Clock Time in Quantum Cosmology

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We consider the conditioning of the timeless solution to the Wheeler-DeWitt equation by a predefined matter clock state in the simple scenario of de Sitter universe. The resulting evolution of the geometrodynamical degree of freedom with respect to clock time is characterized by the "Berry connection" of the reduced geometrodynamical space, which relies on the coupling of the clock with the geometry. When the connection vanishes, the standard Schrödinger equation is obtained for the geometry with respect to clock time. When one considers environment-induced decoherence in the semi-classical limit, this condition is satisfied and clock time coincides with cosmic time. Explicit results for the conditioned wave functions for minimal clocks made up of two quantum harmonic oscillator eigen-states are shown.

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I. INTRODUCTION

In the canonical approach to relativistic and non-relativistic quantum mechanics, where one supposes a background classical geometry, a classical time sticks out in the Schrödinger (functional) equation as a external parameter used by the observer. The apparent lack of an external time for the dynamics of canonical quantum gravity, the so called problem of time put forth by the Wheeler-DeWitt (WD) equation [1]

$$\widehat{H}\Psi = 0\,, \tag{1}$$

has been therefore one of the elements stimulating the research concerning the nature of time and towards understanding how the evolution of the quantum state of the universe Ψ can be described when time and space themselves become dynamical variables [2–6].

Different ways have been produced to try and rescue a effective time variable in the canonical picture of gravity. In one class of such attempts, one tries to extract a physical variable t to be used as a effective time and obtain a Schrödinger-like structure for the Hamiltonian H

$$\widehat{H}\Psi = \left(\widehat{h} - i\partial_t\right)\Psi = 0\,. \tag{2}$$

The structure $\widehat{h} + \widehat{p}_t$, $\widehat{h}$ being a physical Hamiltonian describing the evolution with respect to time t and $\widehat{p}_t$ being the momentum canonically conjugated to time, is characteristic of time-reparametrized Hamiltonians in the classical theory, i.e. Hamiltonians where coordinate time t has been promoted to a dynamical variable $t(\theta)$ dependent on some implicit, unobservable time θ. The absence of a structure like (2) with respect to time in canonical gravity, already at the classical level, is due to the fact that the theory is built on space-time diffeomorphism invariance to begin with and the attempts to try and identify a canonical momentum to use as a generator of time translations is plagued by difficulties of various kinds and degrees (see e.g. [7–9]).

A rather general idea to extract dynamics from a seemingly stationary system has been proposed in what is sometimes called the Page-and-Wootters (PaW) approach or Conditional Probability Interpretation (CPI) [10, 11], where "time" evolution is read, under the condition that the total state Hamiltonian

$$\widehat{H} = \widehat{H}_C \otimes 1_R + 1_C \otimes \widehat{H}_R \tag{3}$$

is constrained, in the quantum correlations between the two partition of total state, i.e. a clock state (C) and the physical state (R) (the "rest of the universe") entangled with it. Time evolution emerges through the measurement of a clock observable of some kind, whose reading conditions the physical state. The attention to this approach seems to have gone through a recent revival, especially after some of the most important criticisms moved to it in the past have been addressed, followed by a experimental illustration of the mechanism [12–14]. The actual application of the CPI to timelessness in canonical quantum gravity, which originally motivated it, still seems to be lacking.

One important requirement that is generally cast on a good clock is that it weakly interacts with the system it describes the evolution of. When we take gravity to partake in the dynamics, though, it couples to all forms of energy, including the clock, and while the coupling might become weak for some chosen cases (e.g. when the clock has conformal coupling to a conformally invariant geometry) or approximations, it may generally be strong or even dominant in the quantum regime. On the other hand, when we consider how classical time emerges from the timeless WD equation (1) in the semi-classical limit, we see that it comes to be defined as the parameter along the classical trajectories of the gravitational degrees of freedom in the WKB approximation. The origin of the classical time of the (functional) Schrödinger equation lies therefore precisely in the coupling between geometry and matter fields.

