TOWARDS A CANONICAL GRAVITY IN TWO TIME AND TWO SPACE DIMENSIONS

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Abstract

We describe a program for developing a canonical gravity in 2+2 dimensions (two time and two space dimensions). Our procedure is similar to the usual canonical gravity but with two times rather than just one time. Our work may be of particular interest as an alternative approach to loop quantum gravity in 2+2 dimensions.

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1.- Introduction

It is well known that self-dual gravity [1]-[3] is one of the key concepts in loop quantum gravity [4]. The general believe is that self-dual gravity makes sense only in four dimensions since in this case the dual of a two form (the curvature) is again a two form. However, there are a number of evidences that self-dual gravity can also be implemented in eight dimensions [5]-[8]. It turns out that even in four dimensions self-dual gravity does not determine the signature of the 'space-time'. In fact, it might be 1+3 or 0+4, as is often considered in most of the current developments of loop quantum gravity, but it might also be 2+2 as has been shown in Ref. [9], where canonical gravity of the splitting type (1)+(1+2) was developed. It is worth mentioning that a canonical approach with a splitting of the type (1+1)+(2) has already be considered (see [10]-[12]and references therein). However, these formalisms are still one time theory since refer to the $diag(-1,+1,+1,+1)$ signature rather than to $diag(-1,-1,+1,+1)$ signature. In this work, we shall show that looking the scenario from the point of view of two time physics one can also consider the splitting (2)+(2) (two time and two space dimensions) of the 'space-time'.

One of the main physical motivation for considering the splitting (2)+(2) of the 'space-time' comes from the possibility of finding a mechanism which may allow to transform canonical gravity in 2+2 dimensions to canonical gravity in 1+3 dimensions. This is equivalent to change one time-like dimension by one space-like dimension and *vice versa*. Surprisingly, this kind of transformation has already be considered in the context of the sigma model (see [13] and references therein) and one wonders whether similar map can be implemented in canonical gravity changing 2+2 dimensions to 1+3 dimensions. Here, we shall give a positive answer to such a question. Specifically, we find an explicit evidence that the splitting of the action associated with 2+2 dimensions may lead to a term which has the typical form of a sigma model action and therefore the transition from 2+2 dimensions to 1+3 dimensions (and *vice versa*) is a viable possibility.

The contents of the paper are as follows: In section 2 and 3, the splittings of the metric and the action associated with a 2+2 manifold are developed, respectively. In section 4, we outline self dual gravity in 2+2 dimensions with special emphasis of the group splitting $Spin(2,2) \sim SU(1,1) \times SU(1,1)$. In section 5, we prove that, in a particular case, the action in 2+2 dimensions leads to a sigma model in which the usual method of changing signature can be applied. Finally, in section 6 we make some final remarks mentioning some topics of future interest.

2. **Splitting the metric associated with a $2+2$ manifold**

We shall assume that the vielbein field $E_{\hat{\mu}}^{\ (\hat{A})} = E_{\hat{\mu}}^{\ (\hat{A})}(t_1, t_2, x_1, x_2) = E_{\hat{\mu}}^{\ (\hat{A})}(\mathbf{t}, \mathbf{x})$ on a $2+2$-manifold M^{2+2}, can be written in the form

$$E_{\hat{\mu}}^{\ (\hat{A})} = \begin{pmatrix} e_{\mu}^{\ (A)} & A_{\mu}^{\ (a)} \\ B_{i}^{\ (A)} & e_{i}^{\ (a)} \end{pmatrix}, \tag{1}$$

where $A_{\mu}^{\ (a)}(\mathbf{t}, \mathbf{x}) \equiv E_{\mu}^{\ (a)}(\mathbf{t}, \mathbf{x})$ and $B_{i}^{\ (A)}(\mathbf{t}, \mathbf{x}) \equiv E_{i}^{\ (A)}(\mathbf{t}, \mathbf{x})$. In (1), the notations $(\hat{A})$ and $\hat{\mu}$ of $E_{\hat{\mu}}^{\ (\hat{A})}$ denote frame and target 'space-time' indices respectively. Of course, this form of $E_{\hat{\mu}}^{\ (\hat{A})}$ resembles a kind of Kaluza-Klein ansatz where one sets $B_{i}^{\ (A)} = 0$ [14]. The inverse $E_{(\hat{A})}^{\ \ \hat{\mu}}(\mathbf{t}, \mathbf{x})$ can be obtained from the relation

$$E_{(\hat{A})}^{\ \ \hat{\mu}} E_{\hat{\nu}}^{\ (\hat{A})} = \delta_{\hat{\nu}}^{\hat{\mu}}. \tag{2}$$

