# A NOTE ON “EINSTEIN’S SPECIAL RELATIVITY BEYOND THE SPEED OF LIGHT” BY JAMES M. HILL AND BARRY J. COX

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Abstract. We show that the transformations J. M. Hill and B. J. Cox introduce between inertial observers moving faster than light with respect to each other is consistent with Einstein’s principle of relativity only if the spacetime is 2 dimensional. However, the mass-change formula predicted by Hill and Cox can be derived without using those transformations.

## 1. Introduction

J. M. Hill and B. J. Cox introduce two transformations to extend Lorentz transformations for inertial observers moving faster than light (FTL) with respect to each other, see equations (3.16) and (3.18) in Hill & Cox (2012). In this paper, for simplicity, we take the speed of light to be 1. With this simplification, the Hill–Cox transformations ($HC_1$ and $HC_2$) become

$$HC_1: \ \ x = \frac{-X+vT}{\sqrt{v^2-1}}, \ \ t = \frac{-T+vX}{\sqrt{v^2-1}}, \ \ y=Y, \ \ z=Z \tag{1}$$

and

$$HC_2: \ \ x = \frac{X-vT}{\sqrt{v^2-1}}, \ \ t = \frac{T-vX}{\sqrt{v^2-1}}, \ \ y=Y, \ \ z=Z \tag{2}$$

where $v$ is the superluminal relative speed of the two observers. In Hill & Cox (2012), these transformations are used to derive some properties of FTL particles. For example, it is shown that the relativistic mass ($m$) depends on the speed ($v$) of a superluminal particle and an observer independent quantity $p_\infty$ as follows:

$$m = \frac{p_\infty}{\sqrt{v^2-1}}. \tag{3}$$

In the present paper, we are going to show that Hill–Cox transformations give a consistent extension of Einstein’s special theory of relativity *only* if the dimension $d$ of spacetime is 2 (i.e., there are 1 space and 1 time dimensions).

This fact does not disprove the derived dynamical formulas of Hill & Cox (2012), it only shows that for $d>2$ the Hill–Cox transformations are not so much convincing for deriving them. However, we claim that (3) can be derived (with another chain of thought) from fewer assumptions consisting of Einstein’s principle of relativity, conservation of relativisitic mass and momentum, as well as some basic

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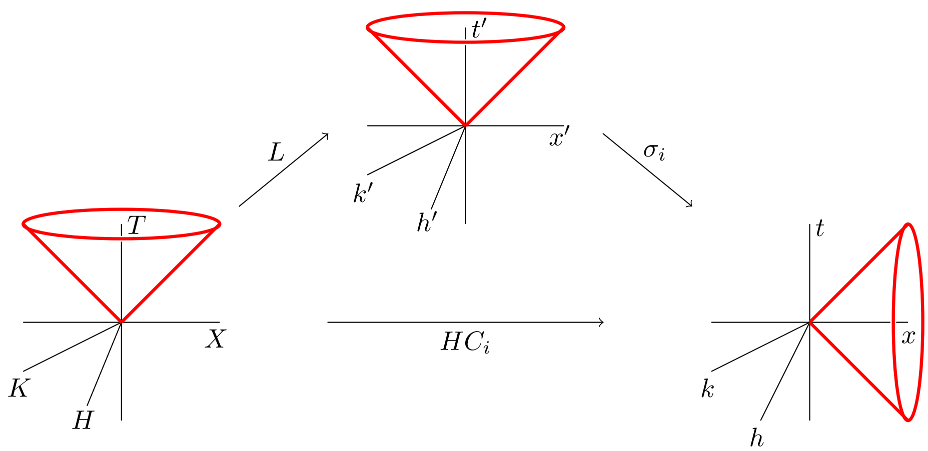


FIGURE 1. Decomposition of $HC_i$ to Lorentz-transformation $L$ and $\sigma_i$ for $i \in \{1, 2\}$.

(usually treated as tacit) axioms, such as observers coordinatize the same events, see Madarász & Székely (2012).

## 2. Consistency of Hill–Cox transformations with the principle of relativity implies $d = 2$

To directly show that Hill–Cox transformations contradict Einstein's principle of relativity (see Einstein 1905) if $d > 2$, first we show that they can be written as the composition of a Lorentz transformation and a transformation exchanging the time axis and a space axis. A straightforward calculation shows that transformation $HC_1$ is the composition of the Lorentz transformation corresponding to speed $1/v$

$$L: \quad x' = \frac{X - T/v}{\sqrt{1 - 1/v^2}}, \quad t' = \frac{T - X/v}{\sqrt{1 - 1/v^2}}, \quad y' = Y, \quad z' = Z. \tag{4}$$

and the following transformation

$$\sigma_1: \quad t = x', \quad x = t', \quad y = y', \quad z = z' \tag{5}$$

i.e., $HC_1 = \sigma_1 \circ L$, see Figure 1. Let us note that $1/v$ is subluminal if $v$ is superluminal.

Transformation $HC_2$ can be decomposed similarly by using in place of $\sigma_1$ the following transformation exchanging the time axis and a space axis:

$$\sigma_2 : t = -x', \quad x = -t', \quad y = y', \quad z = z'. \tag{6}$$

We will show that in the $HC_i$-transformed worldview, there is a unique direction (namely, $x$) in which the speed of light is smallest (namely, 1): in all other directions, either one cannot send out a light signal (there are plenty such directions), or if one can send out a light signal (there are plenty such directions, too), the speed of the light signal is greater (than 1). From this, the FTL-observer belonging to the transformed reference frame can know/"observe" that he is traveling in direction $x$ and is moving FTL as seen by any slower than light observer. (We mean here the observer independent direction belonging to the $x$-axis of the FTL-observer).