In the present work, rather than trying to recover a equation of the Schrödinger type (2) or a notion of time through dynamical observables, we study how, starting from a definition of the clock time, the resulting conditioned state for the geometry evolves with respect to it in the general case where coupling cannot be neglected. We will treat the simple example of the mini-superspace model of a de Sitter universe with a homogeneous massive scalar field minimally coupled to gravity. In Section II we introduce the mini-superspace model and expand the solution of the WD equation in the eigen-states of the matter Hamiltonian. In Section III we discuss how time emerges in the semi-classical limit of gravity and how it appears for the conditioning by a predefined matter clock in the quantum regime. In Section IV we show the explicit solution for minimal working clocks. Final observations and conclusions are provided in Section V.

II. MINI-SUPERSPACE MODEL

We will consider the simple scenario of the minisuperspace for a spatially flat FLRW universe of scale factor a

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \tag{4}$$

with a minimally coupled scalar field ϕ and a cosmological constant Λ. The classical action is [1]

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\,(R - 2\Lambda) + \int d^4x \sqrt{-g} \left(-\frac{1}{2} (\partial_\mu \phi)^2 - U(\phi) \right) . \tag{5}$$

The Lagrangian is

$$L = \frac{V}{\kappa^2} \left(-\frac{3a\dot{a}^2}{N} - N a^3 \Lambda \right) + V a^3 \left(\frac{\dot{\phi}^2}{2N} - NU \right) \tag{6}$$

$$= \frac{1}{\kappa^2} \left(-\frac{\dot{\rho}^2}{3N\rho} - N\rho\Lambda \right) + \rho \left(\frac{\dot{\phi}^2}{2N} - NU \right), \tag{7}$$

where $\dot{}\, := d/dt$, V denotes the comoving volume of the universe and we introduced the physical volume of the universe $\rho := V a^3$ as a dynamical variable. The canonical momenta are

$$\pi_\rho = \frac{\partial L}{\partial \dot{\rho}} = -\frac{1}{\kappa^2} \frac{2\dot{\rho}}{3N\rho}, \quad \pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\rho \dot{\phi}}{N} . \tag{8}$$

The Hamiltonian reads

$$H = \pi_\rho \dot{\rho} + \pi_\phi \dot{\phi} - L = N \left(H_\rho + H_\phi \right), \tag{9}$$

where

$$H_\rho := \rho \left(-\frac{3\kappa^2}{4} \pi_\rho^2 + \frac{\Lambda}{\kappa^2} \right), \quad H_\phi := \frac{\pi_\phi^2}{2\rho} + \rho U(\phi). \tag{10}$$

The metric is

$$ds^2 = -N^2 dt^2 + (\rho/\rho_0)^{2/3} \delta_{ij} dx^i dx^j, \quad \rho_0 := V . \tag{11}$$

The first class Hamiltonian constraint

$$\delta H / \delta N = H_\rho + H_\phi = 0 \tag{12}$$

becomes

$$\frac{1}{3} \left(\frac{\dot{\rho}}{N\rho} \right)^2 = \Lambda + \kappa^2 \left[\frac{\rho^2}{2} \left(\frac{\dot{\phi}}{N} \right)^2 + U(\phi) \right] , \tag{13}$$

which corresponds to the Friedmann equation for the universe with a minimal scalar field if we choose $N = 1$.

Constraint quantization of (12) for a physical state Ψ gives the WD equation

$$(\widehat{H}_\rho + \widehat{H}_\phi)\Psi = 0 . \tag{14}$$

In the representation diagonalizing (ρ, ϕ), we have the WD equation for the wave function $\Psi(\rho, \phi)$:

$$\left[\rho \left(\frac{\partial^2}{\partial \rho^2} + \lambda \right) + \frac{4}{3\kappa^2} \left(-\frac{1}{2\rho} \frac{\partial^2}{\partial \phi^2} + \rho U(\phi) \right) \right] \Psi(\rho, \phi) = 0 , \tag{15}$$

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[1] $\hbar = 1 , \quad \kappa^2 = 8\pi G = \frac{1}{M_p^2}$.

where $\lambda = 4\Lambda/(3\kappa^4)$ and we have assumed appropriate ordering of the operator $\widehat{\pi}_\rho$ for simplicity of analysis.

