Stochastic quantization and the role of time in quantum gravity

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Abstract

We show that the noncritical string field theory developed from two-dimensional quantum gravity in the framework of causal dynamical triangulations can be viewed as arising through a stochastic quantization. This requires that the proper time appearing in the string field theory be identified with the stochastic time of the stochastic formulation. The framework of stochastic quantization gives rise to a natural nonperturbative quantum Hamiltonian, which incorporates a sum over all spacetime topologies. We point out that the external character of stochastic time is a feature that pertains more generally to the proper time or distance appearing in nonperturbative correlation functions in quantum gravity.

1 Introduction

Time is a much-discussed and somewhat enigmatic quantity in classical and even more so in quantum general relativity, where the reparametrization invariance adds to the problem of quantizing the theory. Any attempt to shed additional light on the role of time in a quantized theory of gravity is therefore of interest. Because some of the structural issues concerning time persist also in two spacetime dimensions, one may profitably study toy models of two-dimensional quantum gravity to learn about their resolution. The group of spacetime diffeomorphisms still acts in an analogous fashion to that in four-dimensional general relativity, while the quantization can be carried out without having to deal with the problem of perturbative nonrenormalizability present in the higher-dimensional, physical theory.

The present piece of work is concerned with the two-dimensional quantum gravity known as Lorentzian quantum gravity or quantum gravity based on *causal dynamical triangulations* (CDT). The name refers to the regularization in terms of dynamical, triangulated lattices of the curved spacetimes appearing in the quantum field theory, when formulated as a nonperturbative path integral in Lorentzian signature [1, 2]. It turns out that in two dimensions a continuum limit can be taken analytically. In this paper we will assume this has been done, and work exclusively with the resulting continuum theory. We will show that the string field theory we have developed earlier [3] for the purpose of describing the splitting and joining in time of spatial universes has a natural description as a stochastic quantization of space. Recall that the original (and strictly causal) CDT quantization employs a global proper-time foliation, with respect to which spatial topology changes are forbidden. Generalizing this set-up by allowing isolated causality-violating points, space *can* now split into disconnected components, which may or may not join again at a later time, depending on what processes the model should incorporate. In a (quantum-) gravitational theory, where geometry is defined intrinsically, this raises interesting questions about the global nature of this proper-time variable. We showed in previous work that consistency relations hold among the simply connected quantum amplitudes of the two-dimensional theory, which indicates that a global time interpretation may persist in more complicated situations involving topology change [4, 3].

Here we will demonstrate that the stochastic quantization coincides with the string field theory, and therefore that the global proper time has a natural reinterpretation as the stochastic time arising in a stochastic quantization of (one-dimensional) space.[1] This phenomenon is not unique to the CDT string field theory, but was first observed in [6] in the context of a string field theory developed for non-critical strings [5], after which the CDT construction is modelled. The relation with stochastic quantization was also found independently in a reformulation of matrix

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[1]Note that this differs from a "standard" stochastic quantization of gravity, where stochastic time would appear *in addition* to the time already present as part of the space*time* geometry.

models as collective field theories [7], indicating that we are dealing with a more general phenomenon. Contrary to the rather intricate way in which it enters in two-dimensional *Euclidean* quantum gravity (equivalently, noncritical string theory), the relation is much more straightforward in the case of the Lorentzian CDT string field theory. As we will see in the following, it can be put to use in a constructive manner to find a number of quantum observables nonperturbatively, in the sense of being able to evaluate them on a sum over all genera of two-dimensional spacetime.

