Living Without Supersymmetry – the Conformal Alternative and a Dynamical Higgs Boson

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We show that the key results of supersymmetry can be achieved via conformal symmetry instead. We propose that the Higgs boson be a dynamical fermion-antifermion bound state rather than a fundamental scalar field, so that there is then no quadratically divergent self-energy problem for it and thus no need to invoke supersymmetry to resolve the problem. To obtain such a dynamical Higgs boson we study a conformal invariant gauge theory of interacting fermions and gauge bosons. The conformal invariance of the theory is realized via scaling with anomalous dimensions in the ultraviolet, and by a dynamical symmetry breaking via fermion bilinear condensates in the infrared, a breaking in which the dynamical dimension of the composite operator $\bar{\psi}\psi$ is reduced from three to two. With this reduction we can augment the theory with a then renormalizable four-fermion interaction with dynamical dimension equal to four, and can reinterpret the theory as a renormalizable version of the Nambu-Jona-Lasinio model, with the gauge theory sector with its now massive fermion being a mean-field theory and the four-fermion interaction being the residual interaction. It is this residual interaction and not the mean field that then generates dynamical Goldstone and Higgs states, states that, as noted by Baker and Johnson, the gauge theory sector itself does not possess. The Higgs boson is found to be a narrow resonance just above threshold. We couple the theory to a gravity theory, conformal gravity, that is equally conformal invariant, with the interplay between conformal gravity and the four-fermion interaction taking care of the vacuum energy problem. With conformal gravity being a unitary and renormalizable quantum theory of gravity there is no need for string theory with its supersymmetric underpinnings. With the vacuum energy problem being resolved and with conformal gravity fits to phenomena such as galactic rotation curves and the accelerating universe not needing dark matter, there is no need to introduce supersymmetry for either the vacuum energy problem or to provide a potential dark matter candidate. We propose that it is conformal symmetry rather than supersymmetry that is fundamental, with the theory of nature being a locally conformal, locally gauge invariant, non-Abelian Nambu-Jona-Lasinio theory.

I. INTRODUCTION

The assumption of a supersymmetry between bosons and fermions has been found capable of addressing many key issues in particle physics and gravity (see e.g. [1–4]). In flat space physics an interplay between bosons and fermions can render logarithmically divergent Feynman diagrams finite. Similarly, an interplay between bosons and fermions can cancel the perturbative quadratic divergence that a fundamental Higgs scalar field would possess (the hierarchy problem). In addition, the existence of fermionic supersymmetry generators allows one to evade the Coleman-Mandula theorem [5] that forbids the combining of spacetime and bosonic internal symmetry generators in a common Lie algebra. And with the inclusion of supersymmetry one can potentially achieve a unification of the coupling constants of $SU(3)\times SU(2))_{\rm L}\times U(1)$ at a grand-unified energy scale. In the presence of gravity an interplay between bosons and fermions can cancel the quartic divergence in the vacuum energy. Cancellation of perturbative infinities can also be found in supergravity, the local version of supersymmetry. With supersymmetry one can construct a consistent candidate quantum theory of gravity, string theory, which permits a possible unification of all of the fundamental forces and a metrication (geometrization) of them. Finally, with supersymmetry one has a prime candidate for dark matter.

Despite this quite extensive theoretical inventory, actual experimental detection of any of the required superpartners of the standard fermions and bosons has so far proven elusive. Now until recently one could account for such non-detection by endowing the superpartners with ever higher masses or ever weaker couplings to ordinary matter. However, in order to cancel the quadratic self-energy divergence that a fundamental Higgs field would have, one would need a supersymmetric particle with a mass reasonably close to that of the Higgs boson itself. And now that the Higgs boson has been discovered at the Large Hadron Collider (LHC) and its mass has been determined [6, 7], one should thus anticipate finding a superparticle at the LHC in the same mass region. However, no evidence for any such particle has emerged in an exploration of this mass region, or in decays such as $B^0_{\rm s}\rightarrow\mu^+ +\mu^-$ that were thought to be particularly favorable for supersymmetry [8]. And while the superparticle search at the LHC is still in its early stages, nonetheless the situation is disquieting enough that one should at least contemplate whether it might be possible to dispense with supersymmetry altogether. If one is to consider doing so however, then one must seek an alternative to supersymmetry that has the potential to also achieve its key successes. In this paper we present such a candidate alternative, one that is also based on a symmetry, namely conformal symmetry.

Since the Higgs self-energy problem is currently the most pressing concern for supersymmetry, in this paper we shall concentrate on the issue of generating the Higgs boson dynamically, since one then no longer encounters the quadratic self-energy divergence problem that is associated with a fundamental Higgs scalar field. Moreover, independent of any supersymmetry considerations, now that the Higgs boson has been shown to exist, it anyway becomes imperative to ascertain whether it is fundamental or composite, and determine whether or not a fundamental, double-well Higgs potential is to actually be present in the fundamental action of nature. In the present paper we will show that in a conformal or scale invariant theory as realized via critical scaling with anomalous dimensions there is dynamical symmetry breaking via a fermion bilinear condensate, with a fermionic mass and dynamical Higgs and Goldstone particles being generated, and with the mass of the Higgs boson naturally being of the same order as that of the fermion. Moreover, if conformal symmetry is to be exact at the level of the Lagrangian and to only be broken in the vacuum (giving the dimensionful $\bar{\psi}\psi$ composite operator a vacuum expectation value would break both chiral and scale symmetry), the presence of any dimensionful tachyonic $-\mu^2\phi^2$ term in a fundamental Higgs field Lagrangian would expressly be forbidden.

In order to develop the background needed to establish our results, given that our work is based on earlier work from quite some time ago, for the benefit of the reader and to make the present paper self-contained, in Sec. II we briefly review the antecedents of our current work, antecedents that originated in the work of Johnson, Baker, and Willey [9–14] and the present author [15–17] on critical scaling in quantum electrodynamics in the 1960s and 1970s. In particular, we discuss the Baker-Johnson [12] evasion of the Goldstone theorem in Johnson-Baker-Willey electrodynamics, wherein a self-consistently generated fermion mass is not accompanied by a Goldstone boson. In Sec. III we discuss mass generation in the Nambu-Jona-Lasinio model [18], and in Sec. IV we adapt the analysis to the Johnson-Baker-Willey critical scaling case. In this discussion we follow [15–17] and augment electrodynamics with a four-fermion interaction, one made renormalizable by a reduction in its dynamical dimension from six to four. As originally proposed by the present author this four-fermion interaction was introduced for two reasons: to cancel infinities in the vacuum (zero-point) energy density, infinities that one ordinarily would normal order away, and to facilitate the development of a formalism for treating symmetry breaking by fermion bilinears. (Contemporaneous with the author equivalent results on symmetry breaking by composite operators were obtained by Cornwall, Jackiw, and Tomboulis [19]). The central theme of the present paper is to show that this very same four-fermion interaction also serves to provide a residual interaction that then generates the Goldstone and Higgs bosons that are not present in Johnson-Baker-Willey electrodynamics itself. Augmenting Johnson-Baker-Willey electrodynamics with a four-fermion interaction thus enables us to evade the Baker-Johnson evasion of the Goldstone theorem. Now from the perspective of flat space physics there is no particular need to cancel such vacuum energy infinities since energies are not observable, only energy differences. However, once one couples to gravity one cannot throw away any contribution to the vacuum energy since the hallmark of Einstein's formulation of gravity is that gravity is to couple to every form of energy and not just to energy differences. Thus once we extend conformal invariance to gravity, which we do, we then cannot ignore infinities in the vacuum energy, and they have to be canceled. It is thus gravity that will force the four-fermion interaction and its associated Goldstone and Higgs bosons upon us, and as such the four-fermion interaction would need to have its dynamical dimension be lowered from six to four so that it would not destroy renormalizability. However, to cancel the vacuum energy density infinities completely we will need to include not just the four-fermion contribution but also the contribution of quantum conformal gravity itself. We discuss these vacuum energy issues in Sec. IV and in Sec. V. And in Sec. V we also show that conformal symmetry can achieve all of the key results of supersymmetry, and thus essentially supplant it as a candidate fundamental symmetry of nature.

II. JOHNSON-BAKER-WILLEY ELECTRODYNAMICS

A. Vanishing of the Bare Fermion Mass

In order to explore whether the Higgs field might be dynamical we need a tractable calculational scheme in which one can study dynamical symmetry breaking via fermion bilinear condensates non-perturbatively. To this end we adapt some earlier work of the present author from the 1970s. This earlier work was itself based on even earlier work of Johnson, Baker, and Willey from the 1960s on quantum electrodynamics. The objective of the study of Johnson, Baker, and Willey was to determine whether it might be possible for all the renormalization constants of a quantum field theory to be finite. Quantum electrodynamics was a particularly convenient theory to study since its gauge structure meant that two of its renormalization constants (the fermion-antifermion-gauge boson vertex renormalization constant Z_1 and the fermion wave function renormalization constant Z_2 to which Z_1 is equal) were gauge dependent and could be made finite by an appropriate choice of gauge, with the anomalous dimension of the fermion $\gamma_{\rm F}$ consequently then being zero – and for convenience we set Z_1 and Z_2 equal to one in the following. Johnson, Baker, and Willey were thus left with the gauge boson wave function renormalization constant Z_3 and the fermion bare mass m_0 and its shift δm to address.

Now if one were also to consider the coupling of electrodynamics to gravity, one would then have to address another infinity that electrodynamics possesses, namely that of the zero-point vacuum energy density, and one of the objectives of the present paper is to address this issue. Since Johnson, Baker, and Willey were considering electrodynamics in flat space, the need to address the vacuum energy density infinity did not arise in their study.

As regards Z_3 and m_0, Johnson, Baker and Willey showed that Z_3 would be finite if the fermion-antifermion-gauge boson coupling constant α was at a solution to the Gell-Mann-Low eigenvalue condition. At this eigenvalue they showed that the bare mass scaled as

$$m_0 = m\left(\frac{\Lambda^2}{m^2}\right)^{\gamma(\alpha)/2} \tag{1}$$

where Λ is an ultraviolet cutoff and $m = m_0 + \delta m$ is the renormalized fermion mass. Consequently if the power $\gamma(\alpha)$ is negative (which it perturbatively is), the bare mass would vanish in the limit of infinite cutoff and δm would be finite. As such, the work of Johnson, Baker, and Willey was quite remarkable since it predated the work of Wilson and of Callan and Symanzik on critical scaling and the renormalization group.

Subsequently, Adler and Bardeen [20] recast the work of Johnson, Baker, and Willey in the language of the renormalization group itself, and showed that the renormalized inverse fermion propagator $\tilde{S}^{-1}(p)$ and the renormalized vertex function $\tilde{\Gamma}_{\rm S}(p,p,0)$ associated with the insertion of composite operator $\theta = \bar{\psi}\psi$ with zero momentum into the fermion propagator were related by

$$\left[m\frac{\partial}{m} + \beta(\alpha)\frac{\partial}{\partial\alpha}\right]\tilde{S}^{-1}(p) = -m[1-\gamma_\theta(\alpha)]\tilde{\Gamma}_{\rm S}(p,p,0) \tag{2}$$

in the limit in which the fermion momentum p_μ is deep Euclidean and $\gamma_{\rm F}$ is zero. In the critical scaling situation where $\beta(\alpha) = 0$ this equation admits of an exact asymptotic solution

$$\tilde{S}^{-1}(p) = \not{p} - m\left(\frac{-p^2 - i\epsilon}{m^2}\right)^{\gamma_\theta(\alpha)/2} + i\epsilon,$$
$$\tilde{\Gamma}_{\rm S}(p,p,0) = \left(\frac{-p^2 - i\epsilon}{m^2}\right)^{\gamma_\theta(\alpha)/2}. \tag{3}$$

With the parameter $\gamma(\alpha)$ of (1) thus being identifiable as the anomalous dimension $\gamma_\theta(\alpha)$ of $\bar{\psi}\psi$, and with the full dimension of $\bar{\psi}\psi$ being given by $d_\theta(\alpha) = 3 + \gamma_\theta(\alpha)$, the mechanism for both the finiteness and vanishing of m_0 is to have the dimension of $\bar{\psi}\psi$ be less than canonical.

B. Non-Vanishing of the Physical Fermion Mass and the Baker-Johnson Evasion of the Goldstone Theorem

With a vanishing bare mass the anticommutator of γ_5 with a $Z_2 = 1$ fermion propagator $S(p) = (\not{p} - \Sigma(p))^{-1}$ would obey a self-consistent equation of the form [21]

$$\{\gamma_5, \Sigma(p)\} = \int d^4k K(p,k,0) S(k)\{\gamma_5, \Sigma(k)\} S(k) \tag{4}$$

where $K(p,k,0)$ is the Bethe-Salpeter scattering kernel. This equation could have both trivial and non-trivial solutions for $\Sigma(p)$. Then, if the non-zero solution is chosen, the fermion mass would behave just as dynamical masses behave in self-consistent theories of mass generation, and so one initially would expect the presence of a massless pseudoscalar Goldstone boson associated with the generation of such a fermion mass. However, this turned out not to be the case since there was a hidden renormalization effect in the theory, one associated with the renormalization constant $Z_\theta^{-1/2} = (\Lambda^2/m^2)^{\gamma_\theta(\alpha)/2}$ that renormalizes $\bar{\psi}\psi$ according to $Z_\theta^{-1/2}(\bar{\psi}\psi)_0 = \bar{\psi}\psi$ [20]. ($Z_\theta^{-1/2}$ is also equal to Z_S the vertex renormalization constant for $\Gamma_{\rm S}(p,p,0)$.) With the product $m_0(\bar{\psi}\psi)_0$ being equal to $m_0 Z_\theta^{1/2}\bar{\psi}\psi$, and thus equal to $m\bar{\psi}\psi$ on identifying m to be the finite but non-zero $m = m_0 Z_\theta^{1/2}$, the mass term $m_0(\bar{\psi}\psi)_0 = m\bar{\psi}\psi$ is then a renormalization group invariant, with a non-zero $m_0(\bar{\psi}\psi)_0$ term thus being present in the bare Lagrangian from the outset. Consequently, the chiral symmetry is already broken in the Lagrangian itself and the Goldstone theorem does not apply.

Now if there is to be such a mass term in the Lagrangian one should not actually use (4) where m_0 is absent. Rather, one should use [22]

$$\{\gamma_5, \Sigma(p)\} = \int d^4k K(p,k,0) S(k)\{\gamma_5, \Sigma(k)\} S(k) + 2m_0 \int d^4k K(p,k,0) S(k)\gamma_5 S(k). \tag{5}$$

However, given the fact that m_0 vanishes as Λ^{γ_θ} while the kernel term does not diverge as fast as $\Lambda^{-\gamma_\theta}$ (the photon propagator in the kernel being canonical if $\beta(\alpha) = 0$), the $2m_0\int d^4k K(p,k,0)S(k)\gamma_5 S(k)$ term vanishes in the limit of infinite cutoff and (4) is recovered. There are thus two inequivalent ways to recover (4) from (5), either m_0 identically zero or m_0 vanishing sufficiently rapidly, with inspection of (4) alone not immediately indicating which might be the relevant one.

Now as originally noted in [21], in order to establish the presence of a Goldstone boson it is not sufficient to look at the self-consistent equation for the fermion mass alone. Rather, one must look at the fermion-antifermion scattering amplitude, to see whether there might actually be a massless pole in it, or whether the renormalization procedure might prevent this from occurring. And when Johnson, Baker, and Willey did this in their study of electrodynamics, they found that there was no massless Goldstone pole, with the kernel of the Bethe-Salpeter equation for the scattering amplitude being found (precisely because of the scaling with anomalous dimensions) to be non-compact, so that no pole was generated. Thus having a non-trivial solution to the self-consistent equation for the mass is a necessary but not sufficient condition to secure a Goldstone pole. (The self-consistent mass equation is essentially the self-consistent equation for the residue at the Goldstone pole, and from a study of the equation for the would-be residue alone one cannot establish the presence of the pole itself.) The cause of this lack of a Goldstone boson in the presence of dynamical mass generation was explored by Baker and Johnson and is known as the Baker-Johnson evasion of the Goldstone theorem.

To illustrate the issues involved we note that if there is to be a Goldstone boson it must also appear in $\tilde{\Gamma}_{\rm P}(p,p+q,q)$, the insertion of the pseudoscalar $\bar{\psi}i\gamma_5\psi$ into the fermion propagator. This Green's function obeys

$$\tilde{\Gamma}_{\rm P}(p,p+q,q)=Z_{\rm P}i\gamma^5 + \int d^4k\tilde{K}(p,k,q)\tilde{S}(k)\tilde{\Gamma}_{\rm P}(k,k+q,q)\tilde{S}(k+q). \quad (6)$$

Here the tilde symbol indicates that everything is renormalized, with $Z_{\rm P}$ renormalizing the pseudoscalar vertex function. Since the large p^2 behavior of the theory is not sensitive to mass, this $Z_{\rm P}$ is equal to the previously introduced $Z_{\rm S}$, with $Z_{\rm P}$ vanishing as $(\Lambda^2/m^2)^{\gamma_\theta(\alpha)/2}$. To see if there is a pole we note that we can rewrite (6) by inserting its left-hand side into its right-hand side iteratively, to symbolically then obtain $-i\gamma_5\tilde{\Gamma}_{\rm P}=Z_{\rm P}+Z_{\rm P}\Pi Z_{\rm P}+Z_{\rm P}\Pi Z_{\rm P}\Pi Z_{\rm P}+...$, where Π is an appropriate vacuum polarization term. Thus we obtain

$$-i\gamma_5\tilde{\Gamma}_{\rm P}=\frac{Z_{\rm P}}{1-Z_{\rm P}\Pi}=\frac{1}{Z_{\rm P}^{-1}-\Pi}. \quad (7)$$

Then with $Z_{\rm P}^{-1}$ diverging must faster than Π, no pole is generated. This then is the Baker-Johnson evasion of the Goldstone theorem. And not only would it imply that there is no dynamical pseudoscalar bound state Goldstone particle in the theory, implicit in the analysis is that there would be no dynamical scalar bound state Higgs particle either.

Now, in and of itself, the fact that (6) becomes homogeneous when $Z_{\rm P}$ vanishes does not automatically exclude the possible presence of a pole, since an analogous situation is met in the non-renormalizable but cut-off Nambu-Jona-Lasinio model. As we discuss in more detail in Sec. III, there one introduces a four-fermion coupling $(g/2)(\bar{\psi}i\gamma_5\psi)^2$, and in the pseudoscalar channel of the fermion-antifermion scattering amplitude one obtains a T-matrix of the form

$$T_{\rm P}=\frac{g}{1-g\Pi}=\frac{1}{g^{-1}-\Pi}. \quad (8)$$

Now in the Nambu-Jona-Lasinio case both g^{-1} and Π are divergent in the limit of infinite cutoff and g is zero. However, even though both g^{-1} and Π are divergent, they both diverge at the precisely the same rate, with there then indeed being a massless pole in $T_{\rm P}$. While one would automatically obtain a pole if g and Π are both finite (given dynamical mass generation of course), one could also obtain a pole if g^{-1} and Π diverge, provided they diverge at the same rate. In the renormalizable model we discuss in Sec. IV, we will see that both Goldstone and Higgs bosons will be generated by such a mechanism.

C. Non-Zero Vacuum Expectation Value for $\bar{\psi}\psi$ and the condition $\gamma_\theta(\alpha)=-1$

Now even though the non-trivial solution to the self-consistent fermion mass generating equation given in (4) might not require a Goldstone boson, there was still the issue of determining what would oblige the theory to actually choose the non-trivial solution to it rather than the trivial one in the first place. To this end the present author compared the energy densities of the two solutions to find [15, 16] that if $\gamma_\theta(\alpha)$ took the special value

$$\gamma_\theta(\alpha)=-1, \quad (9)$$

the infrared divergences that would then follow (the theory having been softened so much in the ultraviolet and thus made more and more divergent in the infrared) would then drive the theory into a spontaneously broken vacuum $|\Omega_{\rm m}\rangle$ in which $\langle\Omega_{\rm m}|\bar{\psi}\psi|\Omega_{\rm m}\rangle\neq 0$. In order to take care of the infinities that the energy density contained the present author chose not to normal order them away, but rather to cancel them by a counter-term, with the appropriate one being a four-fermion interaction with coupling constant g. Now for a point-coupled such interaction this counter-term would itself generate new infinities. However with the dimension of $\bar{\psi}\psi$ having been reduced from $d_\theta(\alpha)=3$ to $d_\theta(\alpha)=3+\gamma_\theta(\alpha)=2$, the dimension of $(\bar{\psi}\psi)^2$ was reduced from six to four so that it had become renormalizable. With this specific counter-term the then finite energy density was found to have none other than a double-well potential structure in which $\langle\Omega_{\rm m}|\bar{\psi}\psi|\Omega_{\rm m}\rangle=m/g$ was non-zero. Specifically, in terms of a renormalization group subtraction point μ^2 that we elaborate on in Sec. IV below, the renormalized energy density was given as the double-welled

$$\tilde{\epsilon}(m)=\frac{m^2\mu^2}{16\pi^2}\left[\ln\left(\frac{m^2}{M^2}\right)-1\right], \quad (10)$$

with a local maximum at $m=0$ where $\langle\Omega_0|\bar{\psi}\psi|\Omega_0\rangle=0$ and a degenerate global minimum at $m=M$ where $\langle\Omega_{\rm M}|\bar{\psi}\psi|\Omega_{\rm M}\rangle=M/g$ is non-zero. Mass generation in Johnson-Baker-Willey electrodynamics is thus associated with a vacuum in which $\langle\Omega_{\rm M}|\bar{\psi}\psi|\Omega_{\rm M}\rangle$ is non-zero.

