1

The spin-foam-representation of LQG

Alejandro Perez

Centre de Physique Théorique,
Campus de Luminy, 13288 Marseille, France.
Unité Mixte de Recherche (UMR 6207) du
CNRS et des Universités Aix-Marseille I,
Aix-Marseille II, et du Sud Toulon-Var;
laboratoire afilié à la FRUMAM (FR 2291).

Abstract

The problem of background independent quantum gravity is the problem of defining a quantum field theory of matter and gravity in the absence of an underlying background geometry. Loop quantum gravity (LQG) is a promising proposal for addressing this difficult task. Despite the steady progress of the field, dynamics remains to a large extend an open issue in LQG. Here we present the main ideas behind a series of proposals for addressing the issue of dynamics. We refer to these constructions as the *spin foam representation* of LQG. This set of ideas can be viewed as a systematic attempt at the construction of the path integral representation of LQG.

The *spin foam representation* is mathematically precise in 2+1 dimensions, so we will start this chapter by showing how it arises in the canonical quantization of this simple theory. This toy model will be used to precisely describe the true geometric meaning of the histories that are summed over in the path integral of generally covariant theories.

In four dimensions similar structures appear. We call these constructions *spin foam models* as their definition is incomplete in the sence that at least one of the following issues remains unclear: 1) the connection to a canonical formulation, and 2) regularization independence (renormalizability). In the second part of this chapter we will describe the definition of these models emphasizing the importance of these open issues.

1

1.1 The path integral for generally covariant systems

LQG is based on the canonical (Hamiltonian) quantization of general relativity whose gauge symmetry is diffeomorphism invariance. In the Hamiltonian formulation the presence of gauge symmetries (Dirac P.M.) gives rise to relationships among the phase space variables—schematically $C(p, q) = 0$ for $(p, q) \in \Gamma$—which are referred to as *constraints*. The constraints restrict the set of possible states of the theory by requiring them to lay on the constraint hyper-surface. In addition, through the Poisson bracket, the constraints generate motion associated to gauge transformations on the constraint surface (see Fig. (1.1)). The set of physical states (the so called reduced phase space Γ_{red}) is isomorphic to the space of orbits, i.e., two points on the same gauge orbit represent the same state in Γ_{red} described in different gauges (Fig. 1.1).

In general relativity the absence of a preferred notion of time implies that the Hamiltonian of gravity is a linear combination of constraints. This means that Hamilton equations cannot be interpreted as time evolution and rather correspond to motion along gauge orbits of general relativity. In generally covariant systems conventional time evolution is pure gauge: from an initial data satisfying the constraints one recovers a spacetime by selecting a particular one-parameter family of gauge-transformations (in the standard ADM context this amounts for choosing a particular lapse function $N(t)$ and shift $N^a(t)$).
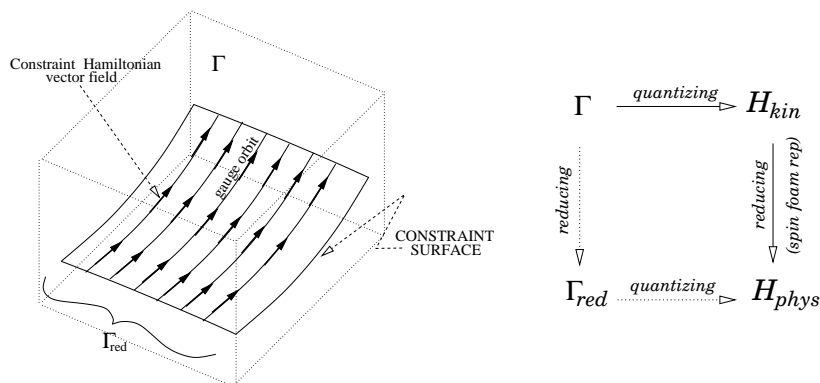

Fig. 1.1. On the left: the geometry of phase space in gauge theories. On the right: the quantization path of LQG (continuous arrows).

From this perspective the notion of spacetime becomes secondary and the dynamical interpretation of the the theory seems problematic (in the quantum theory this is refered to as the *"problem of time"*). A

possible reason for this apparent problem is the central role played by the spacetime representation of classical gravity solutions. However, the reason for this is to a large part due to the applicability of the concept of *test observers* (or more generally *test fields*) in classical general relativity†. Due to the fact that this idealization is a good approximation to the (classical) process of observation the notion of spacetime is useful in classical gravity.

As emphasized by Einstein with his hole argument (see Rovelli C. (2005) for a modern explanation) only the information in relational statements (independent of any spacetime representation) have physical meaning. In classical gravity it remains useful to have a spacetime representation when dealing with idealized test observers. For instance to solve the geodesic equation and then ask *diff-invariant-questions* such as: what is the proper time elapsed on particle 1 between two succesive crossings with particle 2? However, already in the classical theory the advantage of the spacetime picture becomes, by far, less clear if the test particles are replaced by real objects coupling to the gravitational field †.

However, this possibility is no longer available in quantum gravity where at the Planck scale ($\ell_p \approx 10^{-33} cm$) the quantum fluctuations of the gravitational field become so important that there is no way (not even in principle‡) to make observations without affecting the gravitational field. In this context there cannot be any, *a priori*, notion of time and hence no notion of spacetime is possible at the fundamental level. A spacetime picture would only arise in the semi-classical regime with the identification of some subsystems that approximate the notion of test observers.