Of course, the CDT formulation is primarily geared towards solving four-dimensional quantum gravity. In this case the model cannot be solved analytically, but is being investigated by computer simulations, which have already led to a number of new and interesting results [8]. Among them are strong indications that the infrared limit of the theory is just that of classical general relativity. Details of the ultraviolet limit are still under investigation. Candidates for possible UV completions still within a field-theoretical framework are (i) the asymptotic safety scenario with a nontrivial UV fixed point [9, 10], (ii) the scale-invariant gravity model advocated by Shaposhnikov et al. [11, 12], and (iii) the model of Lifshitz gravity suggested by Hořava [13]. To the extent they can be compared, the structural set-up of the latter is reminiscent of that of the CDT approach: one also works with an explicit, global time foliation, and the infrared limit is that of general relativity, while the UV limit – assuming it exists – is highly nonclassical and apparently undergoes a "dynamical dimensional reduction" (also observed in [14]). Interestingly, the construction of the anisotropic Lifshitz gravity models also bears a structural resemblance with that of stochastic quantization, a fact already noted by Hořava [15].

The rest of the paper is organized as follows: in Section 2 we review briefly the formalism of stochastic quantization, closely following reference [16]. In Section 3 we introduce the CDT string field theory and show that it can be viewed as stochastic quantization of space, if CDT proper time is identified with stochastic time. In Section 4 we derive the corresponding nonperturbative Hamiltonian and discuss its properties and interpretation. Section 5 summarizes our results and their possible implications for the nature of time in quantum gravity..

2   Stochastic quantization

This section summarizes the key steps of the stochastic quantization formalism; for more extended textbook treatments see, for example, [16, 17]. The Langevin stochastic differential equation for a single variable x reads

$$\dot{x}^{(\nu)}(t) = -f\left(x^{(\nu)}(t)\right) + \sqrt{\Omega}\,\nu(t), \tag{1}$$

where the dot denotes differentiation with respect to stochastic time t, $\nu(t)$ is a Gaussian white-noise term of unit width and $f(x)$ a drift force. We will only con-

sider the case of dissipative diffusion where the drift force is conservative, that is,

$$f(x) = \frac{\partial S(x)}{\partial x} \tag{2}$$

for some function $S(x)$, a property which ensures the stochastic process satisfies the principle of detailed balance (see e.g. [16]). Without the noise term, (1) reduces to a relaxation equation. In that case – depending on the initial value $x_0 = x(0)$ – $x(t)$ will move towards the "nearest" local minimum of $S(x)$ or run away if there is no minimum which can be reached from x_0 by decreasing $S(x)$. When the noise term is added, $x(t)$ will be kicked around close to a minimum. If there are several local minima, the noise term can kick it from one to another and also to a region of no minimum if it exists. In this manner the noise term creates a probability distribution of $x(t)$, reflecting the assumed stochastic nature of the noise term, with an associated probability distribution

$$P(x, x_0; t) = \left\langle \delta(x - x^{(\nu)}(t; x_0)) \right\rangle_\nu, \tag{3}$$

where the expectation value refers to an average over the Gaussian noise. It can be shown that $P(x, x_0; t)$ satisfies the so-called Fokker-Planck equation

$$\frac{\partial P(x, x_0; t)}{\partial t} = \frac{\partial}{\partial x} \left(\frac{1}{2} \Omega \frac{\partial P(x, x_0; t)}{\partial x} + f(x) P(x, x_0; t) \right). \tag{4}$$

This is an imaginary-time Schrödinger equation, with $\sqrt{\Omega}$ playing a role similar to $\hbar$. It enables us to write P as a propagator for a Hamiltonian operator $\hat{H}$,

$$P(x, x_0; t) = \langle x | e^{-t\hat{H}} | x_0 \rangle, \quad \hat{H} = \frac{1}{2} \Omega \hat{p}^2 + i\hat{p}\, f(\hat{x}), \tag{5}$$

with initial condition $x(t = 0) = x_0$, and $\hat{p} = -i\partial_x$. It follows that by defining

$$\tilde{G}(x_0, x; t) \equiv \frac{\partial}{\partial x_0} P(x, x_0; t) \tag{6}$$

the function $\tilde{G}(x_0, x; t)$ satisfies the differential equation

$$\frac{\partial \tilde{G}(x_0, x; t)}{\partial t} = \frac{\partial}{\partial x_0} \left(\frac{1}{2} \Omega \frac{\partial \tilde{G}(x_0, x; t)}{\partial x_0} - f(x_0)\, \tilde{G}(x_0, x; t) \right). \tag{7}$$

