Nambu-Goto Action and Qubit Theory

In Any Signature and in Higher Dimensions

H. Larraguível∗1, G. V. López∗2 and J. A. Nieto∗3

∗Departamento de Física de la Universidad de Guadalajara, Guadalajara, México.
∗Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma de Sinaloa, 80010, Culiacán Sinaloa, México.

Abstract

We perform an extension of the relation between the Nambu-Goto action and qubit theory. Of course, the Cayley hyperdeterminant is the key mathematical tool in such generalization. Using the Wick rotation we find that in four dimensions such a relation can be established not only in (2+2)-dimensions but also in any signature. We generalize our result to a curved space-time of (2^{2n}+2^{2n})-dimensions and (2^{2n+1}+2^{2n+1})-dimensions.

Keywords: Nambu-Goto action; qubit theory; general relativity
Pacs numbers: 04.60.-m, 04.65.+e, 11.15.-q, 11.30.Ly
March 17, 2016

1helder.larraguivel@red.cucei.udg.mx
2gulopez@udgserv.cencar.udg.mx
3nieto@uas.edu.mx; janieto1@asu.edu.mx

1
Some years ago, Duff [1] discovers hidden new symmetries in the Nambu-Goto action [2]-[3]. It turns out that the key mathematical tool in such a discovery is the Cayley hyperdeterminant [4]. In this pioneer work, however, the target space-time turns out to have an associated \((2 + 2)\)-signature, corresponding to two time and two space dimensions. It was proved in Ref. [5]-[6] that the Duff’s formalism can also be generalized to \((4 + 4)\)-dimensions and \((8 + 8)\)-dimensions. Here, we shall prove that if one introduces a Wick rotations for various coordinates then one can actually extend the Duff’s procedure to any signature in 4-dimensions. Moreover, we also prove that our method can be extended to curved space-time in \((2^{2n} + 2^{2n})\)-dimensions and \((2^{2n+1} + 2^{2n+1})\)-dimensions.

There are a number of physical reasons to be interested on these developments, but perhaps the most important is that eventually our work may be useful on a possible generalization of the remarkable correspondence between black-holes and quantum information theory (see Refs. [7]-[10] and references therein).

Let us start recalling the Duff’s approach on the relation between the Nambu-Goto action and the \((2 + 2)\)-signature. Consider the Nambu-Goto action [2]-[3],

\[
S = \int d\xi^2 \sqrt{\epsilon \det(\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu})}. \tag{1}
\]

Here, the space-time coordinates \(x^\mu\) are real function of two parameters \((\tau, \sigma) = \xi^a\) and \(\eta_{\mu\nu}\) is a flat metric, determining the signature of the target space-time. Moreover, the parameter \(\epsilon\) takes the values +1 or \(-1\), depending whether the signature of \(\eta_{\mu\nu}\) is Euclidean or Lorenziana, respectively.

It turns out that by introducing the world-sheet metric \(g^{ab}\) one can prove that (1) is equivalent to the action [11] (see also Ref. [12] and references therein)

\[
S = \int d\xi^2 \sqrt{-\epsilon \det g_{ab} \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu}}, \tag{2}
\]

which is, of course, the Polyakov action (see Ref. [12] and references therein). In fact, from the expression

\[
\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} - \frac{1}{2} g_{ab} g^{cd} \partial_c x^\mu \partial_d x^\nu \eta_{\mu\nu} = 0, \tag{3}
\]

obtained by varying the action (2) with respect to \(g^{ab}\), it is straightforward to show that from (2) one obtains (1) and \textit{vice versa}. Hence, the actions (1) and (2) are equivalents.

It is convenient to define the induced world-sheet metric
\[ h_{ab} \equiv \partial_a x^\mu \partial_b x'^\nu \eta_{\mu\nu}. \]  

Using this definition, the Nambu-Goto action (1) becomes

\[ S = \int d\xi^2 \sqrt{\epsilon \det(h_{ab})}. \]  

It is not difficult to see that in \((2+2)\)-dimensions the expression (4) can be written as

\[ h_{ab} = \partial_a x^{ij} \partial_b x^{kl} \varepsilon_{ik} \varepsilon_{jl}, \]  

where \(x^{ij}\) denotes a the \(2 \times 2\)-matrix

\[ x^{ij} = \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ -x^2 + x^4 & x^1 - x^3 \end{pmatrix}. \]  

It is important to observe that (7) corresponds to the set \( \mathcal{M}(2, R) \) of any \(2 \times 2\)-matrix. In fact, by introducing the fundamental base matrices

\[ \delta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varepsilon^{ij} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]  

\[ \eta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \lambda^{ij} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

one observes that (7) can be rewritten as the linear combination

\[ x^{ij} = x^1 \delta^{ij} + x^2 \varepsilon^{ij} + x^3 \eta^{ij} + x^4 \lambda^{ij}. \]

