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Dimension and Dimensional Reduction in Quantum Gravity

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Abstract

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A number of very different approaches to quantum gravity contain a common thread, a hint that spacetime at very short distances becomes effectively two dimensional. I review this evidence, starting with a discussion of the physical meaning of "dimension" and concluding with some speculative ideas of what dimensional reduction might mean for physics.

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1 Why Dimensional Reduction?

What is the dimension of spacetime? For most of physics, the answer is straightforward and uncontroversial: we know from everyday experience that we live in a universe with three dimensions of space and one of time. For a condensed matter physicist, say, or an astronomer, this is simply a given. There are a few exceptions—surface states in condensed matter that act two-dimensional, string theory in ten dimensions—but for the most part dimension is simply a fixed, and known, external parameter.

Over the past few years, though, hints have emerged from quantum gravity suggesting that the dimension of spacetime is dynamical and scale-dependent, and shrinks to $d \sim 2$ at very small distances or high energies. The purpose of this review is to summarize this evidence and to discuss some possible implications for physics.

1.1 Dimensional reduction and quantum gravity

As early as 1916, Einstein pointed out that it would probably be necessary to combine the newly formulated general theory of relativity with the emerging ideas of quantum mechanics [1]. In the century since, efforts to quantize gravity have led to many breakthroughs in fundamental physics, from gauge-fixing and ghosts to the background field method to Dirac's analysis of constrained Hamiltonian systems. But the fundamental goal of a complete, consistent quantum theory of gravity still seems distant. In its place, we have a number of interesting but incomplete research programs: most famously string theory and loop quantum gravity, but also group field theory, causal set theory, asymptotic safety, lattice approaches such as causal dynamical triangulations, research based on noncommutative geometry, and various ideas for "emergent" gravity.

In a situation like this, we need to explore many complementary lines of research. One particular strategy is to look for fundamental features that are shared by different quantization programs. There is no guarantee that such features will persist in the "correct" quantum theory of gravity, but such a pattern of recurrence at least makes it more plausible.

We currently have one outstanding example of such a commonality, the predictions of black hole thermodynamics. We have not directly observed Hawking radiation or black hole entropy, but the thermodynamics properties of black holes can be derived in so many different ways, with such a variety of assumptions and approximations [2], that a claim that black holes do not radiate would now seem perverse. Dimensional reduction of spacetime near the Planck scale is a candidate for second such commonality, albeit one that is much less firmly established.

1.2 Dimension as an observable

In most of physics the dimension of space, or spacetime, is taken as a fixed external parameter. While the notion of dimension is ancient—see [3–5] for discussion of the history and philosophy—the mathematical formalism for a space of arbitrary dimension is fairly recent, often attributed to Schäfli's work in the early 1850s [6]. The question of *why* our universe should have the number of dimensions it does was famously discussed in 1917 by Ehrenfest [7], who pointed out that such features as the stability of Newtonian orbits and the duality between electric and magnetic fields are unique to three spatial dimensions. But Ehrenfest also warned that "the questions [of what determines the number of dimensions] have perhaps no sense."

The idea that spacetime might really have more than four dimensions was introduced into physics by Nordström [8], and became more widely known with the work of Kaluza [9] and Klein [10]. The extension beyond five dimensions first appeared, I believe, as an exercise in lecture notes by DeWitt [11]. Kaluza-Klein theory provided a useful illustration of the notion that dimension might be scale-dependent. At small enough distances, the spacetime of Kaluza-Klein theory is an n-dimensional manifold with $n > 4$. At larger scales, though, the compact dimensions can no longer be resolved, and the spacetime becomes effectively four-dimensional, with excitations in the compact directions appearing as towers of massive four-dimensional modes. The converse process of dimensional deconstruction [12], in which appropriate four-dimensional modes effectively "create" extra dimensions at large distances, has more recently become popular in high energy theory.

Here we are interested in the opposite phenomenon, in which the number of effective dimensions *decreases* at short distances or high energies. The first models I know of that exhibited this behavior were introduced by Jourjine [13] , Kaplunovsky and Weinstein [14], Zeilinger and Svozil [15], and Crane and Smolin [16], all in 1985. A year later, Hu and O'Connor observed scale dependence of the effective dimension in anisotropic cosmologies [17], but the wider significance was not fully appreciated. In quantum gravity, the phenomenon of dimensional reduction first appeared in string theory, where a dimension characterizing thermodynamic behavior was found to unexpectedly drop to $d = 2$ at high temperatures, leading Atick and Witten to postulate "a lattice theory with a (1+1)-dimensional field theory on each lattice site" [18]. But it was only with the computation of the flow of spectral dimension in causal dynamical triangulations [19] that the idea really took hold.

To proceed further, though, we will have to address a basic question: how, precisely, do we define "dimension" as a physical observable?

2 Dimensional estimators

We experience space to be three-dimensional. We learn as children that an object has a height, a width, and a depth (this goes back to Aristotle [3]), and, later, that no more than three mutually perpendicular lines can be drawn from a point (this goes back at least to Galileo [20], and perhaps to a lost work by Ptolemy [3]). Later, we learn that a spatial position can be specified by three coordinates, and an event by four; that is, spacetime is homeomorphic to $\mathbb{R}^4$. This picture carries over to classical general relativity, where spacetime is modeled by a four-dimensional manifold, that is, a structure locally diffeomorphic to $\mathbb{R}^4$.

But we also know this is not enough. In mathematics, Cantor's discovery of a one-to-one correspondence between the points of a line segment and the points of a unit square showed the flaw in the intuitive idea that a two-dimensional space has "more points" than a one-dimensional space, and Peano's construction of space-filling curves made it clear that dimension had to be more than a simple counting of the number of parameters needed to specify a point [24]. Moreover, the physical quantity we call dimension clearly depends on scale. We draw a line on a piece of paper and call it one dimensional, but closer up it is three dimensional, and even closer the ink consists of elementary particles that appear point-like (or perhaps string-like). Moving to to quantum mechanics, the question becomes even more problematic. Even for a single particle, the path integral is dominated by nowhere-smooth paths with fractal dimensions [21], with smooth one-dimensional paths appearing only semiclassically. Quantum gravity adds yet another layer of complication: in a spatially compact universe there are no local observables, and it is no longer obvious that a "point" has any real meaning [22, 23]. Ultimately, if our smooth spacetime is somehow emergent at large distances, as many approaches to quantum gravity suggest, then its dimension, too, should be emergent.

How, then, do we decide what "dimension" means? We need "dimensional estimators," physical observables with a simple dependence on dimension that can be generalized to situations in which the meaning of dimension is ambiguous. A number of rather different possibilities exist, and different choices need not always agree. Dimension may depend on exactly what physical question we are asking.

Broadly speaking, dimensional estimators come in two varieties. Some are primarily mathematical, determining the dimension of some class of mathematical spaces that we use to model physics. Others are more immediately physical, starting with a physical quantity that has a simple dimensional dependence and using its value to deduce dimension. The distinction is not always sharp—spectral dimension, for instance, can be viewed as a characteristic of a mathematical random walk or a physical diffusion process—but the distinction is helpful for keeping track of alternatives.

2.1 Geometric dimensions

The mathematical field of "dimension theory" is far too large to discuss in detail in a review article like this one. Readers may want to look at the books [24] or [25] for more formal definitions and derivations, or at [26] for an entertaining mathematical perspective. Here I will restrict myself to brief summaries of the more important mathematical approaches to dimension, some of which have made an appearance in physics and some of which have not.

- **Topology: inductive dimension and covering dimension**

  Three of the common mathematical definitions of dimension are purely topological, depending only on the structure of points and open sets. The inductive dimensions originate from the observation that a line can be divided by a point, a surface by a curve, a three-dimensional space by a surface, etc. More formally, the large inductive dimension of the empty set is -1; the large inductive dimension of a space X is the smallest integer d_I such that any disjoint pair of closed sets $F, H \subset X$ can be separated* by another closed set K of large inductive dimension less than or equal to $d_I - 1$ [25]. The small inductive dimension d_i is defined similarly, but with the closed set H replaced by a single point x.

  The covering dimension originates from Lebesgue's observation that a line can be covered by arbitrarily small intervals in such a way that no point is contained in more than two intervals, a square can be covered by arbitrarily small "bricks" in such a way that no point is contained in more than three bricks, etc. More formally, let $\mathcal{H}$ be an open covering of X, and define the order of $\mathcal{H}$ to be the smallest number n such that each point in X belongs to at most n

*The sets F and H are separated by K if F, H, and K are mutually disjoint and $X\backslash K$ consists of two disjoint sets, one containing F and the other containing H.

sets in $\mathcal{H}$. The covering dimension or Lebesgue dimension d_L is then the minimum value of n such that every open cover of X has a refinement with order less than or equal to $n+1$ [25].