What is the meaning of the path integral in the background independent context? The previous discussion rules out the conventional

† Most (if not all) of the textbook applications of general relativity make use of this concept together with the knowledge of certain exact solutions. In special situations there are even preferred coordinate systems based on this notion which greatly simplify interpretation (e.g. co-moving observers in cosmology, or observers at infinity for isolated systems).

† In this case one would need first to solve the constraints of general relativity in order to find the initial data representing the self-gravitating objects. Then one would have essentially two choices: 1) Fix a lapse $N(t)$ and a shift $N^a(t)$, evolve with the constraints, obtain a spacetime (out of the data) in a particular gauge, and finally ask the *diff-invariant-question*; or 2) try to answer the question by simply studying the data itself (without t-evolution). It is far from obvious whether the first option (the conventional one) is any easier than the second.

‡ In order to make a Planck scale observation we need a Planck energy probe (think of a Planck energy photon). It would be absurd to suppose that one can disregard the interaction of such photon with the gravitational field treating it as test photon.

interpretation of the path integral. There is no meaningful notion of transition amplitude between states at different times $t_1 > t_0$ or equivalently a notion of *"unitary time evolution"* represented by an operator $U(t_1 - t_0)$. Nevertheless, a path integral representation of generally covariant systems arises as a tool for implementing the constraints in the quantum theory as we argue below.

Due to the difficulty associated with the explicit description of the reduced phase space Γ_{red}, in LQG one follows Dirac's prescription. One starts by quantizing unconstrained phase space Γ, representing the canonical variables as self-adjoint operators in a *kinematical Hilbert space* $\mathscr{H}_{kin}$. Poisson brackets are replaced by commutators in the standard way, and the constraints are promoted to self-adjoint operators (see Fig. 1.1). If there are no anomalies the Poisson algebra of classical constraints is represented by the commutator algebra of the associated quantum constraints. In this way the quantum constraints become the infinitesimal generators of gauge transformations in $\mathscr{H}_{kin}$. The physical Hilbert space $\mathscr{H}_{phys}$ is defined as the kernel of the constraints, and hence associated to *gauge invariant* states. Assuming for simplicity that there is only one constraint we have

$$\psi \in \mathscr{H}_{phys} \quad \text{iff} \quad \exp[iN\hat{C}]|\psi\rangle = |\psi\rangle \quad \forall \quad N \in \mathbb{R},$$

where $U(N) = \exp[iN\hat{C}]$ is the unitary operator associated to the gauge transformation generated by the constraint C with parameter N. One can characterize the set of gauge invariant states, and hence construct $\mathscr{H}_{phys}$, by appropriately defining a notion of 'averaging' along the orbits generated by the constraints in $\mathscr{H}_{kin}$. For instance if one can make sense of the *projector*

$$P : \mathscr{H}_{kin} \to \mathscr{H}_{phys} \quad \text{where} \quad P := \int dN \, U(N). \qquad (1.8)$$

It is apparent from the definition that for any $\psi \in \mathscr{H}_{kin}$ then $P\psi \in \mathscr{H}_{phys}$. The path integral representation arises in the representation of the unitary operator $U(N)$ as a sum over *gauge-histories* in a way which is technically analogous to standard path integral in quantum mechanics. The physical interpretation is however quite different as we will show in Sec. 1.2.4. The *spin foam representation* arises naturally as the path integral representation of the field theoretical analog of P in the context of LQG. Needles is to say that many mathematical subtleties appear when one applies the above formal construction to concrete examples (Giulini D. & Marolf D., (1999)).

1.2 *Spin foams* in 3d quantum gravity

Here we derive the *spin foam representation* of LQG in a simple solvable example: 2+1 gravity. For the definition of spin foam models directly in the covariant picture see Freidel (2005), and other approaches to 3d quantum gravity see Carlip S. (1998).

1.2.1 The classical theory

Riemannian gravity in 3 dimensions is a theory with no local degrees of freedom, i.e., a topological theory. Its action (in the first order formalism) is given by

$$S[e, \omega] = \int_M \text{Tr}(e \wedge F(\omega)), \tag{1.9}$$

where $M = \Sigma \times \mathbb{R}$ (for Σ an arbitrary Riemann surface), ω is an $SU(2)$-connection and the triad e is an $su(2)$-valued 1-form. The gauge symmetries of the action are the local $SU(2)$ gauge transformations

$$\delta e = [e, \alpha], \qquad \delta \omega = d_\omega \alpha, \tag{1.10}$$

where α is a $su(2)$-valued 0-form, and the 'topological' gauge transformation

$$\delta e = d_\omega \eta, \qquad \delta \omega = 0, \tag{1.11}$$

where d_ω denotes the covariant exterior derivative and η is a $\mathtt{su(2)}$-valued 0-form. The first invariance is manifest from the form of the action, while the second is a consequence of the Bianchi identity, $d_\omega F(\omega) = 0$. The gauge symmetries are so large that all the solutions to the equations of motion are locally pure gauge. The theory has only global or topological degrees of freedom.

Upon the standard 2+1 decomposition, the phase space in these variables is parametrized by the pull back to Σ of ω and e. In local coordinates one can express them in terms of the 2-dimensional connection A_a^i and the triad field $E_j^b = \epsilon^{bc} e_c^k \delta_{jk}$ where $a = 1, 2$ are space coordinate indices and $i, j = 1, 2, 3$ are $su(2)$ indices. The Poisson bracket is given by

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \, \delta_j^i \, \delta^{(2)}(x, y). \tag{1.12}$$

Local symmetries of the theory are generated by the first class constraints

$$D_b E_j^b = 0, \quad F_{ab}^i(A) = 0, \tag{1.13}$$

which are referred to as the Gauss law and the curvature constraint respectively. This simple theory has been quantized in various ways in the literature, here we will use it to introduce the *spin foam representation*.