Let us now introduce the expression

\[ h = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \partial_a x^i \partial_b x^j h_{ac} h_{bd}. \]  

If one uses (4) one gets

\[ h = \det(h_{ab}). \]  

However, if one considers (6) one obtains

\[ h = \text{Det}(h_{ab}), \]  

where \(\text{Det}(h_{ab})\) denotes the Cayley hyperdeterminant of \(h_{ab}\), namely

\[ \text{Det}(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a x^{ij} \partial_c x^{kl} \partial_b x^{mn} \partial_d x^{rs}. \]
Of course, (11) and (12) imply that
\[ \text{det}(h_{ab}) = \mathcal{D}\text{et}(h_{ab}). \]  
(14)

In turn, (14) means that in (2 + 2)-dimensions the Nambu-Goto action (5) can also be written as
\[ S = \int d\xi^2 \sqrt{\mathcal{D}\text{et}(h_{ab})}. \]  
(15)

Note that, since in this case one is considering the (2 + 2)-signature one must set \( \epsilon = +1 \) in (5).

In (4 + 4)-dimensions the key formula (6) can be generalized as
\[ h_{ab} = \partial_a x^{ijm} \partial_b x^{kl} \varepsilon_{ik} \varepsilon_{jl} \eta_{ms}. \]  
(16)

While in (8 + 8)-dimensions one has
\[ h_{ab} = \partial_a x^{ijmn} \partial_b x^{kl} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{ms} \varepsilon_{nr}. \]  
(17)

(see Refs. [5] and [6] for details). So by considering the real variables \( x^{i_1 \ldots i_n} \) and properly considering the matrices \( \varepsilon_{ij} \) and \( \eta_{ij} \), the previous formalism can be generalized to higher dimensions. Of course, in such cases the Cayley hyperdeterminant \( \mathcal{D}\text{et}(h_{ab}) \) must be modified accordingly.

Observing (7) one wonders whether one can consider in (6) other signatures in 4-dimensions besides the (2 + 2)-signature. It is not difficult to see that using the Wick rotation in any of the coordinates \( x^1, x^2, x^3 \) or \( x^4 \) one can modify the signature. For instance, one can achieve the (1 + 3)-signature if one uses the prescription \( x^2 \rightarrow ix^2 \) in (6). This method lead us inevitable to generalize our method to a complex structure. One simple introduce the complex matrix
\[ z^{ij} = z^1 \delta^{ij} + z^2 \varepsilon^{ij} + z^3 \eta^{ij} + z^4 \lambda^{ij}, \]  
(18)

where the variables \( z^1, z^2, z^3 \) and \( z^4 \) are complex numbers. The expression (6) is generalized accordingly as [13]
\[ h_{ab} = \partial_a z^{ij} \partial_b z^{kl} \varepsilon_{ik} \varepsilon_{jl}. \]  
(19)

Thus, in this case, the Cayley hyperdeterminant becomes
\[ \mathcal{D}\text{et}(h_{ab}) = \frac{1}{2!} \varepsilon^{abcd} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mn} \varepsilon_{rs} \partial_a z^{ij} \partial_b z^{kl} \partial_a z^{mn} \partial_b z^{rs}. \]  
(20)

and consequently the Nambu-Goto action must be written using (20). Of course, the Nambu-Goto action, or the Polyakov action, must be real and
therefore one must choose any of the coordinates $z^1, z^2, z^3$ and $z^4$ in (20) either as pure real or pure imaginary.

Similarly, the generalization to a complex structure can be made by introducing the complex variables $z^{i_1 \ldots i_n}$ and writing

$$
 \text{Det}(h_{ab}) = \frac{1}{2!} e^{ab} e^{cd} \varepsilon_{i_1 j_1 \ldots i_n j_n} \eta_{i_1 \ldots i_n} \eta_{j_1 \ldots j_n} \delta_{i_1 \ldots i_n} \delta_{j_1 \ldots j_n},
$$

or

$$
 \text{Det}(h_{ab}) = \frac{1}{2!} e^{ab} e^{cd} \varepsilon_{i_1 j_1 \ldots i_n j_n} \eta_{i_1 \ldots i_n} \eta_{j_1 \ldots j_n} \delta_{i_1 \ldots i_n} \delta_{j_1 \ldots j_n},
$$

(21)

(22)

depending whether the signature is $(2n + 2n)$ or $(2n + 1 + 2n + 1)$, respectively.