By construction, the large inductive, small inductive, and covering dimensions are integers. For nice enough spaces—for instance, separable metric spaces—they are all equal [24], but there are exotic cases in which they differ. But while these notions are widely used in mathematics, they have seen very little application to physics, except indirectly as the basis for the statement that $\mathbb{R}^n$ is n-dimensional. The problem, I believe, is that they are too primitive and too far removed from observation: how, for instance, do we determine physically whether a set of spacetime events is open?

- **Scaling: box-counting and Hausdorff dimensions**

Once a notion of distance is added, mathematical approaches to dimension become more "physical." In particular, dimension can be determined by the scaling behavior of geometric objects. Suppose, for instance, that a ball of radius r has volume $V(r)$. If at some characteristic length scale $V \sim r^n$, then n is a measure of dimension at that scale. This concept can be extended to graphs, giving upper and lower "internal scaling dimensions" d_{IS} and d_{is} [27].

Alternatively, suppose a ball of radius r can be filled by $N(\epsilon)$ balls of radius ϵ. If $N(\epsilon)$ scales as $(r/\epsilon)^n$, then n is again a measure of dimension. This idea is made precise by the notions of upper and lower box-counting dimension (or "capacity") and Hausdorff dimension. Let $N(X,\epsilon)$ be the smallest number of balls of radius ϵ that can cover the set X. Then if the limit exists, the box-counting dimension of X is [28]

$$d_b(X) = \lim_{\epsilon\to 0} \frac{\ln N(X,\epsilon)}{\ln(1/\epsilon)} \tag{2.1}$$

If the limit does not exist, one can still define upper and lower limits (the supremum and infimum of the limit points), which determine the upper and lower box-counting dimension d_C and d_c.

The Hausdorff dimension is a further refinement, in which the balls are allowed to have different sizes. For a metric space with a distance function $d(x,y)$, the "diameter" $diam(A)$ of a set A is the largest distance between any two elements of A (technically, the supremum $\sup\{d(x,y) : x,y \in A\}$). Consider coverings of a set X by subsets A_i with $diam(A_i) < \epsilon$. For p an arbitrary nonnegative number, define

$$m(X,p) = \lim_{\epsilon\downarrow 0} \inf \sum_i [diam(A_i)]^p \tag{2.2}$$

where the infimum in (2.2) is over all decompositions $X = A_1 \cup A_2 \cup \dots$. As a function of p, $m(X,p)$ is zero for p large, and jumps to infinity for p small. The Hausdorff dimension d_H is the value of p at this threshold; that is

$$d_H(X) = \sup\{p : m(X,p) > 0\} = \inf\{p : m(X,p) < \infty\} \tag{2.3}$$

The Hausdorff dimension is sometimes called the "fractal dimension," as introduced by Mandelbrot [29].

In contrast to the inductive and covering dimensions, box-counting and Hausdorff dimensions need not be integers. Again, though, for "nice" spaces—the differentiable manifolds of general relativity, for example—the scaling dimensions give the expected answer.

For a space with a measure, generalizations of Hausdorff dimension are available [28,30]. The "information dimension," for example, is a variation of the box-counting dimension adjusted to account for the relative probabilities of balls used to cover the set: in analogy to (2.1),

$$d_I(X) = \lim_{\epsilon\to 0}\frac{I(X,\epsilon)}{\ln(1/\epsilon)} \quad \text{with} \quad I(X,\epsilon) = -\sum_{i=1}^{N(\epsilon)} P_i \ln P_i \tag{2.4}$$

where P_i is the probability measure for the ith ball. There are, in fact, infinitely many generalizations, associated with correlations of n points [31]. Yet other dimensions can be defined in terms of Lyapunov numbers of a flow [30]. As far as I know, none of these has been used in investigations of quantum gravity, although the Lyapunov dimension could be relevant to the discussion of geodesics that will appear later.

- **Random walks: spectral and walk dimensions**

If the topological dimensions require too little physical information, the scaling dimensions require perhaps too much. It is not always so easy to define a distance function in a model of quantum gravity, so the physical meaning of a geodesic ball or the diameter of a set can be unclear. An intermediate set of dimensions are based instead on the properties of random walks or diffusion processes, which can be defined in more general settings.

The basic idea is straightforward. In any space on which a random walk can be defined, the resulting diffusion process gradually explores larger and larger regions. More dimensions mean both slower diffusion—there are more "nearby" points—and a slower return to the starting point. Quantitatively, diffusion from an initial position x to a final position x' on a manifold M is described by a heat kernel $K(x,x',s)$ satisfying

$$\left(\frac{\partial}{\partial s} - \Delta_x\right) K(x,x';s) = 0, \qquad \text{with} \qquad K(x,x';0) = \delta(x-x'), \tag{2.5}$$

where Δ_x is the Laplacian on M at x, and s is a measure of the diffusion time. Let $\sigma(x,x')$ be Synge's world function [32], one-half the square of the geodesic distance between x and x'. Then on a manifold of dimension d_S, the heat kernel generically behaves for small s as [11, 33, 34]

$$\begin{aligned} K(x,x';s) &\sim (4\pi s)^{-d_S/2} e^{-\sigma(x,x')/2s}\left(1 + [a_1]s + [a_2]s^2 + \dots\right), \\ K(x,x;s) &\sim (4\pi s)^{-d_S/2}\left(1 + [a_1]s + [a_2]s^2 + \dots\right) \end{aligned} \tag{2.6}$$

where the "HaMiDeW" coefficients $[a_i]$ are known functions of the curvature.

Now consider any space in which a diffusion process or a random walk can be defined. The scale-dependent generalized spectral dimension $d_S(s)$ of a region X is obtained from the "return time,"

$$d_S(s) = -2\frac{d\ln\langle K(x,x;s)\rangle_X}{d\ln s} \tag{2.7}$$

where the angle brackets $\langle\ \rangle_X$ denote the average over points $x \in X$. For a smooth manifold, $\lim_{s\to 0} d_S(s)$ gives the ordinary geometric dimension, while for many fractals, $d_S(s)$ exhibits log periodic oscillations as $s \to 0$ [34]. If s is small but nonzero, $d_S(s)$ gives a scale-dependent dimension, while for s large, $d_S(s)$ begins to probe the global topology.

It is sometimes useful to modify the definition (2.5) of the heat kernel by exchanging the Laplacian Δ_M for some other physically relevant operator. For some interesting cases—for example, theories of quantum gravity defined on manifolds but with simple nonconventional dispersion relations—the spectral dimension is the Hausdorff dimension of momentum space [35, 36]. As a cautionary note, though, a modification of the Laplacian can lead to a "return probability" that is not positive definite; in such cases, a modification of the heat equation (2.5) may be desirable [37].

Another dimension can also be obtained from a diffusion process. The "walk dimension" d_W determines the rate at which a random walker moves away from the origin:

$$d_W = 2\left(\frac{d\ln\langle x^2 K(x,0;s)\rangle}{d\ln s}\right)^{-1} \tag{2.8}$$

The walk dimension is not independent, though: it can be shown [34, 36] that $d_W = 2d_H/d_S$, where d_H is the Hausdorff dimension.

- **Lorentzian dimensions:**

The definitions I have discussed so far have implicitly assumed a Riemannian space, that is, a space with a positive definite metric. The Hausdorff dimension, for instance, treats spacelike and timelike distances equivalently and makes no allowance for null separations. The spectral dimension considers random walks without any limit to causal paths. There are variations, though, that take the Lorentzian nature of spacetime into account.

Like the spectral dimension, the causal spectral dimension d_{cs} [38] is based on the behavior of random walks, but now restricted to causal paths. Obviously, we can no longer use return time, since random walkers now move only forward in time. Instead, a dimension can be obtained from the probability that two paths starting from a shared initial point meet again in a diffusion time s. For a manifold, the asymptotic behavior is obtained from the properties of biased random walks; the resulting causal spectral dimension is again defined by (2.7), but with $\langle K(x,x;s)\rangle$ replaced by the meeting probability $P_{\text{meet}}(s)$.

