partial_a A_{bi} - \partial_b \left({}^{(\gamma,\beta)}\pi^{bi} A_{ai} \right), \tag{B.47}$$

$$\mathcal{H} \quad \rightarrow \quad \mathcal{H} = -\frac{s}{8\sqrt{q}} \left({}^{(\gamma,\beta)}\pi^{[a|ij} {}^{(\gamma,\beta)}\pi^{b]kl} (A - \Gamma)_{bij} (A - \Gamma)_{akl} \right)$$
$$-\frac{s}{4\sqrt{q}} \left({}^{(\gamma,\beta)}\pi^{[a|i} {}^{(\gamma,\beta)}\pi^{b]j} A_{bi} A_{aj} \right) - \sqrt{q} R({}^{(\gamma,\beta)}\pi), \tag{B.48}$$

where we dropped constants in front of the simplicity constraint and in the Hamilton constraint we neglected terms proportional to the simplicity constraint ($\Gamma_{a0i} \approx 0$). Note that in the case without Barbero-Immirzi parameter, the Simplicity constraint $S^{ai} = \epsilon^{ijk}\pi^a_{jk}$ demands the vanishing of π^{aij} and therefore there is no physical information left in the conjugate SU(2) connection A_{aij}. Here, this is not the case and we obtain a genuine connection formulation of General Relativity. Moreover, the other canonical pair $\left\{ A_{ai}, {}^{(\gamma,\beta)}\pi^{bj} \right\}$ has the same structure as $\left\{ K_{ai}, E^{bj} \right\}$. Then, it follows from the know results when extending the ADM phase space to Ashtekar-Barbero variables (cf., e.g., [3]) that there exists a spin connection Γ'_{aij} which annihilates ${}^{(\gamma,\beta)}\pi^{ai}$ and that the transformation $\left\{ A_{ai}, {}^{(\gamma,\beta)}\pi^{bj} \right\} \rightarrow \left\{ A^-_{aij} := \Gamma'_{aij} + \alpha\epsilon_{ijk}A_a^k, \ E_-^{bkl} := \frac{1}{\alpha}\epsilon^{klm} {}^{(\gamma,\beta)}\pi^b_{\ m} \right\}$ for $\alpha \in \mathbb{R}/\{0\}$ is canonical. Defining $A^+_{aij} := A_{aij}$ and $E_+^{aij} := {}^{(\gamma,\beta)}\pi^{aij}$, we obtain the symplectic potential

$$\frac{1}{2} E_+^{aij} \dot{A}^+_{aij} + \frac{1}{2} E_-^{aij} \dot{A}^-_{aij} \tag{B.49}$$

and constraints

$$G^{ij} = D_a^+ E_+^{aij} + D_a^- E_-^{aij}, \tag{B.50}$$

$$\frac{1}{2}\epsilon^{ijk} S_k^a = E_+^{aij} - \frac{\zeta\alpha}{\beta} E_-^{aij}, \tag{B.51}$$

$$\mathcal{H}_a = \frac{1}{2} E_+^{bij} \partial_a A^+_{bij} - \frac{1}{2} \partial_b \left(E_+^{bij} A^+_{aij} \right)$$
$$+ \frac{1}{2} E_-^{bij} \partial_a A^-_{bij} - \frac{1}{2} \partial_b \left(E_-^{bij} A^-_{aij} \right) - \frac{1}{2} E_-^{bij} R'_{abij}(E_-), \tag{B.52}$$

$$\mathcal{H} = -\frac{s}{8\sqrt{q}} \left(E_+^{[a|ij} E_+^{b]kl} \left(A^+ - \Gamma(E_+, E_-) \right)_{bij} \left(A^+ - \Gamma(E_+, E_-) \right)_{akl} \right)$$
$$-\frac{s}{8\sqrt{q}} \left(E_-^{[a|ij} E_-^{b]kl} \left(A^- - \Gamma'(E_-) \right)_{bij} \left(A^- - \Gamma'(E_-) \right)_{akl} \right) - \sqrt{q} R(E_+, E_-), \tag{B.53}$$

where the last term in the spatial diffeomorphism constraint, $E_-{}^{bij}R'_{abij}(E_-)$, vanishes due to the Bianci identity. We made explicit that the spin connection Γ_{aij} in the Hamilton constraint does not annihilate $E_+{}^{aij}$ but the physical combination of $E_+{}^{aij}$ and $E_-{}^{aij}$ (i.e. the combination which remains when solving the simplicity constraint). In this formulation we now have two commuting SU(2) connections A^+_{aij} and A^-_{aij}, which can be interpreted as the two parts SU(2)$^+$ and SU(2)$^-$ of SO(4). They are, however, not uncorrelated and their momenta are multiples of each other, in complete analogy to the relation $K + \gamma L = 0$ of boost- and rotation generators in the new Spin Foam models.

References

[1] N. Bodendorfer, T. Thiemann, and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions I. Hamiltonian Analysis," `arXiv preprint`.

[2] C. Rovelli, *Quantum Gravity*. Cambridge University Press, Cambridge, 2004.

[3] T. Thiemann, *Modern Canonical Quantum General Relativity*. Cambridge University Press, Cambridge, 2007.

[4] S. Holst, "Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action," *Physical Review D* **53** (1996) 5966–5969, `arXiv:gr-qc/9511026`.

[5] L. Freidel, K. Krasnov, and R. Puzio, "BF description of higher-dimensional gravity theories," *Advances in Theoretical and Mathematical Physics* **3** (1999) 1289–1324, `arXiv:hep-th/9901069`.

[6] S. Alexandrov and E. Livine, "SU(2) loop quantum gravity seen from covariant theory," *Physical Review D* **67** (2003) 044009, `arXiv:gr-qc/0209105`.