One can further generalize our procedure by considering a target curved space-time. For this purpose let us introduce the curved space-time metric

$$
 g_{\mu \nu} = e^A_{\mu} e^B_{\nu} \eta_{AB}.
$$

(23)

Here, $e^A_{\mu}$ denotes a vielbein field and $\eta_{AB}$ is a flat metric. The Polyakov action in a curved target space-time becomes

$$
 S = \int d\xi^2 \sqrt{-\epsilon} \det gg_{\mu \nu} \partial_\mu x^a \partial_\nu x^b g_{\mu \nu}.
$$

(24)

Using (23), one sees that this action can be written as

$$
 S = \int d\xi^2 \sqrt{-\epsilon} \det gg^{ab} (\partial_\mu x^\mu e^A_a) (\partial_\nu x^\nu e^B_b) \eta_{AB}.
$$

(25)

So, by defining the quantity

$$
 E^A_a \equiv \partial_\mu x^\mu e^A_a,
$$

(26)

the action in (25) reads as

$$
 S = \int d\xi^2 \sqrt{-\epsilon} \det gg^{ab} E^A_a E^B_b \eta_{AB}.
$$

(27)

Hence, in a target space-time of $(2 + 2)$-dimensions one can write (27) in the form

$$
 S = \int d\xi^2 \sqrt{-\epsilon} \det gg^{ab} E^{ij}_a E^{kl}_b \varepsilon_{ik} \varepsilon_{jl},
$$

(28)

where

$$
 E^{ij}_a \equiv \partial_\mu x^\mu e^{ij}_a.
$$

(29)
Here, we considered the fact that one can always write
\[ e^{ij}_\mu = e^1_\mu \delta^{ij} + e^2_\mu \varepsilon^{ij} + e^3_\mu \eta^{ij} + e^4_\mu \lambda^{ij}. \]  
(30)

Observe that in this development one can consider a generalization of (4) namely
\[ h_{ab} = E^A_a E^B_b \eta_{AB} \]  
(31)

and therefore in \((2 + 2)\)-dimensions this expression becomes
\[ h_{ab} = E^{ij}_a E^{kl}_b \varepsilon_{ik} \varepsilon_{jl}, \]  
(32)

while in \((4 + 4)\)-dimensions and \((8 + 8)\)-dimensions one obtains
\[ h_{ab} = E^{ijm}_a E^{klr}_b \varepsilon_{ik} \varepsilon_{jl} \eta_{mr} \]  
(33)

and
\[ h_{ab} = E^{ijmn}_a E^{klrs}_b \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns}. \]  
(34)

respectively.

At this stage, it is evident that if one wants to generalize the procedure to any signature in a curved space-time one simply substitute in the action (27) either
\[ h_{ab} = \varepsilon_{\hat{i}_1 \ldots \hat{i}_n} \varepsilon_{\hat{j}_1 \ldots \hat{j}_n} \varepsilon_{\hat{i}_{n-1} \hat{j}_{n-1}} \eta_{\hat{i}_n \hat{j}_n} \]  
(35)

or
\[ h_{ab} = \varepsilon_{\hat{i}_1 \ldots \hat{i}_n} \varepsilon_{\hat{j}_1 \ldots \hat{j}_n} \varepsilon_{\hat{i}_{n-1} \hat{j}_{n-1}} \varepsilon_{\hat{i}_n \hat{j}_n}, \]  
(36)

depending whether the signature is \((2^{2n} + 2^{2n})\) or \((2^{2n+1} + 2^{2n+1})\), respectively. Here, we used the prescription \(E^B_a \rightarrow E^B_{\hat{a}}\), with \(E^B_{\hat{a}}\) a complex function.

In order to include \(p\)-branes in our formalism, one notes that the expression (35) and (36) can still be used. In such a case, one allows the indice \(a\) in (35) and (36) to run from 0 to \(p\). Braking such kind of indices as \(a = (\hat{a}_1, \hat{a}_2, \hat{a}_3)\), for a 3-brane, as \(a = (\hat{a}_1, \hat{a}_2, \hat{a}_3)\), for a 5-brane and so on one observes that (35) and (36) can be written as
\[ h_{\hat{a}_1 \ldots \hat{a}_2 \hat{b}_1 \ldots \hat{b}_2} = \varepsilon_{\hat{i}_1 \ldots \hat{i}_p} \varepsilon_{\hat{j}_1 \ldots \hat{j}_p} \varepsilon_{\hat{i}_{p-1} \hat{j}_{p-1}} \eta_{\hat{i}_p \hat{j}_p} \]  
(37)