Two other Lorentzian dimensions, both based on scaling, have their origin in causal set theory [39]. The first, the Myrheim-Meyer dimension [40, 41], is a kind of box-counting dimension in which the "boxes" are Alexandrov neighborhoods, or causal diamonds. We start with d-dimensional Minkowski space, and pick two causally related points p and q with $p \prec q$ (where $\prec$ means "is in the past of"). The Alexandrov interval $I[p,q]$ is the intersection of the future of p and the past of q. A point r in $I[p,q]$ (that is, for which $p \prec r \prec q$) forms two new smaller intervals, $I[p,r]$ and $I[r,q]$. It is not hard to show that these intervals have volumes that obey a scaling relation

$$\frac{\langle \mathrm{Vol}(I[r,q])\rangle_r}{\mathrm{Vol}(I[p,q])} = \frac{\Gamma(d+1)\Gamma(\frac{d}{2})}{4\Gamma(\frac{3d}{2})} \tag{2.9}$$

where the bracket denotes an average over r. Curvature corrections can be added if the manifold is not flat [42]. The right-hand side of (2.9) is monotonically decreasing with d, so the formula can be inverted to determine a dimension. Generalizing to arbitrary spacetimes in which causal relations are defined—including even discrete spacetimes such as causal sets—we obtain a Lorentzian scaling dimension, the Myrheim-Meyer dimension d_{MM}.

The second dimensional estimator coming from causal set theory is the midpoint scaling dimension [43]. Again consider an Alexandrov interval $I(p,q)$, and define the "midpoint" to be the point r for which $I(p,r)$ and $I(r,q)$ are most nearly equal, that is, the point that maximizes $N' = \min\{\mathrm{Vol}(I(p,r)), \mathrm{Vol}(I(r,q))\}$. Then $d_{mid} = \log_2[\mathrm{Vol}(I(p,q))/N']$ gives an estimate of the dimension.

- **Geodesics:**

  A final, more qualitative estimate of dimension can be obtained by examining the behavior of geodesics on a manifold. This idea goes back to Galileo's observation that in a three-dimensional space, at most three perpendicular lines can be drawn from a point [20]. In a d-dimensional Lorentzian manifold, $d-1$ orthogonal timelike geodesics can leave a point. But these geodesics may not behave identically; they may probe very different distances in different directions.

  For instance, let us choose Gaussian normal coordinates, select two random nearby geodesics at $t=0$, and look at their proper distance in each of $d-1$ orthogonal directions at a later time t. In regions of certain manifolds—for instance, in Kasner space for both small and large t [44, 45]—a random geodesic will "see" fewer than the full $d-1$ dimensions; that is, the proper distance will be large in $d_{geod}-1$ dimensions and much smaller in the rest. If we consider a measurement with a finite resolution, the manifold will thus appear to have d_{geod} dimensions. This behavior was first observed by Hu and O'Connor in cosmology. Since they were considering large scale behavior, they called d_{geod} the "infrared dimension," but as we shall see later, a similar phenomenon may happen at small scales.

  Since the heat kernel depends on random walks, one might expect this phenomenon to also appear in the spectral dimension. In a sense, it does. At a given spacetime point, the small s expansion (2.6) is dominated by the first term, which gives the ordinary spectral dimension. But for a fixed s—a fixed "resolution" of the diffusion process—it may happen that higher $[a_i]$ terms dominate in certain regions of spacetime. The Lyapunov dimension of the geodesic flow, mention briefly above, might be helpful in understanding this behavior more quantitatively.

2.2 Physical dimensions

The dimensional estimators of the preceding section were based on characteristics of particular mathematical models of spacetime. We now turn to estimators based on the behavior of matter in spacetime. The distinction is not sharp, of course—the spectral dimension, for instance, can characterize either an abstract random walk or a physical diffusion process—but it can be helpful in clarifying the meanings of certain dimensional estimators.

2.2.1 Thermodynamic dimensions

Many physical dimensional estimators are based on thermodynamic properties. The starting point is the simple observation that the density of states, and therefore the partition function, depends on the phase space volume, and that this, in turn, depends on dimension. In particular, for a free particle at high energy in d spacetime dimensions,

$$\rho(E)dE \sim \frac{V_{d-1}}{\hbar^{2(d-1)}} E^{d-2} dE \tag{2.10}$$

from which the partition function $Z(\beta)$ for a collection of noninteracting particles and all of the associated thermodynamic quantities can be determined.

- **Free energy:**

  For a free field or an ideal gas in a box of volume V_{d-1} in a flat d-dimensional spacetime, the free energy is

$$F(\beta) = -\frac{1}{\beta} \ln Z(\beta) = \pm const.\, V_{d-1} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\beta} \ln\left(1 \mp e^{-\beta\omega(k)}\right) \underset{\beta\to 0}{\sim} V_{d-1} T^d \tag{2.11}$$

  where the top sign is for bosons and the bottom is for fermions. The temperature dependence provides a simple thermodynamic dimensional estimator d_{th1}. It is easy to see that this quantity is sensitive to the physics. For instance, if one changes the dispersion relations—that is, the relationship of $\omega(k)$ to k—this can change d_{th1} [46].

- **Internal energy and equation of state**

  A number of thermodynamic quantities can be derived from the free energy. In particular, the energy density at high temperature is

$$\rho = -\frac{T^2}{V_{d-1}} \frac{\partial}{\partial T} \left(\frac{F}{T}\right) \underset{\beta\to 0}{\sim} (d-1) T^d \tag{2.12}$$

  This is essentially the Stefan-Boltzmann law, and it defines a new estimator d_{th2}. The pressure at high temperature is

$$p = T \frac{\partial}{\partial V_{d-1}} \left(\frac{F}{T}\right) \underset{\beta\to 0}{\sim} T^d \tag{2.13}$$

  and the ratio of pressure to energy density determines the equation of state parameter

$$w \underset{\beta\to 0}{\sim} 1/(d-1) \tag{2.14}$$

  giving another estimator d_{th3}. For more detailed models of dimensional reduction, one can sometimes carry this approach further and compute the effect of fractional dimensions on the full black body spectrum [47]. Although the thermodynamic dimensional estimators agree for ordinary thermodynamics in flat spacetime, there are known examples in which they give different answers [48, 49].

- **Equipartition**

  One more thermodynamic dimensional estimate comes from the equipartition theorem, which tells us that a system at thermal equilibrium has an energy of $\frac{1}{2}T$ per degree of freedom. In particular, an ideal monoatomic gas in d spacetime dimensions has $d-1$ translational degrees of freedom per atom, so

  $$E = \frac{d-1}{2}NT, \qquad C_V = \frac{d-1}{2} \tag{2.15}$$

  where C_V is the heat capacity at constant volume. This gives a final thermodynamic dimensional estimator, d_{th4}, whose meaning is perhaps the most transparent: as a direct count of translational degrees of freedom, it is the thermal analogy of Galileo's counting of perpendicular lines from a point.

2.2.2 Other physical dimensions

While thermodynamic quantities provide convenient estimators of dimension, they are by no means the only physical quantities with simple dimensional dependence. At least three others have been considered.

- **Greens functions**

  As Ehrenfest stressed in his seminal paper on the dimension of space [7], the Newtonian gravitational potential in a d-dimensional spacetime varies as $r^{-(d-3)}$. A slightly more invariant version of this statement is that the short distance Hadamard Greens function for massless particles has the form

  $$G^{(1)}(x,x') \sim \begin{cases} \sigma(x,x')^{-(d-2)/2} & d>2 \\ \ln\sigma(x,x') & d=2 \end{cases} \tag{2.16}$$

  where the world function $\sigma(x,x')$ is half the squared geodesic distance between x and x' [32]. The generalization to an arbitrary spacetime provides a new dimensional estimator d_G. This quantity is closely related to the spectral dimension, since the Greens function is a Laplace transform of the heat kernel, but there can be subtle differences that depend on exactly which Laplacian is used. In causal set theory, for instance, the causal spectral dimension d_{cs} has a behavior quite differently from that of the dimension d_G determined by the Laplacian with the proper classical limit [38, 50, 51].

  This approach forces us to address a puzzling question about dimensional reduction and unitarity. In ordinary quantum field theory, positivity of the spectral function $\rho(\mu)$ in the Källen-Lehman representation of the two-point function implies that the propagator in a four-dimensional spacetime cannot vanish faster than $1/p^2$ at large p [52]. Thus in field theoretic approaches such as asymptotic safety, and in discrete approaches that have something like a Källen-Lehman representation, one must worry that dimensional reduction could indicate the appearance of negative-norm states or a breakdown in quantum field theory [51, 53]. There are several ways to evade this problem, typically involving states that appear in the spectral decomposition but are not present asymptotically, but any theory of quantum gravity that predicts dimensional reduction will eventually have to address this issue.