[7] M. Han, Y. Ma, Y. Ding, and L. Qin, "Hamiltonian analysis of n-dimensional Palatini gravity with matter," *Mod.Phys.Lett.* **A20** (2005) `arXiv:gr-qc/0503024`.

[8] P. Peldán, "Actions for gravity, with generalizations: A Review," *Classical and Quantum Gravity* **11** (1993) 1087–1132, `arXiv:gr-qc/9305011`.

[9] P. A. M. Dirac, *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, Yeshiva University Press, New York, 1964.

[10] P. Peldán, "Ashtekar's variables for arbitrary gauge group," *Physical Review D* **46** (1992) R2279–R2282, `arXiv:hep-th/9204069`.

[11] P. Mitra and R. Rajaraman, "Gauge-invariant reformulation of an anomalous gauge theory," *Physics Letters B* **225** (1989) 267–271.

[12] R. Anishetty and A. S. Vytheeswaran, "Gauge invariance in second-class constrained systems," *Journal of Physics A: Mathematical and General* **26** (1993), no. 20 5613–5619.

[13] A. S. Vytheeswaran, "Gauge unfixing in second-class constrained systems," *Annals of Physics* **236** (1994) 297–324.

[14] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, 1994.

[15] B. Dittrich, "Partial and complete observables for Hamiltonian constrained systems," *General Relativity and Gravitation* **39** (2007) 1891–1927, `arXiv:gr-qc/0411013`.

[16] C. Rovelli, "What is observable in classical and quantum gravity?," *Classical and Quantum Gravity* **8** (1991) 297–316.

[17] C. Rovelli, "Partial observables," *Physical Review D* **65** (2002) 124013, `arXiv:gr-qc/0110035`.

[18] T. Thiemann, "Reduced phase space quantization and Dirac observables," *Classical and Quantum Gravity* **23** (2006) 1163–1180, `arXiv:gr-qc/0411031`.

[19] K. Giesel and T. Thiemann, "Algebraic quantum gravity (AQG): IV. Reduced phase space quantization of loop quantum gravity," *Classical and Quantum Gravity* **27** (2010) 175009, `arXiv:0711.0119 [gr-qc]`.

[20] P. J. Arias and J. Stephany, "Gauge invariance and second class constraints in three-dimensional linearized massive gravity," *Journal of Mathematical Physics* **36** (1995), no. 4 1868–1876, `arXiv:hep-th/9406092`.

[21] N. Barros e Sá, "Hamiltonian analysis of general relativity with the Immirzi parameter," *International Journal of Modern Physics D* **10** (2001) 261–272, `arXiv:gr-qc/0006013`.

[22] A. Ashtekar and C. J. Isham, "Representations of the holonomy algebras of gravity and non-Abelian gauge theories," *Classical and Quantum Gravity* **9** (1992) 1433–1468, `arXiv:hep-th/9202053`.

[23] A. Ashtekar and J. Lewandowski, "Representation Theory of Analytic Holonomy C* Algebras," in *Knots and Quantum Gravity* (J. Baez, ed.), (Oxford), Oxford University Press, 1994. `arXiv:gr-qc/9311010`.

[24] S. Alexandrov and P. Roche, "Critical Overview of Loops and Foams," `arXiv:1009.4475 [gr-qc]`.

[25] S. Alexandrov, "SO (4, C)-covariant Ashtekar-Barbero gravity and the Immirzi parameter," *Classical and Quantum Gravity* **17** (2000) 4255–4268, `arXiv:gr-qc/0005085`.

[26] J. Engle, R. Pereira, and C. Rovelli, "The Loop-Quantum-Gravity Vertex Amplitude," *Physical Review Letters* **99** (2007) 161301, `arXiv:0705.2388 [gr-qc]`.

[27] E. Livine and S. Speziale, "New spinfoam vertex for quantum gravity," *Physical Review D* **76** (2007) 084028, `arXiv:0705.0674 [gr-qc]`.

[28] J. Engle, R. Pereira, and C. Rovelli, "Flipped spinfoam vertex and loop gravity," *Nuclear Physics B* **798** (2008) 251–290, `arXiv:0708.1236 [gr-qc]`.

[29] J. Engle, E. R. Livine, R. Pereira, and C. Rovelli, "LQG vertex with finite Immirzi parameter," *Nuclear Physics B* **799** (2008) 136–149, `arXiv:0711.0146 [gr-qc]`.

[30] L. Freidel and K. Krasnov, "A new spin foam model for 4D gravity," *Classical and Quantum Gravity* **25** (2008) 125018, `arXiv:0708.1595 [gr-qc]`.

[31] E. R. Livine, "Projected spin networks for Lorentz connection: linking spin foams and loop gravity," *Classical and Quantum Gravity* **19** (2002) 5525–5541, `arXiv:gr-qc/0207084`.

[32] N. Bodendorfer, T. Thiemann, and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions III. Quantum Theory," `arXiv preprint`.

[33] N. Bodendorfer, T. Thiemann, and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions IV. Matter Coupling," `arXiv preprint`.

[34] N. Bodendorfer, T. Thiemann, and A. Thurn, "On the Implementation of the Canonical Quantum Simplicity Constraint," `arXiv preprint`.

[35] N. Bodendorfer, T. Thiemann, and A. Thurn, "Towards Loop Quantum Supergravity (LQSG) I. Rarita-Schwinger Sector," `arXiv preprint`.

[36] N. Bodendorfer, T. Thiemann, and A. Thurn, "Towards Loop Quantum Supergravity (LQSG) II. p-Form Sector," `arXiv preprint`.

[37] W. Wieland, "Complex Ashtekar variables and reality conditions for Holst's action," `arXiv:1012.1738 [gr-qc]`.