We find

$$E_{(\hat{A})}^{\ \ \hat{\mu}} = \begin{pmatrix} e_{(A)}^{\ \ \mu} & -A_{(A)}^{\ \ i} \\ -B_{(a)}^{\ \ \mu} & e_{(a)}^{\ \ i} \end{pmatrix}, \tag{3}$$

with

$$A_{(A)}^{\ \ i} \equiv e_{(a)}^{\ \ i} A_{\mu}^{\ (a)} e_{(A)}^{\ \ \mu} \tag{4}$$

and

$$B_{(a)}^{\ \ \mu} \equiv e_{(A)}^{\ \ \mu} B_{i}^{\ (A)} e_{(a)}^{\ \ i}. \tag{5}$$

Here, we are assuming that

$$e_{(A)}^{\ \ \mu} e_{\nu}^{\ (A)} = \delta_{\nu}^{\mu} \tag{6}$$

and

$$e_{(a)}^{\ \ i} e_{j}^{\ (a)} = \delta_{j}^{i}. \tag{7}$$

Moreover, considering (6) and (7) one finds that (3) satisfies (2) provided that the following relations are true:

$$e_{(A)}^{\ \ \mu} B_{i}^{\ (A)} e_{(a)}^{\ \ i} A_{\nu}^{\ (a)} = 0 \tag{8}$$

and

$$e_{(a)}{}^{i}A_{\mu}{}^{(a)}e_{(A)}{}^{\mu}B_{j}{}^{(A)}=0. \tag{9}$$

Let $\eta_{(\hat{A}\hat{B})}$ be a flat $(2+2)$-metric. In general, the metric $\gamma_{\hat{\mu}\hat{\nu}}$ can be defined in terms of $E_{\hat{\mu}}{}^{(\hat{A})}$ in the usual form,

$$\gamma_{\hat{\mu}\hat{\nu}}=E_{\hat{\mu}}{}^{(\hat{A})}E_{\hat{\mu}}{}^{(\hat{B})}\eta_{(\hat{A}\hat{B})}. \tag{10}$$

Using (1) the metric (10) becomes

$$\gamma_{\hat{\mu}\hat{\nu}}=\left(\begin{array}{cc} e_{\mu}{}^{(A)}e_{\nu}{}^{(B)}\eta_{(AB)}+A_{\mu}{}^{(a)}A_{\nu}{}^{(b)}\delta_{(ab)} & e_{\mu}{}^{(A)}B_{j}{}^{(B)}\eta_{(AB)}+A_{\mu}{}^{(a)}e_{j}{}^{(b)}\delta_{(ab)} \\ B_{j}{}^{(B)}e_{\nu}{}^{(A)}\eta_{(AB)}+e_{j}{}^{(b)}A_{\nu}{}^{(a)}\delta_{(ab)} & e_{i}{}^{(a)}e_{j}{}^{(b)}\delta_{(ab)}+B_{i}{}^{(A)}B_{j}{}^{(B)}\eta_{(AB)} \end{array}\right), \tag{11}$$

where $\eta_{(AB)}=diag(-1,-1)$, while $\delta_{(ab)}=diag(+1,+1)$. The expression (11) can also be written as

$$\gamma_{\hat{\mu}\hat{\nu}}=\left(\begin{array}{cc} g_{\mu\nu}+A_{\mu}{}^{i}A_{\nu}{}^{j}g_{ij} & g_{\mu\nu}B_{j}{}^{\nu}+g_{ij}A_{\mu}{}^{j} \\ B_{j}{}^{\mu}g_{\nu\mu}+g_{ji}A_{\nu}{}^{i} & g_{ij}+B_{i}{}^{\mu}B_{j}{}^{\nu}g_{\mu\nu} \end{array}\right). \tag{12}$$

Here, $g_{\mu\nu}=e_{\mu}{}^{(A)}e_{\nu}{}^{(B)}\eta_{(AB)}$, $g_{ij}=e_{i}{}^{(a)}e_{j}{}^{(b)}\delta_{(ab)}$, $A_{\mu}{}^{i}=e_{(a)}{}^{i}A_{\mu}{}^{(a)}$ and $B_{i}{}^{\mu}=e_{(A)}{}^{\mu}B_{i}{}^{(A)}$. Again if $B_{i}{}^{(A)}=0$, (12) looks like a Kaluza-Klein ansatz.