- **Unruh dimension**

  As Unruh showed long ago [54], an accelerated detector will detect particles even in the Minkowski vacuum. The detection rate is determined by the two-point function of the particle being detected. As we have seen, such a two-point function has a characteristic dependence on dimension, giving a slightly different probe of the behavior of Greens functions. Alkofer et al. propose in [55] to use the response function of such a detector—in principle a directly measurably quantity—to define an "Unruh dimension" d_U.

- **Scaling dimensions and anomalous dimensions**

  Physical fields have natural scaling dimensions, which describe their changes under constant rescaling of lengths and masses. A scalar field φ, for instance, has a term in its action of the form $(\partial\varphi)^2$, integrated over a d-dimensional spacetime. For the action to be invariant under constant changes in scale (in units $\hbar = 1$),† φ must scale as $L^{-\Delta_0}$ with $\Delta_0 = \frac{d-2}{2}$. The behavior (2.16) of the two-point function of a scalar field can be read off from this scaling.

  In an interacting quantum field theory, however, the "canonical dimension" (or "engineering dimension") Δ_0 typically receives corrections from renormalization. The full scaling dimension becomes $\Delta = \Delta_0 + \gamma(g)$, where $\{g\}$ are the coupling constants and $\gamma(g)$ is called the anomalous dimension [56]. The anomalous dimension changes with energy scale under the renormalization group flow, so Δ naturally defines a scale-dependent dimension. Note, though, that the anomalous dimension of a composite operator $\mathcal{O}_1\mathcal{O}_2$ is not, in general, just a sum $\Delta(\mathcal{O}_1) + \Delta(\mathcal{O}_2)$, so different choices of operators may give different dimensional estimators. One particularly natural choice is to look at the dimension of a two-point function $\langle 0|\varphi(x)\varphi(x')|0\rangle$ of a scalar field; this gives a quantum version d_{qG} of the Greens function dimension d_G considered above.

3 Evidence for dimensional reduction

Let us now turn to our main topic, the signs of short distance dimensional reduction in quantum gravity. This is a bit tricky, since we do not yet have a complete theory of quantum gravity; nor, as we have seen, do we have a unique way to define dimension. Still, by looking at a variety of approaches to quantum gravity and dimension, let us see how far we can get.

3.1 High temperature string theory

The first indication of dimensional reduction in quantum gravity came from the study of high temperature string theory [18]. As a gas of strings is heated, it undergoes a phase transition, the Hagedorn transition. In 1988, Atick and Witten found that above the Hagedorn temperature, the number of degrees of freedom abruptly drops: the free energy goes as $F/V \sim T^2$, so the thermodynamic temperature d_{th1} of section 2.2.1 falls to $d_{th1} = 2$. Other thermodynamic quantities are not directly computed in [18], but it is easy to check that the dimensions d_{th2} and d_{th3} also fall

†In quantum field theory it is conventional to describe fields in terms of mass dimension rather than length dimension, hence the $-\Delta$ rather than Δ

to $d = 2$. Atick and Witten conclude that "the mysterious system that prevails under distances $\sqrt{\alpha'}$" behaves "as if this system were a (1+1)-dimensional field theory."

This dimensional reduction occurs for strings in a flat background. It may differ in other settings: for instance, the radial dependence of temperature in an AdS black hole background can alter the structure of the transition in a way that changes d_{th1} [57]. So it is interesting to see whether there are other indications of short distance dimensional reduction in string theory. This is not easy: as Gross and Mende have stressed [58], one cannot consistently introduce external probes to explore short distance behavior. If, on the other hand, one uses a string as a probe, one finds that at high energies it sees a region with a size of order $(\ln E)^{1/2}$, limiting the scale that can be explored [59].

It is known that the amplitude for two-string scattering at fixed small angle falls off exponentially with energy [58, 60], much faster than in any ordinary quantum field theory. This *might* be taken as a hint of dimensional reduction. Gross has speculated that this behavior and related high energy symmetries could indicate that the only nontrivial scattering in the high-energy limit is (1+1)-dimensional [61]. As I shall discuss in section 4.2, this may be connected to the phenomenon of "asymptotic silence" that also appears in other approaches to dimensional reduction. It also seems that at leading order in high energy open string scattering, only polarizations in the plane of the scattering are important [62]; as Gross and Mañes put it, this "suggests that the strings are trying to effectively reduce the number of spatial dimensions that they live in at high energy." Still, though, for now these are only hints of interesting behavior.

3.2 Causal dynamical triangulations

The results from string theory were intriguing, but dimensional reduction only became a significant focus of research in 2005, with the discovery of short distance reduction of the spectral dimension in causal dynamical triangulations [19]. Causal dynamical triangulations is a discrete approach to quantum gravity in which curved spacetimes are approximated by piecewise flat simplicial complexes, whose contributions are combined, typically numerically, to form a path integral. The action for a simplicial complex, the Regge action, has been known since 1961 [63], and the idea of combining Regge calculus with Monte Carlo methods to evaluate the path integral on a computer dates back to 1981 [64]. Until rather recently, though, no well-behaved continuum limit had been found; instead, the simulations typically yielded a "crumpled" phase with very high Hausdorff dimension and a two-dimensional "branched polymer" phase [65]. The causal dynamical triangulations program added a new ingredient, a fixed "direction of time,"‡ which controls fluctuations of topology and suppresses the unphysical phases. The program has been remarkably successful, producing not only a de Sitter ground state with the correct volume profile but also the expected spectrum of quantum volume fluctuations [67] and reasonable transition amplitudes [68].

The construction of a four-dimensional spacetime in causal dynamical triangulations starts with four-dimensional simplices. But this does not necessarily determine the large scale spacetime dimension; the path integral may be dominated by configurations that are not manifold-like or that have very different macroscopic appearances, and ultimately the dimension has to be measured.

‡The direction of time is usually introduced by fixing a time-slicing and demanding that all simplices connect one slice to the next, but there is evidence that such a strong slicing condition is not needed as long as a fixed causal structure is maintained [66].

It is easy to define a random walk on a simplicial complex, so a natural choice for a dimensional estimator is the spectral dimension d_S of section 2.1. In 2005, Ambjørn, Jurkiewicz, and Loll found, quite unexpectedly, that while the spectral dimension is $d_S = 4$ at large scales—the model reproduces a four-dimensional universe at large distances—it drops to $d_S \approx 2$ at small scales [19]. Similarly, if one starts with a three-dimensional model the spectral dimension drops from $d_S = 3$ at large scales to $d_S \approx 2$ at small scales [69]. These results have been checked independently in [70], and verified analytically in a simplified toy model [71]. Because of numerical uncertainty, the lower limit is not known exactly, and may be consistent with $d_S = 1.5$ [72], but the occurrence of short distance dimensional reduction in this approach is unambiguous.

3.3 Asymptotic safety

Viewed as an ordinary quantum field theory, general relativity is nonrenormalizable: its effective action has an infinite number of terms, containing arbitrarily high powers of curvature, each with its own coupling constant. Even if all but a few of the coupling constants are set to zero at some scale, they will reappear at other scales under the renormalization group flow. As Weinberg first pointed out, though, the theory might still make sense [73]. For the theory to remain valid at arbitrarily high energies or short distances, the renormalization group flow must have an ultraviolet fixed point. If it does, and if that fixed point has only a finite number of relevant directions, then the infinitely many couplings of the low energy theory are determined by finitely many parameters. Such a theory is "asymptotically safe."

Whether general relativity is asymptotically safe is a deep question, and we are far from a definitive answer. But several pieces of evidence, coming from truncations of renormalization group equations and from exact calculations in dimensionally reduced models, make it plausible that it is [74–76]. If gravity can be described by an asymptotically safe theory with a non-Gaussian ultraviolet fixed point, the anomalous dimensions introduced in section 2.2.2 will flow to fixed values under the renormalization group. For us, the key point is that operators at the fixed point will *necessarily* acquire precisely the anomalous dimensions necessary to make the theory appear two-dimensional [77–79]. In particular, the dimension d_{qG} of section 2.2.2 will flow to $d_{qG} = 2$ at the fixed point.