An explicit example, relevant to the further development of the paper, is given by

$$S(x) = \frac{x^3}{3} - lx. \tag{8}$$

This polynomial function has a local minimum at $x = \sqrt{l}$, a local maximum at $x = -\sqrt{l}$ and is unbounded from below when $x \to -\infty$. It follows that in absence

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of the noise term – corresponding to the classical, unquantized system – the point $\sqrt{l}$ is an attractive fixed point for the classical equation (1) since for all $x_0 > -\sqrt{l}$, $x(t)$ will approach $\sqrt{l}$ as $t \to \infty$. For $x_0 < -\sqrt{l}$ we have a run-away solution and $x(t) \to -\infty$ in a finite time. Omitting the noise term corresponds to taking the limit $\Omega \to 0$. One can then drop the functional average over the noise in (3) to obtain

$$P_{cl}(x, x_0; t) = \delta(x - x(t, x_0)), \quad \tilde{G}_{cl}(x_0, x; t) = \frac{\partial}{\partial x_0} \delta(x - x(t, x_0)). \tag{9}$$

It is readily verified that these functions satisfy eqs. (4) and (7) with $\Omega = 0$, for instance,

$$\frac{\partial \tilde{G}_{cl}(x_0, x; t)}{\partial t} = \frac{\partial}{\partial x_0} \Big((l - x_0^2) \, \tilde{G}_{cl}(x_0, x; t) \Big). \tag{10}$$

3 Quantum dynamics of 2d causal triangulations

Quantum gravity defined through causal dynamical triangulations aims to construct and evaluate the nonperturbative, Lorentzian path integral over spacetime geometries $[g_{\mu\nu}]$, with or without matter coupling. In dimension two, and assuming we already have performed a rotation to Euclidean signature (this is well defined in CDT), this approach gives a definite meaning to the formal (Euclideanized) sum over histories

$$Z(G_{\mathrm{N}}, l) = \int \mathcal{D}[g_{\mu\nu}] \, \mathrm{e}^{-S[g_{\mu\nu}]}, \tag{11}$$

where the (Euclidean) Einstein-Hilbert action is given by

$$S[g_{\mu\nu}] = -\frac{1}{2\pi G_{\mathrm{N}}} \int d^2\xi \sqrt{\det g_{\mu\nu}} \, R + l \int d^2\xi \sqrt{\det g_{\mu\nu}}, \tag{12}$$

with Newton's constant G_{N} and the cosmological constant λ.

One thus proceeds in several steps: first the CDT lattice regularization is used to define the path integral, still with Lorentzian signature. Next, a rotation to Euclidean signature is performed at the level of the individual triangulations. We refer the reader to the original articles [18, 1] or the recent review [19] for details. The resulting real, Euclidean path integral of the form (11) will however differ from a standard one since we insist as part of the kinematical set-up that each path (spacetime history) possess a global time-foliation.[2] One then performs a continuum limit by shrinking the individual triangular building blocks to zero size ("removing the regulator"), while tuning the coupling constant(s) appropriately. This can be done analytically in the original, strictly causal CDT quantum gravity model. Key quantities one can compute in the limit and which contain information

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[2]This is a "remnant" of the corresponding structure in the original Lorentzian spacetimes, which ensures they are well-behaved causally.

