Problem of time and Hamiltonian reduction in the (2+2) formalism

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We apply the Hamiltonian reduction procedure to general spacetimes of 4-dimensions in the (2+2) formalism and find privileged spacetime coordinates in which the physical Hamiltonian is expressed in true degrees of freedom only, namely, the conformal two-metric on the cross section of null hypersurfaces and its conjugate momentum. The physical time is the area element of the cross section of null hypersurface, and the physical radial coordinate is defined by *equipotential* surfaces on a given spacelike hypersurface of constant physical time. The physical Hamiltonian is *constraint-free* and manifestly *positive-definite* in the privileged coordinates. We present the complete set of the Hamilton's equations, and find that they coincide with the Einstein's equations written in the privileged coordinates. This shows that our Hamiltonian reduction is self-consistent and respects the general covariance. This work is a generalization of ADM Hamiltonian reduction of midi-superspace to 4-dimensional spacetimes with no isometries.

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It has been known for a long time that true degrees of freedom of general relativity reside in the conformal two-metric of the cross section of null hypersurfaces[1, 2]. Eliminating unphysical degrees of freedom by identifying arbitrary spacetime coordinates with certain functions in phase space and thereby presenting the theory in terms of physical degrees of freedom only in the privileged coordinates, free from constraints, is known as ADM Hamiltonian reduction[3–5]. K. Kuchař applied this procedure to spacetimes that admit two commuting Killing vector fields, known as midi-superspace[6–8], and showed that the Einstein's theory is equivalent to cylindrical massless scalar field theory propagating in the 1+1 dimensional Minkowski spacetime.

In this Letter, we apply ADM-like Hamiltonian reduction procedure to general spacetimes of 4-dimensions using the (2+2) decomposition[9–13]. The area element of the cross section of null hypersurfaces emerges as the physical time, and the physical radial coordinate is defined by *equipotential* surfaces on a given spacelike hypersurface of constant physical time. We present the fully reduced physical Hamiltonian in these privileged coordinates[14], which turns out to be positive-definite. The momentum constraints are simply the *defining* equations of the momenta of the theory in term of physical degrees of freedoms[6]; hence they are no longer constraints and the theory becomes constraint-free. Moreover, we find that our Hamiltonian reduction is self-consistent because the Hamilton's equations of motion obtained through this Hamiltonian reduction are *identical* to the Ricci-flat equations in the privileged coordinates. As a by-product of this Hamiltonian reduction, we found an independent proof of *topological censorship*[15–20], which follows directly from one of the Einstein's equations in these coordinates.

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Let us recall that the metric in the (2+2) decomposition[1, 2, 9–13] of 4-dimensional spacetimes can be written as

$$ds^2 = 2dudv - 2hdu^2 + \tau\rho_{ab}\left(dy^a + A_+^{\ a}du + A_-^{\ a}dv\right) \times \left(dy^b + A_+^{\ b}du + A_-^{\ b}dv\right). \quad (1)$$

The vector fields $\hat{\partial}_+$ and $\hat{\partial}_-$ defined as

$$\hat{\partial}_+ := \frac{\partial}{\partial u} - A_+^{\ a}\frac{\partial}{\partial y^a},\ \ \hat{\partial}_- := \frac{\partial}{\partial v} - A_-^{\ a}\frac{\partial}{\partial y^a}\ \ (a = 2, 3) \quad (2)$$

are horizontal vector fields orthogonal to the two-dimensional spacelike surface N_2 generated by $\partial_a = \partial/\partial y^a$. The metric on N_2 is given by $\tau\rho_{ab}$ (with $\det\rho_{ab} = 1$), and $\hat{\partial}_-$ is a null vector field with a zero norm. In this Letter, we choose the sign $-2h > 0$ so that $v =$ constant hypersurface is spacelike.

As was shown in [13], the Einstein's equations can be obtained from the variational principle of the following action integral,

$$S = \int dvdud^2y\{\pi_\tau\dot{\tau} + \pi_h\dot{h} + \pi_a\dot{A}_+^{\ a} + \pi^{ab}\dot{\rho}_{ab} - \text{“1”}\cdot C - \text{“0”}\cdot C_+ - A_-^{\ a}C_a\}, \quad (3)$$

where the overdot $\dot{}$ $= \partial_-$, and “1”, “0”, and $A_-^{\ a}$ are Lagrange multipliers that enforce the constraints C, C_+, and C_a, which are given by