As originally introduced by Kadanoff and Wilson, critical scaling described the behavior of a crystal at the critical phase transition temperature where the correlation length is infinite. However at the same critical temperature the order parameter is zero, with it only being non-zero in the ordered phase below the critical temperature. In the case of critical scaling in a quantum field theory, when $\gamma_\theta(\alpha)=-1$ we can have both scaling with anomalous dimensions and a non-zero value for the order parameter $\langle\Omega_M|\bar{\psi}\psi|\Omega_M\rangle$ occur simultaneously. This happens because in a massless theory (analogous to an infinite correlation length) there is no scale, so infrared divergences (needed to generate long range order and an order parameter) are also present; with the effect of $\gamma_\theta(\alpha)=-1$ being to soften the theory so much in the ultraviolet that it becomes sufficiently divergent in the infrared to cause dynamical symmetry breaking to take place. Our work thus provides a framework in which aspects of critical phenomena both at and below the critical temperature are simultaneously present. And in fact one of the motivations for the work of the present author in the 1970s was to try to find such a framework, with it being through the condition $\gamma_\theta(\alpha)=-1$ that it was achieved.

To underscore and illuminate the interplay between mass generation and the spontaneously broken vacuum, a second, independent derivation of the $\gamma_\theta(\alpha)=-1$ condition was also given in [16]. In this derivation the fermion propagator was derived in two separate ways, via the Wilson operator product expansion and via a renormalization group analysis, and compatibility between the two was sought. The Wilson expansion describes the short distance behavior of a massless theory as constructed in a non-spontaneously broken normal vacuum $|\Omega_0\rangle$. The renormalization group describes the short-distance behavior of a theory in which the fermion mass is non-zero. In such a theory we have seen that since the mass is non-zero, at critical scaling the renormalization group describes fluctuations around a spontaneously broken vacuum $|\Omega_{\rm m}\rangle$. To compare the two we thus take matrix elements of the Wilson expansion in $|\Omega_{\rm m}\rangle$. Specifically, in the Wilson operator product expansion at a critical point the leading behavior at short distance of the massless fermion two point function is given by

$$\begin{aligned} T(\psi(x)\bar{\psi}(0)) &= \langle\Omega_0|T(\psi(x)\bar{\psi}(0))|\Omega_0\rangle \\ &+ (\mu^2x^2)^{\gamma_\theta(\alpha)/2} :\psi(0)\bar{\psi}(0): \qquad (11) \end{aligned}$$

where the dots indicate normal ordering with respect to the massless vacuum $|\Omega_0\rangle$ according to $:\psi(0)\bar{\psi}(0):=\psi(0)\bar{\psi}(0)-\langle\Omega_0|\psi(0)\bar{\psi}(0)|\Omega_0\rangle$, $\langle\Omega_0|:\psi(0)\bar{\psi}(0):|\Omega_0\rangle=0$, and μ^2 is an off-shell Green's function subtraction point. If we now take the matrix element of this expansion in the degenerate vacuum $|\Omega_{\rm m}\rangle$ we obtain an asymptotic propagator and inverse propagator that up to coefficients behave as

$$\begin{aligned} \tilde{S}(p) &= \frac{1}{\not p}+(-p^2)^{(-\gamma_\theta(\alpha)/2-2)}, \\ \tilde{S}^{-1}(p) &= \not p-(-p^2)^{(-\gamma_\theta(\alpha)/2-1)}. \end{aligned} \qquad (12)$$

On comparing with (3), (9) follows. (In [16] it was shown that the coefficients match too.) Moreover, not only do we recover (9), we confirm that the relevant vacuum for Johnson-Baker-Willey electrodynamics is indeed a spontaneously broken one. (Some separate discussion of an interplay of the Wilson operator product expansion and vacuum condensates may be found in [23]. Some separate discussion regarding a dynamical dimension four four-fermion interaction in scale invariant quantum electrodynamics may be found in [24, 25].)

D. Evasion of the Baker-Johnson Evasion of the Goldstone Theorem

As we see, the Johnson-Baker-Willey theory has much of the structure of dynamical symmetry breaking and yet has no dynamical Goldstone boson, and thus no dynamical Higgs boson either. Moreover it has much of the structure of the Nambu-Jona-Lasinio model. The essence of the Nambu-Jona-Lasinio model is to rewrite the four-fermion Lagrangian with a strictly massless fermion in terms of a mean-field sector with a massive fermion and a residual interaction sector according to

$$\mathcal{L}=i\bar{\psi}\gamma^\mu\partial_\mu\psi-\frac{g}{2}(\bar{\psi}\psi)^2-\frac{g}{2}(\bar{\psi}i\gamma_5\psi)^2=\mathcal{L}_{\rm MF}+\mathcal{L}_{\rm RI}, \qquad (13)$$

where

$$\begin{aligned} \mathcal{L}_{\rm MF} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi-m\bar{\psi}\psi+\frac{m^2}{2g} \\ \mathcal{L}_{\rm RI} &= -\frac{g}{2}\left(\bar{\psi}\psi-\frac{m}{g}\right)^2-\frac{g}{2}(\bar{\psi}i\gamma_5\psi)^2. \end{aligned} \qquad (14)$$

Even though the full Lagrangian $\mathcal{L}$ is globally chiral symmetric under $\psi\rightarrow e^{i\alpha_5\gamma_5}\psi$ with spacetime independent phase α_5, neither $\mathcal{L}_{\rm MF}$ nor $\mathcal{L}_{\rm RI}$ are themselves separately chirally symmetric. Thus in dynamical symmetry breaking one produces a mean-field theory in which the chiral symmetry is expressly broken at the level of the mean-field Lagrangian. Moreover, no Goldstone boson is present in the mean-field Lagrangian, as could not be the case since the mean-field Lagrangian is expressly not chiral symmetric. Rather, one is generated not by the mean field at all but by the residual interaction, and it is the residual interaction that is needed in order to restore the chiral symmetry that the mean-field sector itself does not possess.

Now, as had been noted by the present author in [15, 16], study of symmetry breaking in Johnson-Baker-Willey electrodynamics can be obtained from the Nambu-Jona-Lasinio model by replacing the point vertex for the insertion of a zero-momentum scalar $\bar{\psi}\psi$ operator into the fermion propagator (viz. $\tilde{\Gamma}_{\rm S}(p,p,0)=1$) by $\tilde{\Gamma}_{\rm S}(p,p,0)=(-p^2/m^2)^{\gamma_\theta(\alpha)/2}$ as given above by the renormalization group equation. Thus, as we show in detail in Sec. IV, we can reinterpret Johnson-Baker-Willey electrodynamics as coupled to a four-fermion interaction to be a mean-field theory, one associated with Lagrangian of the form:

$$\begin{aligned} \mathcal{L}_{\rm QED} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}\gamma^\mu(i\partial_\mu-eA_\mu)\psi \\ &- m\bar{\psi}\psi+\frac{m^2}{2g}, \end{aligned} \qquad (15)$$

where the $m^2/2g$ term is just a constant. And as such, this theory should not contain a Goldstone boson since mean-field theory never does, with the Baker-Johnson evasion of the Goldstone theorem then necessarily having to hold in the mean-field sector. And indeed, for a critical scaling electrodynamics to admit of a mean-field theory structure in the first place, the mean-field theory must thus necessarily be of the Baker-Johnson-evasion type. Nonetheless, as we show below, the related residual interaction will generate a massless pseudoscalar Goldstone boson, and will do so while generating a massive scalar Higgs boson at the same time. Thus by enlarging electrodynamics to include a dynamical four-fermion interaction we are able to evade the Baker-Johnson evasion of the Goldstone theorem, and relate the dynamically generated mass and the spontaneously broken symmetry in the mean-field sector to a Goldstone boson after all. Then, as a very welcome bonus, we obtain a dynamical Higgs boson as well.

While we shall present our derivation of these results below, we note here that all we actually need for the discussion is the behavior of the pure fermion Green's functions containing fermion fields and fermion $\bar{\psi}\psi$ insertions, and for them we only need the assumption of scaling with anomalous dimensions (or equivalently conformal invariance with anomalous dimensions). We do not actually need to specify how the critical scaling was brought about, and thus do not actually need to introduce any explicit coupling to gauge bosons whose associated dynamics could cause coupling constant renormalization beta functions (such as $\beta(\alpha)$) to actually vanish. Our results are thus quite generic, and will continue to hold even if there are many species of fermion (assuming critical scaling of course), so that it thus suffices to discuss a single species of fermion and a single type of symmetry (in our case chiral symmetry) alone. Also we note that we only need to discuss spontaneous breakdown of a global symmetry, since once we have generated a massless Goldstone boson by some dynamical means, Jackiw and Johnson showed [26] that such a dynamically generated Goldstone boson would automatically couple to a massless external gauge field with the relevant quantum numbers, and would put a massless pole in the gauge boson vacuum polarization. This would then cause the gauge boson to become massive, to thus provide an explicit dynamical realization of the Higgs mechanism that was presented in [27–30].

III. THE NAMBU-JONA-LASINIO CHIRAL FOUR-FERMION MODEL

A. Nambu-Jona-Lasinio Model as a Mean-Field Theory

The Nambu-Jona-Lasinio model is a chirally-symmetric four-fermion model of interacting massless fermions with action

$$I_{\rm NJL}=\int d^4x\left[i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-\frac{g}{2}[\bar{\psi}\psi]^2-\frac{g}{2}[\bar{\psi}i\gamma_5\psi]^2\right].\quad(16)$$

As such it is a relativistic generalization of the BCS model. In the mean-field, Hartree-Fock approximation one introduces a trial wave function parameter m that is not in the original action, and then decomposes the action into two pieces, a mean-field piece and a residual interaction according to:

$$\begin{aligned}I_{\rm NJL}&=\int d^4x\left[i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-m\bar{\psi}\psi+\frac{m^2}{2g}\right]\\&+\int d^4x\left[-\frac{g}{2}\left(\bar{\psi}\psi-\frac{m}{g}\right)^2-\frac{g}{2}\left(\bar{\psi}i\gamma_5\psi\right)^2\right]\\&=I_{\rm MF}+I_{\rm RI},\end{aligned}\quad(17)$$

where $I_{\rm MF}$ contains the kinetic energy of a now massive fermion and a self-consistent $m^2/2g$ term. This $m^2/2g$ term acts like a cosmological constant and contributes to the mean-field vacuum energy density. In the mean-field Hartree-Fock approximation one sets

$$\begin{aligned}&\langle\Omega_{\rm m}|\left[\bar{\psi}\psi-\frac{m}{g}\right]^2|\Omega_{\rm m}\rangle=\langle\Omega_{\rm m}|\left[\bar{\psi}\psi-\frac{m}{g}]\right|\Omega_{\rm m}\rangle^2=0,\\&\langle\Omega_{\rm m}|\bar{\psi}\psi|\Omega_{\rm m}\rangle=\frac{m}{g},\quad\langle\Omega_{\rm m}|\bar{\psi}i\gamma_5\psi|\Omega_{\rm m}\rangle=0.\end{aligned}\quad(18)$$

In this approximation one evaluates $\langle\Omega_{\rm m}|\bar{\psi}\psi|\Omega_{\rm m}\rangle$ using $I_{\rm MF}$ alone, with the physical fermion mass M then being the value of m that satisfies the one fermion loop

$$\langle\Omega_{\rm m}|\bar{\psi}\psi|\Omega_{\rm m}\rangle=-i\int\frac{d^4k}{(2\pi)^4}{\rm Tr}\left[\frac{1}{\not{p}-m+i\epsilon}\right]=\frac{m}{g},\quad(19)$$

viz. the gap equation

$$-\frac{M\Lambda^2}{4\pi^2}+\frac{M^3}{4\pi^2}{\rm ln}\left(\frac{\Lambda^2}{M^2}\right)=\frac{M}{g},\quad(20)$$

where Λ is an ultraviolet cutoff, as needed since the Nambu-Jona-Lasinio model is not renormalizable.

Given this gap equation we can calculate the one loop mean-field vacuum energy density $\tilde{\epsilon}(m)=\epsilon(m)-m^2/2g$ as a function of m to obtain

$$\begin{aligned}\tilde{\epsilon}(m)&=i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\ {\rm ln}\left[\frac{\not{p}-m+i\epsilon}{\not{p}+i\epsilon}\right]-\frac{m^2}{2g}\\&=\frac{m^4}{16\pi^2}{\rm ln}\left(\frac{\Lambda^2}{m^2}\right)-\frac{m^2M^2}{8\pi^2}{\rm ln}\left(\frac{\Lambda^2}{M^2}\right)+\frac{m^4}{32\pi^2}.\end{aligned}\quad(21)$$

As we explain below, $\epsilon(m)$ can be constructed as the infinite summation of massless graphs with zero-momentum point $m\bar{\psi}\psi$ insertions (see Fig. 1 below).

We thus see that while the vacuum energy density $i\int d^4p/(2\pi)^4{\rm Tr}\ {\rm ln}[\not{p}-m+i\epsilon]$ has quartic, quadratic and logarithmically divergent pieces, the subtraction of the massless vacuum energy density $i\int d^4p/(2\pi)^4{\rm Tr}\ {\rm ln}[\not{p}+i\epsilon]$

removes the quartic divergence, with the subtraction of the self-consistent induced mean-field term $m^2/2g$ then leaving $\tilde{\epsilon}(m)$ only logarithmically divergent. We shall return to the quartic divergence below when we couple the theory to gravity, but since we are for the moment doing a flat space calculation where only energy differences matter, use of this $\tilde{\epsilon}(m)$ suffices to show that the massive vacuum lies lower than the massless one where $m = 0$. I.e. we recognize the logarithmically divergent $\tilde{\epsilon}(m)$ as having a local maximum at $m = 0$, and a global minimum at $m = M$ where M itself is finite. We thus induce none other than a dynamical double-well potential, and identify M as the matrix element of a fermion bilinear according to $M/g = \langle\Omega_{\rm M}|\bar{\psi}\psi|\Omega_{\rm M}\rangle$.

B. Higgs-Like Lagrangian

While $\tilde{\epsilon}(m)$ has a double-well form familiar from a Higgs model as built out of a Higgs field that is a fundamental, and thus a quantum, field, m itself is not a quantum field. Rather, it is only a c-number matrix element, with $\tilde{\epsilon}(m)$ having a Higgs potential structure even though no fundamental Higgs field is present. As regards a kinetic energy term, we look not at matrix elements in the translationally-invariant vacuum $|\Omega_M\rangle$ but instead at matrix elements in coherent states $|C\rangle$ where $m(x) = \langle C|\bar{\psi}(x)\psi(x)|C\rangle$ is now spacetime dependent. Then we find [31, 32] that the resulting mean-field effective action has a logarithmically divergent part of the form

$$I_{\rm EFF} = \int \frac{d^4x}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)\left[\frac{1}{2}\partial_\mu m(x)\partial^\mu m(x) + m^2(x)M^2 - \frac{1}{2}m^4(x)\right]. \tag{22}$$

If we go further and introduce a coupling $g_{\rm A}\bar{\psi}\gamma_\mu\gamma_5 A_5^\mu\psi$ to an axial gauge field $A_5^\mu(x)$, on setting $\phi = \bar{\psi}(1+\gamma_5)\psi$ the effective action becomes

$$I_{\rm EFF} = \int \frac{d^4x}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)\left[\frac{1}{2}|(\partial_\mu - 2ig_{\rm A}A_{\mu 5})\phi(x)|^2 + |\phi(x)|^2M^2 - \frac{1}{2}|\phi(x)|^4 - \frac{g_{\rm A}^2}{6}F_{\mu\nu 5}F^{\mu\nu 5}\right]. \tag{23}$$

We recognize this action as a double-well Ginzburg-Landau type Higgs Lagrangian with order parameter $\phi(x)$, only now generated dynamically. We thus generalize to the relativistic chiral case Gorkov's derivation of the Ginzburg-Landau order parameter action starting from the BCS four-fermion theory, and see that just as in the theory of superconductivity, there is no need for the Higgs Lagrangian to be built out of a quantized scalar field. In the $I_{\rm EFF}$ effective action associated with the Nambu-Jona-Lasinio model there is a double-well Higgs potential, but since $m(x) = \langle C|\bar{\psi}(x)\psi(x)|C\rangle$ is a c-number, $m(x)$ does not itself represent a q-number scalar field. Rather, as we now show, the q-number fields are to be found as collective modes generated by the residual interaction, with no fundamental scalar field being needed at all.

C. The Collective Tachyon Modes when the Fermion is Massless

To find the collective modes we need to evaluate the vacuum polarizations

$$\begin{aligned}\Pi_{\rm S}(x) &= \langle\Omega|T(\bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0))|\Omega\rangle,\\ \Pi_{\rm P}(x) &= \langle\Omega|T(\bar{\psi}(x)i\gamma_5\psi(x)\bar{\psi}(0)i\gamma_5\psi(0))|\Omega\rangle\end{aligned} \tag{24}$$

associated with the scalar and pseudoscalar sectors, as is appropriate to a chiral-invariant theory. To see why, from the perspective of $\Pi_{\rm S}(x)$ and $\Pi_{\rm P}(x)$, the symmetry needs to be broken, we first evaluate $\Pi_{\rm S}(x)$ and $\Pi_{\rm P}(x)$ on the assumption that the fermion is massless. If we take the fermion to be massless (i.e. setting $|\Omega\rangle = |\Omega_0\rangle$ where $\langle\Omega_0|\bar{\psi}\psi|\Omega_0\rangle = 0$) to one loop order as evaluated using the original $I_{\rm NJL}$ action we obtain

$$\begin{aligned}\Pi_{\rm S}(q^2) &= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\frac{1}{\not{p}+i\epsilon}\frac{1}{\not{p}+\not{q}+i\epsilon}\right],\\ \Pi_{\rm P}(q^2) &= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[i\gamma_5\frac{1}{\not{p}+i\epsilon}i\gamma_5\frac{1}{\not{p}+\not{q}+i\epsilon}\right],\end{aligned} \tag{25}$$

to thus yield

$$\Pi_{\rm S}(q^2) = \Pi_{\rm P}(q^2) = -\frac{\Lambda^2}{4\pi^2} - \frac{q^2}{8\pi^2}\ln\left(\frac{\Lambda^2}{-q^2}\right) - \frac{q^2}{8\pi^2}. \tag{26}$$

The scattering matrices in the two channels are given by iterating the vacuum polarizations according to $T = g + g\Pi g + g\Pi g\Pi g + ...$, to yield

$$\begin{aligned}T_{\rm S}(q^2) &= \frac{g}{1-g\Pi_{\rm S}(q^2)} = \frac{1}{g^{-1}-\Pi_{\rm S}(q^2)},\\ T_{\rm P}(q^2) &= \frac{g}{1-g\Pi_{\rm P}(q^2)} = \frac{1}{g^{-1}-\Pi_{\rm P}(q^2)}.\end{aligned} \tag{27}$$

With g^{-1} being given by the gap equation above, near $q^2 = -2M^2$ both scattering matrices behave as

$$T_{\rm S}(q^2) = T_{\rm P}(q^2) = \frac{Z^{-1}}{(q^2+2M^2)},\ Z = \frac{1}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right) \tag{28}$$

to leading order in the cutoff. We thus obtain degenerate (i.e. chirally symmetric) scalar and pseudoscalar tachyons at $q^2 = -2M^2$ (just like fluctuating around the local maximum in a double-well potential, except that these tachyons are dynamically induced and not put in by hand), with $|\Omega_0\rangle$ thus being unstable. Hence, before determining which vacuum is stable, already we see that if the fermion is massless the theory is unstable.

D. The Collective Goldstone and Higgs Modes when the Fermion is Massive

To find a stable vacuum, we now take the fermion to have non-zero mass M (i.e. we set $|\Omega\rangle = |\Omega_{\rm M}\rangle$). Now we obtain

$$\begin{aligned}&\Pi_{\rm S}(q^2, M)\\&= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\frac{1}{\not{p}-M+i\epsilon}\frac{1}{\not{p}+\not{q}-M+i\epsilon}\right]\\&= -\frac{\Lambda^2}{4\pi^2}+\frac{M^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)+\frac{(4M^2-q^2)}{8\pi^2}\\&+\frac{(4M^2-q^2)}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)-\frac{1}{8\pi^2}\frac{(4M^2-q^2)^{3/2}}{(-q^2)^{1/2}}\\&\times\ln\left(\frac{(4M^2-q^2)^{1/2}+(-q^2)^{1/2}}{(4M^2-q^2)^{1/2}-(-q^2)^{1/2}}\right).\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\Pi_{\rm P}(q^2, M) =\\&-i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[i\gamma_5\frac{1}{\not{p}-M+i\epsilon}i\gamma_5\frac{1}{\not{p}+\not{q}-M+i\epsilon}\right]\\&= -\frac{\Lambda^2}{4\pi^2}+\frac{M^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)-\frac{q^2}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)\\&+\frac{(4M^2-q^2)}{8\pi^2}+\frac{(8M^4-8M^2q^2+q^4)}{8\pi^2(-q^2)^{1/2}(4M^2-q^2)^{1/2}}\\&\times\ln\left(\frac{(4M^2-q^2)^{1/2}+(-q^2)^{1/2}}{(4M^2-q^2)^{1/2}-(-q^2)^{1/2}}\right).\end{aligned} \tag{30}$$

As we see, both $\Pi_{\rm S}(q^2, M)$ and $\Pi_{\rm P}(q^2, M)$ have a branch point at $q^2 = 4M^2$, viz. at the threshold for the creation of a fermion and antifermion pair each with mass M. Given the form for g^{-1}, we find a dynamical pseudoscalar Goldstone boson bound state at $q^2 = 0$ and a dynamical scalar Higgs boson bound state at $q^2 = 4M^2$ ($= -2 \times M^2$(tachyon)), with the two scattering amplitudes behaving near their poles as

$$T_{\rm S}(q^2) = \frac{R_{\rm S}^{-1}}{(q^2-4M^2)}, \qquad T_{\rm P}(q^2) = \frac{R_{\rm P}^{-1}}{q^2}, \tag{31}$$

where

$$R_{\rm S} = R_{\rm P} = \frac{1}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right). \tag{32}$$

(We have labelled the residues $R_{\rm S}^{-1}$ and $R_{\rm P}^{-1}$ to indicate that they are not the previously introduced $Z_{\rm S}^{-1}$ and $Z_{\rm P}^{-1}$.) As we see, the two dynamical bound states are not degenerate in mass (spontaneously broken chiral symmetry), and the dynamical Higgs scalar mass $2M$ is twice the induced mass of the fermion, to thus lie right at the threshold of the fermion-antifermion scattering amplitude.