about the underlying quantum geometry of this continuum theory are so-called "loop amplitudes". An important example is the amplitude denoted by $G_0(l_0, l; t)/l_0$ that (one-dimensional, compactified) space has length l_0 at proper time $t = 0$ and length l at a later proper time t. The quantity $G_0(l_0, l; t)$ without the normalization factor $1/l_0$ has the same interpretation as a transition amplitude, but with a distinguished marked point on the initial spatial loop l_0 (the marking removes the symmetry factor $1/l_0$). It is convenient to introduce the Laplace transform $\tilde{G}_0$ of G_0 by

$$\tilde{G}_0(x_0, x; t) = \int_0^\infty dl_0 \int_0^\infty dl \ e^{-x_0 l_0 - xl} G_0(l_0, l; t), \tag{13}$$

where the variables x_0 and x can be interpreted as boundary cosmological constants. In the original paper on two-dimensional CDT quantum gravity [18] it was shown that $\tilde{G}_0(x_0, x; t)$ satisfies the differential equation

$$\frac{\partial \tilde{G}_0(x_0, x; t)}{\partial t} = \frac{\partial}{\partial x_0} \left((l - x_0^2) \, \tilde{G}_0(x_0, x; t) \right). \tag{14}$$

Note that (up to a minus sign) $\tilde{G}_0(x_0, x; t)$ is obtained from the Laplace transform of $G_0(l_0, l; t)/l_0$ by differentiating with respect to x_0, in the same way as $\tilde{G}(x_0, x; t)$ in eq. (6) was obtained from $P(x, x_0; t)$.

Comparing now eqs. (14) and (10), we see that we can formally re-interpret $\tilde{G}_0(x_0, x; t)$ – an amplitude obtained by nonperturbatively quantizing Lorentzian pure gravity in two dimensions – as the "*classical* probability" $\tilde{G}_{cl}(x_0, x; t)$ corresponding to the action $S(x) = -lx + x^3/3$ of a zero-dimensional system in the context of stochastic quantization. Stochastic quantization of the system amounts to replacing

$$\tilde{G}_0(x_0, x; t) \rightarrow \tilde{G}(x_0, x; t), \tag{15}$$

where $\tilde{G}(x_0, x; t)$ satisfies the differential equation corresponding to eq. (7), namely,

$$\frac{\partial \tilde{G}(x_0, x; t)}{\partial t} = \frac{\partial}{\partial x_0} \left(g \frac{\partial}{\partial x_0} + l - x_0^2 \right) \tilde{G}(x_0, x; t). \tag{16}$$

For reasons which will become apparent below, we have introduced the parameter $g := \Omega/2$. Before turning to the physical interpretation of eq. (16), let us calculate the so-called Hartle-Hawking wave function, which in the CDT *string field theory*[3] is defined as

$$\tilde{W}(x_0) = \int_0^\infty dt \, \tilde{G}(x_0, l = 0; t). \tag{17}$$

The integrand $\tilde{G}(x_0, l; t)$ is obtained from $G(l_0, l; t)$ by making a Laplace transformation (as in (13)) only of l_0 and not of l. By construction, $\tilde{W}(x_0)$ accounts for all

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[3]i.e. isolated branchings and mergings are now allowed

spacetime histories starting with a single spatial loop of any length and ending in "nothing" (the loop of length zero) at an arbitrary later time. For physical reasons we demand that the solution to (16) should obey

$$G(l_0, l; t = 0) = \delta(l - l_0), \qquad G(l_0, l; t = \infty) = 0, \tag{18}$$

conditions which also hold for the pure-gravity amplitude $G_0(l_0, l; t)$ without topology change. Integrating relation (16) from time $t = 0$ to infinity we obtain

$$-1 = \frac{\partial}{\partial x_0} \left(g \frac{\partial}{\partial x_0} + l - x_0^2 \right) \tilde{W}(x_0). \tag{19}$$

This is precisely the differential equation for $\tilde{W}(x_0)$ obtained recently [20] from a matrix model representation of CDT string field theory if g was identified with the string coupling constant, associated with the merging or splitting of spatial universes as a function of time t. Just as in the original pure-gravity CDT model, the parameter t in the string field theory was identified with proper time. We now see that within the extended CDT framework, where topology change is allowed, this time acquires a new interpretation as *stochastic time* and the CDT string field theory that of a stochastic quantization.