$$\text{(i) } C := \frac{1}{2}\pi_h\pi_\tau - \frac{h}{4\tau}\pi_h^2 - \frac{1}{2\tau}\pi_h D_+\tau + \frac{1}{2\tau^2}\rho^{ab}\pi_a\pi_b - \frac{\tau}{8h}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) - \frac{1}{2h\tau}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} - \frac{1}{2h}\pi^{ac}D_+\rho_{ac} - \tau R_{(2)} + D_+\pi_h - \partial_a(\tau^{-1}\rho^{ab}\pi_b) = 0, \quad (4)$$

$$\text{(ii) } C_+ := \pi_\tau D_+\tau + \pi_h D_+ h + \pi^{ab}D_+\rho_{ab} - 2D_+(h\pi_h + D_+\tau) + 2\partial_a(h\tau^{-1}\rho^{ab}\pi_b + \rho^{ab}\partial_b h) = 0, \quad (5)$$

(iii) $C_a := \pi_\tau \partial_a \tau + \pi_h \partial_a h + \pi^{bc}\partial_a \rho_{bc} - 2\partial_b(\rho_{ac}\pi^{bc})$
$\quad -D_+\pi_a - \partial_a(\tau\pi_\tau) = 0.$ (6)

Notice minor changes of sign from [13]. Here $R_{(2)}$ is the scalar curvature of N_2, and the diffN_2-covariant derivative[13] of a tensor density q_{ab} with weight s is defined as

$$D_+q_{ab} := \partial_u q_{ab} - [A_+,\ q]_{\mathrm{L}ab} = \partial_u q_{ab} - A_+^{\ c}\partial_c q_{ab} - q_{cb}\partial_a A_+^{\ c} - q_{ac}\partial_b A_+^{\ c} - s(\partial_c A_+^{\ c})q_{ab}, \tag{7}$$

where $[A_+,\ q]_{\mathrm{L}ab}$ is the Lie derivative of q_{ab} along $A_+ := A_+^{\ a}\partial_a$. Let us define a *potential* function R and rename $A_+^{\ a}$ as w^a such that

$$D_+R := -h\pi_h, \quad w^a = A_+^{\ a}. \tag{8}$$

If we impose the constraint $C_+ = 0$ and $C_a = 0$, then the action (3) becomes

$$S = \int dv du d^2y\{\pi_\tau\dot{\tau} + \pi_R\dot{R} + \pi^w_a\dot{w}^a + \pi^{ab}\dot{\rho}_{ab} - C\} + \text{total derivatives}, \tag{9}$$

where the momenta π_R and π^w_a conjugate to R and w^a are given by

$$\pi_R = -D_+\ln(-h), \tag{10}$$
$$\pi^w_a = \pi_a - (\partial_a R)\ \ln(-h), \tag{11}$$

respectively. The transformation from $\{h, \pi_h; A_+^{\ a}, \pi_a\}$ to $\{R, \pi_R; w^a, \pi^w_a\}$ is clearly a *canonical* transformation, as it changes the action integral by total derivatives only. The Hamiltonian constraint in new variables becomes

$$\begin{aligned} C = &-\frac{1}{2h}\pi_\tau D_+R - \frac{1}{4h\tau}(D_+R)^2 + \frac{1}{2h\tau}(D_+\tau)(D_+R) \\ &-\frac{1}{h}D_+^2R - \frac{1}{h}\pi_R D_+R - \tau R_{(2)} \\ &+\frac{1}{2\tau^2}\rho^{ab}\{\pi^w_a + \ln(-h)\partial_a R\}\{\pi^w_b + \ln(-h)\partial_b R\} \\ &-\frac{1}{2h\tau}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} - \frac{\tau}{8h}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) \\ &-\frac{1}{2h}\pi^{ac}D_+\rho_{ac} - \partial_a[\tau^{-1}\rho^{ab}\{\pi^w_b + \ln(-h)\partial_b R\}] \\ &= 0. \end{aligned} \tag{12}$$

Hamiltonian reduction I: Choose τ as physical time
Notice that the first term in (12) is linear in π_τ, and all the remaining terms are independent of π_τ, since h depends on $\{\pi_R, w^a\}$ only, as is obvious from the equation (10). Thus, the equation of motion of τ is given by

$$\dot{\tau} = \int du d^2y \frac{\delta C}{\delta \pi_\tau} = -\frac{1}{2h}D_+R. \tag{13}$$

Now, recall that $\tau = \tau(v, u, y^a)$. Solving this equation for v, one may view v as a function of (τ, u, y^a) and consequently regard $\{R,\ w^a,\ \rho_{ab}\}$ as functions of (τ, u, y^a).