For the free massive field $U(\phi) = \mu^2\phi^2/2$, the matter Hamiltonian is

$$\widehat{H}_\phi = -\frac{1}{2\rho}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2}\,\rho\,\mu^2\phi^2\,. \tag{16}$$

When we consider the eigen-states of the Hamiltonian $\widehat{H}_\phi$, the variable ρ contained in $\widehat{H}_\phi$ can be treated as an unknown external parameter and the eigenvalue equation is

$$\widehat{H}_\phi\,\chi_n(\phi|\rho) = E_n\,\chi_n(\phi|\rho)\,, \tag{17}$$

where $\chi_n(\phi|\rho)$ are the energy eigen-states of $\widehat{H}_\phi$ for any value of ρ and the energy eigen-states can be determined as

$$\chi_n(\phi|\rho) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,(\mu\rho)^{\frac{1}{4}}\,e^{-\mu\rho\,\phi^2/2}\,H_n\left(\sqrt{\mu\rho}\,\phi\right). \tag{18}$$

$H_n(x)$ are the Hermite polynomials and the associated eigenvalues are

$$E_n = \mu\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \cdots. \tag{19}$$

The eigen-modes χ_n satisfy the orthogonality and completeness relations

$$(\chi_m, \chi_n) := \int_{-\infty}^{\infty} d\phi\,\chi_m^*(\phi|\rho)\,\chi_n(\phi|\rho) = \delta_{mn}\,, \tag{20}$$

$$\sum_k \chi_k^*(\phi_1|\rho)\chi_k(\phi_2|\rho) = \delta(\phi_1 - \phi_2)\,. \tag{21}$$

Then we can expand the universal wave function $\Psi(\rho, \phi)$ as

$$\Psi(\rho, \phi) = \sum_n \psi_n(\rho)\chi_n(\phi|\rho)\,. \tag{22}$$

The function $\psi_n(\rho)$ represents one component of the quantum state of the universe and its form is determined by the substituting (22) in the WD equation (15)

$$\sum_n \left[\rho\left(\partial_\rho^2\psi_n + \lambda\,\psi_n\right)\chi_n + \frac{4}{3\kappa^2}\psi_n\,\widehat{H}_\phi\chi_n\right] = -\sum_n \rho\left[2\partial_\rho\psi_n\,\partial_\rho\chi_n + \psi_n\,\partial_\rho^2\chi_n\right]. \tag{23}$$

After taking inner product with χ_m, the wave function of the universe ψ_n obeys

$$\sum_n \left[\rho\sum_k \widehat{\mathcal{D}}_{mk}\widehat{\mathcal{D}}_{kn} + \left(\rho\lambda + \frac{4E_m}{3\kappa^2}\right)\delta_{mn}\right]\psi_n = 0\,, \tag{24}$$

where the covariant derivative is introduced as

$$\widehat{\mathcal{D}}_{mn} := \delta_{mn}\partial_\rho - iA_{mn}\,, \tag{25}$$

with the "Berry" connection

$$A_{mn} := i(\chi_m, \partial_\rho\chi_n)\,. \tag{26}$$

As a result of the connection, different components ψ_n of the expansion (22) become generally coupled to each other. For each component, the connection leads to a geometric phase and the formal solution of (24) is given by [2]

$$\psi_n(\rho) = \sum_m \left[P\exp\left(i\int d\rho A(\rho)\right)\right]_{nm} b_m G_m(\rho)\,, \tag{27}$$