1.2.2 Spin foams *from the Hamiltonian formulation*

The physical Hilbert space, $\mathscr{H}_{phys}$, is defined by those 'states in $\mathscr{H}_{kin}$' that are annihilated by the constraints. As discussed in Thiemann (2005) (see also Rovelli C. (2005) and Thiemann (2005)), spin network states solve the Gauss constraint—$\widehat{D_a E_i^a}|s\rangle = 0$—as they are manifestly $SU(2)$ gauge invariant. To complete the quantization one needs to characterize the space of solutions of the quantum curvature constraints $\widehat{F}_{ab}^i$, and to provide it with the physical inner product. As discussed in Sec. 1.1 we can achieve this if we can make sense of the following formal expression for the generalized projection operator P:

$$P = \int D[N] \, \exp(i \int_{\Sigma} \text{Tr}[N\widehat{F}(A)]) = \prod_{x \subset \Sigma} \delta[\widehat{F(A)}], \qquad (1.14)$$

where $N(x) \in \text{su}(2)$. Notice that this is just the field theoretical analog of equation (1.8). P will be defined below by its action on a dense subset of test-states called the cylindrical functions $\text{Cyl} \subset \mathscr{H}_{kin}$ (see Ashtekar & Lewandowski (2004)). If P exists then we have

$$\langle sPU[N], s'\rangle = \langle sP, s'\rangle \ \forall \ \ s, s' \in \text{Cyl}, \ N(x) \in su(2) \qquad (1.15)$$

where $U[N] = \exp(i \int \text{Tr}[N\hat{F}(A)])$. P can be viewed as a map $P : \text{Cyl} \to K_F \subset \text{Cyl}^\star$ (the space of linear functionals of Cyl) where K_F denotes the kernel of the curvature constraint. The physical inner product is defined as

$$\langle s', s\rangle_p := \langle s'P, s\rangle, \qquad (1.16)$$

where $\langle, \rangle$ is the inner product in $\mathscr{H}_{kin}$, and the physical Hilbert space as

$$\mathscr{H}_{phys} := \overline{\text{Cyl}/J} \quad \text{for} \quad J := \{s \in \text{Cyl s.t. } \langle s, s\rangle_p = 0\}, \qquad (1.17)$$

where the bar denotes the standard Cauchy completion of the quotient space in the physical norm.

One can make (1.14) a rigorous definition if one introduces a regularization. A regularization is necessary to avoid the naive UV divergences that appear in QFT when one quantizes non-linear expressions of the

canonical fields such as $F(A)$ in this case (or those representing interactions in standard particle physics). A rigorous quantization is achieved if the regulator can be removed without the appearance of infinities, and if the number of ambiguities appearing in this process is under control (more about this in Sec. 1.3.1). We shall see that all this can be done in the simple toy example of this section.
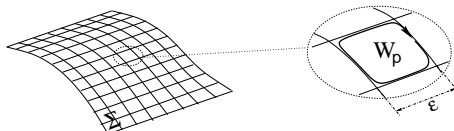

Fig. 1.2. Cellular decomposition of the space manifold Σ (a square lattice of size ϵ in this example), and the infinitesimal plaquette holonomy $W_p[A]$.

We now introduce the regularization. Given a partition of Σ in terms of 2-dimensional plaquettes of coordinate area ϵ^2 (Fig. 1.2) one can write the integral

$$F[N] := \int_\Sigma \text{Tr}[NF(A)] = \lim_{\epsilon \to 0} \sum_p \epsilon^2 \text{Tr}[N_p F_p] \qquad (1.18)$$

as a limit of a Riemann sum, where N_p and F_p are values of the smearing field N and the curvature $\epsilon^{ab} F_{ab}^i[A]$ at some interior point of the plaquette p and ϵ^{ab} is the Levi-Civita tensor. Similarly the holonomy $W_p[A]$ around the boundary of the plaquette p (see Figure 1.2) is given by

$$W_p[A] = \mathbb{1} + \epsilon^2 F_p(A) + \mathcal{O}(\epsilon^2). \qquad (1.19)$$

The previous two equations imply that $F[N] = \lim_{\epsilon \to 0} \sum_p \text{Tr}[N_p W_p]$, and lead to the following definition: given $s, s' \in \text{Cyl}$ (think of *spin network* states) the physical inner product (1.16) is given by

$$\langle s'P, s \rangle := \lim_{\epsilon \to 0} \ \langle s \prod_p \int dN_p \ \exp(i\text{Tr}[N_p W_p]), s \rangle. \qquad (1.20)$$