Intuitively, this happens because the theory becomes scale invariant at a fixed point, and the Einstein-Hilbert action is scale invariant only in two dimensions. The more formal demonstration of this behavior is fairly straightforward [75, 79], although there are a few subtleties [78]. Consider the renormalization group flow of the dimensionless coupling constant $g_N(\mu) = G_N\mu^{d-2}$, where G_N is Newton's constant and μ is the mass scale. Under this flow, we have

$$\mu\frac{\partial g_N}{\partial\mu} = [d-2+\eta_N(g_N,\dots)]g_N, \tag{3.1}$$

where the anomalous dimension η_N depends on both g_N and any other (dimensionless) coupling constants in the theory. It is evident that a free field, or "Gaussian," fixed point can occur at $g_N = 0$. For an additional non-Gaussian fixed point g_N^* to be present, though, the right-hand side of (3.1) must vanish: $\eta_N(g_N^*,\dots) = 2-d$.

But the momentum space propagator for a field with an anomalous dimension η_N goes as $(p^2)^{-1+\eta_N/2}$. For $\eta_N = 2-d$, this becomes p^{-d}, and the associated position space propagator depends logarithmically on distance. As we saw in section 2.2.2, such a logarithmic dependence is characteristic of a two-dimensional conformal field, $d_{qG} = 2$. A variation of this argument shows that matter fields interacting with gravity at a non-Gaussian fixed point exhibit two-dimensional behavior as well [75].

Renormalization group methods can also be applied to the spectral dimension near the fixed point by computing the flow of the return probability $K(x, x; s)$ of eqn. (2.7) [80,81]. The flow effectively gives a scale dependence to the metric in the heat equation. Using two different truncations of the effective action, it has been confirmed that three scaling regimes appear: a classical regime with $d_S = 4$, a quantum regime near the fixed point with $d_S = 2$, and an intermediate semiclassical regime with a spectral dimension that depends on the truncation and the renormalization group trajectory. Efforts are now underway to evaluate the anomalous dimensions of length and volume operators [82], which might eventually lead to results for the scaling dimensions of section 2.1 near the fixed point.

3.4 Causal set theory

Yet another sign of dimensional reduction comes from a completely different approach to quantum gravity. Causal set theory [83] starts with perhaps the most primitive picture of a spacetime, a collection of discrete points whose only relations are those of causality. Most such sets are not at all manifold-like, but the hope is that a suitable dynamical principle will pick out the physically relevant class. For those causal sets that *do* approximate manifolds, the Lorentzian metric structure is determined by the causal structure, which fixes the conformal class of the metric [84], and the number of points in a region, which determines the conformal factor.

Causal sets are intrinsically Lorentzian, and one should use dimensional estimators that respect this property. One such choice is the Myrheim-Meyer dimension d_{MM} described in section 2.1. It has been shown that for very small causal sets—those with four, five, or six elements—the average Myrheim-Meyer dimension is $d_{MM} \approx 2$ [50]. Work on extending this computation to larger sets is in progress. Further evidence comes from "generic" causal sets, which have a structure known as a Kleitman-Rothschild order [85]. Such sets are not manifold-like, and must be dynamically suppressed at large scales, but they arguably remain important at small distances. They have a Myrheim-Meyer dimension of $d_{MM} = 2.38$. There are also indications that causal sets obtained by randomly "sprinkling" points in Minkowski space have Myrheim-Meyer dimensions that fall to $d_{MM} \approx 2$ for small subspaces [39]. Again, work on this question is in progress.

One can also consider the causal spectral dimension d_{cs} of section 2.1. Here, rather dramatically, it has been shown that the dimension of a manifold-like causal set *increase* at short distances [38]. However, the d'Alembertian implicit in the definition of d_{cs} also fails to reproduce the standard flat spacetime d'Alembertian on causal sets that approximate Minkowski space. If one instead computes the spectral dimension from the "right" d'Alembertian, one obtains a result that shows the usual pattern of dimensional reduction to $d_{cs} = 2$ at short distances [51]. It also seems that the Greens function dimension d_G of section 2.2.2 falls to $d_G = 2$ at short distances, although there are some ambiguities in a choice of regularization [50, 51].

3.5 Loop quantum gravity

Another place that we might look for evidence for dimensional reduction is loop quantum gravity, a popular canonical quantization scheme based on a particular choice of connection variables and a consequent nonstandard inner product [86]. At the kinematical level, loop quantum gravity has been very successful, with a rigorously defined Hilbert space of spin network states and a collection of interesting geometrical operators. The dynamics, as described by the Hamiltonian constraint, is much more poorly understood—in particular, the full physical Hilbert space is not known—and we shall see that this may be important for our questions.

In loop quantum gravity the evidence is mixed, but there are at least some indications of lower-dimensional behavior at short distances. This possibility was first noted by Modesto [87], who pointed out that the average area in loop quantum gravity could be written in the form

$$\langle \hat{A}_\ell \rangle = \frac{\sqrt{\ell^2(\ell^2+\ell_p^2)}}{\sqrt{\ell_0^2(\ell_0^2+\ell_p^2)}}\langle \hat{A}_{\ell_0}\rangle \tag{3.2}$$

where ℓ is a (quantized) length and ℓ_p is the Planck length. For large ℓ this is just the ordinary scaling of area, but for small ℓ it differs, suggesting a change in scaling dimensions of section 2.1. If we translate this behavior into a scale-dependent effective metric and, as in the asymptotic safety program, use this metric to determine a heat kernel and a spectral dimension, we find a flow from $d_S = 4$ at large scales to $d_S = 2$ at small scales. This argument has also been extended to include spin foams, the "paths" in the loop quantum gravity version of a path integral [88].

On the other hand, it is also possible to define a heat kernel and both Hausdorff and spectral dimension directly on a discrete complex such as a spin network or a spin foam [89]. Here the results are more ambiguous. It seems that pure spin network states do not exhibit dimensional flow [90], and while superpositions that exhibit dimensional reduction can be constructed, the short-distance dimension depends on details of the superposition [91]. The conclusion for physics may thus depend on the dynamics—exactly which superpositions are allowed by the Hamiltonian constraint—rather than merely on the kinematics. As a possible first step in this direction, dimensional reduction has been investigated in loop quantum cosmology [92, 93], where modifications of the full constraint algebra lead to modified dispersion relations (see section 3.7). The results are still somewhat ambiguous, depending on model-dependent choices for particular quantum corrections, but the simple choices yield spectral and a thermodynamic dimensions $d_S = d_{th2} = 2.5$ [92] or $d_S = 1$ [93] at short distances.

3.6 The short distance Wheeler-DeWitt equation

Early attempts to canonically quantize general relativity led to the Wheeler-DeWitt equation [94],

$$\left\{16\pi\ell_p^2 G_{ijkl}\frac{\delta}{\delta q_{ij}}\frac{\delta}{\delta q_{kl}} - \frac{1}{16\pi\ell_p^2}\sqrt{q}\,{}^{(3)}\!R\right\}\Psi[q] = 0 \tag{3.3}$$

where q_{ij} is the spatial metric at a fixed time and $G_{ijkl} = \frac{1}{2}q^{-1/2}\left(q_{ik}q_{jl} + q_{il}q_{jk} - q_{ij}q_{kl}\right)$ is the DeWitt supermetric. This is almost certainly not quite the right way to quantize gravity—we don't know how to make real sense of the operators or to construct a properly gauge-fixed inner product—but it is generally believed that the eventual quantum theory will contain something close to the Wheeler-DeWitt equation, at least as an approximation. So if quantum gravity leads to short distance dimensional reduction, that fact ought to be evident here as well.

The wave function $\Psi[q]$ contains information about the metric at all scales. To focus in on small scales, we can look at the strong coupling limit $\ell_p \to \infty$. As Isham first pointed out [95], this is also an ultralocal limit: the only term involving spatial derivatives of the metric drops out.[§] This limit was studied extensively in the 1980s (see [45] for references). As the Planck length becomes large, particle horizons shrink and light cones collapse to timelike lines, leading to the decoupling of neighboring points and the consequent ultralocal behavior [96]. In the completely decoupled limit, the solution at each point is essentially a Kasner space [97], while for large but finite ℓ_p solutions exhibit BKL behavior [98–100], looking locally Kasner but with chaotic "bounces" that change the Kasner axes and exponents.

Now, Kasner space is certainly four-dimensional. But as noted in section 2.1, geodesics at early times essentially see only one of the spatial dimensions, and the dimensional estimator d_{geod} shrinks to $d_{geod} = 2$ [44, 45]. This behavior has also been noticed in a completely different astrophysical context [101]. As discussed in [45], it is plausible that the spectral dimension exhibits a similar behavior. The heat kernel for Kasner space is of the form [102, 103]

$$K(x, x; s) \sim \frac{1}{4\pi s^2}\left[1 + \frac{a}{t^2}\, s + \dots\right]. \tag{3.4}$$

For a fixed time t, one can always find s small enough that the first term dominates, giving a "microscopic" spectral dimension $d_S = 4$. But for a fixed return time s, that is, a fixed scale at which one is measuring the dimension, there is always a time t small enough that the second term dominates, giving an effective spectral dimension of $d_S = 2$.