E. Fixing the Wick Contour for the Vacuum Energy Density

For what is to follow below, we need to make one further comment regarding the evaluation of the vacuum energy density. As discussed for instance in [16], what we have been calling $\epsilon(m)$ is not the energy density of the vacuum. Rather, according to the Gell-Mann-Low adiabatic switching procedure, it is actually an energy density difference

$$\epsilon(m) = \langle\Omega_m|H_m|\Omega_m\rangle - \langle\Omega_0|H_0|\Omega_0\rangle, \tag{33}$$

where $H_m = H_0 + m\bar{\psi}\psi$ is the Hamiltonian density in the presence of the $m\bar{\psi}\psi$ term, while H_0 is the Hamiltonian density in its absence. Specifically, in the adiabatic switching procedure one starts with a Hamiltonian H_0 at time $t = -\infty$ and ground state $|\Omega_0^-\rangle$, switches on a source term such as $m\bar{\psi}\psi$, and then switches the source off at $t = +\infty$, to then return H_0 to its ground state only now in the state $|\Omega_0^+\rangle$. The two states $|\Omega_0^-\rangle$ and $|\Omega_0^+\rangle$ can only differ by a phase, and that phase is given by the energy density difference given in (33). As constructed, this phase could not know what the ground state energy density of H_0 itself might be (it is only gravity that could know), and thus $\epsilon(m)$ could only be an energy density difference. Given (33), we note that since $\langle\Omega_0|\bar{\psi}\psi|\Omega_0\rangle = 0$, we can rewrite $\epsilon(m)$ as

$$\epsilon(m) = \langle\Omega_m|H_m|\Omega_m\rangle - \langle\Omega_0|H_m|\Omega_0\rangle, \tag{34}$$

to put it in the form relevant to the dynamical symmetry breaking of interest to us here.

Diagramatically, $\epsilon(m)$ generates the Green's functions associated with zero-momentum insertions of $\bar{\psi}\psi$, and can be written as

$$\epsilon(m) = \sum\frac{1}{n!}G_0^{(n)}(q_\mu = 0, m = 0)m^n \tag{35}$$

Here $G_0^{(n)}$ is the $\bar{\psi}\psi$ Green's function with n insertions as calculated in the massless H_0 theory, with the $G_0^{(2)}$ and $G_0^{(4)}$ terms for instance being given by

$$\begin{aligned}&G^{(2)}(q_\mu = 0, m = 0) = -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\frac{1}{\not{p}+i\epsilon}\frac{1}{\not{p}+i\epsilon}\right]\\&G^{(4)}(q_\mu = 0, m = 0) =\\&-i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\frac{1}{\not{p}+i\epsilon}\frac{1}{\not{p}+i\epsilon}\frac{1}{\not{p}+i\epsilon}\frac{1}{\not{p}+i\epsilon}\right]\end{aligned} \tag{36}$$

in the Nambu-Jona-Lasinio mean-field case. Formally, the infinite series for $\epsilon(m)$ given in Fig. 1 can be summed, to give

$$\begin{aligned}\epsilon(m) &= i\int\frac{d^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{(-1)}{2n}{\rm Tr}\left[(-i)^2\left(\frac{i}{\not{p}+i\epsilon}\right)^2m^2\right]^n\\&= i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\ln\left[\frac{\not{p}-m+i\epsilon}{\not{p}+i\epsilon}\right],\end{aligned} \tag{37}$$

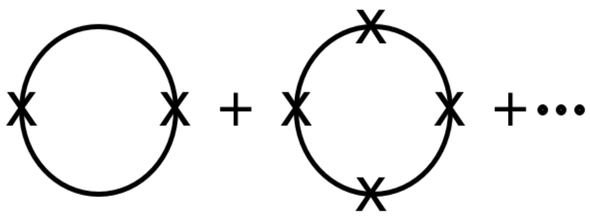

FIG. 1: Vacuum energy density $\epsilon(m)$ via an infinite summation of massless graphs with zero-momentum point $m\bar{\psi}\psi$ insertions.

just as needed for (21).

Despite this, we note that the contour for the p_0 integration in each of the $G_0^{(n)}$ is that associated with a massless Feynman propagator, and not that associated with a massive one. However, for both the $1/(\not{p}+i\epsilon)$ and $1/(\not{p}+m+i\epsilon)$ propagators the poles are located below the real p_0 axis when p_0 is positive, and above the real p_0 axis when p_0 is negative (upper left and lower right quadrants of the complex p_0 plane). Thus in both the cases we can make the same Wick rotation along quarter circles in the upper right and lower left quadrants, to obtain a Wick contour loop that contains no poles within it, to thus yield

$$\int_{-\infty}^{\infty} dp_0 + \int_{\infty}^{i\infty} dp_0 + \int_{i\infty}^{-i\infty} dp_0 + \int_{-i\infty}^{-\infty} dp_0 = 0. \quad (38)$$

Then, if we can drop the two quarter circle at infinity terms we obtain

$$-i\int_{-\infty}^{\infty} dp_0 = -i\int_{-i\infty}^{i\infty} dp_0 = \int_{-\infty}^{\infty} dp_4 \quad (39)$$

where $p_4 = -ip_0$. As constructed, for determining $\epsilon(m)$ we should in general use (35) with its massless fermion propagator contour, and even if we can do the infinite sum and obtain some function of a massive fermion propagator, we should continue to use the same massless fermion contour, i.e. we should Wick rotate every term in (35) before doing the summation. Now for the Nambu-Jona-Lasinio case it did not matter since the massless and massive Wick contour loops coincided, and neither contained any poles. However, as we show below, in the Johnson-Baker-Willey electrodynamics case the two Wick rotations do not coincide (for general $\Sigma(p)$ the $1/(\not{p}-\Sigma(p)+i\epsilon)$ propagator can have a much more complicated structure in the complex p_0 plane), and we must use the massless Wick contour loop since that is what (35) requires. Moreover, for the Johnson-Baker-Willey case the great utility of (35) is that while the scaling solution given in (3) only applies for $p^2 \gg m^2$, if there is to be critical scaling in the massless theory, then scaling forms would hold at all momenta as there is no mass scale in the massless theory. Thus, even if a theory with a mass is only scale invariant for large momenta, its $\bar{\psi}\psi$ Green's functions can be constructed by an infinite summation of graphs all of which are scale invariant for all momenta, and all of which use the massless theory Feynman propagator contour.

F. General Requirements for the Generation of Goldstone and Higgs Bosons

To summarize, given the Baker-Johnson evasion of the Goldstone theorem and the constraints that the renormalization process can produce, we see that in order to generate a Goldstone boson in a renormalizable quantum field theory via dynamical symmetry breaking four conditions need to be met. First, we need to show that the unbroken vacuum possesses a tachyonic instability. Second, we need to show that the fermion mass obeys a self-consistent gap type equation. Third, we need to show that the vacuum associated with the non-trivial solution to the self-consistent gap type equation has lower energy density than the vacuum associated with the trivial solution. And fourth, we need to show that there is in fact a massless pole in the fermion-antifermion scattering amplitude. In Sec. IV we shall show that in Johnson-Baker-Willey electrodynamics coupled to a four-fermion interaction all four of these criteria are met when $\gamma_\theta(\alpha) = -1$.

As regards the Higgs boson, if we can produce a pseudoscalar bound state at all, then in a chirally symmetric theory we must get a scalar bound state as well. The two states will necessarily be degenerate in mass if the symmetry is unbroken. However, when the symmetry is broken, the mass degeneracy of the two states will be lifted, with the scalar bound state necessarily acquiring a mass of order the symmetry breaking scale, so that there is then no hierarchy problem for it, and no need to utilize the breaking of scale invariance to control its mass.

IV. JOHNSON-BAKER-WILLEY ELECTRODYNAMICS COUPLED TO A FOUR-FERMION INTERACTION

A. Vacuum Energy Density for Arbitrary $\gamma_\theta(\alpha)$

As described above, we decompose the QED plus four-fermion Lagrangian $\mathcal{L}_{\rm QED-FF}$ into mean-field and residual interaction pieces according to

$$\begin{aligned}
\mathcal{L}_{\rm QED-FF} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi \\
&-\frac{g}{2}[\bar{\psi}\psi]^2 - \frac{g}{2}[\bar{\psi}i\gamma_5\psi]^2 \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\bar{\psi}\psi + \frac{m^2}{2g} \\
&-\frac{g}{2}\left(\bar{\psi}\psi - \frac{m}{g}\right)^2 - \frac{g}{2}\left(\bar{\psi}i\gamma_5\psi\right)^2 \\
&= \mathcal{L}_{\rm QED-MF} + \mathcal{L}_{\rm QED-RI}.
\end{aligned} \quad (40)$$

According to (35), in order to determine the $\epsilon(m)$ associated with $\mathcal{L}_{\rm QED-MF}$ we need to sum an infinite number of massless theory graphs. With our assumption of critical scaling, as noted in [15, 16] these massless graphs can be obtained from the Nambu-Jona-Lasinio point vertex graphs by replacing point vertices with $\Gamma_{\rm S}(p,p,0) = 1$ by the fully dressed and renormalized $\tilde{\Gamma}_{\rm S}(p,p,0)$. However, since these needed vacuum energy density graphs are massless theory graphs we need the massless theory $\tilde{\Gamma}_{\rm S}(p,p,0)$, and in order to renormalize it we shall use an off-shell renormalization with a parameter μ^2. Then, since there is no scale in the massless theory, the assumption of critical scaling with anomalous dimensions allows us to set

$$\begin{aligned}\tilde{S}^{-1}(p, m=0) &= \not{p} + i\epsilon, \\ \tilde{\Gamma}_{\rm S}(p,p,0) &= \left(\frac{-p^2 - i\epsilon}{\mu^2}\right)^{\gamma_\theta(\alpha)/2}\end{aligned} \qquad (41)$$

for all momenta in the massless theory. Moreover, a critical scaling massless theory will not be just scale invariant, it will be conformal invariant too. Thus as well as requiring all massless theory Green's functions to be constrained by scale invariance and the renormalization group equations, for two- and three-point functions conformal invariance fixes their form completely. Thus in the massless theory we can write the exact relation

$$\begin{aligned}&\langle \Omega_0 | T(\psi(x) : \bar{\psi}(z)\psi(z) : \bar{\psi}(y)) | \Omega_0 \rangle \\ &= \frac{\mu^{-\gamma_\theta}(\not{y} - \not{z})(\not{z} - \not{x})}{[(y-z)^2(z-x)^2]^{(1+d_\theta)/2}[(x-y)^2]^{(3-d_\theta)/2}},\end{aligned} \qquad (42)$$

with the form for $\tilde{\Gamma}_{\rm S}(p,p,0)$ given in (41) then following upon a Fourier transform and an amputation of the external fermion legs [16]. In (42) we should note that the normal ordering is with respect to $|\Omega_0\rangle$, and we can use this normal ordering prescription for all Green's functions other than those related to the vacuum energy density, as the vacuum energy density plays no role in the standard Dyson-Wick expansion of Green's functions. Now we could normalize μ so that it is equal to the eventual dynamical mass M right away, but for tracking where everything comes from it is more convenient to keep it as is until the end.

Given (41), we replace (37) by

$$\begin{aligned}\epsilon(m) &= i\int \frac{d^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{(-1)}{2n} \\ &\times {\rm Tr}\left[(-i)^2\left(\frac{-p^2 - i\epsilon}{\mu^2}\right)^{\gamma_\theta(\alpha)}\left(\frac{i}{\not{p} + i\epsilon}\right)^2 m^2\right]^n \\ &= \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}{\rm Tr}\ \ln\left[1 - \frac{m^2}{p^2 + i\epsilon}\left(\frac{-p^2 - i\epsilon}{\mu^2}\right)^{\gamma_\theta(\alpha)}\right],\end{aligned} \qquad (43)$$

with the infinite summation of massless graphs in Fig. 1 being replaced by the infinite summation in Fig. 2.

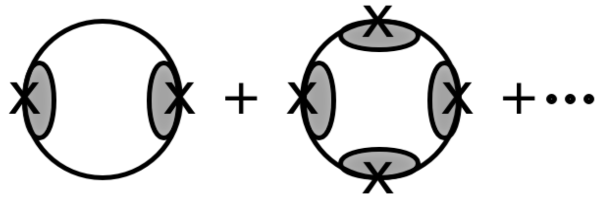

FIG. 2: Vacuum energy density $\epsilon(m)$ via an infinite summation of massless graphs with zero-momentum dressed $m\bar{\psi}\psi$ insertions.

In terms of the quantity

$$\tilde{S}_\mu^{-1}(p) = \not{p} - m\left(\frac{-p^2 - i\epsilon}{\mu^2}\right)^{\gamma_\theta(\alpha)/2} + i\epsilon, \qquad (44)$$

we can rewrite $\epsilon(m)$ as

$$\epsilon(m) = i\int \frac{d^4p}{(2\pi)^4}\left[{\rm Tr}\ \ln(\tilde{S}_\mu^{-1}(p)) - {\rm Tr}\ \ln(\not{p} + i\epsilon)\right]. \qquad (45)$$

Given the form of (45), on comparing with (3) it is suggestive to think of $\tilde{S}_\mu^{-1}(p)$ as the massive theory propagator. However, it cannot be, since, as constructed, (3) only gives the asymptotic form for the massive propagator. Moreover, in the massive theory the renormalization group equation for the Green's function involving a further $\bar{\psi}\psi$ insertion is of the form [20]

$$\begin{aligned}&\left[m\frac{\partial}{m} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_\theta(\alpha)\right]\tilde{\Gamma}_{\rm S}(p,p,0) \\ &= m(1 - \gamma_\theta(\alpha)]\tilde{\Gamma}_{\rm SS}(p,p,0),\end{aligned} \qquad (46)$$

where $\tilde{\Gamma}_{\rm SS}$ contains two zero-momentum $\bar{\psi}\psi$ insertions. Since $\tilde{\Gamma}_{\rm SS}$ is not zero identically, $\tilde{\Gamma}_{\rm S}$ and thus $\tilde{S}^{-1}(p)$ of (3) must have non-leading terms beyond those exhibited in (3). Since on power counting grounds $[m\partial_m + \beta(\alpha)\partial_\alpha + 2\gamma_\theta(\alpha)]\tilde{\Gamma}_{\rm SS}(p,p,0)$ will be related to a $\tilde{\Gamma}_{\rm SSS}$ that contains contains three zero-momentum insertions, $\tilde{\Gamma}_{\rm SS}(p,p,0)$ will acquire a leading term of the form $(-p^2/m^2)^{\gamma_\theta(\alpha)}$, while $\tilde{\Gamma}_{\rm S}(p,p,0)$ will acquire a non-leading term of the form $m(-p^2/m^2)^{\gamma_\theta(\alpha)}$. Consequently, $\tilde{S}^{-1}(p)$ wiil then behave as $\tilde{S}^{-1}(p) = \not{p} - m(-p^2/m^2)^{\gamma_\theta(\alpha)/2} - m^2(-p^2/m^2)^{\gamma_\theta(\alpha)}$. Further non-leading terms would then be generated via the renormalization group for Green's functions with even more insertions. Thus $\tilde{S}_\mu^{-1}(p)$ is not the full fermion propagator for Johnson-Baker-Willey electrodynamics. However, it can serve as the propagator for evaluating Feynman graphs associated with the mean-field Lagrangian of interest to us here, since the vertex function $\tilde{\Gamma}_{\rm S}(p,p,0) = (-p^2/\mu^2)^{\gamma_\theta(\alpha)/2}$ is the exact vertex function in a critical scaling massless theory. However, even in this mean-field theory we cannot take $\tilde{S}_\mu^{-1}(p)$ to be its propagator since it has poles in the complex p_0 plane. These poles would contribute in a Wick rotation. However, as we noted above, the p_0 contour is fixed not by the massive theory but by the underlying massless one. Thus, as we elaborate on in more detail below, we must evaluate (43) using the massless theory Wick contour just as given in (39). Nonetheless, we will continue to utilize the $\tilde{S}_\mu^{-1}(p)$ propagator since it is very convenient for bookkeeping purposes.

B. Vacuum Energy Density for $\gamma_\theta(\alpha) = -1$

When $\gamma_\theta(\alpha) = -1$, evaluation of $\epsilon(m)$ is straightforward, and yields [15, 16]

$$\epsilon(m) = -\frac{m^2\mu^2}{8\pi^2}\left[\ln\left(\frac{\Lambda^2}{m\mu}\right) + \frac{1}{2}\right]. \tag{47}$$

With $\epsilon'(m)$ being equal to $\langle\Omega_m|\bar{\psi}\psi|\Omega_m\rangle$, in the Hartree-Fock approximation we obtain (Fig. 3)

$$\begin{aligned}\langle\Omega_m|\bar{\psi}\psi|\Omega_m\rangle &= \epsilon'(m)\\ &= -i\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[\tilde{\Gamma}_{\mathrm{S}}(p,p,0)\tilde{S}_\mu(p)]\\ &= 4i\int\frac{d^4p}{(2\pi)^4}\frac{m\mu^2}{(p^2+i\epsilon)^2+m^2\mu^2}\\ &= -\frac{m\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{m\mu}\right) = \frac{m}{g}. \end{aligned}\tag{48}$$

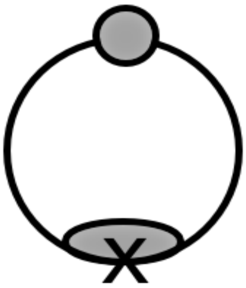

FIG. 3: The $\gamma_\theta(\alpha) = -1$ tadpole graph for $\langle\Omega_m|\bar{\psi}\psi|\Omega_m\rangle$ with a zero-momentum dressed $m\bar{\psi}\psi$ insertion and a dressed $\tilde{S}_\mu(p)$ propagator.

We thus identify the physical mass as the one that satisfies (48) according to the manifestly non-perturbative gap-type equation

$$-\frac{\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{M\mu}\right) = \frac{1}{g}, \qquad M = \frac{\Lambda^2}{\mu}\exp\left(\frac{4\pi^2}{\mu^2 g}\right). \tag{49}$$

Finally, recalling the $m^2/2g$ counter-term in (40), we can write the renormalized mean-field vacuum energy density just as previously given in (10), viz. (see Fig. 4)

$$\tilde{\epsilon}(m) = \epsilon(m) - \frac{m^2}{2g} = \frac{m^2\mu^2}{16\pi^2}\left[\ln\left(\frac{m^2}{M^2}\right) - 1\right], \tag{50}$$

with its local maximum at $m = 0$ and its global minimum at $m = M$. Quite remarkably, with only the one counter-term, $m^2/2g$, as expressly provided by the mean-field theory, we find that $\tilde{\epsilon}(m)$ is completely finite. This then is the power of dynamical symmetry breaking, it generates appropriate counter-terms automatically.

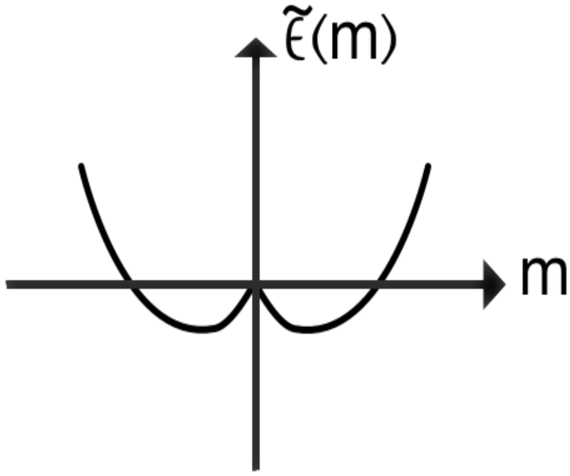

FIG. 4: Dynamically generated double-well potential for the renormalized vacuum energy density when $\gamma_\theta(\alpha) = -1$.

C. Higgs-Like Lagrangian

To develop an analog of a kinetic energy term to add on to $\tilde{\epsilon}(m)$, we need to determine the massive theory $\Pi_{\mathrm{S}}(x, m)$ as defined in (24). In the massless theory first, we can use conformal invariance to determine $\Pi_{\mathrm{S}}(x, m = 0)$ exactly. Thus we set

$$\begin{aligned}&\langle\Omega_0|T(:\bar{\psi}(x)\psi(x)::\bar{\psi}(y)\psi(y):)|\Omega_0\rangle\\ &= \frac{\mu^{-2\gamma_\theta}\mathrm{Tr}[(\not{x}-\not{y})(\not{y}-\not{x})]}{[(x-y)^2]^{(d_\theta+1)/2}[(y-x)^2]^{(d_\theta+1)/2}}.\end{aligned}\tag{51}$$

With an appropriate normalization Fourier transforming then gives

$$\begin{aligned}\Pi_{\mathrm{S}}(q^2, m=0) &= -i\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big[[p^2(p+q)^2]^{\gamma_\theta(\alpha)/4}\\ &\times\frac{1}{\not{p}}[p^2(p+q)^2]^{\gamma_\theta(\alpha)/4}\frac{1}{\not{p}+\not{q}}\Big].\end{aligned}\tag{52}$$

As well as construct $\Pi_{\mathrm{S}}(x, m = 0)$ via conformal invariance we can start with its definition as $\langle\Omega_0|T(:\bar{\psi}(x)\psi(x)::\bar{\psi}(y)\psi(y):)|\Omega_0\rangle$ and make a Dyson-Wick contraction between the fields at x_μ and y_μ. At the one-loop level this then yields

$$\begin{aligned}\Pi_{\mathrm{S}}(q^2, m=0) = -i\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big[\tilde{\Gamma}_{\mathrm{S}}(p+q,p,-q)\\ \times\tilde{S}_\mu(p,m=0)\tilde{\Gamma}_{\mathrm{S}}(p,p+q,q)\tilde{S}_\mu(p+q,m=0)\Big],\end{aligned}\tag{53}$$

where the massless $\tilde{S}_\mu(p, m = 0)$ is given in (41), and where we have introduced

$$\tilde{\Gamma}_{\rm S}(p, p+q, q) = \left[\frac{(-p^2)}{\mu^2}\frac{(-(p+q)^2)}{\mu^2}\right]^{\gamma_\theta(\alpha)/4}. \qquad (54)$$

Now that we know the $\tilde{\Gamma}_{\rm S}(p, p+q, q)$ vertex needed for $\Pi_{\rm S}(q^2, m = 0)$, just as with the infinite summation of massless theory graphs associated with the generation of the massive theory $\epsilon(m)$, the massive theory $\Pi_{\rm S}(q^2, m)$ is also given by an infinite summation. In this summation, apart from the two $\bar{\psi}(x)\psi(x)$ insertions that carry momentum q_μ, all other insertions carry zero-momentum and couple with vertices that are given by (41). The summation thus results in massless fermion propagators being replaced by massive ones according to [17]

$$\Pi_{\rm S}(q^2, m) = -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\tilde{\Gamma}_{\rm S}(p+q, p, -q)$$
$$\times\tilde{S}_\mu(p)\tilde{\Gamma}_{\rm S}(p, p+q, q)\tilde{S}_\mu(p+q)\Big], \qquad (55)$$

where the massive theory $\tilde{S}_\mu(p)$ is given in (44).