The partition is chosen so that the links of the underlying spin network graphs border the plaquettes. One can easily perform the integration over the N_p using the identity (Peter-Weyl theorem)

$$\int dN \ \exp(i\text{Tr}[NW]) = \sum_j (2j + 1) \ \text{Tr}[\overset{j}{\Pi}(W)], \qquad (1.21)$$

where $\overset{j}{\Pi}(W)$ is the spin j unitary irreducible representation of $SU(2)$. Using the previous equation

$$\langle s'P, s\rangle := \lim_{\epsilon \to 0} \prod_p^{n_p(\epsilon)} \sum_{j_p}(2j_p + 1) \ \langle s' \ \mathrm{Tr}[\overset{j_p}{\Pi}(W_p)]), s\rangle, \qquad (1.22)$$

where the spin j_p is associated to the p-th plaquette, and $n_p(\epsilon)$ is the number of plaquettes. Since the elements of the set of Wilson loop operators $\{W_p\}$ commute, the ordering of plaquette-operators in the previous product does not matter. The limit $\epsilon \to 0$ exists and one can give a closed expression for the physical inner product. That the regulator can be removed follows from the orthonormality of $SU(2)$ irreducible representations which implies that the two spin sums associated to the action of two neighboring plaquettes collapses into a single sum over the action of the *fusion* of the corresponding plaquettes (see Fig 1.3). One can also show that it is finite†, and satisfies all the properties of an inner product (Noui K. & Perez A. (2005)).

$$\sum_{jk}(2j + 1)(2k + 1) \quad \boxed{\begin{array}{|c|c|} \hline j & k \\ \hline \end{array}} \quad = \sum_k (2k + 1) \quad \boxed{\begin{array}{|c|} \hline k \\ \hline \end{array}}$$

Fig. 1.3. In two dimensions the action of two neighboring plaquette-sums on the vacuum is equivalent to the action of a single larger plaquette action obtained from the fusion of the original ones. This implies the trivial *scaling* of the physical inner product under refinement of the regulator and the existence of a well defined limit $\epsilon \to 0$.

1.2.3 The spin foam representation

Each $\mathrm{Tr}[\overset{j_p}{\Pi}(W_p)]$ in (1.22) acts in $\mathscr{H}_{kin}$ by creating a closed loop in the j_p representation at the boundary of the corresponding plaquette (Figs. 1.4 and 1.6). Now, in order to obtain the *spin foam representation* we introduce a non-physical (*coordinate time*) as follows: Instead of working

† The physical inner product between spin network states satisfies the following inequality

$$|\langle s, s'\rangle_p| \leq C \sum_j (2j + 1)^{2-2g},$$

for some positive constant C. The convergence of the sum for genus $g \geq 2$ follows

$$\text{Tr}[\overset{k}{\Pi}(W_p)] \, \triangleright \qquad \boxed{j \quad (P)} \quad = \sum_m N_{j,m,k} \quad \boxed{j \quad \overset{k}{} \atop m}$$

Fig. 1.4. Graphical notation representing the action of one plaquette holonomy on a *spin network* state. On the right is the result written in terms of the *spin network* basis. The amplitude $N_{j,m,k}$ can be expressed in terms of Clebsch-Gordan coefficients.

with one copy of the space manifold Σ we consider $n_p(\epsilon)$ copies as a discrete foliation $\{\Sigma_p\}_{p=1}^{n_p(\epsilon)}$. Next we represent each of the $\text{Tr}[\overset{j_p}{\Pi}(W_p)]$ in (1.22) on the corresponding Σ_p. If one inserts the resolution of unity in $\mathscr{H}_{kin}$ between the slices, graphically

$$\mathbb{1} = \sum_{\gamma \subset \Sigma, \{j\}_\gamma} |\gamma, \{j\}\rangle\langle\gamma, \{j\}| \qquad (1.23)$$

where the sum is over the complete basis of *spin network* states $\{|\gamma, \{j\}\rangle\}$—based on all graphs $\gamma \subset \Sigma$ and with all possible spin labelling—one arrives at a sum over spin-network histories representation of $\langle s, s'\rangle_p$. More precisely, $\langle s', s\rangle_p$ can be expressed as a sum over amplitudes corresponding to a series of transitions that can be viewed as the 'time evolution' between the 'initial' *spin network* s' and the 'final' *spin network* s. This is illustrated in the two simple examples of Figs. 1.5 and 1.7); on the r.h.s. we illustrate the continuum spin foam picture obtained when the regulator is removed in the limit $\epsilon \to 0$.

Spin network nodes evolve into edges while *spin network* links evolve into 2-dimensional faces. Edges inherit the intertwiners associated to the nodes and faces inherit the spins associated to links. Therefore, the

directly. The case of the sphere $g = 0$ and the torus $g = 1$ can be treated individually (Noui K. & Perez A. (2005)).

Fig. 1.5. A set of discrete transitions in the loop-to-loop physical inner product obtained by a series of transitions as in Figure 1.4. On the right, the continuous *spin foam representation* in the limit $\epsilon \to 0$.

$$\text{Tr}[\overset{n}{\Pi}(W_p)] \rhd \quad\quad = \sum_{o,p} \frac{1}{\Delta_n \Delta_j \Delta_k \Delta_m} \left\{ \begin{array}{ccc} j & k & m \\ n & o & p \end{array} \right\}$$

Fig. 1.6. Graphical notation representing the action of one plaquette holonomy on a *spin network* vertex. The object in brackets ({}) is a $6j$-symbol and $\Delta_j := 2j + 1$.