Let $\Gamma^{\hat{\alpha}}_{\hat{\mu}\hat{\nu}}=\Gamma^{\hat{\alpha}}_{\hat{\nu}\hat{\mu}}$ and $\omega_{\hat{\mu}}{}^{(\hat{A}\hat{B})}=-\omega_{\hat{\mu}}{}^{(\hat{B}\hat{A})}$ be the Christoffel symbols and the spin connection, respectively. We shall assume that $E_{\hat{\mu}}{}^{(\hat{A})}$ satisfies the formula

$$\partial_{\hat{\mu}}E_{\hat{\nu}}{}^{(\hat{A})}-\Gamma^{\hat{\alpha}}_{\hat{\mu}\hat{\nu}}E_{\hat{\alpha}}{}^{(\hat{A})}+\omega_{\hat{\mu}}{}^{(\hat{A}\hat{B})}E_{\hat{\nu}(\hat{B})}=0. \tag{13}$$

Using (13) it is not difficult to see that $\omega_{(\hat{A}\hat{B}\hat{C})}=E_{(\hat{A})}{}^{\hat{\mu}}\omega_{\hat{\mu}(\hat{B}\hat{C})}=-\,\omega_{(\hat{A}\hat{C}\hat{B})}$ can be written in terms of

$$\Omega_{\hat{\mu}\hat{\nu}}{}^{(\hat{A})}=\partial_{\hat{\mu}}E_{\hat{\nu}}{}^{(\hat{A})}-\partial_{\hat{\nu}}E_{\hat{\mu}}{}^{(\hat{A})}, \tag{14}$$

in the form

$$\omega_{(\hat{A}\hat{B}\hat{C})}=\frac{1}{2}\left[\Omega_{(\hat{A}\hat{B}\hat{C})}+\Omega_{(\hat{C}\hat{A}\hat{B})}-\Omega_{(\hat{B}\hat{C}\hat{A})}\right]. \tag{15}$$

3. **Splitting the action associated with a $2+2$ manifold**

After some manipulation one can show that up to total derivative the action

$$S = \frac{1}{4}\int_{M^{2+2}} \sqrt{\gamma} R, \tag{16}$$

is reduced to [14]

$$S = -\frac{1}{4}\int_{M^{2+2}} E(\Omega_{(\hat{A}\hat{B}\hat{C})}\Omega^{(\hat{A}\hat{B}\hat{C})} + 2\Omega_{(\hat{A}\hat{B}\hat{C})}\Omega^{(\hat{A}\hat{C}\hat{B})} - 4\Omega_{(\hat{A}\hat{B})}{}^{(\hat{B})}\Omega^{(\hat{A}\hat{C})}{}_{(\hat{C})}), \tag{17}$$

where $E = \det E_{\hat{\mu}}^{(\hat{A})}$. By using the splitting (1) one may try to compute (16) via (17), but perhaps a simpler alternative may be achieved by introducing the non-coordinate basis [15]

$$D_\mu = \partial_\mu - A^i_\mu \partial_i \tag{18}$$

and

$$D_i = \partial_i - B^\mu_i \partial_\mu. \tag{19}$$

The advantage of this basis is that brings the metric (11) into the block diagonal form

$$\gamma_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}. \tag{20}$$

The case in which $B^\mu_i = 0$ has already been considered by the authors of the Ref [16]. They obtain that up to total derivative, the action (16) becomes

$$\begin{array}{c} S = \frac{1}{4}\int_{M^{2+2}} \sqrt{-\det g_{\mu\nu}}\sqrt{\det g_{ij}} \\ \times\{g^{\mu\nu}\hat{R}_{\mu\nu} + g^{ij}\tilde{R}_{ij} + \frac{1}{4}g_{ij}F^i_{\mu\nu}F^{\mu\nu j} \\ \\ +\frac{1}{4}g^{\mu\nu}g^{ij}g^{kl}[\mathcal{D}_\mu g_{ik}\mathcal{D}_\nu g_{jl} - \mathcal{D}_\mu g_{ij}\mathcal{D}_\nu g_{kl}] \\ \\ +\frac{1}{4}g^{ij}g^{\mu\nu}g^{\alpha\beta}[\partial_i g_{\mu\alpha}\partial_j g_{\nu\beta} - \partial_i g_{\mu\nu}\partial_j g_{\alpha\beta}]\}. \end{array} \tag{21}$$