As discussed in [32], to now get the coefficient of the kinetic energy associated with a coherent state in which $\langle C|\bar{\psi}\psi|C\rangle = m(x)$ with spacetime dependent $m(x)$, we need to calculate the derivative of $\Pi_{\rm S}(q^2, m(x))$ at $q^2 = 0$. Now even though the massive $\Pi_{\rm S}(-\partial_\mu\partial^\mu, m(x))$ depends on the spacetime coordinates when $m(x)$ depends on the spacetime coordinates, we note that if we develop $\Pi_{\rm S}(-\partial_\mu\partial^\mu, m(x))$ as an infinite sum of massless graphs, for each of those graphs there is no spacetime dependence and we can use momentum space Feynman diagrams. Graphically, in the Nambu-Jona-Lasinio case first we evaluate $\Pi_{\rm S}(-\partial_\mu\partial^\mu, m(x))$ using the summation in Fig. 5, and then in the Johnson-Baker-Willey case we use the summation given in Fig. 6.

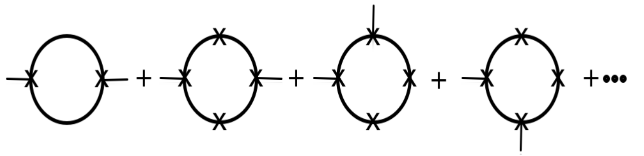

FIG. 5: $\Pi_{\rm S}(q^2, m(x))$ developed as an infinite summation of massless graphs, each with two point $m\bar{\psi}\psi$ insertions carrying momentum q_μ (shown as external lines), with all other point $m\bar{\psi}\psi$ insertions carrying zero momentum.

Via the summation in the Nambu-Jona-Lasino case the kinetic energy term given in (23) was obtained in [16]. In the Johnson-Baker-Willey case with $\gamma_\theta(\alpha) = -1$, an expansion of $\Pi_{\rm S}(q^2 = 0, m)$ around $q^2 = 0$ is algebraically found to give the q^2 derivative $\Pi_{\rm S}'(q^2, m) = -3\mu/128\pi m$ at $q^2 = 0$. This then yields an effective Higgs-like La-

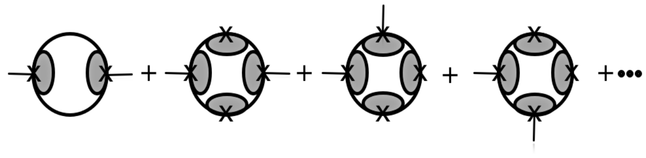

FIG. 6: $\Pi_{\rm S}(q^2, m(x))$ developed as an infinite summation of massless graphs, each with two dressed $m\bar{\psi}\psi$ insertions carrying momentum q_μ (shown as external lines), with all other dressed $m\bar{\psi}\psi$ insertions carrying zero momentum.

grangian of the form [17]

$$\begin{aligned}\mathcal{L}_{\rm EFF} &= -\tilde{\epsilon}(m(x))\\ &-\frac{1}{2}m(x)[\Pi_{\rm S}(-\partial_\mu\partial^\mu, m(x)) - \Pi_{\rm S}(0, m(x))]m(x) + ...\\ &= -\frac{m^2(x)\mu^2}{16\pi^2}\left[\ln\left(\frac{m^2(x)}{M^2}\right) - 1\right]\\ &+\frac{3\mu}{256\pi m(x)}\partial_\mu m(x)\partial^\mu m(x) + ... \qquad (56)\end{aligned}$$

Here the dots denote higher gradient terms, and there is no reason to be concerned about their presence since $m(x)$ is only a c-number, and thus (56) would not be associated with a non-renormalizable or non-local field theory if the higher gradient terms are included. Rather, (56) is generated by dynamical symmetry breaking in a local, renormalizable field theory, one which leads to the expansion given in (56) in which every term is automatically finite. In this way, without every introducing any input fundamental tachyonic mass term, we can generate an effective double-well Higgs Lagrangian, one which could readily be coupled to a gauge field just as in (23).

D. The Collective Tachyon Modes when the Fermion is Massless

To test for tachyons we need to evaluate the massless theory $\Pi_{\rm S}(q^2, m = 0)$ given above and also the pseudoscalar $\Pi_{\rm P}(q^2, m = 0)$, which is given by

$$\begin{aligned}\Pi_{\rm P}(q^2, m = 0) &= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[[p^2(p+q)^2]^{\gamma_\theta(\alpha)/4}i\gamma_5\\ &\times\frac{1}{\not{p}}[p^2(p+q)^2]^{\gamma_\theta(\alpha)/4}i\gamma_5\frac{1}{\not{p}+\not{q}}\Big]. \qquad (57)\end{aligned}$$

When $\gamma_\theta(\alpha) = -1$ a straightforward Wick rotation with spacelike q^2 yields

$$\begin{aligned}\Pi_{\rm S}(q^2, m = 0) &= \Pi_{\rm P}(q^2, m = 0)\\ &= -\frac{\mu^2}{4\pi^2}\left[\ln\left(\frac{\Lambda^2}{(-q^2)}\right) - 3 + 4\ \ln 2\right]. \qquad (58)\end{aligned}$$

With g^{-1} being given in (49), we thus see that both $g^{-1} - \Pi_{\rm S}(q^2, m = 0)$ and $g^{-1} - \Pi_{\rm P}(q^2, m = 0)$ are finite, with the four-fermion interaction thus supplying just the needed counter-term to make both the massless $\Pi_{\rm S}(q^2, m=0)$ and the massless $\Pi_{\rm P}(q^2, m=0)$ be finite.

With the T matrix being given by

$$T_{\rm S}(q^2) = \frac{g}{1-g\Pi_{\rm S}(q^2)} = \frac{1}{g^{-1}-\Pi_{\rm S}(q^2)},$$
$$T_{\rm P}(q^2) = \frac{g}{1-g\Pi_{\rm P}(q^2)} = \frac{1}{g^{-1}-\Pi_{\rm P}(q^2)}, \qquad (59)$$

we see that both the scalar and pseudoscalar scattering matrices have a spacelike pole at

$$q^2 = -M\mu e^{4\ln 2-3} = -0.797M\mu, \qquad (60)$$

with both amplitudes behaving as

$$T_{\rm S}(q^2) = T_{\rm P}(q^2) = \frac{31.448M\mu}{(q^2+0.797M\mu)} \qquad (61)$$

near the tachyonic poles. We thus confirm that the massless vacuum is unstable.

E. The Collective Goldstone Mode when the Fermion is Massive

With the massive theory $\Pi_{\rm S}(q^2, m)$ being given by (55), because of the chiral invariance of the massless theory vertices the analogous massive theory $\Pi_{\rm P}(q^2, m)$ is given by

$$\Pi_{\rm P}(q^2, m) = -i\int \frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\tilde{\Gamma}_{\rm S}(p+q, p, -q)$$
$$\times i\gamma_5\tilde{S}_\mu(p)\tilde{\Gamma}_{\rm S}(p, p+q, q)i\gamma_5\tilde{S}_\mu(p+q)\Big]. \qquad (62)$$

Both the massive $\Pi_{\rm S}(q^2, m)$ and the massive $\Pi_{\rm P}(q^2, m)$ are logarithmically divergent when $\gamma_\theta(\alpha) = -1$, with the divergence being the same as that of the massless $\Pi_{\rm S}(q^2, m=0)$ and $\Pi_{\rm P}(q^2, m=0)$ since the large momentum behavior of the Green's functions is not sensitive to the fermion mass. Consequently, both $g^{-1} - \Pi_{\rm S}(q^2, m)$ and $g^{-1} - \Pi_{\rm P}(q^2, m)$ are finite, with the four-fermion interaction thus supplying just the needed counter-term to make both the massive $\Pi_{\rm S}(q^2, m)$ and the massive $\Pi_{\rm P}(q^2, m)$ be finite.

When $\gamma_\theta(\alpha) = -1$, $\Pi_{\rm S}(q^2, m)$ and $\Pi_{\rm P}(q^2, m)$ evaluate to

$$\Pi_{\rm S}(q^2, m) = -4i\mu^2\int \frac{d^4p}{(2\pi)^4}\frac{N(q,p)+m^2\mu^2}{D(q,p,m)},$$
$$\Pi_{\rm P}(q^2, m) = -4i\mu^2\int \frac{d^4p}{(2\pi)^4}\frac{N(q,p)-m^2\mu^2}{D(q,p,m)}, \qquad (63)$$

where

$$N(q,p) = (p^2+i\epsilon-q^2/4)$$
$$\times(-(p-q/2)^2-i\epsilon)^{1/2}(-(p+q/2)^2-i\epsilon)^{1/2},$$
$$D(q,p,m) = (((p-q/2)^2+i\epsilon)^2+m^2\mu^2)$$
$$\times(((p+q/2)^2+i\epsilon)^2+m^2\mu^2). \qquad (64)$$

(In (63) and (64) we have conveniently translated p_μ to $p_\mu - q_\mu/2$.)

On now evaluating $\Pi_{\rm P}(q^2, m)$ at $q^2 = 0$ we obtain

$$\Pi_{\rm P}(q^2=0, m) = -4i\mu^2\int \frac{d^4p}{(2\pi)^4}\frac{(p^2)(-p^2)-m^2\mu^2}{((p^2+i\epsilon)^2+m^2\mu^2)^2}.$$
$$= 4i\mu^2\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+i\epsilon)^2+m^2\mu^2}. \qquad (65)$$

On comparing with (48) and (49) we see that when m is equal to M, $\Pi_{\rm P}(q^2 = 0, M)$ is equal to none other than g^{-1}. In the pseudoscalar $T_{\rm P}(q^2)$ channel we thus obtain our sought-after massless pseudoscalar Goldstone boson. Finally, with an expansion of $\Pi_{\rm P}(q^2 = 0, M)$ around $q^2 = 0$ algebraically being found to give the q^2 derivative $\Pi'_{\rm P}(q^2, M) = -7\mu/128\pi M$ at $q^2 = 0$, near the Goldstone pole $T_{\rm P}(q^2)$ is found to evaluate to

$$T_{\rm P}(q^2) = \frac{128\pi M}{7\mu q^2} = \frac{57.446M}{\mu q^2}. \qquad (66)$$

Now at $\gamma_\theta(\alpha) = -1$ the quantity g^{-1} is infinite (as counter-terms need to be) and g itself is zero. Thus even though the $\Pi_{\rm P}$-independent homogeneous term in $T_{\rm P} = g + g\Pi_{\rm P}g + g\Pi_{\rm P}g\Pi_{\rm P}g + ... = g/(1-g\Pi_{\rm P})$ would be zero, nonetheless the interplay between the numerator and the denominator still enables a pole to be generated. Thus, as we had noted above, even if the homogeneous term in a scattering amplitude iteration vanishes there still could be a pole. Thus in conclusion we note that even though Johnson-Baker-Willey electrodynamics does not on its own have a Goldstone boson pole, when it is coupled to the four-fermion interaction it then does.

F. The Collective Higgs Mode when the Fermion is Massive – the Needed Contour

Because we were able to show that $\Pi_{\rm P}(q^2 = 0, M)$ and g^{-1} were identically equal, we did not actually need to explicitly evaluate either quantity, and thus to establish the presence of a Goldstone pole we did not need to explicitly specify the contour needed for the p_0 integration. To show that there is a Higgs boson pole in the scalar channel we will need to specify the contour and will need to evaluate the $\Pi_{\rm S}(q^2, M)$ integral explicitly, since, unlike in the Goldstone case where there is an axial-vector Ward identity, there appears to be no general theorem or relevant Ward identity that would tell us a priori what value the mass of a dynamical Higgs boson should be. Since each massive fermion graph is an infinite sum of massless fermion graphs, as noted above, the massive theory inherits its contour from the massless one. For the massless case we note that on translating p_μ to $p_\mu - q_\mu/2$ the massless $\Pi_{\rm S}(q^2, m=0)$ given in (52) evaluates to

$$\Pi_{\rm S}(q^2, m=0) = -4i\mu^2\int \frac{d^4p}{(2\pi)^4}\frac{N(q,p)}{D(q,p,m=0)}, \qquad (67)$$

when $\gamma_\theta(\alpha) = -1$, with $N(q,p)$ and $D(q,p)$ being given in (64). The integrand in $\Pi_{\rm S}(q^2, m=0)$ has both poles and branch points, the poles coming from the zeroes of $D(q,p,m=0)$ and the branch points from the zeroes of $N(q,p)$.

For spacelike q_μ we set $q_\mu = (0,0,0,q_3)$ and find that all poles and branch points are in the lower right and upper left quadrants in the complex p_0 plane. Consequently, for spacelike q_μ we can use the Wick contour loop given in (39) as is since there are no poles within the loop, and indeed we already did so when we tested for tachyons.

For timelike q_μ we set $q_\mu = (q_0,0,0,0)$ with $q_0 \geq 0$, to find poles at

$$p_0 = q_0/2 + p - i\epsilon, \qquad p_0 = -q_0/2 + p - i\epsilon,$$
$$p_0 = q_0/2 - p + i\epsilon, \qquad p_0 = -q_0/2 - p + i\epsilon. \quad (68)$$

The $p_0 = q_0/2 + p - i\epsilon$ pole is always in the lower right quadrant in the complex p_0 plane, and the $p_0 = -q_0/2 - p + i\epsilon$ pole is always in the upper left quadrant. If $p > q_0/2$ the $p_0 = -q_0/2 + p - i\epsilon$ pole is in the lower right quadrant and the $p_0 = q_0/2 - p + i\epsilon$ pole is in the upper left quadrant. However, if $p < q_0/2$ the $p_0 = -q_0/2 + p - i\epsilon$ pole migrates to the lower left quadrant and the $p_0 = q_0/2 - p + i\epsilon$ pole migrates to the upper right quadrant.

The pattern of $N(q,p) = 0$ branch points completely follows the same pattern as that of the poles. By taking branch cuts to terminate at either end at branch points, we will have two branch cuts in total. We shall take one branch cut to run between the two branch points in the upper half p_0 plane and the other to run between the two branch points in the lower half plane. Thus for $p > q_0/2$ all poles and branch cuts are in the upper left and lower right quadrants, and so we can make the standard Wick rotation given in (39) as is as per Fig. 7. However, for $p < q_0/2$ we will in addition need to circumnavigate the branch points and poles that have migrated to the upper right and lower left half planes. Since the branch points and poles have migrated from the upper left and lower right planes into the upper right and lower left planes, as they migrate we must deform the Wick contour loop so that no singularities enter the loop as per Fig. 8.

For timelike q_μ the full Wick contour loop is then the standard one given in (38) and (39), as augmented with an integration above the cut from $p_0 = 0$ to the branch point at $p_0 = q_0/2 - p$, then round this branch point followed by an integration to the branch point at $p_0 = -q_0/2 + p$, then round this branch point and back to $p_0 = 0$. This contour does not enclose any of the poles (they have also been circumnavigated), and thus we can write

$$-i\int_{-\infty}^{\infty} dp_0 = \int_{-\infty}^{\infty} dp_4 + I_{\rm cut}. \quad (69)$$

Consequently, the full cut contribution is given by four times the first section, viz.

$$I_{\rm cut} = -\frac{4i\mu^2}{\pi^3}\int_0^{q_0/2} dp p^2 \int_0^{q_0/2-p} dp_0 \frac{N(q_0,p,p_0)}{D(q_0,p,p_0,m)}. \quad (70)$$

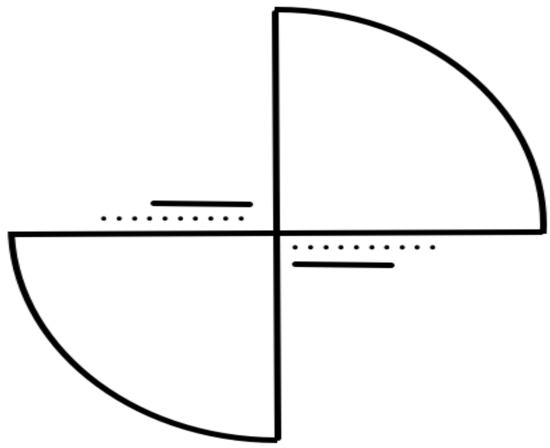

FIG. 7: The standard Wick contour. The branch cuts are shown as lines and the poles as dots.

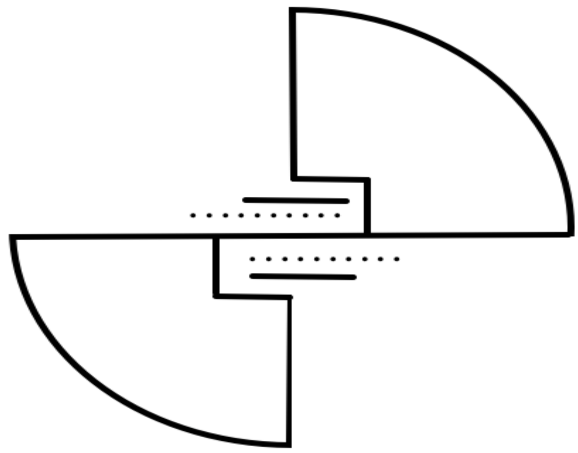

FIG. 8: The migrated Wick contour. The branch cuts are shown as lines and the poles as dots.

The imaginary p_0 axis contribution, $I_{\rm Wick}$, is given by

$$I_{\rm Wick} = \frac{\mu^2}{\pi^3}\int_0^{\infty} dp p^2 \int_{\infty}^{\infty} dp_4 \frac{N(q_0,p,p_4) + m^2\mu^2}{D(q_0,p,p_4,m)}. \quad (71)$$

Now while p^2 is spacelike along the p_4 axis, in the cut region both $(p_0 - q_0/2)^2 - p^2$ and $(p_0 + q_0/2)^2 - p^2$ are timelike. Thus while we recognize both $[-((p_0 - q_0/2)^2 - p^2)]^{1/2}$ and $[-((p_0 + q_0/2)^2 - p^2)]^{1/2}$ as being real and positive on the p_4 axis, given that $\tilde{\Gamma}_{\rm S}(p,p,0) = (-p^2/\mu^2)^{\gamma_\theta(\alpha)/2}$, each of the two square roots should be interpreted with an extra factor of i in the timelike case. Thus while the net square root factor in $N(q_0,p,p_4)$ is positive definite, the net square root factor in $N(q_0,p,p_0)$ possesses an overall minus sign.

To appreciate the nature and sense of the contour it is instructive to change the location of the branch cut in the massless theory $\tilde{\Gamma}_{\rm S}(p,p,0)$ as given in (41) by replacing it by

$$\tilde{\Gamma}_{\rm S}(p,p,0) \;=\; \left(\frac{p^2+i\epsilon}{\nu^2}\right)^{\gamma_\theta(\alpha)/2} \tag{72}$$

In this case the mean-field theory effective propagator given in (44) would be replaced by

$$\tilde{S}_\nu^{-1}(p) \;=\; \not{p} - m\left(\frac{p^2+i\epsilon}{\nu^2}\right)^{\gamma_\theta(\alpha)/2} + i\epsilon. \tag{73}$$

As a function of a complex variable, $\tilde{S}_\nu(p)$ has poles at $p_0^2-p^2+i\epsilon = m\nu$ and at the tachyonic $p_0^2-p^2+i\epsilon = -m\nu$ when $\gamma_\theta(\alpha) = -1$. All the poles in $p_0^2-p^2+i\epsilon = m\nu$ lie in the lower right and upper left quadrants in the complex p_0 plane, as do all the poles in $p_0^2-p^2+i\epsilon = -m\nu$ if $p > (m\nu)^{1/2}$. However for $p < (m\nu)^{1/2}$ the poles migrate to $p_0 = \pm i(m\nu - p^2)^{1/2} \mp \epsilon$. While these poles lie on the imaginary axis they are slightly displaced from it into the upper left and lower right quadrants. Consequently, none of the poles in $\tilde{S}_\nu(p)$ lie inside the standard Wick contour loop. For this propagator we can thus Wick rotate as per (38) and (39). Suppose we now continue back from ν^2 to μ^2. When we do so the poles in $\tilde{S}_\nu(p)$ will move into the complex plane according to $p_0^2-p^2+i\epsilon \pm im\mu = 0$, and in particular some will move into the upper right and lower left quadrants. Thus when we make this continuation we must at the same time deform the Wick contour loop so that it continues to contain no poles. We thus consider the upper right and left quadrant poles to be in a zone of avoidance. To specify this zone exactly we note that the poles of $p_0^2 = p^2 \pm im\mu$ are given as

$$\begin{aligned} p_0 \;&=\; \frac{1}{2^{1/2}}\left[(p^4+m^2\mu^2)^{1/2}+p^2)\right]^{1/2} \\ &\pm\; \frac{i}{2^{1/2}}\left[(p^4+m^2\mu^2)^{1/2}-p^2)\right]^{1/2} \\ p_0 \;&=\; -\frac{1}{2^{1/2}}\left[(p^4+m^2\mu^2)^{1/2}+p^2)\right]^{1/2} \\ &\mp\; \frac{i}{2^{1/2}}\left[(p^4+m^2\mu^2)^{1/2}-p^2)\right]^{1/2} \end{aligned} \tag{74}$$

The poles in the upper right quadrant thus lie in a region that begins at $p=0$ where $p_0 = (m\mu)^{1/2}(1+i)/2^{1/2}$, with an imaginary part that falls off as p increases, reaching zero at $p=\infty$ where the real part of the location of the pole becomes infinite, with the zone of avoidance thus being wedge shaped. An analogous situation exists for the poles in the lower left quadrant. Thus if we want to define a contour for the massive theory with the $\tilde{S}_\mu(p)$ propagator, for Green's functions such as $\Pi_{\rm S}(q^2,m)$ we must define the p_0 integration to run not along the real axis, but rather to skirt the zones of avoidance in the lower left and upper right quadrants as per Fig. 9 by going around them so that no poles are then picked up in the Wick contour loop. In this way the complex p_0 plane poles in $\tilde{S}_\mu(p)$ do not play a physical role in the Wick contour loop needed for $\Pi_{\rm S}(q^2,m)$.