series of transitions can be represented by a 2-complex whose 1-cells are labelled by intertwiners and whose 2-cells are labelled by spins. The places where the action of the plaquette loop operators create new links (Figs. 1.6 and 1.7) define 0-cells or vertices. These foam-like structures are the so-called *spin foams*. The *spin foam* amplitudes are purely combinatorial and can be explicitly computed from the simple action of the loop operator in $\mathscr{H}_{kin}$. The physical inner product takes the standard Ponzano-Regge form when the *spin network* states s and s' have only 3-valent nodes. Explicitly,

$$\langle s, s' \rangle_p = \sum_{F_{s \to s'}} \prod_{f \subset F_{s \to s'}} (2j_f + 1)^{\frac{\nu_f}{2}} \prod_{v \subset F_{s \to s'}} \quad , \quad (1.24)$$

where the sum is over all the spin foams interpolating between s and s' (denoted $F_{s \to s'}$, see Fig. 1.10), $f \subset F_{s \to s'}$ denotes the faces of the spin

foam (labeled by the spins j_f), $v \subset F_{s \to s'}$ denotes vertices, and $\nu_f = 0$ if $f \cap s \neq 0 \land f \cap s' \neq 0$, $\nu_f = 1$ if $f \cap s \neq 0 \lor f \cap s' \neq 0$, and $\nu_f = 2$ if $f \cap s = 0 \land f \cap s' = 0$. The tetrahedral diagram denotes a $6j$-symbol: the amplitude obtained by means of the natural contraction of the four intertwiners corresponding to the 1-cells converging at a vertex. More generally, for arbitrary *spin networks*, the vertex amplitude corresponds to $3nj$-symbols, and $\langle s, s' \rangle_p$ takes the same general form.
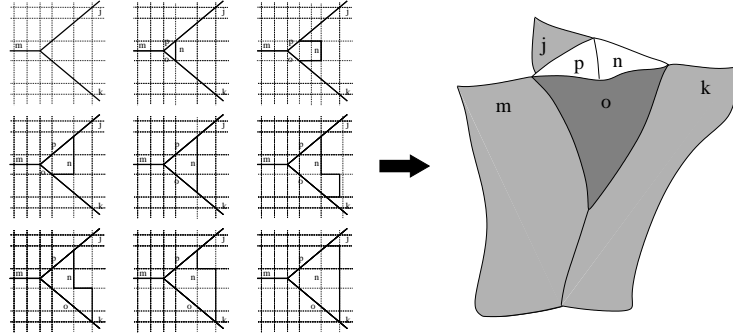
Fig. 1.7. A set of discrete transitions representing one of the contributing histories at a fixed value of the regulator. On the right, the continuous *spin foam representation* when the regulator is removed.

Even though the ordering of the plaquette actions does not affect the amplitudes, the *spin foam representation* of the terms in the sum (1.24) is highly dependent on that ordering. This is represented in Fig 1.8 where a spin foam equivalent to that of Fig 1.5 is obtained by choosing an ordering of plaquettes where those of the central region act first. One can see this freedom of representation as an analogy of the gauge freedom in the spacetime representation in the classical theory.
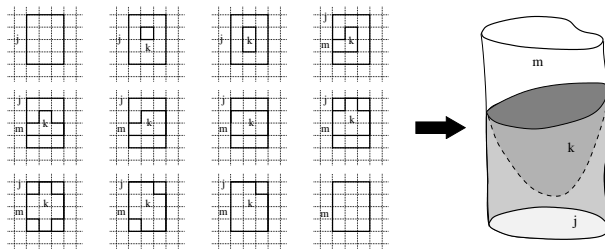

Fig. 1.8. A different representation of the transition of figure 1.5. This spin-foam is obtained by a different ordering choice in (1.22).

One can in fact explicitly construct a basis of $\mathscr{H}_{phys}$ by choosing
an linearly independent set of representatives of the equivalence classes
defined in (1.17). One of these basis is illustrated in Fig. 1.9. The
number of quantum numbers necessary to label the basis element is
$6g - 6$ corresponding to the dimension of the moduli space of $SU(2)$
flat connections on a Riemann surface of genus g. This is the number of
degrees of freedom of the classical theory. In this way we arrive at a fully
combinatorial definition of the standard $\mathscr{H}_{phys}$ by reducing the infinite
degrees of freedom of the kinematical phase space to finitely many by
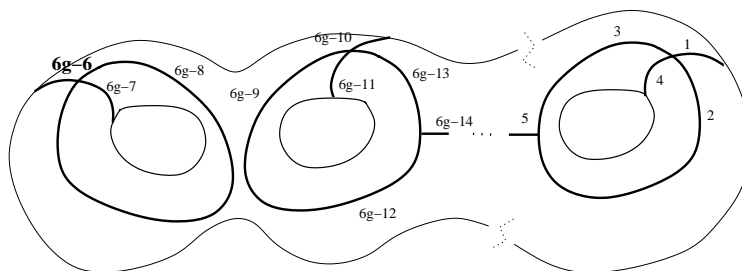the action of the generalized projection operator P.

Fig. 1.9. A spin-network basis of physical states for an arbitrary genus g Rie-
mann surface. There are $6g - 6$ spins labels (recall that 4-valent nodes carry
an intertwiner quantum number).

1.2.4 Quantum spacetime as gauge-histories

What is the geometric meaning of the spin foam configurations? Can we
identify the spin foams with *"quantum spacetime configurations"*? The
answer to the above questions is, strictly speaking, in the negative in
agreement with our discussion at the end of Sec. 1.1. This conclusion
can be best illustrated by looking first at the simple example in 2+1 grav-
ity where $M = S^2 \times \mathbb{R}$ ($g = 0$). In this case the spin foam configurations
appearing in the transition amplitudes look locally the same to those
appearing in the representation of P for any other topology. However, a
close look at the physical inner product defined by P permits to conclude
that the physical Hilbert space is one dimensional—the classical theory
has zero degree of freedom and so there is no non-trivial Dirac observ-
able in the quantum theory. This means that the sum over spin foams in
(1.24) is nothing else but a sum over *pure gauge degrees of freedom* and
hence no physical interpretation can be associated to it. The spins la-