Note that eqs. (16) and (19) are highly nonperturbative in the sense of describing a third-quantized system of geometry, incorporating topology changes of space. Eq. (19) for $\tilde{W}(x_0)$ was originally derived in a matrix model representation of the CDT string field theory.[4] What we have done here is to derive these expressions by applying "blindly" the rules of stochastic quantization, treating "x" as an ordinary variable, like the position of a particle, whereas in reality x is the boundary cosmological constant introduced by the Laplace transformation (13). A variable with a more direct physical interpretation is the conjugate length variable l of the boundary, measuring the size of the spatial universe. The Hamiltonian as a function of this physical length can be obtained by an inverse Laplace transform from the "classical" Hamiltonian $\hat{H}_0 = -\mathrm{d}/\mathrm{d}x(l - x^2)$ from (5) with $\Omega = 0$, leading to

$$\hat{H}_0(l) = -l \frac{\mathrm{d}^2}{\mathrm{d}l^2} + l\, l. \tag{20}$$

This is a standard Hermitian operator on wave functions $\psi(l)$ on the positive real axis, which are square-integrable with respect to the scalar product

$$\langle \psi_1 | \psi_2 \rangle = \int_0^\infty \frac{\mathrm{d}l}{l}\, \psi_1^*(l) \psi_2(l). \tag{21}$$

The scalar product is fixed uniquely by requiring appropriate composition properties of the propagator $G_0(l_0, l; t)$ [1]. The eigenfunctions $\psi_n(l)$ of $\hat{H}_0(l)$ are the states

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[4]for a rescaled version in terms of dimensionless parameters; c.f. eq. (30) of ref. [20]

of the spatial universe which are propagated unchanged by $\hat{G}_0 = e^{-t\hat{H}_0}$ with kernel $\hat{G}_0(l_1, l_2; t)$. When we Laplace-transform this to x-space the scalar product to be used is the one inherited from l-space. For instance, the Laplace transform of $\delta(l_1 - l_2)$ is $1/(x + y)$, which acts like the appropriate δ-function in x-space. In other words, the physically motivated boundary conditions are different from the ones one would choose if x were a standard configuration space variable. Likewise, an acceptable eigenfunction of $\hat{H}_0(x)$ is not a standard square-integrable function on the real x-axis. Consequently, instead of (9) we should use

$$\tilde{G}_0(x_0, x; t) = \frac{d}{dx_0}\left(\frac{1}{x(t, x_0) + x}\right) = -\frac{x^2(t, x_0) - l}{x_0^2 - l} \frac{1}{(x(t, x_0) + x)^2}. \quad (22)$$

Despite these differences compared to the situation in "ordinary" x-space the formal derivation of stochastic quantization is unchanged. A neat geometric interpretation of how stochastic quantization can capture topologically nontrivial amplitudes has been given in [6]. Applied to the present case, we can view the propagation in stochastic time t for a given noise term $\nu(t)$ as classical in the sense that solving the Langevin equation (1) for $x^{(\nu)}(t)$ iteratively gives precisely the tree diagrams with one external leg corresponding to the action $S(x)$ (and including the derivative term $\dot{x}^{(\nu)}(t)$), with the noise term acting as a source term. Performing the functional integration over the Gaussian noise term corresponds to integrating out the sources and creating loops, or, if we have several independent trees, to merging these trees and creating diagrams with several external legs. If the dynamics of the quantum states of the spatial universe takes place via the strictly causal CDT-propagator $\hat{G}_0 = e^{-t\hat{H}_0}$, a single spatial universe of length l cannot split into two spatial universes. Similarly, no two spatial universes are allowed to merge as a function of stochastic time. However, introducing the noise term *and* subsequently performing a functional integration over it makes these processes possible. This explains how the stochastic quantization can automatically generate the amplitudes which are introduced by hand in a string field theory, be it of Euclidean character as described in [6], or within the framework of CDT.