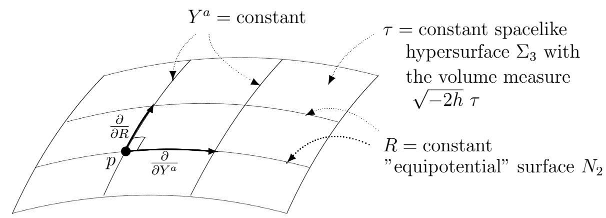

FIG. 1: On $R =$ constant "equipotential" surface N_2 on Σ_3, Y^a are introduced such that $Y^a =$ constant is normal to N_2 at each point $p \in N_2$. Then the "shift" vector w^a becomes zero at p.

Therefore, it follows that

$$\dot{R} = \dot{\tau}\partial_\tau R,\ \dot{w}^a = \dot{\tau}\partial_\tau w^a,\ \dot{\rho}_{ab} = \dot{\tau}\partial_\tau\rho_{ab}, \quad C = -(\frac{2h}{D_+R})\dot{\tau}C, \tag{14}$$

since $\partial u/\partial v = \partial y^a/\partial v = 0$. Then the action (9) becomes

$$\begin{aligned} S &= \int dv du d^2y \dot{\tau}\{\pi_\tau + \pi_R\partial_\tau R + \pi^w_a\partial_\tau w^a + \pi^{ab}\partial_\tau\rho_{ab} + (\frac{2h}{D_+R})C\} \\ &= \int d\tau du d^2y\{\pi_R\partial_\tau R + \pi^w_a\partial_\tau w^a + \pi^{ab}\partial_\tau\rho_{ab} - C^{(1)}\}, \end{aligned} \tag{15}$$

where we replaced $dv\dot{\tau}$ by $d\tau$ in the second line, and $C^{(1)}$ is defined as

$$C^{(1)} = -(\frac{2h}{D_+R})C - \pi_\tau. \tag{16}$$

Hamiltonian reduction II: Choose R as physical radius and fix arbitrary coordinates y^a on $R =$ constant subspace such that $w^a = 0$
The second step in the Hamiltonian reduction consists of *identifying* arbitrary coordinate u as R and *choosing* $y^a = Y^a$ such that the "shift" vector w^a on $R =$ constant subspace is zero,

$$w^a = 0. \tag{17}$$

Then, it follows from (13) that

$$\dot{\tau} = -\frac{1}{2h} \geq 0, \tag{18}$$

which means that τ increases monotonically along the out-going null vector field. Thus, $\{R, Y^a\}$ are the privileged coordinates on $\tau =$ constant hypersurface Σ_3, and therefore the following equations are trivially true,

$$\partial_a\tau = \partial_R\tau = 0. \tag{19}$$

But recall that τ is a scalar density with weight 1, rather than a scalar function, with respect to diffN_2 transformations. Thus, the covariant derivative $D_R\tau$ is given by

$$D_R\tau = \partial_R\tau - w^a\partial_a\tau - (\partial_a w^a)\tau. \tag{20}$$

By the condition (17) and (19), τ is covariantly constant on Σ_3,

$$D_R\tau = 0, \tag{21}$$

which means that our choice of τ as a physical time is consistently defined over Σ_3, even though τ is a scalar density rather than a scalar function. The Hamilton's equations of motion follows from the variational principle of the action integral (15)

$$\partial_\tau q^I = \int_{\Sigma_3} dud^2y \frac{\delta C^{(1)}}{\delta \pi_I}|_{u=R, y^a=Y^a}, \tag{22}$$

$$\partial_\tau \pi_I = -\int_{\Sigma_3} dud^2y \frac{\delta C^{(1)}}{\delta q^I}|_{u=R, y^a=Y^a}, \tag{23}$$

for $q^I = \{R, w^a, \rho_{ab}\}$, and $\pi_I = \{\pi_R, \pi^w_a, \pi^{ab}\}$. In the following we present the main results of this Letter without derivation:

Einstein's evolution equations

$$1. \frac{\partial R}{\partial \tau} = 0 \Rightarrow$$

$$\tau R_{(2)} = \frac{1}{2}\tau^{-2}\rho^{ab}\pi^w_a\pi^w_b - \partial_a(\tau^{-1}\rho^{ab}\pi^w_b) \tag{24}$$