belling the intermediate spin foams do not correspond to any measurable quantity. For any other topology this still holds true, the true degrees of freedom being of a global topological character. This means that in general (even when local excitations are present as in 4d) the spacetime geometric interpretation of the spin foam configurations is subtle. This is an important point that is often overlooked in the literature: one cannot interpret the spin foam sum of (1.24) as a sum over geometries in any obvious way. Its true meaning instead comes from the averaging over the gauge orbits generated by the quantum constraints that defines P—recall the classical picture Fig. 1.1, the discussion around eq. (1.8), and the concrete implementation in 2+1 where $U(N)$ in (1.15) is the unitary transformation representing the orbits generated by F. Spin foams represent a *gauge history* of a kinematical state. A sum over gauge histories is what defines P as a means for extracting the true degrees of freedom from those which are encoded in the kinematical boundary states.
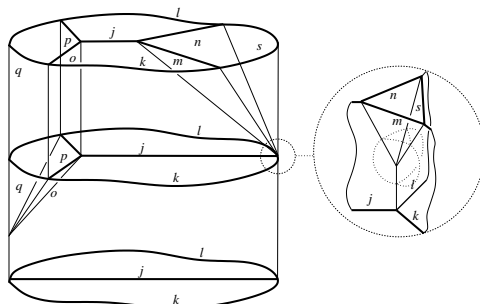

Fig. 1.10. A *spin foam* as the 'colored' 2-complex representing the transition between three different *spin network* states. A transition vertex is magnified on the right.

Here we studied interpretation of the *spin foam representation* in the precise context of our toy example; however, the validity of the conclusion is of general character and holds true in the case of physical interest four dimensional LQG. Although, the quantum numbers labelling the spin foam configurations correspond to eigenvalues of *kinematical* geometric quantities such as length (in 2+1) or area (in 3+1) LQG, their physical meaning and *measurability* depend on dynamical considerations (for instance the naive interpretation of the spins in 2+1 gravity as quanta of physical length is shown here to be of no physical relevance). Quantitative notions such as time, or distance as well as qualitative statements about causal structure or time ordering are misleading (at

best) if they are naively constructed in terms of notions arising from an interpretation of *spin foams* as quantum spacetime configurations†.

1.3 *Spin foam models* in four dimensions

We have studied 2+1 gravity in order to introduce the qualitative features of the *spin foam representation* in a precise setting. Now we discuss some of the ideas that are pursued for the physical case of 3+1 LQG.

Spin foam representation of canonical LQG

There is no complete construction of the physical inner product of LQG in four dimensions. The *spin foam representation* as a device for its definition was originally introduced in the canonical formulation by Rovelli (see Rovelli C. (2005)). In 4-dimensional LQG difficulties in understanding dynamics are centered around understanding the space of solutions of the quantum scalar constraint $\hat{S}$. The physical inner product formally becomes

$$\langle Ps, s' \rangle_{\mathrm{diff}} = \int D[N] \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \left[\int_{\Sigma} N(x)\widehat{S}(x) \right]^n s, s' \rangle_{\mathrm{diff}}, \quad (1.25)$$

where $\langle \ , \ \rangle_{\mathrm{diff}}$ denotes the inner product in the Hilbert space of diff-invariant states, and the exponential in (the field theoretical analog of) (1.8) has been expanded in powers.

Smooth loop states are naturally annihilated by $\widehat{S}$ (independently of any quantization ambiguity (Jacobson T. & Smolin L. (1988) and Smolin L. & Rovelli, C. (1990))). Consequently, $\widehat{S}$ acts only on *spin network* nodes. Generically, it does so by creating new links and nodes modifying the underlying graph of the *spin network* states (Figure 1.11).
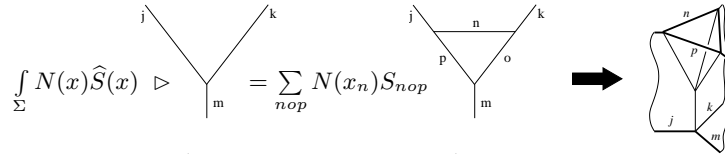

Fig. 1.11. The action of the scalar constraint and its *spin foam representation*. $N(x_n)$ is the value of N at the node and S_{nop} are the matrix elements of $\widehat{S}$.

In a way that is qualitatively similar to what we found in the concrete

† The discussion of this section is a direct consequence of Dirac's perspective applied to the *spin foam representation*.

implementation of the curvature constraint in 2+1 gravity, each term in the sum (1.25) represents a series of transitions—given by the local action of $\widehat{S}$ at *spin network* nodes—through different *spin network* states interpolating the boundary states s and s' respectively. The action of $\hat{S}$ can be visualized as an 'interaction vertex' in the 'time' evolution of the node (Figure 1.11). As in 2+1 dimensions, equation (1.25) can be pictured as sum over 'histories' of *spin networks* pictured as a system of branching surfaces described by a 2-complex whose elements inherit the representation labels on the intermediate states (see Fig. 1.10). The value of the 'transition' amplitudes is controlled by the matrix elements of $\widehat{S}$.