The key feature in this analysis is "asymptotic silence" [100], the collapse of light cones and the corresponding decoupling of neighboring points. We will return to this issue in section 4.2, but for now it is worth noting that the same phenomenon appears in a variety of other approaches to quantum gravity, from loop quantum cosmology to noncommutative geometry.

3.7 Modified dispersion relations and noncommutative geometry

The thermodynamic properties of a gas of particles depend on the dispersion relations $E = E(p)$. There are many reasons one might try to alter the standard dispersion relations, most of them unrelated to gravity. But some research directions in quantum gravity—the application of noncommutative geometry, for instance, or generalizations of the uncertainty principle—lead naturally to such modifications.

It has been known for years that modified dispersion relations can change thermodynamics [104]. Inspired by the string theory results of section 3.1, several investigations in the early 2000s briefly mentioned the possibility of dimensional reduction [105–107]. A concentrated focus on dimensional flow came more recently, starting, I believe, with an analysis by Husain, Seahra, and Webster

[§]Strictly speaking, the theory is not quite ultralocal even in this limit, since the diffeomorphism constraint, which *does* involve spatial derivatives, must still be imposed.

of “polymer quantization,” a loop-quantum-gravity-inspired modification of the ordinary rules of quantum mechanics [49]. Here, the thermodynamic dimension d_{th2} was shown to drop to $d_{th2} = 5/2$ at high temperatures. Not long after, thermodynamic behavior was investigated for a particular noncommutative spacetime, Snyder space [46], where it was shown that at high temperatures, $d_{th2} = d_{th3} = d_{th4} = 2$. Thermodynamic dimensions were analyzed more systematically in [48] for a variety of dispersion relations, and it was demonstrated that the various dimensions of section 2.2.1 could all differ from each other, and could differ from spectral dimension as well.

Given a set of modified dispersion relations, the spectral dimension is closely related to the thermodynamic dimensions, although they are not quite equivalent [48]. Roughly speaking, a dispersion relation $E^2 - f(p^2) = 0$ translated into a d'Alembertian ${\partial_t}^2 - f(\nabla^2)$, which can be Wick rotated and inserted into (2.5) to obtain a spectral dimension. Conversely, given *any* energy dependence of a spectral dimension, one can reconstruct a dispersion relation that reproduces that dependence [108]. This gives us an uncomfortable amount of freedom—we can choose any scale dependence of dimension we like, and build by hand a suitable dispersion relation—and it compels us to focus on dispersion relations that have a strong independent rationale.

One such rationale is the polymer quantization of [49], which starts from the nonstandard inner product of loop quantum gravity.¶ Another is the use of simple noncommutative geometries, as in [46], and their implied dispersion relations. A variation of “doubly special relativity,” a deformation of special relativity that preserves frame independence while introducing an invariant energy scale, leads to a spectral dimension that falls to $d_S = 2$ at high energies, but the result depends on a free parameter [110]. A similar reduction of spectral dimension has been studied another common model of noncommutative geometry, κ-Minkowski space. Here, too, the dimension decreases at small scales, but with a limiting value that depends on a nonunique choice of Laplacian [111, 112]. It has recently been shown that the static potential between two charges becomes constant at very small distances in κ-Minkowski space, corresponding roughly to a Greens function dimension $d_G \leq 3$ [113]. Another model with a group-valued momentum space, inspired by exact results from three-dimensional gravity, also gives $d_S = 2$ at high energies [114].

On the other hand, we can instead start with the more formal approach to noncommutative geometry based on the spectral action developed Connes and his collaborators [115]. Here it appears that the spectral dimension is $d_S = 0$ [116, 117]. A variation of the spectral action exists, though, that leads to a spectral dimension that fall from $d_S = 4$ at large scales to $d_S = 2$ at small scales, the “conventional” behavior we have seen elsewhere [118].

3.8 Minimum length

A frequent theme in quantum gravity is the possible existence of a minimum length [119, 120]. Heuristically, if one tries to probe too small a length, one needs so much energy that the region being probed collapses into a black hole and becomes unobservable. While a minimum length is not a universal feature of quantum gravity, it is a common one, so it is interesting to see whether this is enough to imply some sort of short distance dimensional reduction.

Modesto and Nicolini [121] have argued that one effect of a minimum length should be to smear out the starting point of the diffusion process described by (2.5), replacing the initial condition

¶Note, though, that different approaches to polymer quantization may not lead to dimensional reduction [109].

$K(x, x'; 0) = \delta(x - x')$ with a Gaussian distribution

$$K(x, x'; 0) = \left(\frac{1}{4\pi\ell^2}\right)^{d/2} e^{-|x-x'|^2/4\ell^2} \tag{3.5}$$

where ℓ is the minimum length. The resulting spectral dimension drops from $d_S = d$ at large scales to $d_S = d/2$ at the minimum length scale. Padmanabhan, Chakraborty, and Kothawala [122] have looked at scaling dimensions for a geodesic ball with a "quantum metric" that incorporates a minimum length, finding that for any large-scale dimension, the box-counting dimension of section 2.1 falls to $d_b = 2$ at small scales. A minimum length may also be incorporated through a "generalized uncertainty principle" [120], which alters the Heisenberg commutation relations. These changes lead to modified dispersion relations, as in section 3.7, so, not surprisingly, dimensional reduction at short distances occurs [49, 110]. Maziashvili has also argued that the existence of a minimum length causes the box-counting dimension d_c of section 2.1 to decrease at small distances [123].

3.9 Modified gravity

Yet another approach to quantum gravity is to modify the Einstein-Hilbert action to make the theory renormalizable. One popular example is "Hořava-Lifshitz gravity" [124], a model that sacrifices Lorentz invariance in exchange for (possible) renormalizability. Here, a generalization of the spectral dimension flows from $d_S = 4$ at low energies to $d_S = 2$ at high energies [125]. Similarly, in curvature-squared models, which are renormalizable but very probably nonunitary, a generalized spectral dimension falls to $d_S = 2$ [127]. In certain nonlocal renormalizable models, the spectral dimension again falls at high energies, now by an amount that depends on particular choices of form factors [128].

While these results may be taken as further evidence for short distance dimensional reduction, they may also serve as cautionary notes. The models I have described are defined on ordinary four-dimensional manifolds, and while the reduction of spectral dimension takes place at high energies, it is essentially classical. The same might be said for some of the models of modified dispersion relations of section 3.7. The moral, perhaps, is once again that one must be very careful about exactly what one means by "dimension"—different choices of how to measure dimension can give different results, which may not always reflect our intuition of what the dimension "really" is.

3.10 Spacetime foam

Many years ago, Wheeler suggested that quantum fluctuations of the geometry and perhaps the topology of spacetime should lead to a "foamy" structure at the Planck scale [129]. A complete understanding of such behavior would certainly require a full quantum theory of gravity, but in the spirit of "stochastic gravity" [130] we might try to approximate spacetime foam by stochastic classical fluctuations. Several attempts in this direction have hinted at small scale dimensional reduction.

One of the earliest proposals, due to Crane and Smolin [16, 131], models spacetime foam as a collection of virtual black holes. If the distribution of such black holes is scale invariant and sufficiently dense, the spacetime outside their horizons becomes fractal, with a Hausdorff dimension and a scaling dimension for Greens functions of the form $d_H = d_G = 4 - \varepsilon$, where ε depends on the (unknown) details of the distribution. A few years later, Haba looked at the behavior of scalar Greens functions in a spacetime with a stochastically fluctuating metric with a scale-invariant spectrum of fluctuations [132, 133]. The result is again a reduction of d_G by an amount that depends on the details of the distribution. Both proposals were motivated in part by the hope that metric fluctuations might tame the infinities of quantum field theory, and both have at least a heuristic connection to the asymptotic safety program of section 3.3. The minimum length scenario of Padmanabhan et al. [122] is in some sense a variation of the Crane-Smolin model, since the "quantum metric" can be viewed as generating a foamy structure of "holes" in spacetime.