In the expression (21) the following definitions are considered:

$$\hat{R}_{\mu\nu} = D_\mu\Gamma^\alpha_{\alpha\nu} - D_\alpha\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu} - \Gamma^\beta_{\beta\alpha}\Gamma^\alpha_{\mu\nu}, \tag{22}$$

$$\tilde{R}_{ij} = \partial_i\Gamma^k_{kj} - \partial_k\Gamma^k_{ij} + \Gamma^k_{il}\Gamma^l_{kj} - \Gamma^l_{lk}\Gamma^k_{ij}, \tag{23}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - A^j_\mu \partial_j A^i_\nu + A^j_\nu \partial_j A^i_\mu \tag{24}$$

and

$$\mathcal{D}_\mu g_{ij} = \partial_\mu g_{ij} - [A^k_\mu \partial_k g_{ij} + \partial_i A^k_\nu g_{kj} + \partial_j A^k_\nu g_{ki}]. \tag{25}$$

By symmetry, one may expect that in the most general case with $B^\mu_i \neq 0$, the action

$$\begin{array}{c} S = \frac{1}{4}\int_{M^{2+2}} \sqrt{-\det g_{\mu\nu}}\sqrt{\det g_{ij}} \\ \times\{g^{\mu\nu}\hat{R}_{\mu\nu} + g^{ij}\tilde{R}_{ij} + \frac{1}{4}g_{ij}F^i_{\mu\nu}F^{\mu\nu j} + \frac{1}{4}g_{\mu\nu}H^\mu_{ij}H^{\nu ij} \\ +\frac{1}{4}g^{\mu\nu}g^{ij}g^{kl}[\mathcal{D}_\mu g_{ik}\mathcal{D}_\nu g_{jl} - \mathcal{D}_\mu g_{ij}\mathcal{D}_\nu g_{kl}] \\ +\frac{1}{4}g^{ij}g^{\mu\nu}g^{\alpha\beta}[\mathcal{D}_i g_{\mu\alpha}\mathcal{D}_j g_{\nu\beta} - \mathcal{D}_i g_{\mu\nu}\mathcal{D}_j g_{\alpha\beta}]\}, \end{array} \tag{26}$$

generalizes (21). Here, we used the definitions

$$H^\mu_{ij} = \partial_i B^\mu_j - \partial_j B^\mu_i - B^\alpha_i \partial_\alpha B^\mu_j + B^\alpha_j \partial_\alpha B^\mu_i, \tag{27}$$

$$\tilde{R}_{ij} = D_i \Gamma^k_{kj} - D_k \Gamma^k_{ij} + \Gamma^k_{il}\Gamma^l_{kj} - \Gamma^l_{lk}\Gamma^k_{ij}, \tag{28}$$

and

$$\mathcal{D}_i g_{\mu\nu} = \partial_i g_{\mu\nu} - [B^\alpha_i \partial_\alpha g_{\mu\nu} + \partial_\mu B^\alpha_i g_{\alpha\nu} + \partial_\nu B^\alpha_i g_{\alpha\mu}]. \tag{29}$$

In principle, as in Ref. [16] has been mentioned, the above method is independent of the signature of the space-time. So, exactly the same result can be obtained in the case of $m+n$-dimensional manifold which locally looks like $M \times N$. In this context, the action (21) admits an interpretation of a generally invariant gauge theory of $DiffN$ interacting with gauged gravity and non-linear sigma field based on M. When N corresponds to a group space G the theory may admit an interpretation of Kaluza-Klein type theory. In fact, in such a case one requires that G is an isometry of the $m+n$-dimensional metric and the resulting theory becomes the Einstein-Yang-Mills-Sigma theory. In principle, in the case of the generalized action (26) one can make a similar analysis. However, now one needs to combine two possible interpretations. In fact, the action (26) describes both a general invariant gauge theory of $DiffN$ based on M and a general invariant gauge theory of $DiffM$ based on N.

For our purpose it is convenient to recall that for $m = 1$ and $n = 3$ the action (21) reduces to the canonical $1 + 3$ decomposition of four-dimensional gravity. Since this scenario is generalized by Ashtekar formalism one becomes motivated to look for a similar generalization for both (21) and (26) actions.

For $m=2$ and $n=2$, there are a number of works related to (21), but not to (26). For instance, in Ref. [17] the self-dual Einstein gravity is identified with $m=2$-dimensional sigma model with gauge symmetry $SDiffN^2$, the area preserving diffeomorphism of N^2. However, the original signature of the metric is of the form $diag(-1,+1,+1,+1)$ rather than $diag(-1,-1,+1,+1)$, as it is our interest in this work.