Spin foam representation in the Master Constraint Program

The previous discussion is formal. One runs into technical difficulties if one tries to implement the construction of the 2+1 gravity in this case. The main reason for this is the fact that the constrain algebra does not close with structure constants in the case of 3+1 gravity†. In order to circumvent this problem Thiemann recently proposed to impose one single *master* constraint defined as

$$M = \int\limits_{\Sigma} dx^3 \ \frac{S^2(x) - q^{ab}V_a(x)V_b(x)}{\sqrt{\det q(x)}}, \tag{1.26}$$

where q^{ab} is the space metric and $V_a(x)$ is the vector constraint. Using techniques developed by Thiemann this constraint can indeed be promoted to a quantum operator acting on $\mathscr{H}_{kin}$. The physical inner product could then be defined as

$$\langle s, s'\rangle_p := \lim_{T\to\infty} \langle s, \int\limits_{-T}^{T} dt \ e^{it\widehat{M}} s'\rangle. \tag{1.27}$$

A spin-foam-representation of the previous expression is obtained by splitting the t-parameter in discrete steps and writing

$$e^{it\widehat{M}} = \lim_{n\to\infty} [e^{it\widehat{M}/n}]^n = \lim_{n\to\infty} [1 + it\widehat{M}/n]^n. \tag{1.28}$$

† In 2+1 gravity the constraint algebra correspond to the Lie algebra of $ISO(3)$ (isometries of Euclidean flat spacetime). There are no local degrees of freedom and the underlying gauge symmetry has a non dynamical structure. In 3+1 gravity the presence of gravitons changes that. The fact that the constraint algebra closes with structure functions means that the gauge symmetry structure is dynamical or field dependent. This is the key difficulty in translating the simple results of 2+1 into 3+1 dimensions.

The *spin foam representation* follows from the fact that the action of the basic operator $1 + it\widehat{M}/n$ on a *spin network* can be written as a linear combination of new *spin networks* whose graphs and labels have been modified by the creation of new nodes (in a way qualitatively analogous to the local action shown in Figure 1.11). An explicit derivation of the physical inner product of 4d LQG along these lines is under current investigation.

Spin foam representation: the covariant perspective

In four dimensions the *spin foam representation* of LQG has also been motivated by lattice discretizations of the path integral of gravity in the covariant formulation (for recent reviews see Perez A. (2003) and Oriti D. (2001)). In four dimensions there are two main lines of approach; both are based on classical formulations of gravity based on modifications of the BF-theory action.

The first approach is best represented by the Barrett-Crane model (Barrett J.W. & Crane L. (1998)) and corresponds to the quantization attempt of the classical formulation of gravity based on the Plebanski action

$$S[B, A, \lambda] = \int \mathrm{Tr}\left[B \wedge F(A) + \lambda \ B \wedge B\right], \qquad (1.29)$$

where B is an $so(3, 1)$-valued two-form λ is a Lagrange multiplier imposing a quadratic constraint on the B's whose solutions include the sector $B = \star(e \wedge e)$, for a tetrad e, corresponding to gravity in the tetrad formulation. The key idea in the definition of the model is that the path integral for BF-theory, whose action is $S[B, A, 0]$,

$$P_{topo} = \int \ D[B]D[A] \exp\left[i \int \mathrm{Tr}\left[B \wedge F\right]\right] \qquad (1.30)$$

can be defined in terms of *spin foams* by a simple generalization of the construction of Sec. 1.2. Notice that the formal structure of the action $S[B, A, 0]$ is analogous to that of the action of 2+1 gravity (1.9) (see Baez J.C. (2000)). The Barrett-Crane model aims at providing a definition of the path integral of gravity formally written as

$$P_{GR} = \int \ D[B]D[A] \ \delta\left[B \to \star(e \wedge e)\right] \ \exp\left[i \int \mathrm{Tr}\left[B \wedge F\right]\right], \quad (1.31)$$

where the measure $D[B]D[A]\delta[B \to \star(e \wedge e)]$ restricts the sum in (1.30) to those configurations of the topological theory satisfying the constraints $B = \star(e \wedge e)$ for some tetrad e. The remarkable fact is that the constraint

$B = \star(e \wedge e)$ can be directly implemented on the *spin foam* configurations of P_{topo} by appropriate restriction on the allowed spin labels and intertwiners. All this is possible if a regularization is provided, consisting of a cellular decomposition of the spacetime manifold. The key open issue is however how to get rid of this regulator. A proposal for a regulator independent definition is that of the *group field theory formulation* (Oriti D. (2005)).

A second proposal is the one recently introduced by Freidel and Starodubtsev 2005 based on the formulation McDowell-Mansouri action of Riemannian gravity given by

$$S[B, A] = \int \mathrm{Tr}[B \wedge F(A) - \frac{\alpha}{4} B \wedge B \gamma_5], \qquad (1.32)$$

where B is an $so(5)$-valued two-form, A an $so(5)$ connection, $\alpha = G\Lambda/3 \approx 10^{-120}$ a coupling constant, and the γ_5 in the last term produces the symmetry braking $SO(5) \to SO(4)$. The idea is to define P_{GR} as a power series in α, namely

$$P_{GR} = \sum_{n=0}^{\infty} \frac{(-i\alpha)^n}{4^n n!} \int D[B] D[A] (\mathrm{Tr}[B \wedge B \gamma_5])^n \exp\left[i \int \mathrm{Tr}[B \wedge F]\right].$$

Notice that each term in the sum is the expectation value of a certain power of B's in the well understood topological BF field theory. A regulator in the form of a cellular decomposition of the spacetime manifold is necessary to give a meaning to the former expression. Due to the absence of local degrees of freedom of BF-theory it is expected that the regulator can be removed in analogy to the 2+1 gravity case. It is important to show that removing the regulator does not produce an uncontrollable set of ambiguities (see remarks below regarding *renormalizability*).