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[2] For a function $\psi = BG$, $(\partial_\rho - iA)\psi = (\partial_\rho B - iAB)G + B\partial_\rho G = B\partial_\rho G$ for $\partial_\rho B = iAB$. The solution of this condition is $B = P\exp\left(i\int d\rho A(\rho)\right)$.

where b_m are constants, the symbol P denotes a path ordering and the functions G_n satisfy

$$-\widehat{H}_\rho G_n = E_n G_n \tag{28}$$

which gives

$$G_n = \rho\, e^{-i\sqrt{\lambda}\,\rho}\,{}_1F_1[1 + i\beta_n/\sqrt{\lambda}, 2, 2i\sqrt{\lambda}\rho]\,, \quad \beta_n = \frac{4\mu}{3\kappa^2}(n+1/2)\,, \quad n = 0\,, 1\,, 2\,, \ldots \tag{29}$$

${}_1F_1$ being Kummer's confluent hypergeometric function.

III. EMERGENCE OF TIME

A. WKB time

In the semi-classical limit, time naturally appears as a parameter along the superspace trajectories of the spatial geometry. One considers the WKB ansatz for the wave function [15, 16]

$$\Psi(\rho,\phi) = \exp\left[i\sum_{n=0}^{\infty}\left(\frac{3\kappa^2}{2}\right)^{n-1} S_n(\rho,\phi)\right]. \tag{30}$$

By substituting in (15) and equating each order of $\left(3\kappa^2/2\right)^p$ one obtains

$$[p=-2]: \quad \partial_\phi S_0 = 0\,, \tag{31}$$

$$[p=-1]: \quad \left(\partial_\rho S_0\right)^2 = 3\Lambda\,, \tag{32}$$

$$[\ p=0\ ]: \quad \left(-\frac{1}{2\rho}\frac{\partial^2}{\partial\phi^2} + \rho U\right) e^{iS_1} = \rho\partial_\rho S_0 \partial_\rho S_1\ , \tag{33}$$

$$\ldots$$

Equations (31) and (32) give $S_0 = S_0(\rho)$ and the Einstein-Hamilton-Jacobi equation for de Sitter space, respectively. Then introducing the WKB time as a parameter along the classical trajectories of $\rho(\tau)$

$$\partial_\tau := -\rho(\partial_\rho S_0)\partial_\rho = -\rho\pi_\rho\partial_\rho\,, \tag{34}$$

equation (33) gives the functional Schrödinger equation

$$\mp i\partial_\tau\chi = \widehat{H}_\phi\chi \tag{35}$$

for the matter wave functional $\chi := e^{iS_1}$ on the classical de Sitter background. Ambiguity in the sign of (34), which determines the direction of the cosmological arrow of time, corresponds to the choice of sign in (32), where positive sign gives the contracting de Sitter universe and negative sign the expanding one. These correspond also to the choice of G_n rather than G_n^* as a fundamental solution in (28). In the semi-classical approximation, the total wave function is

$$\Psi(\rho,\phi) \propto e^{\pm i\sqrt{\lambda}\rho}\chi(\rho,\phi)\,. \tag{36}$$

B. Scalar field as a clock

Classical (WKB) time (34) relies on the coupling between geometry and matter and on the existence of trajectories for the geometrodynamical degrees of freedom. Therefore it cannot be extended straightforwardly to the full quantum regime. In the present approach we start with the definition of a clock time through the scalar field ϕ, which we regard as a clock. More specifically, given the Hamiltonian $\widehat{H}_\phi$, we define a "clock time" T as a parameter of the following normalized clock wave function

$$\widetilde{\chi}(T,\phi|\rho) = \sum_n c_n\, e^{-iE_nT}\chi_n(\phi|\rho)\,, \quad \sum_n c_n^* c_n = 1\,, \tag{37}$$

which satisfies the standard Schrödinger equation

$$i\partial_T \widetilde{\chi} = \widehat{H}_\phi\, \widetilde{\chi}\,. \tag{38}$$