A rather different stochastic treatment of vacuum fluctuations appears in [134]. As we have recently come to understand [135, 136], quantum fluctuations of the vacuum have a highly non-Gaussian distribution, with a "fat tail" of large positive energy fluctuations. These positive fluctuations focus null geodesics, and can be amplified by the nonlinearities of the Raychaudhuri equation. The result is a phenomenon known in statistics as "gambler's ruin": large positive fluctuations are rare, but once a large enough fluctuation occurs, subsequent negative fluctuations are too small to reverse its effect. In two spacetime dimensions, it was shown in [134] that these fluctuations collapse light cones at scales near the Planck length, arguably leading to a kind of universal short distance asymptotic silence of the sort discussed in section 3.6. It seems likely that this result can be extended to higher dimensions; see [137] for a first step.

3.11 Multifractional geometry

Multifractional geometry is not so much a model of quantum gravity that predicts dimensional reduction, but rather a broad mathematical framework that naturally incorporates theories with scale-dependent dimensions. The formalism, based on fractional calculus, has been developed extensively by Calcagni and collaborators [36, 138, 139], and can provide a setting for writing down particular models of dimensional reduction that capture key features of other quantization programs. As we shall see in section 5, a class of multifractional models currently offer the strongest observational constraints on dimensional reduction.

4 Common threads?

We have seen that many approaches to quantum gravity show indications of dimensional reduction near the Planck scale. Taken individually, each of these hints remains quite speculative. Perhaps the most persuasive evidence comes from asymptotic safety, in which the argument for two-dimensional behavior at the ultraviolet fixed point is compelling, and causal dynamical triangulations, in which the evidence for flow of the spectral dimension is extremely strong. But for this evidence to be truly convincing, we would have to actually know that quantum gravity has an interacting ultraviolet fixed point, as required by asymptotic safety, or that causal dynamical triangulations has the right continuum limit; that is, we would have to know how to quantize gravity.

Taken as a body, though, these hints become quite a bit stronger. It seems rather unlikely that so many different approaches to quantum gravity would converge on the same result merely by accident. If this agreement is more than coincidence, however, it ought to be possible to to find a common thread, a single origin for dimensional reduction that is shared by all of the various approaches.

This is not easy. Even for the much more mature subject of black hole thermodynamics, we don't really understand why so many approaches to quantization lead to the same expressions for temperature and entropy, although there is some evidence that this behavior can be traced back to a shared symmetry [140, 141]. Moreover, as we have seen, different hints of dimensional flow employ different definitions of dimension, which need not be equivalent. If we are willing to be speculative, though, there have been suggestions of two possible common threads that could explain the "universality" of dimensional reduction.

4.1 Scale invariance

The fundamental premise of asymptotic safety is that the renormalization group flow of quantum gravity has a non-Gaussian (that is, interacting) ultraviolet fixed point. Such a fixed point is, by definition, a point at which the theory becomes scale invariant. This, in turn, implies dimensional reduction, essentially because it is only in two dimensions that Newton's constant is dimensionless.

Causal dynamical triangulations arguably exhibits a similar scale invariance. There is growing evidence that a continuum limit occurs at a second order phase transition [142, 143], and at such a transition any theory becomes scale invariant [144]. This result may connect back to asymptotic safety: it has been argued that the continuum limit may only exist if the renormalization group flow for causal dynamical triangulations itself has an ultraviolet fixed point [145].

Similarly, the two terms in the Wheeler-DeWitt equation (3.3) scale differently under constant rescalings of the metric, but in the short distance limit only one of these remains, leading to a global scale invariance.[‖] The Crane-Smolin "foam" model discussed in section 3.10 can be implemented with a scale-invariant distribution of virtual black holes, again giving scale invariance at short distances. For causal sets and loop quantum gravity, the situation is less clear, but Dittrich has proposed that loop quantum gravity and similar background-independent formulations should obey a set of consistency conditions under coarse-graining that are similar to the requirement of an interacting ultraviolet fixed point in asymptotic safety [147].

Amelino-Camelia, Arzano, Gubitosi, and Magueijo have argued forcefully for the fundamentality of scale invariance [148]. The central feature of the observed fluctuations of the cosmic microwave background (CMB) is its nearly scale-invariant spectrum. This is usually taken as evidence for inflation, but Amelino-Camelia et al. propose that it is instead a sign that at short enough distances and high enough energies, the universe really is scale invariant. Such a scale invariance may also be useful elsewhere, addressing some mysteries in particle physics; see, for example, [149].

4.2 Asymptotic silence

The Planck length $\ell_p = \sqrt{\hbar G/c^3}$ depends on Planck's constant, Newton's constant, and the speed of light. The $\ell_p \to \infty$ limit of section 3.6 is often described as a strong coupling limit, but it can equally well be viewed as an "anti-Newtonian" $c \to 0$ limit [96]. In this limit, light cones collapse

[‖] This invariance applies only for *constant* rescalings. In quantum field theory, such scale invariance plus Lorentz invariance usually implies full conformal invariance [146], but it is not clear whether this holds for quantum gravity.

to lines, and nearby points become causally disconnected. This kind of behavior was first noticed in cosmology, where it was given the name "asymptotic silence" [150, 151]. In a sense, this is a reduction to one dimension, but the process goes by way of two dimensions, since, as described in section 3.6, the decoupling of neighboring points leads to BKL behavior with $d_{geod} \sim 2$.

What makes this behavior intriguing is that it appears in many different contexts in quantum gravity. It has been suggested that short distance asymptotic silence arises directly from quantum fluctuations of the vacuum stress-energy tensor [134], as discussed in section 3.10. Eichhorn et al. have shown that causal sets that approximate continuum spacetimes exhibit something akin to asymptotic silence, with space breaking up at short distances into disconnected "islands" [152]. Sorkin (cited in [152]) has also observed that in an asymptotically silent spacetime, one might expect drastically reduced scattering at high energies, since nearby worldlines do not meet. This does, indeed, happen in causal set theory, but also in string theory scattering at fixed small angle [58, 60]. The apparent reduction of the spectral dimension to $d_S = 0$ in the Connes spectral action, as described in section 3.7, has been conjectured to be related to asymptotic silence, and a similar result occurs in a more "Lorentzian" approach to formal noncommutative geometry as well [153, 154]. And in loop quantum cosmology, there is evidence that the quantum deformation of the constraint algebra leads to asymptotic silence at high curvatures and high energies [155].

5 Can dimensional reduction be tested?

Ideally, we would resolve the question of dimensional flow by simply looking at observational evidence. In practice, this seems very difficult: in most approaches that exhibit this kind of behavior, dimensional reduction takes place near the Planck scale, well outside the range of any ordinary experiment. On the other hand, we *do* have results for Planck scale searches for other phenomena such as broken Lorentz invariance [156–158], so the possibility of observational tests is not completely outrageous.

Indeed, broken Lorentz invariance is the first place we might look for the effect of a reduction in the number of spacetime dimensions. Unfortunately, though, the strongest existing limits come from searches for "systematic" symmetry-breaking [156,158]—that is, patterns coming from a single preferred frame—while the effects of dimensional reduction are typically "nonsystematic," varying rapidly in space and time. In asymptotic silence, for instance, the preferred Kasner axis that picks the local two-dimensional spacetime at any given point varies chaotically [159, 160]. There are some observational limits on nonsystematic violations of Lorentz invariance—see [158] for a recent review—but these have not yet been applied to the question of dimensional reduction.

But while we do not yet have observational evidence, several lines of investigation are currently in progress.

- **Cosmology:**

  The expansion of the universe—especially if it includes an inflationary epoch—can stretch out signals at the Planck length to sizes that might become observationally accessible. Cosmology is thus a natural place to look for signs of Planck-scale dimensional reduction. In the absence of a clear set of quantum gravitational predictions, we can use various simplified models to describe dimensional reduction; ultimately, we will have to understand the extent to which any predictions depend on the choice of model.

As noted in section 3.7, a scale-dependent spectral dimension can be translated into modified dispersion relations [108]. Using this observation, Amelino-Camelia et al. show in [161] that a spectral dimension $d_S = 2$ at short distances naturally leads to a scale-invariant spectrum of fluctuations in the CMB even without inflation. In more restrictive contexts, related ideas appeared earlier in [162, 163], and similar results come from asymptotically safe cosmology [164]. Typical models of this sort give predictions of the running of the spectral index of CMB fluctuations that may be testable [148, 165, 166], and a start has been made on investigating other properties of various models [167, 168]. It is not yet known, however, whether there are features that would uniquely distinguish models with dimensional reduction from more conventional cosmologies.