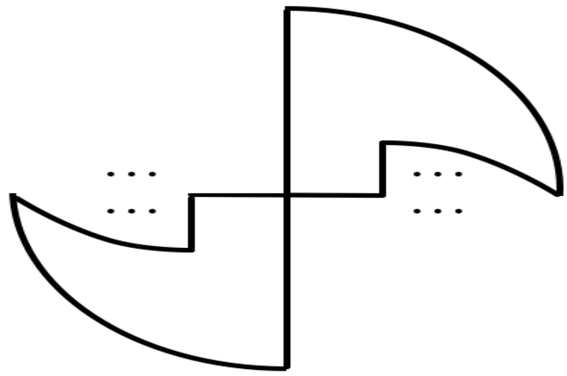

FIG. 9: The deformed Wick contour needed for $\tilde{S}_\nu(p)$. Poles are shown as dots.

The complex p_0 plane poles in $\tilde{S}_\mu(p)$ would however play a role if we want to integrate using a Feynman contour. For $\tilde{S}_\nu(p)$ first, the Feynman contour is obtained by closing below the real p_0 axis and integrating along a semicircle in the lower half plane. This contour would then include all poles with $\mathrm{Re}[p_0] > 0$, and for $\tilde{S}_\nu(p)$ all would have $\mathrm{Im}[p_0] < 0$. If we now continue to $\tilde{S}_\mu(p)$ we would continue to include all poles with $\mathrm{Re}[p_0] > 0$. This would require us to include the zone of avoidance in $\mathrm{Re}[p_0] > 0$ but not include the zone of avoidance with $\mathrm{Re}[p_0] < 0$. Thus for $\mathrm{Re}[p_0] > 0$ the Feynman contour is the compliment of the Wick contour, while for $\mathrm{Re}[p_0] < 0$ the Feynman contour is the same as the Wick contour

The general rule then for all of the cases described above is that the Wick contour loop integration is always to be defined as being the contour that contains no poles and circumnavigates all upper right and lower left quadrant cuts. For all the cases this will always yield (69). Similarly, the Feynman contour is to always be defined as the contour that includes all poles with $\mathrm{Re}[p_0] > 0$.

Finally, since the p_0 contours are different for spacelike and timelike q_μ, we cannot first evaluate $\Pi_{\rm S}(q^2,m)$ for spacelike q_μ (say using Feynman parameters for amplitudes with Euclidean p_μ and q_μ) and then continue the resulting answer to timelike q_μ since we would miss the migrated cuts, with the spacelike and timelike q_μ Wick contour loops being different. We will thus need to evaluate the timelike q_μ case directly.

G. The Collective Higgs Mode when the Fermion is Massive – Results

For timelike q_μ we shall explicitly evaluate $I_{\rm Wick}$ and $I_{\rm cut}$ in detail in the appendix, and will show there that as a function of $q_0^2 = q^2$, $I_{\rm Wick}$ has a branch point at $q^2 = 2m\mu$. While we will show thus explicitly in the appendix, it may be understood heuristically by noting that at $p=0$ the massive $\tilde{S}_\mu(p)$ propagator has poles at $p_0 = (1+i)(m\mu)^{1/2}/2^{1/2}$ and at $p_0 = (1-i)(m\mu)^{1/2}/2^{1/2}$, and thus a particle-antiparticle threshold at $q^2 = ((1+$

$i)(m\mu)^{1/2}/2^{1/2} + (1-i)(m\mu)^{1/2}/2^{1/2})^2 = 2m\mu$. For q^2 below this threshold the integrands in both $I_{\rm Wick}$ and $I_{\rm cut}$ as given in (71) and (70) are real. Consequently, with g^{-1} being real and with $I_{\rm cut}$ itself possessing an overall factor of i, there cannot be any bound state Higgs boson pole at or below $q^2 = 2m\mu$ (unless both $I_{\rm Wick} - 1/g$ and $I_{\rm cut}$ just happen to vanish at some common value of q^2 in that region – though this turns out not to be the case). However, while the integrand in $I_{\rm cut}$ remains real above the $q^2 = 2m\mu$ threshold so that $I_{\rm cut}$ itself remains pure imaginary, the integrand in $I_{\rm Wick}$ becomes complex, and then one can find a pole. Any solution above the threshold must thus satisfy $g^{-1} - {\rm Re}[I_{\rm Wick}] - {\rm Im}[I_{\rm Wick}] - I_{\rm cut} = 0$. The Higgs boson must thus be a resonance, with its width then being fixed by $I_{\rm cut}$. With the actual integrals only being doable numerically, in the appendix we show that there is an explicit solution, with our sought-after dynamical massive scalar Higgs boson being a narrow resonance lying below the real axis in the complex q^2 plane, with parameters

$$\begin{aligned} q_0({\rm Higgs}) &= (1.480 - 0.017i)(M\mu)^{1/2}, \\ q^2({\rm Higgs}) &= (2.189 - 0.051i)M\mu. \end{aligned} \tag{75}$$

We had noted above that we always had the freedom to normalize μ to M. On now doing so, the Higgs boson parameters become

$$\begin{aligned} q_0({\rm Higgs}) &= (1.480 - 0.017i)M, \\ q^2({\rm Higgs}) &= (2.189 - 0.051i)M^2, \end{aligned} \tag{76}$$

to thus naturally be of order the fermion mass scale. Thus even though the $\Pi_{\rm S}$-independent homogeneous term in $T_{\rm S} = g + g\Pi_{\rm S}g + g\Pi_{\rm S}g\Pi_{\rm S}g + ... = g/(1-g\Pi_{\rm S})$ is zero (g^{-1} being divergent according to (49)), nonetheless we again see that the vanishing of the homogeneous term in the scattering amplitude need not prevent the presence of a pole.

In the literature attention has focussed on the fact that in the Nambu-Jona-Lasinio model the dynamical Higgs boson is a stable bound state that lies right at the particle-antiparticle threshold with a mass twice that of the dynamical fermion. However, as we see, this is not a generic feature of dynamical symmetry breaking, and in fact it could only possibly occur if the scattering amplitude is purely real at the threshold. For a point coupled theory such as the Nambu-Jona-Lasinio model, this is in fact the case. However, once we give the coupling some momentum dependence the dynamical Higgs boson could move away from the particle-antiparticle threshold, and could potentially become a resonance rather than a stable bound state.

H. Distinguishing a Dynamical Higgs Boson from a Fundamental One

If the Higgs boson is to be dynamical, it would be very instructive to identify some way to distinguish it from a fundamental Higgs boson. Also we would need to account for the fact that a fundamental Higgs field theory works so well in weak interactions. To this end let us consider the path integral representation of the generator $Z(\bar{\eta},\eta)$ of fermion Green's functions associated with the fermion sector of the $\mathcal{L}_{\rm QED-FF}$ Lagrangian given in (40), viz.

$$\begin{aligned} Z(\bar{\eta},\eta) &= \int [d\bar{\eta}d\eta] \exp\Bigg[i\int d^4x \bigg(\bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi \\ &- \frac{g}{2}(\bar{\psi}\psi)^2 + \bar{\eta}\psi + \bar{\psi}\eta \bigg)\Bigg], \end{aligned} \tag{77}$$

with Grassmann sources η and $\bar{\eta}$. (For simplicity we have left out the $(g/2)(\bar{\psi}i\gamma_5\psi)^2$ term present in (40), though it could be incorporated via a dummy pseudoscaler field if desired.) Via Gaussian path integration on a dummy scalar field variable σ, $Z(\bar{\eta},\eta)$ can be rewritten as

$$\begin{aligned} Z(\bar{\eta},\eta) &= \int [d\bar{\eta}d\eta d\sigma] \exp\Bigg[i\int d^4x \bigg(\bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi \\ &- \frac{g}{2}(\bar{\psi}\psi)^2 + \frac{g}{2}\left(\frac{\sigma}{g} - \bar{\psi}\psi\right)^2 + \bar{\eta}\psi + \bar{\psi}\eta \bigg)\Bigg], \end{aligned} \tag{78}$$

and thus as

$$\begin{aligned} Z(\bar{\eta},\eta) &= \int [d\bar{\eta}d\eta d\sigma] \exp\Bigg[i\int d^4x \bigg(\bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi \\ &- \sigma\bar{\psi}\psi + \frac{\sigma^2}{2g} + \bar{\eta}\psi + \bar{\psi}\eta \bigg)\Bigg]. \end{aligned} \tag{79}$$

We recognize (79) as having the same structure as the mean-field Lagrangian $\mathcal{L}_{\rm QED-MF}$ given in (40). Thus the fermion Green's functions of the $\mathcal{L}_{\rm QED-FF}$ theory of interest to us in this paper are given as the fermion Green's functions of a Yukawa-coupled scalar field theory. In consequence, diagramatically the perturbative expansions associated with (79) and with a theory with a fundamental scalar field are in one to one correspondence. However, in (79) there is no source term $J\sigma$ for the scalar field (in a true fundamental Higgs Lagrangian there would be such a source term), and thus (79) only generates Green's functions with external fermion legs and does not generate any Green's functions with external scalar field legs. Thus in the dynamical Higgs case one can generate the fermion Green's functions using a scalar field theory in which the only role of the scalar field is to contribute internally in Feynman diagrams and to never appear in any external legs. From the perspective of (79) it would be the all-order iteration of internal σ exchange diagrams in $Z(\bar{\eta},\eta)$ that then generates the dynamical Higgs and Goldstone poles that we have found in $T_{\rm S}(q^2)$ and $T_{\rm P}(q^2)$. The only distinction between (79) and a fundamental Higgs field theory would be in those weak interaction processes in which the Higgs boson goes on shell. While beyond the scope of the present paper, it would be very instructive to determine what such differences might then look like.

V. CONFORMAL SYMMETRY CHALLENGES SUPERSYMMETRY

A. Cancellation of Infinities

Because of the Fermi statistics of half-integer spin particles and the Bose statistics of integer spin particles, the Feynman diagrams of closed fermion loops and closed boson loops have opposite overall signs. Consequently, they are able to cancel each others' perturbative infinities to some degree. This can occur not just in supersymmetry but also in supergravity, its local extension (a recent review of cancellations in the supergravity case may be found in [33]).

To compare and contrast with conformal symmetry, we note that with critical scaling there is also a cancellation of infinities. However, it does not occur order by order in perturbation theory. Rather, it is only achieved non-perturbatively via an infinite summation of diagrams. In this paper we have encountered four examples of this, the finiteness of the gauge boson wave function renormalization constant Z_3, the form for m_0 given in (1), the structure of $T_{\rm S}(q^2)$ and $T_{\rm P}(q^2)$ in both the massless and massive cases, and the form for $\tilde{\epsilon}(m)$ as given in (50).

For Z_3 the finiteness is achieved immediately just by being at a critical point where $\beta(\alpha) = 0$. For m_0 it is instructive to expand (1) as

$$m_0 = m\left[1+\frac{\gamma_\theta(\alpha)}{2}\ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{\gamma_\theta^2(\alpha)}{8}\ln^2\left(\frac{\Lambda^2}{m^2}\right)+....\right]. \qquad (80)$$

In this expansion all the radiative correction terms individually diverge. However because of critical scaling the coefficients of these terms are such that their non-perturbative sum exponentiates, with the sum itself then being finite if $\gamma_\theta(\alpha)$ is negative. Thus if one were to write (80) as some low-order perturbative term plus a counter-term, the counter-term would then represent the rest of the series. Thus in the language of perturbation theory, critical scaling uniquely fixes the needed counter-term. Since the cancellation is really a cancellation of infinities in the vertex renormalization constant $Z_{\rm S} = Z_\theta^{-1/2} = (\Lambda^2/\mu^2)^{\gamma_\theta(\alpha)/2}$ that multiplicatively renormalizes the massless theory $\Gamma_{\rm S}(p,p,0) = (\Lambda^2/p^2)^{-\gamma_\theta(\alpha)/2}$ to give $\tilde{\Gamma}_{\rm S}(p,p,0) = (p^2/\mu^2)^{\gamma_\theta(\alpha)/2}$, it is a purely ultraviolet effect. Thus it can occur in either a massless theory or in the short-distance behavior of a massive theory, with it not being sensitive to any mass generation that might be taking place in the infrared.

As regards $T_{\rm S}$ and $T_{\rm P}$ we note that in the series $T = g + g\Pi g + g\Pi g\Pi g + ...$ every term is divergent, and quadratically so in the Nambu-Jona-Lasinio case as per (20) and (26). If $\gamma_\theta(\alpha) = -1$, then as per (49) and (63), term by term in the series the divergence is brought down to logarithmic. The total sum however is finite. With g^{-1} having been fixed as the Hartree-Fock condition $\langle\Omega_{\rm M}|\bar{\psi}\psi|\Omega_{\rm M}\rangle = M/g$, the cancellation involves an interplay between short-distance and long-distance effects, and is expressly sensitive to the mass generation mechanism. Thus the $\gamma_\theta(\alpha) = -1$ condition reduces the divergences in $\Pi_{\rm S}(q^2, M)$, $\Pi_{\rm P}(q^2, M)$, and $\langle\Omega_M|\bar{\psi}\psi|\Omega_M\rangle$ to logarithmic, with the infrared Hartree-Fock condition $\langle\Omega_M|\bar{\psi}\psi|\Omega_M\rangle = M/g$ then leading to completely finite scattering amplitudes $T_{\rm S}$ and $T_{\rm P}$. Thus in the language of perturbation theory, critical scaling plus symmetry breaking uniquely fixes the needed counter-terms.

Exactly the same set of cancellations is found to occur for $\tilde{\epsilon}(m)$ as well. As evidenced in (50), the $\gamma_\theta(\alpha) = -1$ condition reduces the divergence in $\epsilon(m)$ from quadratic to logarithmic, with the symmetry breaking then generating precisely the needed $m^2/2g$ counter-term to make $\tilde{\epsilon}(m)$ completely finite. Welcome as this is, nonetheless, left out from this discussion is the vacuum energy density quartic divergence to which we alluded before. And so it is to this issue that we now turn.

B. Supersymmetry Treatment of the Vacuum Energy Density

There are two separate issues for the vacuum energy density. First, simply because a matter field energy-momentum tensor is composed of products of quantum fields at the same spacetime point, there is a zero-point problem. This problem already occurs in a massless theory with a normal vacuum. And second, when one generates mass via symmetry breaking, not only does the zero-point vacuum energy density change, in addition a cosmological constant term is produced.

To illustrate the issues that are involved, it is convenient to first look at the vacuum expectation value of the energy-momentum tensor

$$T^{\mu\nu}_{\rm M} = i\hbar\bar{\psi}\gamma^\mu\partial^\nu\psi \qquad (81)$$

of a free fermion matter field of mass $m = 0$ in flat, four-dimensional spacetime, with the fermion obeying the massless Dirac equation. With $k^\mu = (\omega_k, \bar{k})$ where $\omega_k = k$, following a Feynman contour integration in the complex frequency plane the vacuum matrix element evaluates to

$$\langle\Omega_0|T^{\mu\nu}_{\rm M}|\Omega_0\rangle = -\frac{2\hbar}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\frac{k^\mu k^\nu}{\omega_k}. \qquad (82)$$

With its $k^\mu k^\nu$ structure $\langle\Omega_0|T^{\mu\nu}_{\rm M}|\Omega_0\rangle$ has the generic form of a perfect fluid with a timelike fluid velocity vector $U^\mu = (1, 0, 0, 0)$, viz.

$$\langle\Omega_0|T^{\mu\nu}_{\rm M}|\Omega_0\rangle = (\rho_{\rm M} + p_{\rm M})U^\mu U^\nu + p\eta^{\mu\nu}, \qquad (83)$$

where

$$\rho_{\rm M} = \langle\Omega_0|T^{00}_{\rm M}|\Omega_0\rangle = -\frac{2\hbar}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\omega_k, \qquad (84)$$

$$p_{\rm M} = \langle\Omega_0|T^{11}_{\rm M}|\Omega_0\rangle = \langle\Omega_0|T^{22}_{\rm M}|\Omega_0\rangle = \langle\Omega_0|T^{33}_{\rm M}|\Omega_0\rangle = -\frac{2\hbar}{3(2\pi)^3}\int_{-\infty}^{\infty}d^3k\frac{k^2}{\omega_k}. \quad (85)$$

The zero-point energy density $\rho_{\rm M}$ and the zero-point pressure $p_{\rm M}$ are related by the tracelessness condition

$$\eta_{\mu\nu}\langle\Omega_0|T^{\mu\nu}_{\rm M}|\Omega_0\rangle = 3p_{\rm M} - \rho_{\rm M} = 0 \quad (86)$$

since $\eta_{\mu\nu}k^\mu k^\nu = 0$. (We use diag$[\eta_{\mu\nu}] = (-1,1,1,1)$ here and in the discussion of gravity below.) Since $p_{\rm M}$ is not equal to $-\rho_{\rm M}$, the zero-point energy-momentum tensor does not have the form of a cosmological constant term, to underscore that fact that the zero-point problem is distinct from the cosmological constant problem.

With both $\rho_{\rm M}$ and $p_{\rm M}$ being divergent, in terms of a 3-momentum cutoff K the divergences can be parametrized as the quartic divergences

$$\rho_{\rm M} = -\frac{\hbar K^4}{4\pi^2}, \qquad p_{\rm M} = -\frac{\hbar K^4}{12\pi^2}. \quad (87)$$

Cancellation of these mass-independent quartic divergences is readily achieved in supersymmetry since a massless boson loop has the opposite sign to a massless fermion loop.

However, the situation changes once the fermion acquires mass. For a free massive fermion in flat spacetime with vacuum $|\Omega_{\rm M}\rangle$ the form of the energy-momentum tensor remains unchanged but the Dirac equation becomes that of a massive fermion. Then, with $k^\mu = ((k^2+m^2/\hbar^2)^{1/2},\bar{k})$, $\rho_{\rm M}$ and $p_{\rm M}$ now evaluate to

$$\rho_{\rm M} = -\frac{\hbar K^4}{4\pi^2} - \frac{m^2K^2}{4\pi^2\hbar} + \frac{m^4}{16\pi^2\hbar^3}\ln\left(\frac{4\hbar^2K^2}{m^2}\right) - \frac{m^4}{32\pi^2\hbar^3},$$
$$p_{\rm M} = -\frac{\hbar K^4}{12\pi^2} + \frac{m^2K^2}{12\pi^2\hbar} - \frac{m^4}{16\pi^2\hbar^3}\ln\left(\frac{4\hbar^2K^2}{m^2}\right) + \frac{7m^4}{96\pi^2\hbar^3}, \quad (88)$$

and while $3p_{\rm M} - \rho_{\rm M}$ is no longer zero, $p_{\rm M}$ remains unequal to $-\rho_{\rm M}$. In (88) we encounter quadratic and logarithmic divergences. Since both of these divergences are mass dependent, they cannot be canceled by an interplay between fermions and bosons unless the fermions and bosons are degenerate in mass. Since no supersymmetric partners of the ordinary particles have been detected to date, we know that the masses of the superparticles are far from being degenerate with those of the ordinary particles, with supersymmetry thus leaving $\langle\Omega_{\rm M}|T^{\mu\nu}_{\rm M}|\Omega_{\rm M}\rangle$ quadratically divergent. In fact the situation is similar to that met with a fundamental scalar Higgs field self-energy since it too has a quadratic divergence (the contribution due to a fermion that is Yukawa-coupled to the Higgs scalar field is equal to the quantity $\Pi_{\rm S}(q^2,M)$ given in (29)). And it too can only be canceled via supersymmetry if there is a superparticle in the same mass region as the Higgs particle itself, and this appears not to be the case.

Finally, as regards the cosmological constant, as long as the supersymmetry is unbroken, the cosmological constant is zero. Specifically, in a supersymmetric theory one has a generic anticommutator of the form $\{Q^\alpha, Q^\dagger_\alpha\} = H$, where the Q_α are Grassmann supercharges and H is the Hamiltonian. If the supercharges annihilate $|\Omega_0\rangle$ (viz. unbroken supersymmetry), then $\langle\Omega_0|H|\Omega_0\rangle$ is zero, the energy of the vacuum is zero, and the cosmological constant is thus zero too. However if the Grassmann charges do not annihilate the vacuum $|\Omega_{\rm M}\rangle$ then $\langle\Omega_{\rm M}|H|\Omega_{\rm M}\rangle$ is non-zero and a non-zero cosmological constant is induced, one whose magnitude would be as big as the supersymmetry breaking scale. Since this scale is known to be no smaller than the largest currently accessible energy at the LHC, this would give a cosmological constant contribution to standard Einstein-gravity based cosmology that would be at least 60 or so orders of magnitude larger than allowable by current Hubble plot data.

C. Conformal Gravity Treatment of the Vacuum Energy Density

If the quartic divergence given in (87) is not to be canceled by a boson loop associated with a superparticle, then the only apparent remaining option is for it to be canceled by gravity itself, as the gravitational field $g_{\mu\nu}$ is itself bosonic. And indeed in any quantum gravitational theory one would encounter products of gravitational fields, with quantum gravity thus having a zero-point problem of its own. Now one cannot make the needed cancellation using standard Einstein gravity itself since it is not renormalizable at the quantum level. However, one can do so in conformal gravity since it is a consistent quantum theory, being renormalizable, unitary, and ghost free [34–37].