What is new in the CDT string field theory considered here is that we can use the corresponding stochastic field theory to solve the model, since we arrive at closed equations valid to all orders in the genus expansion. Equations (16) and (19) are such examples. Translating them to l-space and using the boundary conditions $W(l = 0) = 1$ and $G(l_0 = 0, x; t) = 0$ (because the loop of length l_0 is marked), we obtain from (19)

$$\hat{H}(l)W(l) = 0, \quad (23)$$

which is a Wheeler-deWitt type equation for the spatial universe. In addition, we have

$$\frac{\partial G(l_0, l; t)}{\partial t} = -\hat{H}(l_0)\, G(l_0, l; t), \quad (24)$$

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where the *extended* Hamiltonian

$$\hat{H}(l) = -l\frac{\partial^2}{\partial l^2} + ll - gl^2 \tag{25}$$

now has an extra potential term coming from the inclusion of branching points.[5] Eq. (23) is readily solved in terms of the Airy function Bi, namely,

$$W(l) = \frac{\mathrm{Bi}\left(\frac{l}{g^{2/3}} - g^{1/3}l\right)}{\mathrm{Bi}\left(\frac{l}{g^{2/3}}\right)}, \tag{26}$$

while

$$G(l_0, l; t) = \langle l|e^{-t\hat{H}(l)}|l_0\rangle \tag{27}$$

describes the nonperturbative propagation of a spatial loop of length l_0 to a spatial loop of length l in proper (or stochastic) time t, now including the summation over all genera. The Hamiltonian $\hat{H}(l)$ is a well-defined Hermitian operator with respect to the measure (21).

4   The extended Hamiltonian

Let us recap the results of the original CDT model, where space was not allowed to split into disconnected parts [18, 1, 19]. We have a Hamiltonian $\hat{H}_0(l)$ and a corresponding eigenvalue equation

$$\hat{H}_0(l)\psi_n(l) = E_n\psi_n^{(0)}(l), \quad \hat{H}_0(l) = -l\frac{\partial^2}{\partial l^2} + ll. \tag{28}$$

The eigenfunctions and eigenvalues are given by

$$\psi_n^{(0)}(l) = p_n(l)\,\mathrm{e}^{-\sqrt{l}l}, \quad E_n = 2n\sqrt{l}, \quad n = 1, 2, \ldots, \tag{29}$$

where the $p_n(l)$ are polynomials in $\sqrt{l}l$ and $p_n(0) = 0$. Furthermore, we have

$$\hat{H}_0(l)W_0(l) = 0, \quad W_0(l) = \mathrm{e}^{-\sqrt{l}l}, \tag{30}$$

where $W_0(l)$ is the Hartle-Hawking wave function of the original CDT model and relations (30) are the counterparts of (23) and (26) when $g = 0$. Formally, the amplitude $W_0(l)$ is a solution to eq. (28) with eigenvalue $E = 0$. However, $E = 0$

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[5]Since in the derivation of $\hat{H}(l)$ we considered only loop-loop amplitudes (as opposed to arbitrary multi-loop amplitudes), this Hamiltonian seems to capture only a sector of the full dynamics of the string field theory. To what extent $\hat{H}(l)$ already incorporates the complete dynamics in some "effective" way – as suggested by the fact that it does contain an infinite genus summation – is an issue that remains to be understood better.

does not belong to the spectrum of $\hat{H}_0$ since $W_0(l)$ is not integrable at zero with respect to the measure (21). Exactly the same is true for the extended Hamiltonian $\hat{H}(l)$ and the corresponding Hartle-Hawking amplitude $W(l)$. – In order to analyze the spectrum of $\hat{H}(l)$, it is convenient to put the differential operator into standard form. After a change of variables

$$l = \frac{1}{2}z^2, \qquad \psi(l) = \sqrt{z}\phi(z), \tag{31}$$

the eigenvalue equation becomes

$$\hat{H}(z)\phi(z) = E\phi(z), \quad \hat{H}(z) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{1}{2}lz^2 + \frac{3}{8z^2} - \frac{g}{4}z^4. \tag{32}$$