In a different approach to modeling dimensional reduction, two dimensions of spacetime can be initially "frozen" and then joined by mode-matching to a fully dynamical four-dimensional spacetime [169]. Here, observational signatures in the CMB seemed to be washed out by an ensuing period of inflation. On the other hand, certain multifractional models predict log-periodic oscillations, and these are now strongly constrained by CMB measurements [170].

In a rather different context, there has been recent interest in the cosmological implications of quantum tunneling from a two-dimensional spacetime to four dimensions. In the language of Lorentz symmetry breaking, this is a "systematic" change, one with a single global choice of preferred dimensions. As such, it is not directly applicable to most of the dimensional flow scenarios discussed here, but the results might still have some relevance. Tunneling events of this sort lead to anisotropic spacetimes, although the anisotropy can be diluted by subsequent inflation. Signatures of such an anisotropy in the CMB have been investigated in [171–174].

In addition to primordial quantum fluctuations of the CMB , we can look directly at the CMB spectrum itself. This spectrum probes much larger length scales and lower energies—we are looking at the surface of last scattering rather than the primordial fluctuations—but it tests a different kind of dimension. In one of the earliest application of the notion of thermodynamic dimension, Caruso and Oguri showed in 2009 that the dimension of space at this scale differs from $d_{th} = 4$ by at most a part in 10^{-5} [47].

- **Particle physics:**

  Short-distance dimensional reduction can have implications throughout particle physics. The problem, as usual, is that the Planck scale is so remote from any scale we can directly access. To have any possibility of obtaining useful results, we must look at high energies, long time spans for effects to accumulate, and precise measurements. Even with these tricks, we have not yet managed to probe Planck scale effects, but in some models we are getting closer:

  - The first applications of particle physics to determine dimension were designed merely to measure the Hausdorff dimension of spacetime at "ordinary" scales. By considering the anomalous magnetic moment of the electron in a spacetime of fractal dimension, Svozil and Zeilinger concluded that our spacetime must have a dimension $d_H > 4-5\times10^{-7}$ [15]. A similar calculation of the Lamb shift gives $d_H > 4 - 3.6 \times 10^{-11}$ [175].

– One might hope that processes dominated by loops in Feynman diagram might be more sensitive to dimensional reduction, since they involve virtual particles of arbitrarily high energies. By looking at B-$\bar{B}$ oscillations in spacetimes with scale-dependent Hausdorff dimensions, Shevchenko has shown that observations are almost completely insensitive to values $2 < d_H < 5$ at energies of 300-400 GeV [176], corresponding to distances on the order of 10^{-18} m (still, of course, much larger than the Planck length).

– The most extensive studies have looked at certain classes of multifractional models [177, 178]. The strongest bounds come from modifications of the Lamb shift and the anomalous magnetic moment of the electron, both of which are measured with great precision. Such observations constrain the characteristic length scale ℓ_* for dimensional reduction in these models: for a general class of models, Calcagni et al. find $\ell_* < 10^{-17}$ m, and for a specific choice of parameters, $\ell_* < 10^{-27}$ m. (For a review, see [36].)

– Tests of Lorentz invariance achieve remarkable sensitivity by allowing possible violations to accumulate over very long distances, for instance by looking at photons from distant galaxies. The asymptotic silence picture of section 4.2 suggests the same approach: the passage through successive random Kasner-like spaces might have an observable effect on photon propagation. An investigation of this possibility is in progress.

- **TeV-scale gravity:**

  The basic reason observational searches for dimensional flow are so hard, of course, is that the Planck energy so high. One exotic possibility is that in the correct theory of gravity, the true Planck energy might be much lower, perhaps near the TeV scale. Such modifications of general relativity were originally introduced to solve puzzles in high energy physics, and typically require extra dimensions [179, 180], seemingly the opposite of our focus here. But the hints of dimensional reduction in section 3 are largely independent of the number of starting dimensions, so we can begin with more than four dimensions and still ask about dimensional reduction to $d = 2$. Alternatively, we might simply postulate that dimensional reduction takes place for unknown reasons at energies less than E_p, a scenario that might also help with some problems in high energy physics [181, 182].

  With this assumptions, tests become much easier, and some of the limits described above become significant. In fact, there is even a claim of observational evidence for dimensional reduction, in the form of unexpectedly high planarity in secondary particles from cosmic ray collisions in the atmosphere [183]. Accelerator tests have been proposed [182–185], based on several different models of dimensional reduction, though so far no relevant deviations from standard physics have been found.

6 Implications for physics

I will close with some speculation about the possible implications of short distance dimensional reduction for the future. Let us assume for the moment that the phenomenon is real, and that it takes place near the Planck scale. What does this mean for physics?

Of course, much of the answer depends on the details. The two possible common threads of section 4, scale invariance and asymptotic silence, suggest rather different directions for physics. Still, there are a few "universal" characteristics.

First, dimensional reduction would almost certainly have important implications for cosmology. One can invent exceptions: a contracting universe that "bounces" well before reaching Planck scale, for instance, might never reach a scale at which dimensional reduction is important. In most cosmological models, though, the very early universe would have a dimension smaller than four. This would be reflected not only in the geometry, but in the quantum vacuum, equations of state, and thermodynamic relations. Whether these effects would now be observable would depend on the amount of inflation—enough inflation can wash out almost any microscopic signature—but certainly any cosmology that hopes to deal with the birth of the universe would have to incorporate the fact that the newborn universe started with a different number of dimensions.

Second, dimensional reduction would almost certainly affect our understanding of black holes [186–188]. In particular, black hole thermodynamics depends strongly on dimension, and the dynamics of evaporation would certainly be affected by dimensional reduction near the horizon. This could have implications for astronomy, influencing estimates of the densities of primordial black holes. It would certainly have implications for fundamental issues like the information loss problem, which depend strongly on the dimension-dependent behavior of black holes in their final stages of evaporation.

Third, it is conceivable that dimensional reduction could profoundly affect our understanding of quantum field theory. The divergences of QFT are dimension-dependent, and there have been suggestions that dimensional reduction to $d = 2$ at high energies could eliminate ultraviolet divergences [132, 133, 181, 189]. Like most work in this area, these ideas have not yet been developed very deeply, but they could prove important.

Finally, dimensional reduction would surely affect the way we look at quantum gravity. At the least, it would suggest new formulations in which our four (or more) macroscopic dimensions are split into two sectors with different characteristic length scales. If, in addition, the dimension at the smallest scales is $d = 2$, as suggested by many of the approaches of section 3, this might allow us to employ the powerful methods of two-dimensional conformal field theory [190].

As noted in [191], a number of authors have looked at the effect of such a splitting of spacetime, although in the rather different setting of high energy (eikonal) scattering [192–194]. They begin with a metric that can be written locally as

$$ds^2 = \ell_\|^2 g_{\mu\nu} dx^\mu dx^\nu + \ell_\perp^2 h_{ij} dx^i dx^j \tag{6.1}$$

with the added condition that "transverse" derivatives ∂_i are small. The Einstein-Hilbert action is then approximately

$$I \sim \frac{\ell_\perp^2}{\ell_p^2} \int d^2x d^2y \sqrt{h} \left(\sqrt{-g} R_g + \frac{1}{4} \sqrt{-g} g^{\mu\nu} \partial_\mu h_{ij} \partial_\nu h_{kl} \epsilon^{ik} \epsilon^{jl} \right) , \tag{6.2}$$

an expression that looks very much like a two-dimensional action for the transverse metric h_{ij}. The action (6.2) is not quite conformally invariant: the trace of its effective stress-energy tensor is $T \sim \Box\sqrt{h}$, while conformal invariance would require $T = 0$. But the deviation can be small, for instance falling to zero at small t in Kasner space. Using a trick introduced by Solodhukin [195], we can extract an effective scalar field by setting $h_{ij} = (1+\varphi)\sigma_{ij}$ with $\det\sigma_{ij} = 1$. To lowest order, φ then turns out to be a two-dimensional Liouville field with a central charge $c \sim A_\perp/\ell_p^2$. This is still far from the whole story—in the language of section 5, the metric (6.1) exhibits a "systematic" breaking of Lorentz invariance, with a global choice of longitudinal and transverse directions—but it offers an interesting place to start. A more general "2+2" splitting can be found in [196], though the dual length scales have not yet been incorporated. It may also be possible to use the general multifractional techniques of Calcagni described in section 3.11.

Or perhaps we should be more radical. In [148] it is proposed that we should start with a fundamental conformally invariant two-dimensional theory, from which the added dimensions of spacetime emerge after conformal symmetry-breaking. This is so far only a dream (though it is oddly reminiscent of the Polyakov approach to string theory [197]), but it could be a dream worth pursuing.

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