Conformal gravity assumes invariance under local conformal transformations of the form $g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)}g_{\mu\nu}(x)$. In the pure gravitational sector of the theory it is thus required to possess the Weyl tensor based action

$$I_{\rm W} = -\alpha_g\int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} \equiv -2\alpha_g\int d^4x(-g)^{1/2}\left[R_{\mu\kappa}R^{\mu\kappa} - \frac{1}{3}(R^\alpha{}_\alpha)^2\right], \quad (89)$$

where α_g is a dimensionless gravitational coupling constant. Functional variation of this action with respect to the metric defines a gravitational tensor

$$W^{\mu\nu} = \frac{1}{2}g^{\mu\nu}(R^\alpha{}_\alpha)^{;\beta}{}_{;\beta} + R^{\mu\nu;\beta}{}_{;\beta} - R^{\mu\beta;\nu}{}_{;\beta} - R^{\nu\beta;\mu}{}_{;\beta} - 2R^{\mu\beta}R^\nu{}_\beta + \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}g^{\mu\nu}(R^\alpha{}_\alpha)^{;\beta}{}_{;\beta} + \frac{2}{3}(R^\alpha{}_\alpha)^{;\mu;\nu} + \frac{2}{3}R^\alpha{}_\alpha R^{\mu\nu} - \frac{1}{6}g^{\mu\nu}(R^\alpha{}_\alpha)^2, \quad (90)$$

and a fourth-order derivative equation of motion of the form

$$-4\alpha_g W^{\mu\nu}+T^{\mu\nu}_{\rm M}=0, \qquad (91)$$

when the theory is coupled to a conformal invariant matter sector. If we define $-4\alpha_g W^{\mu\nu}$ to be the energy-momentum tensor $T^{\mu\nu}_{\rm GRAV}$ of gravity, and introduce a energy-momentum tensor for the universe as a whole we can rewrite (91) as

$$T^{\mu\nu}_{\rm UNIV}=T^{\mu\nu}_{\rm GRAV}+T^{\mu\nu}_{\rm M}=0, \qquad (92)$$

to thus put the gravity and matter sectors on an equal footing, while showing that the total energy-momentum tensor of the universe is zero.

Given the conformal symmetry, no dimensionful parameters are allowed in the conformal action. Thus both the Einstein-Hilbert action

$$I_{\rm EH}=-\frac{1}{16\pi G}\int d^4x(-g)^{1/2}R^{\alpha}{}_{\alpha} \qquad (93)$$

and a cosmological constant action

$$I_{\Lambda}=-\int d^4x(-g)^{1/2}\Lambda \qquad (94)$$

are forbidden. Thus just like supersymmetry, conformal symmetry forbids the presence of any fundamental cosmological constant at the level of the Lagrangian.

If we now quantize the gravity sector of the conformal theory to lowest order in Planck's constant around flat (viz. the first quantum correction), and take the vacuum expectation value of $T^{\mu\nu}_{\rm GRAV}$ in the massless vacuum $|\Omega_0\rangle$ we obtain a zero-point energy density in the gravity sector of the form [37]

$$\langle\Omega_0|T^{\mu\nu}_{\rm GRAV}|\Omega_0\rangle=\frac{2\hbar}{(2\pi)^3}\int_{-\infty}^{\infty}d^3k\frac{Z(k)k^{\mu}k^{\nu}}{\omega_k}, \qquad (95)$$

where $Z(k=|\bar{k}|)$ is the gravitational field wave function renormalization constant, as defined [36, 37] as the coefficient of the delta function in canonical commutation relations for the momentum modes of the gravitational field. Inserting (95) and (87) into (92) then yields

$$Z(k)=1. \qquad (96)$$

Thus, we can effect a complete cancellation of the quartically divergent zero-point terms. Moreover, we do not need to introduce any regulators to separately define either $\langle\Omega_0|T^{\mu\nu}_{\rm GRAV}|\Omega_0\rangle$ or $\langle\Omega_0|T^{\mu\nu}_{\rm M}|\Omega_0\rangle$, as each term regulates the other as needed to maintain the stationarity condition $\langle\Omega_0|T^{\mu\nu}_{\rm UNIV}|\Omega_0\rangle=0$, with the cancellation being done mode by mode and not mode sum by mode sum. As long as (92) is maintained order by order in perturbation theory (which it is since both the gravity and matter sectors are renormalizable when conformal), then the mode by mode cancellation will persist, with matrix elements of $T^{\mu\nu}_{\rm UNIV}$ never having a zero-point problem. In addition, we note that we not only do not need to specify $Z(k)$ a priori, we actually cannot in fact do so. Rather, $Z(k)$ is determined entirely by the coupling of gravity to matter, with the quantization of matter enforcing the quantization of gravity since the condition $Z(k)=0$ is not consistent with (92). To underscore that $Z(k)$ cannot be assigned independently but is determined by the structure of the matter source to which gravity is coupled, we note that if the gravitational source consists of M massless gauge bosons and N two-component fermions, the vanishing of $\langle\Omega_0|T^{\mu\nu}_{\rm UNIV}|\Omega_0\rangle$ then entails that $2Z(k)+M-N=0$ [36], with gravity adjusting to whatever its source is.

Moreover, since we do not need to introduce any regulators we do not obtain any anomalies such as the trace anomaly. Specifically, while scale invariance Ward identities would be violated by anomalies, to thus give both $\langle\Omega_0|T^{\mu\nu}_{\rm GRAV}|\Omega_0\rangle$ and $\langle\Omega_0|T^{\mu\nu}_{\rm M}|\Omega_0\rangle$ trace anomalies, the vanishing of $\langle\Omega_0|T^{\mu\nu}_{\rm UNIV}|\Omega_0\rangle$ is not a Ward identity condition but a stationarity condition. Since $\langle\Omega_0|T^{\mu\nu}_{\rm UNIV}|\Omega_0\rangle$ is thus anomaly free, anomalies in $\langle\Omega_0|T^{\mu\nu}_{\rm GRAV}|\Omega_0\rangle$ and $\langle\Omega_0|T^{\mu\nu}_{\rm M}|\Omega_0\rangle$ must cancel each other identically. However since we can effect a mode by mode cancellation without needing to look for a regulated sum of modes by regulated sum of modes cancelation, we never have to deal with the trace anomaly at all, and can treat both $T^{\mu\nu}_{\rm GRAV}$ and $T^{\mu\nu}_{\rm M}$ as continuing to retain the tracelessness required by conformal invariance.

The conformal gravity cancellation of zero-point energies described above is not quite the same as the supersymmetry cancellation, since that cancellation did not address the gravitational zero-point energy problem, to thus leave the issue open. To clarify the issue, consider the second-order derivative Einstein gravity equation of motion

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}\right)=T^{\mu\nu}_{\rm M}. \qquad (97)$$

If (97) is to be an operator identity, then the two sides of it are to both be quantum-mechanical or to both be classical. However, since the gravity side is not well-defined quantum-mechanically, one takes it to be classical. But since the matter side is built out of quantum fields, the matter side is quantum-mechanical. To get round this one replaces (97) by a hybrid

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}\right)_{\rm CL}=\langle\Omega|T^{\mu\nu}_{\rm M}|\Omega\rangle. \qquad (98)$$

However, since the matter term in (98) has a zero-point problem, one must find a mechanism to cancel it, and must do so via the matter side alone. Now while unbroken supersymmetry actually achieves this, as we noted above, broken supersymmetry does not. However, since the gravity side of (98) is finite it cannot be equal to something that is infinite. Thus, in the literature one commonly ignores the fact that gravity is to couple to all forms of energy rather than only to energy differences, and subtracts off the zero-point infinity by hand and replaces (98) by

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}{}_{\alpha}\right)_{\rm CL}=\langle\Omega|T^{\mu\nu}_{\rm M}|\Omega\rangle_{\rm FIN}. \quad (99)$$

Thus in treating the contribution of the Fermi sea to the stability of white dwarfs or in evaluating the contribution of the cosmic microwave background to cosmology, one uses an energy operator of the generic form $H=\sum(a^{\dagger}(\bar{k})a(\bar{k})+1/2)\hbar\omega_k$, and then by hand discards the $H=\sum\hbar\omega_k/2$ term. And then, after all this is done, the finite part of $\langle\Omega|T^{\mu\nu}_{\rm M}|\Omega\rangle$ still has an uncanceled and as yet uncontrolled cosmological constant contribution that still needs to be dealt with. The present author is not aware of any formal derivation of (99) starting from a consistent quantum gravity theory, and notes that since it is (99) that is conventionally used in astrophysics and cosmology, it would not appear to yet be on a fully secure footing.

In the conformal case the situation is somewhat different. In the event of dynamical symmetry breaking, critical scaling and $\gamma_{\theta}(\alpha)=-1$, one has to take matrix elements of (92) in the self-consistent, Hartree-Fock vacuum $|\Omega_{\rm M}\rangle$. The quantity $\langle\Omega_{\rm M}|T^{00}_{\rm M}|\Omega_{\rm M}\rangle$ consists of the previously introduced $\tilde{\epsilon}(M)$ as given in (50), together with the quartically divergent $\rho_{\rm M}$ as given in (87), as it had originally been removed in (43). The vanishing of $T^{\mu\nu}_{\rm UNIV}$ then entails that at the minimum where $m=M$

$$\langle\Omega_M|T^{00}_{\rm GRAV}|\Omega_M\rangle-\frac{\hbar K^4}{4\pi^2}-\frac{M^4}{16\pi^2\hbar^3}=0. \quad (100)$$

From (100) it follows that $Z(k)$ is given by [37]

$$kZ(k)=(k^2+iM^2/\hbar^2)^{1/2}-\frac{iM^2}{4\hbar^2(k^2+iM^2/\hbar^2)^{1/2}}$$
$$+(k^2-iM^2/\hbar^2)^{1/2}+\frac{iM^2}{4\hbar^2(k^2-iM^2/\hbar^2)^{1/2}}. \quad (101)$$

(In [37] (101) was originally derived via a Feynman contour using the $S_{\nu}(p)$ propagator given in (73). Continuation to the $S_{\mu}(p)$ propagator given in (44) yields (101).) As we see, $Z(k)$ is again determined by the dynamics, and even though the gravitational modes remain massless, $Z(k)$ adjusts to the fact that the fermion has mass.

With (100) and (101), we see that when the symmetry is broken, $\langle\Omega_{\rm M}|T^{00}_{\rm M}|\Omega_{\rm M}\rangle$ adjusts from the purely quartic (87) to acquire a logarithmic divergence in (50). This logarithmic divergence is then automatically canceled by the induced and thus dynamically determined cosmological constant term $M^2/2g$ (dynamical in the sense that it depends on the state in which matrix elements are taken), with gravity then automatically canceling the quartic divergence and the residual finite part, $-M^4/16\pi^2\hbar^3$, of $\langle\Omega_{\rm M}|T^{00}_{\rm M}|\Omega_{\rm M}\rangle$. Moreover, the cancellation works no matter how big M^4 might be, and none of it is observable since it all occurs in the vacuum, i.e. it is due entirely to the occupied negative energy states in the Dirac sea. Specifically, what one measures in actual astrophysical phenomena is not the vacuum but the behavior of the positive energy modes that can be excited out of it.

To be more specific, we note that since all of the infinities in $T^{\mu\nu}_{\rm GRAV}$ and $T^{\mu\nu}_{\rm M}$ are in the vacuum sector, if we decompose them into finite and divergent parts according to $T^{\mu\nu}_{\rm GRAV}=(T^{\mu\nu}_{\rm GRAV})_{\rm FIN}+(T^{\mu\nu}_{\rm GRAV})_{\rm DIV}$, $T^{\mu\nu}_{\rm M}=(T^{\mu\nu}_{\rm M})_{\rm FIN}+(T^{\mu\nu}_{\rm M})_{\rm DIV}$, (92) will decompose into

$$(T^{\mu\nu}_{\rm GRAV})_{\rm DIV}+(T^{\mu\nu}_{\rm M})_{\rm DIV}=0, \quad (102)$$

$$(T^{\mu\nu}_{\rm GRAV})_{\rm FIN}+(T^{\mu\nu}_{\rm M})_{\rm FIN}=0. \quad (103)$$

All of the infinities are taken care of by (102), and for astrophysics and cosmology we can then use the completely infinity-free (103). In this way for studying white dwarfs or the cosmic microwave background we can now use $H=\sum a^{\dagger}(\bar{k})a(\bar{k})\hbar\omega_k$ alone, as the zero-point contribution has already been taken care of by gravity itself and does not appear in (103) at all. Moreover, when we do excite positive energy modes out of the vacuum we will generate a new cosmological constant contribution, and it this term that is measured in cosmology. Cosmology thus only sees the change in the vacuum energy density due to adding in positive energy modes and does not see the full negative energy mode vacuum energy density itself, i.e. in (103) one is sensitive not to $\langle\Omega_{\rm M}|T^{\mu\nu}_{\rm M}|\Omega_{\rm M}\rangle$, and not to $\langle\Omega_{\rm M}|bT^{\mu\nu}_{\rm M}b^{\dagger}|\Omega_{\rm M}\rangle$, but only to their difference $\langle\Omega_{\rm M}|bT^{\mu\nu}_{\rm M}b^{\dagger}|\Omega_{\rm M}\rangle-\langle\Omega_{\rm M}|T^{\mu\nu}_{\rm M}|\Omega_{\rm M}\rangle$. Also gravity sees this effect mode by mode, i.e. gravity mode by fermion mode. In contrast, if one uses (99), then gravity sees an entire sum over fermion modes, which is one of the reasons why in the standard Einstein theory the cosmological constant effect is so big. To summarize, if one wants to take care of the cosmological constant problem, one has to take care of the zero-point problem, and when one has a renormalizable theory of gravity, via an interplay with gravity itself one is then able to do so.

D. Conformal Gravity as a Consistent Quantum Gravitational Theory

As a quantum theory conformal gravity had long been known to be renormalizable (α_g being dimensionless), but being fourth order it had long been thought to possess negative norm ghost states that would violate unitarity. This view of conformal gravity is suggested by writing the massless fourth-order propagator $1/k^4$ as the $M^2\to 0$ limit

$$\frac{1}{k^4}=\lim\left[\frac{1}{M^2}\left(\frac{1}{k^2}-\frac{1}{k^2+M^2}\right)\right]. \quad (104)$$

With the second term in (104) having a negative coefficient one immediately anticipates that the theory has states with negative norm. However, from inspection of a c-number propagator alone one cannot determine what quantum-mechanical Green's function the propagator is to correspond to. For this one has to quantize the theory, construct the appropriate Hilbert space and then construct the propagator. When Bender and Mannheim did this they found [34, 35] that the quantum Hamiltonian was not Hermitian, but that it instead was PT symmetric. In such a situation the correct Hilbert space norm is given by the overlap not of the right-eigenvectors of the Hamiltonian with their Dirac conjugates, viz. the Dirac norm $\langle R|R\rangle$, but rather by the overlap of the right-eigenvectors of the Hamiltonian with its left-eigenvectors, viz. $\langle L|R\rangle$, with the left-eigenvectors being related to the PT conjugates of the right-eigenvectors. And with $\langle L|R\rangle$ being found to not be negative in the fourth-order case, when one uses the PT-theory norm one can associate (104) with a unitary theory. (Since one can write $\langle L| = \langle R|A$ with an appropriate operator A, it is through this A that the minus sign in (104) is generated, rather than through properties of the states themselves.) Thus by recognizing conformal gravity to be a PT theory rather than a Hermitian one, its unitarity can then be secured.

In addition, with the $1/M^2$ prefactor in (104) actually blowing up in the $M^2 \to 0$ limit, Bender and Mannheim found that $M^2 \to 0$ limit was singular, with the Hamiltonian associated with the pure $1/k^4$ propagator actually not being diagonalizable, but being of Jordan-block form instead. Since the Hamiltonian is not diagonalizable, it manifestly could not be Hermitian. Thus the ghost problem in fourth-order theories only arose because one tried to treat the theory as though it was a Hermitian theory and as though one could use the standard Dirac norm. Thus the apparent generation of negative Dirac norm states indicates not that the theory violates conservation of probability, but that the Hamiltonian is not Hermitian and the Dirac norm is not the appropriate norm.

With conformal gravity thus being a consistent theory of gravity, one expressly constructed in the four spacetime dimensions for which there is observational evidence, one does not need to resort to string theory. Thus one has no need for supersymmetry (or for extra dimensions for that matter) that are so key to string theory. Also, since conformal gravity has no need to utilize the interplay between spacetime and the fermionic supercharges of supersymmetry that is central to string theory, it has no need to find a way to evade the Coleman-Mandula theorem that would forbid any such interplay for bosonic charges.

E. Conformal Gravity and the Cosmological Constant Problem

If one takes the mean-field Lagrangian and couples it to geometry, then just as in (23) where the mean field was coupled to an axial gauge field, one finds [38–40] that (cf. (125) below) the coupling in this case is that of a conformally coupled field, viz. $\partial_\mu m(x)\partial^\mu m(x)/2 - m^2(x)R^\alpha{}_\alpha/12$. However, since we are in a conformal theory, we can make a conformal transform that would bring $m(x)$ in the effective Higgs Lagrangian of (56) to a constant. Thus at $m = M = \mu$, and with $\hbar = 1$, when coupled to geometry the effective Higgs Lagrangian of (56) takes the form

$$\mathcal{L}_{\rm EFF} = \frac{M^4}{16\pi^2} - \frac{M^2}{512\pi}R^\alpha{}_\alpha. \qquad (105)$$

In its coupling to M^2 the Ricci scalar appears with the opposite sign to the sign that appears in the Einstein-Hilbert action (compare (93) and (125) below). This then leads to repulsive rather than attractive gravity. In the conformal theory attractive Newtonian gravity arises not from this term but from the $W_{\mu\nu}$ term [41, 42]. Since the Weyl tensor $C_{\lambda\mu\nu\kappa}$ and $W_{\mu\nu}$ both vanish in geometries such as Robertson-Walker that are conformal to flat, $W_{\mu\nu}$ plays no role in cosmology, to thus allow cosmological gravity to be repulsive and local gravity to be attractive. This fact was capitalized on in [43] to show that a repulsive cosmological gravity would have no flatness problem. And in [44] it was shown that the theory would have no horizon problem, and that in such a cosmology there would be cosmic repulsion and it would lower the current era value q_0 of the deceleration parameter with respect to its value in standard attractive gravity. Specifically in [44] it was shown that even without the M^4 term this would reduce q_0 from its pure matter inflationary universe value of $q_0 = 1/2$ to $q_0 = 0$. When the cosmological term is included the matter energy-momentum tensor takes the form

$$\begin{aligned} T^{\mu\nu}_{\rm M} &= i\hbar\bar{\psi}\gamma^\mu\partial^\nu\psi - \frac{M^2}{256\pi}\left(R^{\mu\nu} - \frac{g^{\mu\nu}}{2}R^\alpha{}_\alpha\right) \\ &- g^{\mu\nu}\frac{M^4}{16\pi^2}. \end{aligned} \qquad (106)$$

In (106) it is understood that, as discussed above, now only the positive frequency components of the fields are to appear, and not the full vacuum contribution – i.e. just the finite part of the energy-momentum tensor as given in (103). Then, no matter how big M might be, it was shown [45, 46] that q_0 was obliged to lie in the narrow range $-1 \leq q_0 \leq 0$, with the associated luminosity distance $d_{\rm L}$ versus redshift z relation being of the form

$$d_L = -\frac{c}{H_0}\frac{(1+z)^2}{q_0}\left(1 - \left[1 + q_0 - \frac{q_0}{(1+z)^2}\right]^{1/2}\right), \qquad (107)$$

where H_0 is the current value of the Hubble parameter. With $q_0 = -0.37$ (107) was found [46] to provide every bit as good a fit to the accelerating universe data [47–49] as the standard $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$ paradigm. However the fit provided by (107) requires no dark matter or fine tuning at all, with the acceleration coming from the negative effective Newton constant and the negative spatial curvature k that conformal cosmology possesses. (More technically, it is not that dark matter is excluded, it is just that the contribution of any matter, dark or even luminous, to current Hubble plot era cosmic evolution is highly suppressed in conformal cosmology.) Moreover, conformal cosmology continues to be accelerating at higher redshift and thus requires none of the fine tuning that would make the standard cosmology only be accelerating at late redshifts. Thus at higher redshift the Hubble plots associated with conformal cosmology and standard cosmology will differ markedly, a potentially testable diagnostic. Finally, we note that with there being no need for dark matter in conformal cosmology, there is no need for supersymmetry to provide any dark matter candidates (not that supersymmetry is currently known to naturally lead to $\Omega_{\rm M} = 0.3$, or to $\Omega_\Lambda = 0.7$ for that matter when it does so).

F. Conformal Gravity and the Dark Matter Problem

While the Weyl tensor vanishes in geometries that are homogeneous and isotropic, as soon as one introduces localized sources the homogeneity is lost and $W^{\mu\nu}$ of (90) is no longer zero. Despite its somewhat formidable appearance Mannheim and Kazanas [42] were able to determine its form exactly and to all orders in classical geometries that are only spherically symmetric about a single point. In particular they found that $B(r) = -g_{00}(r)$ exactly obeys the fourth-order poisson equation

$$\nabla^4 B(r) = \frac{3}{4\alpha_g B(r)}(T^0{}_0 - T^r{}_r) = f(r). \qquad (108)$$

The general solution to this equation is given by

$$\begin{aligned} B(r) &= -\frac{1}{6}\int_0^r dr' f(r')\left(3r'^2 r + \frac{r'^4}{r}\right) \\ &- \frac{1}{6}\int_r^\infty dr' f(r')(3r'^3 + r' r^2) + B_0(r), \end{aligned} \qquad (109)$$

where $B_0(r)$ obeys $\nabla^4 B_0(r) = 0$. Since the integration in (109) extends all the way to $r = \infty$, the $B(r)$ potential receives contributions from material both inside and outside any system of interest. According to (109), a star of radius r_0 produces an exterior potential of the form $V^*(r > r_0) = -\beta^* c^2/r + \gamma^* c^2 r/2$ per unit solar mass of star. We thus recover the Newtonian potential while finding that the potential gets modified at large distances, i.e. at precisely the distances where one has to resort to dark matter. Integrating the $V^*(r)$ potential over a thin disk with a surface brightness $\Sigma(R) = \Sigma_0 \exp(-R/R_0)$ with scale length R_0 (the typical configuration for the stars in a spiral galaxy) yields the net local potential produced by the stars in the galaxy itself, and leads to a locally generated contribution to galactic circular velocities of the form [50]

$$\begin{aligned} v^2_{\rm LOC} &= \frac{N^*\beta^* c^2 R^2}{2R_0^3}\left[I_0\left(\frac{R}{2R_0}\right)K_0\left(\frac{R}{2R_0}\right) \right. \\ &- \left. I_1\left(\frac{R}{2R_0}\right)K_1\left(\frac{R}{2R_0}\right)\right] \\ &+ \frac{N^*\gamma^* c^2 R^2}{2R_0} I_1\left(\frac{R}{2R_0}\right)K_1\left(\frac{R}{2R_0}\right), \end{aligned} \qquad (110)$$

where N^* is the number of stars in the galaxy.