1.3.1 The UV problem in the background independent context

In the *spin foam representation*, the functional integral for gravity is replaced by a sum over amplitudes of combinatorial objects given by foam-like configurations (*spin foams*). This is a direct consequence of the background independent treatment of the gravitational field degrees of freedom. As a result there is no place for the UV divergences that plague standard quantum field theory. The combinatorial nature of the fundamental degrees of freedom of geometry appears as a regulator of all the interactions. This seem to be a common feature of all the formulations referred to in this chapter. Does it mean that the UV problem in LQG

is resolved? The answer to this question remains open for the following reason. All the definitions of spin foams models require the introduction of some kind of regulator generically represented by a space (e.g., in the canonical formulation of 2+1 gravity or in the master-constraint program) or spacetime lattice (e.g. in the Barrett-Crane model or in the Freidel-Starodubtsev prescription). This lattice plays a role of a UV regulator in more or less the same sense as a UV cut-off (Λ) in standard QFT. The UV problem in standard QFT is often associated to divergences in the amplitudes when the limit $\Lambda \to \infty$ is taken. The standard renormalization procedure consists of taking that limit while appropriately tuning the bare parameters of the theory so that UV divergences cancel to give a finite answer. Associated to this process there is an intrinsic ambiguity as to what values certain amplitudes should take. These must be fixed by appropriate comparison with experiments (renormalization conditions). If only a finite number or renormalization conditions are required the theory is said to be *renormalizable*. The ambiguity of the process of removing the regulator is an intrinsic feature of QFT.

The background independent treatment of gravity in LQG or the *spin foam models* we have described here do not escape to this general considerations (Perez A. (2005)). Therefore, even though no UV divergences can arise as a consequence of the combinatorial structure of the gravitational field, the heart of the UV problem is now to be found in the potential ambiguities associated with the elimination of the regulator. This remains an open problem for all the attempts of quantization of gravity in 3+1 dimensions. The problem takes the following form in each of the approaches presented in this chapter:

- The removal of the regulator in the 2+1 case is free of ambiguities and hence free of any UV problem (Perez A. (2005)).
- In the case of the master constraint program one can explicitly show that there is a large degree of ambiguity associated to the regularization procedure (Perez A. (2005)). It remains to be shown whether this ambiguity is reduced or disappears when the regulators are removed in the definition of P.
- The Barrett-Crane model is discretization dependent. No clear-cut prescription for the elimination of the triangulation dependence is known.
- The Freidel-Starodubtsev prescription suffers (in principle) from the ambiguities associated to the definition of the expectation value of the

B-monomials appearing in (1.33) before the regulator is removed†. It is hoped that the close relationship with a topological theory might cure these ambiguities although this remains to be shown.

Progress in the resolution of this issue in any of these approaches would represent a major breakthrough in LQG.

Acknowledgements

I would like to thank M. Mondragon and A. Mustatea for careful reading of the manuscript and D. Oriti for his effort and support in this project.

References

Dirac P.M. (). Lectures on Quantum Mechanics

Rovelli C. (2005). *Quantum Gravity*. Cambridge: Cambridge Univ. Press.

Giulini D. & Marolf D., (1999). On the generality of refined algebraic quantization. *Class. Quant. Grav.*, **16**, 2479.

Thiemann T. (2005). . Cambridge: Cambridge Univ. Press.

Ashtekar A. & Lewandowski J. (2004). Background independent quantum gravity: A status report. *Class. Quant. Grav.*, **21** R53.

Thiemann, T. (2005) in *Towards Quantum Gravity*, Cambridge: Cambridge Univ. Press. (to appear)

Carlip S. (1998). *Quantum gravity in 2+1 dimensions*. Cambridge: Cambridge Univ. Press.

Noui K. & Perez A. (2005). Three dimensional loop quantum gravity: Physical scalar product and spin foam models. *Class. Quant. Grav*, **22**, 4489–4514.

Perez A. (2003). Spin foam models for quantum gravity. *Class. Quant. Grav*, **20**, R43.

Oriti D. (2001). Spacetime geometry from algebra: Spin foam models for non- perturbative quantum gravity. *Rept. Prog. Phys.*, **64**, 1489–1544.

Jacobson T. & Smolin L. (1988). Nonperturbative quantum geometries. *Nucl. Phys.*, B299, 295.

Smolin L. & Rovelli, C. (1990). Loop space representation of quantum general relativity. *Nucl. Phys.*, B331, 80.

Barrett J.W. & Crane L. (1998). Relativistic spin networks and quantum gravity. *J.Math.Phys.*, **39**, 3296-3302.

Baez J.C. (2000). An Introduction to Spin Foam Models of Quantum Gravity and BF Theory. *Lect.Notes Phys.*, **543**, 25–94.

Freidel L. & Starodubtsev A. (2005). Quantum gravity in terms of topological observables. *arXiv:hep-th/0501191*.

† There are various prescriptions in the literature on how to define this monomials. They are basically constructed in terms of the insertion of appropriate sources to construct a BF generating function. All of them are intrinsically ambiguous, and the degree of ambiguity grows with the order of the monomial. The main source of ambiguity resides in the issue of where in the discrete lattice to act with the functional source derivatives.

Oriti D. (2005) in *Towards Quantum Gravity*, Cambridge: Cambridge Univ.
    Press. (to appear)
Perez A. (2005). On the regularization ambiguities in loop quantum gravity.
    arXiv:gr-qc/0509118.