This shows that the potential is unbounded from below, but such that the eigenvalue spectrum is still discrete[6]. For small g, there is a large barrier of height $l^2/(2g)$ separating the unbounded region for $l > l/g$ from the region $0 \leq l \leq l/(2g)$ where the potential grows. This situation is perfectly suited to applying a standard WKB analysis. For energies less than $l^2/(2g)$, the eigenfunctions of $\hat{H}_0(l)$ will be good approximations to those of $\hat{H}(l)$. However, when $l > l/g$ the exponential fall-off of $\psi_n^{(0)}(l)$ will be replaced by an oscillatory behaviour, with the wave function falling off only like $1/l^{1/4}$. The corresponding $\psi_n(l)$ is still square-integrable since we have to use the measure (21). For energies larger than $l^2/(2g)$, the solutions will be entirely oscillatory, but still square-integrable.

What follows from our analysis is that a dramatic change has occurred in the quantum behaviour of the one-dimensional universe as a consequence of allowing topology changes. In the original, strictly causal quantum gravity model an eigenstate $\psi_n^{(0)}(l)$ of the spatial universe had an average size of order $1/\sqrt{l}$, increasing as a function of energy. Allowing for topology changes (and assuming g suitably small and n not too large), only the large-l tail of $\psi_n^{(0)}(l)$ will change. As a result, the probability $|\psi_n(l)|^2/l$ for finding a universe with size in the interval $[l, l + dl]$ is almost unchanged as long as $l < l/g$. However, the average size of the universe is now infinite! We see now that the oscillatory behaviour of the amplitude $W(l)$ for $l > l/g$ already observed in [20] can be understood as a consequence of l lying in the region where the potential in $\hat{H}(l)$ is unbounded below.

We still need to choose a selfadjoint extension of $\hat{H}(l)$ such that the spectrum of $\hat{H}(l)$ can be determined unambiguously. One way of doing this is to appeal again to stochastic quantization, following the strategy used by Greensite and Halpern [22],

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[6]Whenever the potential is unbounded below with fall-off faster than $-z^2$, the spectrum is discrete, reflecting the fact that the classical escape time to infinity is finite. In this way, the unbounded potential behaves effectively like a finite box. In addition, like in the case of a box, there exists a one-parameter family of selfadjoint Hamiltonians, depending on the specific choice of boundary condition one imposes at infinity. See reference [21] for a more detailed discussion relevant to the present situation.

which was applied to the double-scaling limit of matrix models in [23, 24, 21]. The Hamiltonian (5) corresponding to the Fokker-Planck equation (16), namely,

$$\hat{H}(x)\psi(x) = -g\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}S(x)}{\mathrm{d}x}\psi(x)\right), \quad S(x) = \left(\frac{x^3}{3} - l\,x\right), \qquad (33)$$

is not Hermitian if we view x as an ordinary real variable and wave functions $\psi(x)$ as endowed with the standard scalar product on the real line. However, by a similarity transformation one can transform $\hat{H}(x)$ to a new operator

$$\tilde{H}(x) = \mathrm{e}^{-S(x)/2g}\,\hat{H}(x)\,\mathrm{e}^{S(x)/2g}; \quad \tilde{\psi}(x) = \mathrm{e}^{-S(x)/2g}\psi(x), \qquad (34)$$

which *is* Hermitian on $L^2(R, dx)$. We have

$$\tilde{H}(x) = -g\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{1}{4g}\left(\frac{\mathrm{d}S(x)}{\mathrm{d}x}\right)^2 + \frac{1}{2}\frac{\mathrm{d}^2 S(x)}{\mathrm{d}x^2}\right), \qquad (35)$$

which after substitution of the explicit form of the action becomes

$$\tilde{H}(x) = -g\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \quad V(x) = \frac{1}{4g}(l - x^2)^2 + x. \qquad (36)$$

The fact that one can write

$$\tilde{H}(x) = \hat{R}^\dagger \hat{R}, \quad \hat{R} = -\sqrt{g}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2\sqrt{g}}\frac{\mathrm{d}S(x)}{\mathrm{d}x} \qquad (37)$$

implies that the spectrum of $\tilde{H}(x)$ is positive, discrete and unambiguous. We conclude that the formalism of stochastic quantization has provided us with a nonperturbative definition of the CDT string field theory.