There are two contributions due to material outside the galaxy, i.e. due to the rest of the universe. The first is a linear potential term with coefficient $\gamma_0/2 = (-k)^{1/2}$ coming from cosmology (associated with the $B_0(r)$ term, and due to writing a comoving Robertson-Walker geometry with negative curvature in the rest frame coordinate system of the galaxy). The second arises from the integral from r to ∞ term in (109) due to cosmological inhomogeneities such as clusters of galaxies, and is of a quadratic potential form with coefficient κ. When all these contributions are combined, the total circular velocities are given by

$$v^2_{\rm TOT} = v^2_{\rm LOC} + \frac{\gamma_0 c^2 R}{2} - \kappa c^2 R^2. \qquad (111)$$

Mannheim and O'Brien [51–54] have applied this formula to the rotation curves of a set of 141 different galaxies and found very good fitting with parameters

$$\begin{aligned} \gamma^* &= 5.42 \times 10^{-41}{\rm cm}^{-1},\ \gamma_0 = 3.06\times 10^{-30}{\rm cm}^{-1}, \\ \kappa &= 9.54\times 10^{-54}{\rm cm}^{-2}, \end{aligned} \qquad (112)$$

with no dark matter being needed. Thus even though there is only one free parameter per galaxy, viz. N^*, a parameter that is common to all galactic rotation curve fits, and even though there is basically no flexibility, (111) fully captures the essence of the data.

We should note that it was not the dark matter problem the first got the present author interested in conformal gravity. Rather, it was because conformal gravity possessed a symmetry that forbade the presence of any the cosmological constant term at the level of the starting Lagrangian [55]. Moreover, Mannheim and Kazanas set out with the quite limited objective of trying to see whether a theory that was not based on the Einstein-Hilbert action could still lead to a Newtonian potential. It was only on solving the conformal gravity theory in a static, spherically symmetric geometry that they discovered that the theory not only did indeed support a Newtonian potential, it was accompanied by a linear potential term that they had not anticipated. That this linear potential could then be used to eliminate the need for galactic dark matter is therefore quite non-trivial.

We should also note that in contrast to the conformal gravity fits, dark matter fits to this same set of galaxies requires 282 additional free parameters, viz. two free parameters for each galactic dark matter halo. Now dark matter theory does provide generic forms for the shapes of the halos [56, 57], but each halo has two free numerical parameters, parameters which for the moment have to be phenomenologically determined by the fitting itself. Thus, with there being no need for dark matter in conformal gravity fits to galactic rotation curves, we again note that there is no need for supersymmetry to provide any dark matter candidates (not that supersymmetry is anyway currently known to naturally lead to values for any of the 282 free halo parameters). Finally, since both (107) and (111) do capture the essence of the data, then, if supersymmetry, dark matter theory, and even string theory, are to be correct, they should be able to derive these formulas for themselves.

We would also like to note that even if a supersymmetric particle is discovered at the LHC, this would not necessarily solve the dark matter problem. Specifically, the so far unsuccessful underground dark matter searches have identified a fairly large exclusion zone in supersymmetric cross section versus supersymmetric mass plots. For any supersymmetric particles discovered at the LHC to be dark matter they would have to not fall in this exclusion zone, and would, of course, then have to be found in the allowed region.

As regards conformal gravity, if it is to supplant dark matter then it will have to successfully describe astrophysical phenomena such as gravitational lensing and the anisotropy structure of the cosmic microwave background. The study of conformal cosmological fluctuation theory given in [58] provides a first step in this direction.

G. Conformal Invariance and the Metrication and Unification of the Fundamental Forces

With string theory with its supersymmetric underpinnings being capable of addressing both a metrication of all the fundamental forces and a unification of them, it is of interest to see how conformal symmetry fares on these issues where there is no supersymmetry to appeal to and no way to evade the constraints of the Coleman-Mandula theorem. We discuss first metrication, and we shall follow the recent discussion given in [59] where the effects of some generalized geometric connections were considered.

In the presence of some generalized geometric connection $\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Lambda^{\lambda}{}_{\mu\nu} + \delta\Gamma^{\lambda}{}_{\mu\nu}$ where $\Lambda^{\lambda}{}_{\mu\nu}$ is the standard Levi-Civita connection

$$\Lambda^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_{\mu}g_{\nu\alpha} + \partial_{\nu}g_{\mu\alpha} - \partial_{\alpha}g_{\nu\mu}), \qquad (113)$$

one introduces a generalized spin connection of the form

$$-\tilde{\omega}_{\mu}^{ab} = -\omega_{\mu}^{ab} + V^{b}_{\lambda}\delta\Gamma^{\lambda}{}_{\nu\mu}V^{a\nu}, \qquad (114)$$

where the $V^{a\nu}$ are vierbeins and ω_{μ}^{ab} is given by

$$-\omega_{\mu}^{ab} = V^{b}_{\nu}\partial_{\mu}V^{a\nu} + V^{b}_{\lambda}\Lambda^{\lambda}{}_{\mu\nu}V^{a\nu} = \omega_{\mu}^{ba}. \qquad (115)$$

In terms of this generalized connection the Dirac action for a massless fermion takes the form

$$I_{\rm D} = \frac{1}{2}\int d^4x(-g)^{1/2}i\bar{\psi}\gamma^{a}V^{\mu}_{a}(\partial_{\mu} + \Sigma_{bc}\tilde{\omega}_{\mu}^{bc})\psi + \text{H. c.}, \qquad (116)$$

where $\Sigma_{ab} = (1/8)(\gamma_a\gamma_b - \gamma_b\gamma_a)$ and the γ_a refer to a fixed frame.

Consider now a $\delta\Gamma^{\lambda}{}_{\mu\nu}$ of the form

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = -\frac{2i}{3}g^{\lambda\alpha}\left(g_{\nu\alpha}A_{\mu} + g_{\mu\alpha}A_{\nu} - g_{\nu\mu}A_{\alpha}\right) + \frac{1}{2}g^{\lambda\alpha}(Q_{\mu\nu\alpha} + Q_{\nu\mu\alpha} - Q_{\alpha\nu\mu}). \qquad (117)$$

Here $Q^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} = -Q^{\lambda}{}_{\nu\mu}$ is the antisymmetric Cartan torsion tensor. With A_{μ} being a vector field, the A_{μ}-dependent connection term is essentially the connection first introduced by Weyl, differing from it only through the presence of the additional factor of i, a factor that enforces PT symmetry and is crucial for metrication [59]. Following some algebra the insertion of the full $\tilde{\omega}_{\mu}^{ab}$ into the Dirac action is found to lead to the action [59]

$$I_{\rm D} = \int d^4x(-g)^{1/2}i\bar{\psi}\gamma^{a}V^{\mu}_{a}(\partial_{\mu} + \Sigma_{bc}\omega_{\mu}^{bc} - iA_{\mu} - i\gamma^{5}S_{\mu})\psi, \qquad (118)$$

where

$$S^{\mu} = \frac{1}{8}(-g)^{-1/2}\epsilon^{\mu\alpha\beta\gamma}Q_{\alpha\beta\gamma}. \qquad (119)$$

We recognize $I_{\rm D}$ as describing none other than a fermion coupled to a standard Levi-Civita based spin connection and to chiral electromagnetism. Thus through the use of the generalized spin connection we are able to provide a purely geometric origin for both vector and axial-vector gauge fields.

Apart from possessing full local vector gauge symmetry $[\psi(x) \to e^{i\alpha(x)}\psi(x),\ A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu}\alpha(x)]$ and full local axial-vector gauge symmetry $[\psi(x) \to e^{i\gamma_5\alpha(x)}\psi(x),\ S_{\mu}(x) \to S_{\mu}(x) + \partial_{\mu}\alpha(x)]$, the action in (118) has another local invariance, namely local conformal invariance, with it being left invariant under $g_{\mu\nu}(x) \to e^{2\alpha(x)}g_{\mu\nu}(x)$, $V^{a}_{\mu}(x) \to e^{\alpha(x)}V^{a}_{\mu}(x)$, $\psi(x) \to e^{-3\alpha(x)/2}\psi(x)$, $A_{\mu}(x) \to A_{\mu}(x)$, $S_{\mu}(x) \to S_{\mu}(x)$. (Each of these local transformations has its own $\alpha(x)$ of course.)

In addition, as noted in [59], the action also possess two discrete symmetries, namely PT and CPT symmetry, with the factor of i in (117) being needed to secure these invariances for the A_{μ}-dependent sector. (As noted in [59], if we were not to include the factor of i in (117), the A_{μ}-dependent piece of the connection would not couple in the generalized Dirac action at all.)

The extension to the non-Abelian case is direct. If for instance we put the fermions into the fundamental representation of $SU(N)\times SU(N)$ with $SU(N)$ generators T^{i}

that obey $[T^i, T^j] = if^{ijk}T^k$, replace A_μ by $g_V T^i A^i_\mu$, replace $Q_{\alpha\beta\gamma}$ by $g_A T^i Q^i_{\alpha\beta\gamma}$, and thus replace S_μ by $g_A T^i S^i_\mu$ in the connections, we obtain a locally $SU(N) \times SU(N)$ invariant Dirac action of the form

$$I_{\rm D} = \int d^4x(-g)^{1/2} i\bar{\psi}\gamma^a V^\mu_a(\partial_\mu + \Sigma_{bc}\omega^{bc}_\mu - ig_V T^i A^i_\mu - ig_A\gamma^5 T^i S^i_\mu)\psi. \qquad (120)$$

This action is precisely a local chiral Yang-Mills action, and remains locally conformally invariant under $g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)}g_{\mu\nu}(x)$, $V^a_\mu(x) \rightarrow e^{\alpha(x)}V^a_\mu(x)$, $\psi(x) \rightarrow e^{-3\alpha(x)/2}\psi(x)$, $A^i_\mu(x) \rightarrow A^i_\mu(x)$, $S^i_\mu(x) \rightarrow S^i_\mu(x)$, while still being PT and CPT invariant as well. Since the action given in (120) is the standard action that is used to describe the coupling of fermions to Yang-Mills fields and to standard Riemannian geometry, it is the action that is used in particle physics all the time. It thus has a dual characterization – it can be generated via local gauge invariance or via a generalized geometric connection.

To obtain the form of the kinetic energy operator for the gauge fields and the metric we perform a path integration over the fermion fields (equivalent to a one fermion loop Feynman diagram) using the above Dirac action, to obtain an effective action whose leading term is

$$I_{\rm EFF} = \int d^4x(-g)^{1/2}C\left[\frac{1}{20}\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}(R^\alpha{}_\alpha)^2\right] + \frac{1}{3}G^i_{\mu\nu}G^{\mu\nu}_i + \frac{1}{3}S^i_{\mu\nu}S^{\mu\nu}_i\right], \qquad (121)$$

where C is a log divergent constant. (The vector piece of $I_{\rm EFF}$ may be found in [38] and the axial-vector piece may be found in [60] and in (23) above.) In (121) we recognize the conformal gravity action with the $R_{\mu\nu}R^{\mu\nu} - (1/3)(R^\alpha{}_\alpha)^2$ term being evaluated with the Levi-Civita connection alone, and with the rest of the generalized connection emerging as the gauge field sector of a chiral Yang-Mills action. Thus even though we start with a non-Riemannian connection we finish up with a strictly Riemannian geometry, with all of the non-Riemannian structure being buried in the gauge fields. As noted in [59], the reason for this is that a generalized Riemann or Weyl tensor built out of the generalized connection would not be locally conformal invariant, since neither $A_\mu(x)$ nor $S_\mu(x)$ transform at all under a local conformal transformation. Hence the only allowed action in the pure geometric sector is that based on the Weyl tensor as constructed from the standard Levi-Civita connection alone, with the fermion path integration with a conformal invariant Dirac action having no choice but to produce it in (121).

Moreover, now that we have established the generic form needed for the gauge and metric sectors of the theory, and have seen that in this sector there are no cross-terms between any of the various connections in $\tilde{\Gamma}^\lambda{}_{\mu\nu}$, we now augment the Dirac action with a fundamental Yang-Mills gauge field ($I_{\rm YM}$) action and a conformal ($I_{\rm W}$) metric sector action of the form

$$I_{\rm W} + I_{\rm YM} = \int d^4x(-g)^{1/2}\left[-2\alpha_g\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}(R^\alpha{}_\alpha)^2\right) - \frac{1}{4}G^i_{\mu\nu}G^{\mu\nu}_i - \frac{1}{4}S^i_{\mu\nu}S^{\mu\nu}_i\right]. \qquad (122)$$

This action not only respects full conformal and gauge symmetry, like $I_{\rm D}$ it has a dual characterization – it can be generated via local gauge invariance or via a generalized geometric connection. Finally, on adding an $SU(N) \times SU(N)$ invariant four-fermion action

$$I_{\rm FF} = -\int d^4x(-g)^{1/2}\frac{g_{\rm FF}}{2}\left[\bar{\psi}T^i\psi\bar{\psi}T^i\psi + \bar{\psi}i\gamma_5 T^i\psi\bar{\psi}i\gamma_5 T^i\psi\right], \qquad (123)$$

with coupling $g_{\rm FF}$, we can write down the fundamental action for a conformal invariant universe, viz.

$$I_{\rm UNIV} = I_{\rm D} + I_{\rm W} + I_{\rm YM} + I_{\rm FF}. \qquad (124)$$

If the dynamics associated with (124) leads to critical scaling and an $I_{\rm FF}$ with dynamical dimension equal to four, the $I_{\rm UNIV}$ action will then provide a fully renormalizable and consistent action for the universe in which all mass is generated in the vacuum by dynamical symmetry breaking.

In addition, we noted that since (120) is the standard action used in physics, the effective action given in (121) must always appear in particle physics at the one fermion loop level. Now, as noted in [38], radiative loops due to other standard fields such as scalars and gauge bosons yield a log divergence of the same sign, and thus the fermionically-generated $I_{\rm EFF}$ could not be canceled by other fundamental fields. The infinity in (121) is thus an infinity that supersymmetry could not cancel.

However, this infinity could be canceled via conformal invariance, and this can be done in two ways. Specifically, since $I_{\rm W} + I_{\rm YM}$ is fully renormalizable, one could cancel the C term directly by a renormalization counter-term. However, the C term could also be cancelled non-perturbatively if there is critical scaling. Specifically, we recall that, unlike the scalar $\Pi_{\rm S}(x) = \langle\Omega|T(\bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0))|\Omega\rangle$, in quantum electrodynamics higher order radiative corrections to the vacuum polarization $\Pi_{\mu\nu}(x) = \langle\Omega|T(\bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(0)\gamma_\nu\psi(0))|\Omega\rangle$ do not generate higher powers of $\ln(\Lambda^2)$, but are all equally linear in $\ln(\Lambda^2)$. Hence if the respective coefficients of all these $\ln(\Lambda^2)$ terms sum to zero, this divergence will be canceled completely. The condition that the coefficients do sum to zero requires the coupling constant to be a solution to the Gell-Mann-Low eigenvalue condition, viz. the critical scaling condition. And indeed this is precisely how Johnson, Baker, and Willey were able to make Z_3 finite. Thus in the language of perturbation theory, critical scaling uniquely fixes the needed counter-term.

Now while the same analysis would equally apply if there is critical scaling in the axial-vector sector, we have not made a similar analysis for conformal gravity. However, we note that the generation of (121) from (120) involves matrix elements of fermion loops not with scalar insertions of fermion bilinears but with vector, axial-vector and tensor insertions instead. Now all of these particular insertions are associated with conservation conditions, and it is thus plausible that the cancellation would hold for conformal gravity too. Then, should it indeed hold, the dynamics associated with $I_{\rm UNIV}$ would not only be renormalizable, non-perturbatively it would even be completely finite.

Moreover, if one goes further and even breaks the conformal invariance by adding a spacetime-dependent mass term $-\int d^4x(-g)^{1/2}\bar{\psi}(x)M(x)\psi(x)$ to the Dirac action, the above $I_{\rm EFF}$ remains intact while being augmented by the "mean-field" action [38–40], [37]

$$I_{\rm MF} = \int d^4x(-g)^{1/2}C\bigg[-M^4(x)+\frac{1}{6}M^2(x)R^{\alpha}{}_{\alpha} - (\partial_\mu+iA_\mu)M(x)(\partial^\mu-iA^\mu)M(x)\bigg]. \qquad (125)$$

Here C is the same log divergent constant as before, with two last terms in (125) needing to appear jointly in order to maintain local conformal invariance. Again, it does not appear possible for supersymmetry to cancel this infinity as the superpartners are not degenerate with the regular particles. However, as noted above, with critical scaling and $\gamma_\theta=-1$, the infinity in $I_{\rm MF}$ will be canceled.

Now while it is nice to obtain an action such as (122), as given it could not describe the real world since there are no massless axial photons. The chiral symmetry thus must be broken spontaneously, and as we have shown in this paper, that is precisely what critical scaling does when $\gamma_\theta(\alpha)=-1$. Thus starting from a generalized torsion connection we are led not only to axial-vector gauge bosons, we are forced to break the associated symmetries dynamically. Similarly, since there is only one photon, any other vector symmetries must be spontaneously broken also, with (122) then being augmented by the mean-field terms that accompany dynamical mass generation.

The extension of our ideas to grand-unified theories of the strong, electromagnetic and weak interactions is direct since one could endow a generalized connection with all of the needed internal quantum numbers, while not generating any quantum number dependence in the pure metric sector as it is automatically based on the Levi-Civita connection alone. There is however a caveat. The conformal group that underlies conformal invariance is $SO(4,2)$ and its covering group is $SU(2,2)$. The fundamental representation of $SU(2,2)$ is a 4-dimensional spinor representation. Thus, in a conformal invariant world all fermions must be four-component, with there thus having to be right-handed neutrinos and not just left-handed ones. Families of quarks and leptons must thus contain 16 fundamental two-component spinors and not just 15. Hence the smallest grand unified group allowed would be the anomaly-free $SO(10)$, with its fundamental spinor representation being 16-dimensional.

If one is to have a chance to achieve coupling constant unification without supersymmetry, one needs some reasonably low lying mass scale beyond those of the standard $SU(3)\times SU(2)_{\rm L}\times U(1)$. Depending on how it is broken the grand-unified group could provide such a scale, though we should note that the $B^0_{\rm s}\to\mu^++\mu^-$ data of [8] leave little room for any physics beyond the standard model of any description at current energies. However, we should also note that this whole issue would be moot if the renormalized coupling constant of the grand-unified group is itself at a renormalization group fixed point away from the origin. However, the coupling constants would be able to depend on the running scale if the theory has a non-trivial fixed point for some value of the coupling constant other than the physical one, with the theory tracking to the origin at high energies because of its asymptotic freedom, while tracking to the non-trivial fixed point and spontaneously breaking the symmetry in the infrared. (An alternate possibility was noted in [16] – when $\gamma_\theta(\alpha)=-1$, the fluctuations produced by the renormalizable four-fermion residual interaction are themselves asymptotically free.)

Without super charges one is constrained by the Coleman-Mandula theorem, and so one could not unify the strong, electromagnetic and weak interactions with gravity by embedding spacetime and internal symmetries in a common Lie algebra. However, with conformal invariance there is an alternate way to extend unification to include gravity as well. Specifically, with fermions being in the fundamental representation of the conformal group, consider some general complex transformation on a fermion of the form $\psi\to\exp(\alpha_{\rm R}+i\alpha_{\rm I})\psi$, with only $\alpha_{\rm I}$ carrying internal quantum numbers. Then gauging $\alpha_{\rm I}$ gives Yang-Mills while gauging $\alpha_{\rm R}$ gives conformal gravity. Thus starting from the kinetic energy of a free massless fermion in flat spacetime, on imposing all these local gaugings we obtain none other than the Dirac action given in (120). Hence while Yang-Mills theories are obtained by gauging the imaginary part of the phase of the fermion field, gravity is obtained by gauging its real part. In this way spacetime and gravity can be unified, with it being (124) that should be considered as the fundamental action for physics, an action that can be obtained either by local gauging or by geometry, an action that could serve as a candidate theory of everything.

H. Final Comments

In this paper we have presented arguments to show that conformal invariance can do as well as supersymmetry in addressing some key concerns in particle physics. We thus advocate that conformal symmetry be regarded as a symmetry that is every bit as fundamental to physics as Lorentz invariance and Poincare invariance. Specifically, conformal symmetry is the full symmetry of the light cone, and in the absence of mass all particles must move on the light cone, with conformal symmetry thus being an exact symmetry at the level of the Lagrangian if all mass generation is to come solely from the vacuum. In [37] we have made the case for local conformal gravity, while in [61] 't Hooft has made the case for local conformal symmetry.

Moreover, we noted above that the action given in (120) is locally conformal invariant, and that a fermion path integration automatically generates the conformal gravity action. However, the action given in (120) is the standard fermion action that is used in particle physics. Thus both conformal invariance and conformal gravity cannot be avoided, and must play some role in physics.