From the point of view of the signature $diag(-1,-1,+1,+1)$ there is not a particular reason for assuming a $DiffN$ based on M rather than $DiffM$ based on N. For this reason it is reasonable to consider the generalized action (26) instead of (21). In this context, one observes that in addition to the invariance of (26) under both $DiffN$ and $DiffM$, an immediate symmetry of (26) is a kind of dual symmetry consist in the interchange of both $g_{\mu\nu} \leftrightarrow g_{ij}$ and $A^i_\mu \leftrightarrow B^\mu_i$. One can continue analyzing further properties of the action (26), but here we are more interested in describing an outline for its possible generalization in the context of Ashtekar formalism.

4. **Selfdual gravity in $2+2$ dimensions**

For our purpose, we recall that the self-dual curvature

$$^{+}R^{(\hat{A}\hat{B})}_{\hat{\mu}\hat{\nu}} = (R^{(\hat{A}\hat{B})}_{\hat{\mu}\hat{\nu}} + \frac{i}{2}\varepsilon^{(\hat{A}\hat{B})}{}_{(\hat{C}\hat{D})} R^{(\hat{C}\hat{D})}_{\hat{\mu}\hat{\nu}}) = -\frac{i}{2}\varepsilon^{(\hat{A}\hat{B})}{}_{(\hat{C}\hat{D})}\, {}^{+}R^{(\hat{C}\hat{D})}_{\hat{\mu}\hat{\nu}}, \tag{30}$$

where $\varepsilon^{(\hat{A}\hat{B})}{}_{(\hat{C}\hat{D})}$ is the completely antisymmetric density tensor, plays a central role in the development of the Ashtekar formalism. Complex factor i in (30) is linked to the Lorenziana signature $diag(-1,+1,+1,+1)$. In the case of Euclidean signature $diag(+1,+1,+1,+1)$ the imaginary factor i in (30) can be dropped:

$$^{+}R^{(\hat{A}\hat{B})}_{\hat{\mu}\hat{\nu}} = (R^{(\hat{A}\hat{B})}_{\hat{\mu}\hat{\nu}} + \frac{1}{2}\varepsilon^{(\hat{A}\hat{B})}{}_{(\hat{C}\hat{D})} R^{(\hat{C}\hat{D})}_{\hat{\mu}\hat{\nu}}) = \frac{1}{2}\varepsilon^{(\hat{A}\hat{B})}{}_{(\hat{C}\hat{D})}\, {}^{+}R^{(\hat{C}\hat{D})}_{\hat{\mu}\hat{\nu}}. \tag{31}$$

It turns out that in the signature $diag(-1,-1,+1,+1)$ one can also use (31). Here, we would like to see what are the consequences of (31) in a canonical approach. In the case of both Euclidean and Lorenziana signature, (30) and (31) give $^{+}R^{(ab)}_{\hat{\mu}\hat{\nu}} = -i\varepsilon^{(ab)}{}_{(c)}\, {}^{+}R^{(c0)}_{\hat{\mu}\hat{\nu}}$ and $^{+}R^{(ab)}_{\hat{\mu}\hat{\nu}} = \varepsilon^{(ab)}{}_{(c)}\, {}^{+}R^{(c0)}_{\hat{\mu}\hat{\nu}}$ respectively and therefore one observes that in both cases the $^{+}R^{(cd)}_{\hat{\mu}\hat{\nu}}$ component can be written in terms of $^{+}R^{(a0)}_{\hat{\mu}\hat{\nu}}$. Moreover, symbolically one has $^{+}R^{(a0)}_{\hat{\mu}\hat{\nu}} \sim \partial_{\hat{\mu}}\, {}^{+}\omega^{(a0)}_{\hat{\nu}} - \partial_{\hat{\nu}}\, {}^{+}\omega^{(a0)}_{\hat{\mu}} + \ldots$ and thus one can consider $F^a_{\hat{\mu}\hat{\nu}} \equiv {}^{+}R^{(a0)}_{\hat{\mu}\hat{\nu}}$ as the Yang-Mills field strength and $A^a_{\hat{\mu}} = {}^+\omega^{(a0)}_{\hat{\mu}}$ as the gauge field, with $SU(2)$ as a gauge group. Roughly speaking, these observations are some of the key reasons behind the success of the Ashtekar formalism. However, in the case of the signature $diag(-1,-1,+1,+1)$ the scenario seems to be different. This is because in such case there is not a particular reason for considering the splitting of (31) in terms of only one time coordinate (see Ref. [9]) instead of two times coordinates. Specifically, in this case one has the following splitting of (31):