(topological censorship)

$$2. \frac{\partial w^a}{\partial \tau} = 0 \Rightarrow$$

$$\tau^{-1}\pi^w_a = -\partial_a \ln(-h) \ \text{(superpotential)} \tag{25}$$

$$3. \frac{\partial \tau}{\partial \tau} = 1 \ \text{(trivial)} \tag{26}$$

$$4. \frac{\partial \ln(-h)}{\partial \tau} = H_* - \frac{1}{\tau} \ \text{(superpotential)} \tag{27}$$

$$5. \frac{\partial \pi^w_a}{\partial \tau} = 2\tau^{-1}\pi^w_a + (\pi^{bc} + \frac{1}{2}\rho^{bd}\rho^{ce}\partial_R\rho_{de})\partial_a\rho_{bc}$$

$$-\partial_b(2\pi^{bc}\rho_{ac} + \rho^{bc}\partial_R\rho_{ac}) \tag{28}$$

(evolution equation of π^w_a)

$$6. \frac{\partial \pi_\tau}{\partial \tau} = \frac{1}{2}\tau^{-2} + \tau^{-2}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd}$$

$$-\frac{1}{4}\rho^{ab}\rho^{cd}(\partial_R\rho_{ac})(\partial_R\rho_{bd}) - 2\tau^{-2}\partial_a(h\rho^{ab}\pi^w_b) \tag{29}$$

(evolution equation of π_τ)

where H_* is defined by

$$H_* = \frac{1}{\tau}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{4}\tau\rho^{ab}\rho^{cd}(\partial_R\rho_{ac})(\partial_R\rho_{bd})$$

$$+\pi^{ac}\partial_R\rho_{ac} + \frac{1}{2\tau} \geq \frac{1}{2\tau}. \tag{30}$$

Einstein's constraint equations

$$7. \ C = 0 \Rightarrow \pi_\tau = -H_* - 2\pi_R \ \text{(def. of } \pi_\tau) \tag{31}$$

$$8. \ C_+ = 0 \Rightarrow \pi_R = -\pi^{ab}\partial_R\rho_{ab} \tag{32}$$

(def. of radial momentum)

$$9. \ C_a = 0 \Rightarrow \tau^{-1}\pi^w_a = -\pi^{bc}\partial_a\rho_{bc} + 2\partial_b(\pi^{bc}\rho_{ac})$$

$$-\tau\partial_a(H_* + \pi_R) \ \text{(def. of angular momentum)} \tag{33}$$

Superpotential $\ln(-h)$

$$10. \ \partial_\tau \ln(-h) = H_* - \tau^{-1} \tag{34}$$

$$11. \ -\partial_R \ln(-h) = \pi_R \tag{35}$$

$$12. \ -\partial_a \ln(-h) = \tau^{-1}\pi^w_a \tag{36}$$

Integrability conditions

$$13. \ \partial_R(\tau^{-1}\pi^w_a) = \partial_a\pi_R \tag{37}$$

$$14. \ \partial_\tau\pi_R = -\partial_R H_* \tag{38}$$

$$15. \ \partial_\tau(\tau^{-1}\pi^w_a) = -\partial_a H_* \tag{39}$$

The evolution equations of ρ_{ab} and π^{ab} can be found from the reduced action principle,

$$S_* = \int_{\Sigma_3} dRd^2Y\{\pi^{ab}\partial_\tau\rho_{ab} - C^{(1)}_*\}, \tag{40}$$

where $C^{(1)}_*$ is the restriction of $C^{(1)}$ by the coordinate conditions $u = R$ and $y^a = Y^a$,

$$C^{(1)}_* := C^{(1)}|_{u=R, y^a=Y^a}$$

$$= H_* + 2\pi_R + 2h\{\tau R_{(2)} - \frac{1}{2}\tau^{-2}\rho^{ab}\pi^w_a\pi^w_b$$

$$+\partial_a(\tau^{-1}\rho^{ab}\pi^w_b)\}. \tag{41}$$

Here π_R is a total derivative given by (35). Variation of the reduced action S_* with respect to h reproduces *topological censorship constraint* (24), so the superpotential h in (41) is in fact a Lagrange multiplier enforcing the constraint (24), but this constraint *does* contribute to the following equations of motion:

Evolution equations of ρ_{ab} and π^{ab}

$$16. \frac{\partial \rho_{ab}}{\partial \tau} = \int_{\Sigma_3} dRd^2Y \frac{\delta C^{(1)}_*}{\delta \pi^{ab}} \tag{42}$$

$$17. \frac{\partial \pi^{ab}}{\partial \tau} = -\int_{\Sigma_3} dRd^2Y \frac{\delta C^{(1)}_*}{\delta \rho_{ab}}. \tag{43}$$

The spacetime metric in these privileged coordinates becomes

$$ds^2 = -4hdRd\tau - 2hdR^2 + \tau\rho_{ab}dY^adY^b. \tag{44}$$

On $\tau =$ constant spacelike hypersurface Σ_3, the volume measure of Σ_3 is given by

$$\sqrt{g} = \sqrt{-2h}\ \tau, \tag{45}$$

which increases monotonically in τ, as one finds that

$$\partial_\tau \ln\sqrt{g} = \frac{1}{2}H_* + \frac{1}{2\tau} \geq 0, \tag{46}$$

where we used the equation (34). Let us discuss some key features of above equations.