In this paper we have shown that conformal gravity can address the quantum gravity, the cosmological constant, and the dark matter problems. That one theory can address three problems might seem surprising. However, all of these problems have a common origin, namely the extrapolation of the standard Einstein equations beyond their solar system origins. Specifically, if we extrapolate the Einstein equations to galactic distances and beyond we get the dark matter problem, if we extrapolate to cosmology we get the cosmological constant problem, and if we quantize the theory and extrapolate to short distances far off the mass shell we get renormalization and zero-point problems. Since all three problems have a common origin, they can have a common solution, with conformal gravity potentially being that solution since it provides a very different extrapolation.

When the present author in the 1970s found that one could make the four-fermion interaction renormalizable via dynamical dimensions, it appeared to have the potential to provide a solution to the four-fermion theory of weak interactions that would be an alternative to the spontaneously broken gauge theory solution. However, now we see that when Yang-Mills theories are coupled to gravity, we need both, namely we need Yang-Mills for scattering amplitudes and we need a renormalizable four-fermion interaction for the vacuum energy density. Then, when we interplay the two, with critical scaling we find that we can generate dynamical Goldstone and dynamical Higgs bosons, just as needed for a spontaneously broken gauge theory of weak interactions, with there being no need for any fundamental Higgs fields at all. Acknowledgment: The author wishes to thank Michael Mannheim for his help in the preparation of the figures.

Appendix A: The Collective Higgs Mode when the Fermion is Massive – the Calculation

1. The Basic Equations

In this appendix we evaluate $I_{\rm cut}$ and $I_{\rm Wick}$ as given in (70) and (71). For $I_{\rm cut}$ first it is convenient to remove the q_0 dependence from the range of integration, and so we set $p_0 = q_0\lambda/2$, $p = q_0\sigma/2$. Following some straightforward algebra, and recalling the extra minus sign in $N(q_0, p, p_0)$ as discussed above, we then obtain

$$\begin{aligned}
I_{\rm cut} &= -\frac{4i\mu^2}{\pi^3}\int_0^1 d\sigma\sigma^2\int_0^{1-\sigma} d\lambda\frac{N_{\rm cut}}{D_{\rm cut}},\\
N_{\rm cut} &= -(\lambda^2-\sigma^2-1)[(\lambda^2-\sigma^2+1)^2-4\sigma^2]^{1/2}q_0^8,\\
D_{\rm cut} &= 256m^4\mu^4+32m^2\mu^2[(\lambda^2-\sigma^2+1)^2+4\sigma^2]q_0^4\\
&+ [(\lambda^2-\sigma^2+1)^2-4\sigma^2]^2q_0^8.
\end{aligned}\qquad\text{(A1)}$$

With $I_{\rm cut}$ thus behaving as q_0^8 at small q_0, $I_{\rm cut}$ makes no contribution to the derivative at $q^2 = 0$ of either $\Pi_{\rm S}(q^2, m)$ (as would be needed for (56)) or $\Pi_{\rm P}(q^2, m)$ (as would be needed for (66)). With the $\lambda^2 - \sigma^2 - 1$ factor in $N_{\rm cut}$ always being negative in the $I_{\rm cut}$ integration range, $I_{\rm cut}$ is then expressly proportional to a negative number times i. The evaluation of $I_{\rm cut}$ at arbitrary q_0 can be done numerically, and because of its q_0 behavior $I_{\rm cut}$ is quite small, taking the value $-0.000291\mu^2 i$ at the threshold where $q_0^2 = 2m\mu$. Numerically $I_{\rm cut}$ is found to be a monotonic function of q_0, and because $I_{\rm cut}$ is so so small, we can immediately anticipate that the eventual Higgs boson resonance that we will find above the threshold will have a very narrow width.

With $I_{\rm Wick}$ being evaluated on the p_4 axis, for it we set

$$\begin{aligned}
N(q,p) &= -(p_4^2+p^2+q_0^2/4)\\
&\times [(p^2+p_4^2-q_0^2/4)^2+p_4^2q_0^2]^{1/2},\\
D(q,p) &= [(p^2+p_4^2-q_0^2/4)^2+p_4^2q_0^2-m^2\mu^2]^2\\
&+ 4m^2\mu^2(p^2+p_4^2-q_0^2/4)^2.
\end{aligned}\qquad\text{(A2)}$$

On setting $p_4 = r\cos\theta$, $p = r\sin\theta$, and on then setting $\cos\theta = z$, for $I_{\rm Wick}$ we obtain

$$\begin{aligned}
I_{\rm Wick} &= \frac{2\mu^2}{\pi^3}\int_0^\infty drr^3\int_0^1 dz(1-z^2)^{1/2}\\
&\times \left[\frac{N(q,r,z)+m^2\mu^2}{D(q,r,z)}\right],\\
N(q,r,z) &= -(r^2+q_0^2/4)[(r^2-q_0^2/4)^2+r^2z^2q_0^2]^{1/2},\\
D(q,r,z) &= [(r^2-q_0^2/4)^2+r^2z^2q_0^2-m^2\mu^2]^2\\
&+ 4m^2\mu^2(r^2-q_0^2/4)^2.
\end{aligned}\qquad\text{(A3)}$$

Inspection of $D(q,r,z)$ now shows that $D(q,r,z)$ will vanish if $r = q_0/2$, $z = m\mu/rq_0$, i.e. if $z = 2m\mu/q_0^2$. Since z is less than one there will always be some r and some z for which $D(q,r,z)$ will vanish if $q_0^2 \geq 2m\mu$. We thus identify $q_0^2 = q^2 = 2m\mu$ as a threshold, and anticipate a discontinuity in $I_{\rm Wick}$ if $q^2 \geq 2m\mu$. Below we will calculate the discontinuity and show that $I_{\rm Wick}$ with its seemingly real integrand actually develops an imaginary part when $q^2 \geq 2m\mu$, and it is this imaginary part that will then cancel the pure imaginary $I_{\rm cut}$.

Given (A3) we can calculate its q^2 derivative at $q^2 = 0$ algebraically. The $N(q,r,z)/D(q,r,z)$ term yields $-5\mu/128\pi m$ while the $m^2\mu^2/D(q,r,z)$ term yields

$+2\mu/128\pi m$. This then yields the values $-3\mu/128\pi m$ for $\Pi'_{\rm S}(q^2=0)$ and $-7\mu/128\pi m$ for $\Pi'_{\rm P}(q^2=0)$ that were used in (56) and (66) above.

To simplify the writing it is convenient to introduce

$$\alpha = \frac{(r^2-q_0^2/4)^2-m^2\mu^2}{r^2q^2},$$
$$\beta = \frac{(r^2-q_0^2/4)^2+m^2\mu^2}{r^2q^2}, \quad \text{(A4)}$$

so that we can set

$$N(q,r,z) = -(r^2+q_0^2/4)rq_0(z^2+\alpha/2+\beta/2)^{1/2},$$
$$D(q,r,z) = r^4q_0^4(z^4+2\alpha z^2+\beta^2). \quad \text{(A5)}$$

The substitution $z=y/(1+y^2)^{1/2}$ enables us to evaluate the needed z integrations, according to

$$I_2 = \int_0^1 dz \frac{(1-z^2)^{1/2}}{z^4+2\alpha z^2+\beta^2}$$
$$= \int_0^\infty dy \frac{1}{y^4(1+2\alpha+\beta^2)+2y^2(\alpha+\beta^2)+\beta^2}$$
$$= \frac{\pi}{4\beta(\alpha^2-\beta^2)^{1/2}}[(\alpha+\beta^2+(\alpha^2-\beta^2)^{1/2})^{1/2}$$
$$-(\alpha+\beta^2-(\alpha^2-\beta^2)^{1/2})^{1/2}], \quad \text{(A6)}$$

and

$$I_1 = \int_0^1 dz \frac{(1-z^2)^{1/2}(z^2+\alpha/2+\beta/2)^{1/2}}{z^4+2\alpha z^2+\beta^2}$$
$$= \int_0^\infty dy \frac{1}{2^{1/2}(1+y^2)^{1/2}}$$
$$\times \frac{(2y^2+(\alpha+\beta)(1+y^2))^{1/2}}{y^4(1+2\alpha+\beta^2)+2y^2(\alpha+\beta^2)+\beta^2}$$
$$= -\frac{i}{8^{1/2}\beta^2(\alpha+\beta)(\alpha-\beta)^{1/2}}$$
$$\times[(\alpha+\beta)(\alpha^2-\beta^2)^{1/2}(F_-+F_+)$$
$$+(\alpha^2+\alpha\beta-2\beta^2)(F_--F_+)], \quad \text{(A7)}$$

where

$$F_\pm = \int_0^\phi d\theta \frac{1}{(1-j_\pm \sin^2\theta)(1-k\sin^2\theta)^{1/2}}, \quad \text{(A8)}$$

with

$$j_\pm = \frac{1+2\alpha+\beta^2}{\alpha+\beta^2\pm(\alpha^2-\beta^2)^{1/2}},$$
$$\phi = i\ \text{arcsinh}(\infty), \quad k = \frac{2+\alpha+\beta}{\alpha+\beta}. \quad \text{(A9)}$$

The substitution $\sin\theta = i\tan\nu$ allows us to rewrite (A8) as

$$F_\pm = i\int_0^{\pi/2} \frac{d\nu}{\cos\nu}\frac{1}{(1+j_\pm\tan^2\nu)(1+k\tan^2\nu)^{1/2}}$$
$$= -\frac{i}{j_\pm-1}K(1-k)+\frac{ij_\pm}{j_\pm-1}E(1-n_\pm,1-k), \quad \text{(A10)}$$

where $K(1-k)$ and $E(1-n_\pm,1-k)$ are the complete elliptic integrals

$$K(1-k) = \int_0^{\pi/2} d\nu \frac{1}{(1-(1-k)\sin^2\nu)^{1/2}}$$
$$E(1-n_\pm,1-k) = \int_0^{\pi/2} d\nu$$
$$\times\frac{1}{(1-(1-n_\pm)\sin^2\nu)(1-(1-k)\sin^2\nu)^{1/2}}. \quad \text{(A11)}$$

Finally, in terms of all these expressions we can write $\Pi_{\rm S}(q^2,m)$ as

$$\Pi_{\rm S}(q^2,m) = \frac{2\mu^2}{\pi^3}\int_0^\infty drr^3\Big[-\frac{(r^2+q_0^2/4)I_1}{r^3q_0^3}$$
$$+\frac{m^2\mu^2 I_2}{r^4q_0^4}\Big]. \quad \text{(A12)}$$

Then with the physical mass being given by M, to find any Higgs boson we need to look for zeros of the finite

$$\hat{\Pi}_{\rm S}(q^2,M) = \Pi_{\rm S}(q^2,M)-g^{-1}. \quad \text{(A13)}$$

where g^{-1} is given in (49).

While it does not appear to be possible to do the integration in (A12) analytically, the utility of (A12) is that we can extract an analytic expression for the discontinuity from it. However, before doing so we first evaluate $\hat{\Pi}_{\rm S}(q^2,M)$ below the $q^2=2M\mu$ threshold. At $q^2=0$ we can evaluate $\hat{\Pi}_{\rm S}(q^2=0,M)$ analytically to obtain the value $\mu^2/4\pi^2=0.025330\mu^2$. As we increase q^2, via numerical integration we find that ${\rm Re}[\hat{\Pi}_{\rm S}(q^2,M)]$ decreases monotonically, reaching a value of $0.003373\mu^2$ at $q^2=2M\mu$. We can thus anticipate that it will vanish a little beyond the threshold.

2. The Discontinuity

As we had noted earlier, above the threshold the integral in (A12) becomes undefined at $r=q_0/2$. To avoid this we must either move q_0 off the real axis or keep q_0 real and deform the r-integration contour. To implement the former we look for $\hat{\Pi}_{\rm S}(q^2,M)$ to vanish at some $q_0=q_{\rm R}-i\Gamma$, with a necessarily positive Γ of dimension $(M\mu)^{1/2}$ if the Higgs boson is indeed to be a resonance. With the quantity $r^2-q_0^2/4$ that appears in α and β then becoming $r^2-(q_{\rm R}-i\Gamma)^2/4$ near the resonance, to implement the more convenient latter procedure, for $q_0>q_{\rm R}$ only we split the (A12) integral into two parts, a real part, $I_{\rm R}$, that consists of an integration involving two intervals $r\in(0,q_{\rm R}/2-\Gamma)$ and $r\in(q_{\rm R}/2+\Gamma,\infty)$, and a complex part $I_{\rm COM}$ along a semicircle in the upper half r plane of radius Γ in which $r=q_{\rm R}/2+\Gamma e^{i\theta}$ where $\theta\in(\pi,0)$. Then we can solve for the real and imaginary parts of $\hat{\Pi}_{\rm S}(q^2,M)=0$ to fix both the position and the width of the resonance at some $q_0=q_{\rm R}-i\Gamma$, $q^2=q_{\rm R}^2-\Gamma^2-2iq_{\rm R}\Gamma$.

Now before we solve for the location of the Higgs boson we do not know whether it will in fact turn out to be a narrow resonance. Thus we shall take Γ to be small, and then self-consistently discover that the solution is one in which it is in fact small. With small Γ we can evaluate (A12) on the semicircle in the upper half r plane by making a Taylor series expansion. With the measure for the integration on the semicircle being given by $dr = i\Gamma e^{i\theta}d\theta$, to lowest order in Γ we only need to evaluate the integrand in (A12) to zeroth order in Γ. On inserting $r = q_{\rm R}/2 + \Gamma e^{i\theta}$ in (A7) this yields

$$
\begin{aligned}
I_1 &\to \int_0^\infty dy \frac{q_{\rm R}^8 y}{(1+y^2)^{1/2}(y^2(q_{\rm R}^4 - 4M^2\mu^2) - 4M^2\mu^2)^2} \\
&= \Bigg(\frac{q_{\rm R}^4(1+y^2)^{1/2}}{2(4M^2\mu^2 + (4M^2\mu^2 - q_{\rm R}^4)y^2)} \\
&+ \frac{q_{\rm R}^2}{4(q_{\rm R}^4 - 4M^2\mu^2)^{1/2}} \\
&\times \ln\left(\frac{q_{\rm R}^2 + (q_{\rm R}^4 - 4M^2\mu^2)^{1/2}(1+y^2)^{1/2}}{q_{\rm R}^2 - (q_{\rm R}^4 - 4M^2\mu^2)^{1/2}(1+y^2)^{1/2}} \right) \Bigg)\Bigg|_0^\infty \\
&= \frac{i\pi q_{\rm R}^2}{4(q_{\rm R}^4 - 4M^2\mu^2)^{1/2}} - \frac{q_{\rm R}^4}{8M^2\mu^2} - \frac{q_{\rm R}^2}{4(q_{\rm R}^4 - 4M^2\mu^2)^{1/2}} \\
&\times \ln\left(\frac{q_{\rm R}^2 + (q_{\rm R}^4 - 4M^2\mu^2)^{1/2}}{q_{\rm R}^2 - (q_{\rm R}^4 - 4M^2\mu^2)^{1/2}} \right). \qquad \text{(A14)}
\end{aligned}
$$

Similarly, for (A6) we obtain

$$
\begin{aligned}
I_2 &\to \int_0^\infty dy \frac{q_{\rm R}^8}{(y^2(q_{\rm R}^4 - 4M^2\mu^2) - 4M^2\mu^2)^2} \\
&= \Bigg(\frac{q_{\rm R}^8 y}{8M^2\mu^2(4M^2\mu^2 + (4M^2\mu^2 - q_{\rm R}^4)y^2)} \\
&+ \frac{q_{\rm R}^8}{32M^3\mu^3(q_{\rm R}^4 - 4M^2\mu^2)^{1/2}} \\
&\times \ln\left(\frac{2M\mu + (q_{\rm R}^4 - 4M^2\mu^2)^{1/2} y}{2M\mu - (q_{\rm R}^4 - 4M^2\mu^2)^{1/2} y} \right) \Bigg)\Bigg|_0^\infty \\
&= \frac{i\pi q_{\rm R}^8}{32M^3\mu^3(q_{\rm R}^4 - 4M^2\mu^2)^{1/2}}. \qquad \text{(A15)}
\end{aligned}
$$

Finally, on inserting (A14) and (A15) into (A12) and doing the now trivial θ integration from $\theta = \pi$ to $\theta = 0$, the real part of $I_{\rm COM}$ is found to evaluate to

$$
\begin{aligned}
{\rm Re}[I_{\rm COM}] &= \frac{q_{\rm R}^3\Gamma}{4\pi^3 M^2} + \frac{\mu^2 q_{\rm R}\Gamma}{2\pi^3(q_{\rm R}^4 - 4M^2\mu^2)^{1/2}} \\
&\times \ln\left(\frac{q_{\rm R}^2 + (q_{\rm R}^4 - 4M^2\mu^2)^{1/2}}{q_{\rm R}^2 - (q_{\rm R}^4 - 4M^2\mu^2)^{1/2}} \right), \qquad \text{(A16)}
\end{aligned}
$$

while the imaginary part evaluates to

$$
{\rm Im}[I_{\rm COM}] = \frac{i\mu\Gamma q_{\rm R}(q_{\rm R}^2 - 2M\mu)^{1/2}}{4\pi^2 M(q_{\rm R}^2 + 2M\mu)^{1/2}}. \qquad \text{(A17)}
$$

As we see, there is an explicit branch point at $q_{\rm R}^2 = 2M\mu$ in $\Pi_{\rm S}(q^2, M)$ just as we had anticipated. (There is no branch point at $q^2 = -2M\mu$ since (A12), (A16), and (A17) only hold for timelike q_μ.) The discontinuity structure exhibited in (A16) is reminiscent of that obtained for $\Pi_{\rm S}(q^2, M)$ in the Nambu-Jona-Lasinio model as given in (29), where there is also a threshold branch point. Also we note that even though the imaginary parts given in (A14) and (A15) are actually singular at the branch point, their coefficients are such that when they combine in (A17) the singularity is canceled. Since singularities of this sort are not allowed, their cancellation in (A17) provides a nice internal check on our calculation. With this cancellation, rather than diverge at the threshold ${\rm Im}[I_{\rm COM}]$ actually vanishes there. With $I_{\rm CUT}$ not vanishing there the Higgs boson must thus lie above threshold. With $I_{\rm CUT}$ and ${\rm Im}[I_{\rm COM}]$ having opposite signs, a cancellation between them can thus be effected above threshold, with the resulting sign of Γ then indeed being the positive one required by unitarity. With $q_{\rm R}$ being fixed by a cancellation between ${\rm Re}[I_{\rm WICK}]$ and ${\rm Re}[I_{\rm COM}]$, the imaginary part cancellation then fixes the magnitude of Γ.

3. Numerical Results

For the actual numerical work we must evaluate not $\Pi_{\rm S}(q^2, M)$ itself but $\hat{\Pi}_{\rm S}(q^2, M) = \Pi_{\rm S}(q^2, M) - g^{-1}$, as only the latter quantity is finite. We shall use a hat notation to indicate that we now refer quantities to $\hat{\Pi}_{\rm S}(q^2, M)$ rather than to $\Pi_{\rm S}(q^2, M)$. Since we have broken the evaluation of $\hat{\Pi}_{\rm S}(q^2, M)$ into a low section, $\hat{I}_{\rm WICK}({\rm LOW})$, where $r < q_{\rm R}/2 - \Gamma$, a high section, $\hat{I}_{\rm WICK}({\rm HIGH})$, where $r > q_{\rm R}/2 + \Gamma$, and a semicircle section $I_{\rm COM}$, then since the integration in (48) involves the full $r \in (0, \infty)$ range, we need to include the contribution, $\hat{I}_{\rm GGAP}$, to g^{-1} of the gap region $r \in (q_{\rm R}/2 - \Gamma, q_{\rm R}/2 + \Gamma)$. This contribution is readily found to evaluate to

$$
\hat{I}_{\rm GGAP} = \frac{2\mu^2 q_{\rm R}^3 \Gamma}{\pi^2(q_{\rm R}^4 + 16M^2\mu^2)}. \qquad \text{(A18)}
$$

With this addition the full $\hat{\Pi}_{\rm S}(q^2, M)$ is given by

$$
\begin{aligned}
\hat{\Pi}_{\rm S}(q^2, M) &= \hat{I}_{\rm WICK}({\rm LOW}) + \hat{I}_{\rm WICK}({\rm HIGH}) \\
&+ \hat{I}_{\rm GGAP} + {\rm Re}[I_{\rm COM}] + {\rm Im}[I_{\rm COM}] + I_{\rm CUT}. \qquad \text{(A19)}
\end{aligned}
$$

With everything now being well-defined, we can proceed to solve the condition $\hat{\Pi}_{\rm S}(q^2, M) = 0$, and numerically find that $\hat{\Pi}_{\rm S}(q^2, M)$ vanishes at

$$
\begin{aligned}
q_{\rm R} &= 1.480(M\mu)^{1/2}, \quad \Gamma = 0.017i(M\mu)^{1/2} \\
q^2 &= (2.189 - 0.051i)M\mu. \qquad \text{(A20)}
\end{aligned}
$$

In this solution the six terms in (A19) respectively evaluate to 0.004710, −0.008832, 0.001610, 0.002517, $0.000406i$, $-0.000400i$ (in units of μ^2), as given to six decimal places, with $\hat{\Pi}_{\rm S}(q^2, M)$ thus vanishing to five.

We thus self-consistently confirm that $q_{\rm R}$ is indeed close to threshold where $q_{\rm R} = 1.414(M\mu)^{1/2}$, and that Γ is indeed small and that its sign had correctly been chosen. Near the resonance pole $\hat{\Pi}_{\rm S}(q^2, M)$ behaves as

$$\hat{\Pi}_{\rm S}(q^2, M) = (q^2 - (q_{\rm R} - i\Gamma)^2)(-0.021662 + 0.000484i), \qquad \text{(A21)}$$

with $T_{\rm S}(q^2) = 1/(g^{-1} - \Pi_{\rm S}(q^2))$ thus having the Breit-Wigner structure

$$T_{\rm S}(q^2) = \frac{46.141 + 1.030i}{q^2 - 2.2189M\mu + 0.051iM\mu}. \qquad \text{(A22)}$$

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