$$ {}^+R^{(AB)}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\varepsilon^{(AB)}{}_{(cd)}{}^+R^{(cd)}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\varepsilon^{(AB)}\varepsilon_{(cd)}\ {}^+R^{(cd)}_{\hat{\mu}\hat{\nu}}, \tag{32} $$

$$ {}^+R^{(Aa)}_{\hat{\mu}\hat{\nu}} = \varepsilon^{(Aa)}{}_{(Bb)}{}^+R^{(Bb)}_{\hat{\mu}\hat{\nu}} = \varepsilon^{(A)}{}_{(B)}\varepsilon^{(a)}{}_{(b)}{}^+R^{(Bb)}_{\hat{\mu}\hat{\nu}}, \tag{33} $$

and

$$ {}^+R^{(ab)}_{\hat{\mu}\hat{\nu}} = \varepsilon^{(ab)}{}_{(AB)}{}^+R^{(AB)}_{\hat{\mu}\hat{\nu}} = \varepsilon^{(ab)}\varepsilon_{(AB)}\ {}^+R^{(AB)}_{\hat{\mu}\hat{\nu}}. \tag{34} $$

Clearly, (32) and (34) are equivalent expressions. The formula (33) simply seems a indices relation, between the different components of the frame indices of the object ${}^+R^{(Aa)}_{\hat{\mu}\hat{\nu}}$. However, one can verify that (33) reduces the four frame indices components of ${}^+R^{(Aa)}_{\hat{\mu}\hat{\nu}}$ to only two independent components. Finally, one notes that (32) determines ${}^+R^{(AB)}_{\hat{\mu}\hat{\nu}}$ in terms of ${}^+R^{(cd)}_{\hat{\mu}\hat{\nu}}$ and vice versa. But in two dimensions one can write ${}^+R^{(AB)}_{\hat{\mu}\hat{\nu}} = \varepsilon^{(AB)}\ {}^+R_{\hat{\mu}\hat{\nu}}$, where ${}^+R_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\varepsilon_{(CD)}{}^+R^{CD}_{\hat{\nu}}$. So, symbolically, in this case, one expects to have ${}^+R_{\hat{\mu}\hat{\nu}} \sim \partial_{\hat{\mu}}{}^+\omega_{\hat{\nu}} - \partial_{\hat{\nu}}{}^+\omega_{\hat{\mu}}$, where $\omega_{\hat{\mu}} = \frac{1}{2}\varepsilon_{(CD)}\ {}^+\omega^{(CD)}_{\hat{\mu}}$, and therefore ${}^+\omega_{\hat{\nu}}$ can be understood as an Abelian gauge field. Similarly, we can write ${}^+R^{(Aa)}_{\hat{\mu}\hat{\nu}} \sim \partial_{\hat{\mu}}{}^+\omega^{(Aa)}_{\hat{\nu}} - \partial_{\hat{\mu}}{}^+\omega^{(Aa)}_{\hat{\nu}} + ...$, with ${}^+\omega^{(Aa)}_{\hat{\nu}}$ corresponding to only two additional independents gauge fields.

It may be helpful to analyze the above scenario from the point of view of group splitting. In general the splitting of the curvature can be related to the splitting of the connection. In turn the splitting of the connection is related to group algebra splitting. In the case of Euclidean signature the splitting of the curvature $R^{(\hat{A}\hat{B})}_{\hat{\mu}\hat{\nu}}$ in terms of self-antiself dual parts has its origins in the splitting $Spin(4) \sim Spin(3) \times Spin(3)$, while in the case of Lorenzian signature one has $SO(1,3) \sim SU(2) \times SU(2)$ (see Refs. [18]-[20]. In Ref. [9] is mentioned that in an scenario of $2+2$ dimensions one may consider the splitting $SO(2,2) \sim SL(2,R) \times SL(2,R)$. This observation may in principle be extended to an splitting of the form $Spin(2,2) \sim SU(1,1) \times SU(1,1)$. This is because there exist the isomorphism $SL(2,R) \sim SU(1,1)$. However, one should mention that these kind of splittings are not sufficient for the a consistent splitting of the curvature. In fact, one still needs to verify that at the level of the corresponding algebra the self-dual ${}^{+}\omega$ and antiself-dual ${}^{-}\omega$ parts of the connection ω are in fact connections of the corresponding group: $SL(2,R)$ or $SU(1,1)$ in our case. This can be accomplish by splitting the $SU(2,2)$ gauge transformation into two $SU(1,1)$ gauge transformations and checking that the self-dual and antiself dual connections behave properly under the reduced gauge transformations associated with $SU(1,1)$.