(i) First of all, let us mention that, without derivation, the whole set of the above equations are *identical to vacuum Einstein's equations* $R_{AB} = 0$ for spacetimes whose

metric is given by (44). Thus, the whole procedure of the Hamiltonian reduction proposed in this Letter respects the general covariance, as it must, even though the final theory is written in the privileged coordinates.
(ii) The integral of (24) over a closed two-surface N_2 becomes

$$\int_{N_2} d^2Y \tau^{-2} \rho^{ab} \pi_a^w \pi_b^w = 16\pi(1-g) \geq 0, \tag{47}$$

where g is the genus of N_2. This identity states that, as long as the out-going null hypersurface forms a congruence of null geodesics which admits a cross section, the spatial topology of that null hypersurface is either a two-sphere or a torus. This is an astonishingly simple proof of *topological censorship*, as it does not rely on assumptions such as global hyperbolicity, asymptotic conditions, energy conditions, and so on, which are normally assumed in the literature[15–20].
(iii) The spatial integral of $C_*^{(1)}$ defined in (41) is the sought-for *physical* Hamiltonian of vacuum spacetimes. If we impose the topological censorship constraint (24), the true Hamiltonian becomes

$$\tilde{H}_* := \int_{\Sigma_3} dR d^2Y H_* \geq 0, \tag{48}$$

which is *positive-definite* in the privileged coordinate.
(iv) The logarithm of the conformal factor in the (τ, R) subspace is the *superpotential* $\ln(-h)$, whose gradients yield $(H_*, \pi_R, \tau^{-1}\pi_a^w)$ through (34), (35), and (36), respectively. The superpotential[6] is a local function of x, determined by the line integral

$$\ln\frac{h(x)}{h(x_0)} = \int_{x_0, C}^{x} \{(H_* - \tau^{-1}) d\tau - \pi_R dR - \tau^{-1}\pi_a^w dY^a\} \tag{49}$$

along *any* contour C from x_0 to x in a given spacetime.
(v) The integrability conditions (37), (38), and (39) are the consistency conditions, which follow from the very definition of the superpotential $\ln(-h)$.
(vi) The momentum constraints are just the *defining* equations of the radial momentum density π_R and angular momentum density $\tau^{-1}\pi_a^w$ in terms of true gravitational degrees of freedom through (32) and (33). They do not restrict the theory in any way, so that the theory becomes *constraint-free*. The last two terms on the right hand side of (33), which are total derivatives, represent coordinate effects as they do not contribute when integrated over a closed two-surface N_2. If we define the total linear momentum $\tilde{\Pi}_R$ and total angular momentum $\tilde{\Pi}_a$ as

$$\tilde{\Pi}_R := \int_{\Sigma_3} dR d^2Y \pi_R, \quad \tilde{\Pi}_a := \int_{\Sigma_3} dR d^2Y \tau^{-1} \pi_a^w, \tag{50}$$

then one finds that

$$\tilde{\Pi}_R = -\int_{\Sigma_3} dR d^2Y \pi^{bc} \partial_R \rho_{bc}, \tag{51}$$

$$\tilde{\Pi}_a = -\int_{\Sigma_3} dR d^2Y \pi^{bc} \partial_a \rho_{bc}. \tag{52}$$

These equations show that $\tilde{\Pi}_R$ and $\tilde{\Pi}_a$ are the generating functions of translations of ρ_{ab} and π^{ab} along ∂_R and ∂_a. This justifies our interpretation of $\tilde{\Pi}_R$ and $\tilde{\Pi}_a$ as total linear and angular momentum carried by physical degrees of freedom, respectively. Moreover, if suitable boundary conditions on ρ_{ab} and π^{ab} are assumed, then, by virtue of the integrability conditions (38) and (39), $\tilde{\Pi}_R$ and $\tilde{\Pi}_a$ are *conserved* in τ time!

Details of this work and applications of this Hamiltonian formalism to quantum theory will be published in forthcoming papers.

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