or
\[ h_{\hat{a}_1 \ldots \hat{a}_2 \hat{b}_1 \ldots \hat{b}_2} = \varepsilon_{\hat{i}_1 \ldots \hat{i}_p} \varepsilon_{\hat{j}_1 \ldots \hat{j}_p} \varepsilon_{\hat{i}_{p-1} \hat{j}_{p-1}} \varepsilon_{\hat{i}_p \hat{j}_p}, \]  
(38)
respectively. The analogue of Cayley hyperdeterminant in this case will be

\[
\hat{\text{Det}}(h_{\hat{a}_1\ldots\hat{a}_2\hat{b}_1\ldots\hat{b}_2}) =
\]

\[= \varepsilon_{\hat{a}_1\hat{b}_1} \cdots \varepsilon_{\hat{a}_p \hat{b}_p} E^{i_1 \ldots i_p} \varepsilon^{i_{k_1} \ldots i_{k_p}} E_{j_1 \ldots j_p} \varepsilon_{i_1 \ldots i_{p-1}} \varepsilon_{j_1 \ldots j_{p-1}} \varepsilon_{i_p j_p},
\]

and therefore the corresponding Nambu-Goto action becomes

\[
S = \int d\xi^{p+1} \sqrt{c \hat{\text{Det}}(h_{\hat{a}_1\ldots\hat{a}_2\hat{b}_1\ldots\hat{b}_2})}.
\]

Summarizing, we have generalized the Duff’s procedure concerning the combination of the Nambu-Goto action and the Cayley hyperdeterminant in target space-time of (2+2)-dimensions. Such a generalization first corresponds to a curved worlds with \((2^{2n} + 2^{2n})\)-signature or \((2^{2n+1} + 2^{2n+1})\)-signature. Using complex structure we may be able to extend the procedure to any signature. Further, we generalize the method to \(p\)-branes.

It turns out that these generalization may be useful in a number of physical scenario beyond string theory and \(p\)-branes. In fact, since the quantity \(z^{j_1 \ldots j_n}\) can be identified with a \(n\)-qubit one may be interested in the route leading to oriented matroid theory [14] (see also Ref. [15]-[16]). In this direction, using the phi trope concept (see Ref. [17] and references therein), which is a complex generalization of the concept of chirotope in oriented matroid theory, a link between super \(p\)-branes and qubit theory has already been established [17]. Thus, it may be interesting for further developments to explore the connection between the results of the present work and supersymmetry via the Grassmann-Plücker relations (see Refs. [8]-[9] and references therein). It is worth mentioning that such relations are natural mathematical notions in information theory linked to \(n\)-qubit entanglement. Indeed, in such a case, the Hilbert space can be broken in the form \(C^{2n} = C^L \otimes C^l\) with \(L = 2n - 1\) and \(l = 2\). This allows a geometric interpretation in terms of the complex Grassmannian variety \(Gr(L, l)\) of 2-planes in \(C^{2n}\) via the Plücker embedding. In this context, the Plücker coordinates of Grassmannians \(Gr(L, l)\) are natural invariants of the theory (see Ref. [9] for details). However, it has been mentioned in Ref. [18], and proved in Refs. [19] and [20], that for normalized qubits the complex 1-qubit, 2-qubit and the 3-qubit are deeply related to division algebras via the Hopf maps, \(S^3 \overset{S^1}{\rightarrow} S^2\), \(S^7 \overset{S^3}{\rightarrow} S^4\) and \(S^{15} \overset{S^7}{\rightarrow} S^8\), respectively. In order to clarify the possible application of these observations in the context of our formalism let us consider the general complex state \(|\psi\rangle \in C^{2n}\),

\[
|\psi\rangle = \sum_{i_1 i_2 \ldots i_n = 0}^{1} z^{i_1 i_2 \ldots i_n} |i_1 i_2 \ldots i_n\rangle,
\]

(41)
where $|i_1i_2...i_n> = |i_1> \otimes |i_2> \otimes ... \otimes |i_n>$ correspond to a standard basis of the $n$-qubit. It is interesting to make the following observations. First, let us denote a $n$-rebit system (real $n$-qubit) by $x^{i_1i_2...i_n}$. So, one finds that a 3-rebit and 4-rebit have 8 and 16 real degrees of freedom, respectively. Thus, one learns that the 4-rebit can be associated with the 16 degrees of freedom of a 3-qubit. It turns out that this is the kind of embedding discussed in Ref. [9]. In this context, one sees that in the Nambu-Goto context one may consider the 16-dimensions target space-time as the maximum dimension required by division algebras via the Hopf map $S^{15} \to S^8$.

Acknowledgments

J. A. Nieto would like to thank to P. A. Nieto for helpful comments. This work was partially supported by PROFAPI/2007 and PIFI 3.3.

References