5   Summary and discussion

In this paper we have shown that there is an alternative derivation, using stochastic quantization, of the CDT string field theory introduced earlier in [3, 20]. The stochastic quantization is not performed for the initial path integral over all space-time geometries, but at the level of the effective continuum dynamics of the spatial geometry of the universe, which for the case of gravity in 1+1 dimensions is described by a single variable, the universe's size or length. Interestingly, in order for the equivalence to hold, we had to identify the stochastic time of the construction with the proper time of the original CDT model. As a bonus, the stochastic quantization naturally led us to *a* nonperturbative definition of the CDT string field theory. This is nontrivial, because the theory contains a sum over all spacetime topologies. Our construction mirrored that of Kawai et al. [6], who were the first to

observe (in a Euclidean context) that the noncritical string field theory developed by them could be viewed as a stochastic quantization of space, with stochastic time playing the role of proper time in the corresponding two-dimensional quantum gravity theory. Physically, the two string field theories are of course different. In the CDT case we were able to push the formalism further to obtain an explicit quantum Hamiltonian and analyze its spectral properties.

At first sight, it may seem curious that stochastic time – usually thought of as a fictitious, external parameter – makes an appearance as the "time" of a quantum-gravitational theory. However, it may be argued that the external character of this particular *distance parameter* is something found more generally in the construction of diffeomorphism-invariant correlation functions in nonperturbative quantum gravity. As a simple example, consider the case of a scalar field ϕ coupled to (Euclidean) quantum gravity in two dimensions. A diffeomorphism-invariant definition of a two-point correlator can be obtained by integrating over all pairs of insertion points of the matter fields which are a geodesic distance R apart, that is,

$$\langle \phi(R)\phi(0) \rangle = \int \mathcal{D}[g_{\mu\nu}(\xi)] \int \mathcal{D}\phi(\xi) \, \mathrm{e}^{-S(g_{\mu\nu},\phi)} \tag{38}$$

$$\times \int \mathrm{d}^2\xi_1 \sqrt{\det g(\xi_1)} \int \mathrm{d}^2\xi_2 \sqrt{\det g(\xi_2)} \, \phi(\xi_1)\phi(\xi_2) \, \delta(D_g(\xi_1,\xi_2) - R). \tag{39}$$

The function $D_g(\xi_1,\xi_2)$ appearing in the argument of the δ-function denotes the geodesic distance between the points labelled ξ_1 and ξ_2. As indicated by the notation, this distance depends on the other dynamical field variable, the metric $g_{\mu\nu}(\xi)$.

In this construction, the geodesic distance R is fixed outside the functional integral, and therefore may be regarded as external. It does not refer to any particular metric, but is the geodesic distance in *all* geometries entering in the functional integral simultaneously. From this point of view it is of course intimately related to the dynamical quantum properties of the ensemble. In particular, R can have genuine quantum properties, for example, it can scale noncanonically. The proper time appearing in the description of the "world sheets" of the string field theories has a similar status. It is a notion of time which is defined invariantly (in this case as the "geodesic distance to a one-dimensional boundary"), and superimposed on an ensemble of geometries.

It is precisely this notion of proper time which in both Euclidean and Lorentzian two-dimensional quantum gravity with topology change (a.k.a. string field theory in zero-dimensional target space) apparently is equivalent to stochastic time. Although in our present derivation the third-quantized nature of the construction appeared in an essential way, the argument about the "external" nature of this time in correlation functions we made above appealed neither to the inclusion of nontrivial topology nor the dimensionality of spacetime. This suggests that stochastic time may play a role in these more general situations too, a line of enquiry that is currently under investigation.

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