5. **From canonical gravity in** $2+2$ to $1+3$ **dimensions**

Here, we shall give an outline of the possibility to apply a map to the action (26) in such a way that one can go from gravity in $2+2$ dimensions to gravity in $1+3$ dimensions. Our mechanism is similar to the one used in a sigma model theory (see Ref. [13] and references therein).

Let us start by recalling how starting from the action (21) one can obtain the usual canonical gravity in 1+3 dimensions. In this case one obtains exactly the same action as (21) but with M^{1+3} instead of M^{2+2}. Thus, we take the index $m=1$ and the index $n=2,3$ and 4. One discovers that the action (21) is reduced to

$$S=\frac{1}{4}\int_{M^{1+3}}\sqrt{\det(g_{ij})}\{g^{ij}\tilde{R}_{ij}++\frac{1}{4}g^{\mu\nu}g^{ij}g^{kl}[\mathcal{D}_\mu g_{ik}\mathcal{D}_\nu g_{jl}-\mathcal{D}_\mu g_{ij}\mathcal{D}_\nu g_{kl}]\}, \tag{35}$$

with

$$\mathcal{D}_1 g_{ij}=\partial_1 g_{ij}-[A^k_1\partial_k g_{ij}+\partial_i A^k_1 g_{kj}+\partial_j A^k_1 g_{ki}]. \tag{36}$$

Defining the extrinsic curvature as

$$K_{ij}\equiv\mathcal{D}_1 g_{ij}, \tag{37}$$

one sees that (35) can be rewritten in the form

$$S=\frac{1}{4}\int_{M^{1+3}}\sqrt{\det(g_{ij})}\{g^{ij}\tilde{R}_{ij}+\frac{1}{4}g^{ij}g^{kl}[K_{ik}K_{jl}-K_{ij}K_{kl}]\}, \tag{38}$$

which is the typical action for canonical gravity in 1+3 dimensions.

In view of the above review we see that besides the curvature term $g^{ij}\tilde{R}_{ij}$ the relevant term is the second term in (35). For this reason we shall focus in the term:

$$S=\frac{1}{16}\int_{M^{2+2}}\sqrt{\det(g_{ij})}g^{\mu\nu}g^{ij}g^{kl}[\mathcal{D}_\mu g_{ik}\mathcal{D}_\nu g_{jl}-\mathcal{D}_\mu g_{ij}\mathcal{D}_\nu g_{kl}]. \tag{39}$$

Four our purpose, we shall take $\mathcal{D}_\mu g_{ik} = \partial_\mu g_{ik}$. Moreover, we shall call $\Phi^{(p)}$ with $(p) = 1, 2, 3$ the three degrees of freedom associated with the the two dimensional metric g_{ik}. So the action (39) becomes

$$S = \frac{1}{16} \int_{M^{2+2}} \sqrt{\det(g_{\alpha\beta})} g^{\mu\nu} \partial_\mu \Phi^{(p)} \partial_\mu \Phi^{(q)} h_{(pq)}, \tag{40}$$

where

$$h_{(pq)} = g^{ij} g^{kl} \left(\frac{\partial g_{ik}}{\partial \Phi^{(p)}} \frac{\partial g_{jl}}{\partial \Phi^{(q)}} - \frac{\partial g_{ij}}{\partial \Phi^{(p)}} \frac{\partial g_{kl}}{\partial \Phi^{(q)}} \right). \tag{41}$$

We recognize in (40) a sigma model type action. Since in principle, g_{ik} is different when one is considering a theory with 2+2 signature or with 1+3 signature, one can consider the fact that the metric h_{pq} is signature dependent. To illustrate how duality may work by starting with the action (39) we shall further simplify the scenario. Let us assume that $g_{\mu\nu}$ and $h_{(pq)}$ are flat metrics $\delta_{\mu\nu}$ and $\eta_{(pq)}$ respectively. So, (40) becomes

$$S = \frac{1}{16} \int_{M^{2+2}} \delta^{\mu\nu} \partial_\mu \Phi^{(p)} \partial_\nu \Phi^{(q)} \eta_{(pq)}. \tag{42}$$