This is quite a strong violation of Einstein's principle of relativity (e.g., because the space of each slower than light observer is isotropic while this is not so for the FTL-observers).

The idea of our proof is depicted in Figure 1. From the figure, it can be seen that the $HC_1$-image of the light cone in any 3-dimensional subspace containing the $x$-axis is a flipped-over light cone, and the $x$-axis is in the middle of this "lying" light cone. In more detail:

Assume $d > 2$. In this proof, we identify directions with straight lines in the subspace orthogonal to the time-axis, and going through the origin. So, let $k$ be any direction in the transformed worldview, let $h$ be the direction orthogonal to the $tx$-plane in the 3-dimensional subspace generated by $k$ together with the $tx$-plane. Note that $HC_1$ is a linear bijection, it takes the $TX$-plane to the $tx$-plane (itself), and it is the identity on the subspace orthogonal to the $TX$-plane. Now, $HC_1 = \sigma_1 \circ L$. Thus, $L$ takes the light cone (emanating from the origin) to itself, and $\sigma_1$ flips over this light cone so that that $x$, the $HC_1$-image of the time axis $T$, is in the middle of this flipped-over light cone, see Figure 1. Then the speed of light in direction $k$ (i.e., the slant of the intersection of the flipped-over light cone with the $tk$-plane) is 1 if $k$ is $x$, it gets bigger on both sides as $k$ approaches $h$, it becomes infinite on both sides half-way towards $h$, and from then the flipped-over light cone does not intersect the $tk$-plane any more. (After $k$ passes the $th$-plane, the same happens in the opposite order, by symmetry.) This proves our claim for $HC_1$. The proof is the same for $HC_2$, only we have to use $\sigma_2$ in place of $\sigma_1$.

The fact that Hill–Cox transformations do not work if $d > 2$ is not surprising. It can be shown in a strictly axiomatic framework, with using only a few assumptions of special relativity theory that inertial observers cannot move faster than the speed of light if $d > 2$, see, e.g., Theorem 2.1 in Andréka *et al.* (2012). By observers we mean reference frames as, e.g., the standard relativity book d'Inverno (1992) does. So the difference between particles and observers is that particles do not need to have worldviews (frames of reference), hence dealing with particles does not require dealing with worldview transformations.

For $d = 2$, transformations $HC_1$ and $HC_2$ are perfectly consistent with Einstein's special relativity. In this case, exchanging time and space is the usual way for constructing models satisfying the axioms of special relativity in which there are FTL observers. This construction is investigated in section 2.4 in Andréka *et al.* (2002).

## 3. Do we need FTL observers in a theory of FTL particles?

The existence of particles moving with the speed of light (photons) does not imply the existence of observers moving with the speed of light. The same way, the existence of FTL particles does not imply (logically) the existence of FTL observers. This fact suggests that in order to elaborate a theory of superluminal particles, we do not necessarily have to introduce superluminal observers.

Indeed, even though observers cannot move FTL if $d > 2$, the superluminal motion of particles is consistent with the kinematics of special relativity, see Székely (2012). The dynamical results of Hill and Cox can also be proved to hold in higher dimensions, without using FTL observers. For example, their formula (3) can be derived because a natural, consistent axiom system of special relativistic particle

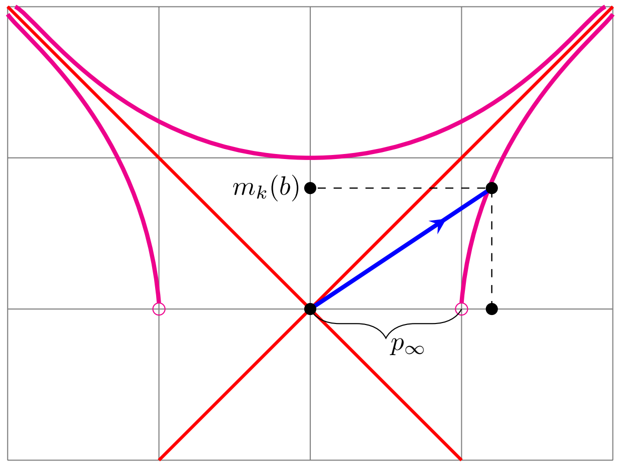


FIGURE 2. Illustration for equations (7) and (8)

dynamics containing Einstein's principle of relativity implies that

$$m_k(b)\sqrt{|1-v_k^2(b)|} = m_h(b)\sqrt{|1-v_h^2(b)|}, \tag{7}$$

where $m_k(b)$ and $m_h(b)$ are the relativistic masses and $v_k(b)$ and $v_h(b)$ are the speeds of a (possibly FTL) particle $b$ with respect to (ordinary slower than light) inertial observers $k$ and $h$. This is done in Madarász & Székely (2012), relying on Andréka *et al* (2008). We get formula (3) by introducing observer independent quantity for FTL particle $b$ as

$$p_\infty(b) := m_k(b)\sqrt{v_k^2(b)-1}. \tag{8}$$

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