We shall also assume that $\delta_{\mu\nu}$ refers to the Euclidean sector of both 2+2 and 1+3 signatures. This means that $\eta_{(pq)}$ will depends on the two times associated with 2+2 signature, or one time and one space in the case of the 1 +3 signature. In other words, we shall assume that in the case of 2+2 signature $\eta_{(pq)}$ takes the form $\eta_{(pq)} = diag(1, 1, 1)$, while in the sector of 1+3 signature $\eta_{(pq)}$ is given by $\eta_{(pq)} = diag(-1, 1, 1)$. Thus, our task is to see how one can go from $\eta_{(11)} = 1$ in the case of 2+2 dimensions to $\eta_{(11)} = -1$ in the case of 1+3 dimensions. Therefore we focus in the reduced action

$$S = \frac{1}{16} \int_{M^{2+2}} \delta^{\mu\nu} \partial_\mu \Phi^{(1)} \partial_\nu \Phi^{(1)} \eta_{11} = \frac{1}{16} \int_{M^{2+2}} \delta^{\mu\nu} \partial_\mu \Phi^{(1)} \partial_\nu \Phi^{(1)}. \tag{43}$$

The next step is a standard procedure. We introduce an auxiliary gauge field $\mathcal{A}_\mu$ and add to (43) a term $\varepsilon^{\mu\nu} \mathcal{A}_\mu \partial_\nu \Psi^{(1)}$, where $\Psi^{(1)}$ is a dual field. Thus, (43) becomes

$$S = \frac{1}{8} \int_{M^{2+2}} \frac{1}{2} \delta^{\mu\nu} (\partial_\mu - \mathcal{A}_\mu) \Phi^{(1)} (\partial_\nu - \mathcal{A}_\nu) \Phi^{(1)} + \varepsilon^{\mu\nu} \mathcal{A}_\mu \partial_\nu \Psi^{(1)}. \tag{44}$$

The symmetries of the theory allows us to set $\mathcal{A}_\mu = 0$ or $\Phi^{(1)} = 0$. In the first case the action (44) is reduced to (43). While in the second case by setting $\Phi^1 = 0$ in (44) one gets

$$S = \frac{1}{8} \int_{M^{2+2}} \frac{1}{2} \delta^{\mu\nu} \mathcal{A}_\mu \mathcal{A}_\nu + \varepsilon^{\mu\nu} \mathcal{A}_\mu \partial_\nu \Psi^{(1)}. \tag{45}$$

Solving (45) for $\mathcal{A}_\mu$, one obtains

$$\mathcal{A}^\mu + \varepsilon^{\mu\nu}\partial_\nu \Psi^{(1)} = 0. \tag{46}$$

Substituting this result into (45) yields the dual action

$$S = \tfrac{1}{16}\int_{M^{1+3}} (-1)\delta^{\mu\nu}\partial_\mu \Psi^{(1)}\partial_\nu \Psi^{(1)}. \tag{47}$$

The minus sign in (47) means that we have be able to change the value of η_{11} from 1 to -1 as expected. In turn this means that the original flat metric $\eta_{(pq)} = diag(1,1,1)$ corresponding to the $2+2$ signature becomes, in the dual sector, the flat metric $\eta_{(pq)} = diag(-1,1,1)$ associated with the $1+3$ signature. Presumably, this procedure can be, of course, generalized for non-flat metrics $g_{\mu\nu}$ and $h_{(pq)}$, but this will require some additional computations.

It turns out that signature changes can be connected with topology changes [21]. So, it may be interesting to relate our present procedure of signature change with that of topological change.

6. **Final remarks**

Summarizing we have described a self-dual gravitational theory in which the signature corresponds to two time and two space dimensions, that is in the signature $diag(-1,-1,+1,+1)$. Our preliminary analysis indicates that an action of the form

$$S = \frac{1}{4}\int_{M^{2+2}} E\ E^{\hat{\mu}}_{(\hat{A})} E^{\hat{\nu}}_{(\hat{B})}\ {}^{+}R^{(\hat{A}\hat{B})}_{\hat{\mu}\hat{\nu}}, \tag{48}$$

will describe a self-dual gravitational gauge theory with a gauge field with only three degrees of freedom. Of course, in order to have a complete theory one needs to develop (48) in full details, but in this sense our proposed action (26) surely may provide an important mathematical tool for that purpose.

Finally, it is worth mentioning that one of the main motivations in Ref. [9] was the idea of establishing a connection between Ashtekar formalism in $diag(-1,-1,+1+1)$ signature and oriented matroid matroid theory [22] (see also Ref. [23]-[24] and references therein). We believe that the present work can be also useful in such a quest.

Note added: While we were preparing this paper we became aware of the Refs. [25]-[26], where new variables for classical and quantum gravity in all dimensions are discussed. It will be interesting for further research to see whether there is a connection between the present work and such references.

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