By quantum estimation of the time parameter T of the clock $\widetilde{\chi}(T,\phi|\rho)$, the quantum state of the geometry conditioned by the clock reading T will effectively be described by

$$\widetilde{\psi}(\rho,T) := (\widetilde{\chi}(T,\phi|\rho), \Psi(\rho,\phi,T)) = \sum_m c_m^* e^{iE_m T}\psi_m(\rho) = \sum_{m,n} c_m^* b_n e^{iE_m T} B_{mn} G_n(\rho) \tag{39}$$

where we have defined for simplicity of notation the matrix of elements

$$B_{mn}(\rho) := \left[P \exp\left(i\int d\rho A(\rho)\right)\right]_{mn}\,. \tag{40}$$

The exact form of the connection is

$$A_{mn}(\rho) = i(\chi_m, \partial_\rho \chi_n) = i\frac{\alpha_{mn}}{4\rho}\,, \tag{41}$$

where

$$\begin{aligned}\alpha_{mn} &:= 4\rho \int_{-\infty}^{\infty} d\phi\, \chi_m^*(\phi|\rho)\, \partial_\rho \chi_n(\phi|\rho) \\ &= \sqrt{n(n-1)}\, \delta_{m,n-2} - \sqrt{(n+2)(n+1)}\, \delta_{m,n+2}\,. \end{aligned}\tag{42}$$

Notice that for the present choice of Hamiltonian the energy eigenvalues are independent on the parameter ρ and therefore are not affected by the derivative in the connection. The integral of the connection is

$$\int_{\rho_0}^{\rho} d\rho' A_{mn}(\rho') = i\frac{\alpha_{mn}}{4}\ln(\rho/\rho_0)\,. \tag{43}$$

where we may identify the arbitrary scale ρ_0 with the comoving volume of the universe introduced before. If we set this scale, for example, as the Planck scale, then $\rho < \rho_0$ belong to the sub-Planckian, strong quantum regime.

C. Evolution equation

The time evolution of state $\widetilde{\psi}(\rho,T) = \sum_n c_n^* e^{iE_n T}\psi_n$ is determined implicitly through its conditioning by the clock and is not generally of the Schrödinger type with the pure geometrodynamical Hamiltonian H_ρ. To derive the explicit dynamic law, one may start by noticing that

$$\frac{1}{\Delta T}\int_T^{T+\Delta T} dT' e^{i\mu n T'} = \delta_{n0}\,, \quad \Delta T := \frac{2\pi}{\mu}\,, \tag{44}$$

and therefore

$$c_n^*\psi_n(\rho) = \frac{1}{\Delta T}\int_T^{T+\Delta T} dT' e^{-iE_n T'}\widetilde{\psi}(\rho,T')\,, \tag{45}$$

$$c_n^* E_n \psi_n(\rho) = -\frac{i}{\Delta T}\int_T^{T+\Delta T} dT' e^{-iE_n T'}\widetilde{\psi}(\rho,T')\partial_{T'}\widetilde{\psi}(\rho,T')\,. \tag{46}$$

This allows to write

$$\begin{aligned}\widetilde{\psi}(\rho,T) &= \sum_n c_n^* e^{iE_n T}\psi_n &(47)\\ &= \frac{1}{\Delta T}\sum_n \int_T^{T+\Delta T} dT' e^{-iE_n(T-T')}\widetilde{\psi}(\rho,T')\,. &(48)\end{aligned}$$

A representation of the delta function in the interval $[T, T+\Delta T]$ is therefore

$$\delta(T'-T) := \frac{1}{\Delta T}\sum_n e^{-iE_n(T'-T)}\,. \tag{49}$$

Using the previous relations and assuming $c_n \neq 0\ \forall n$, the WD equation (24) gives the evolution equation of $\widetilde{\psi}$

$$i\partial_T\widetilde{\psi}(\rho,T) = \widehat{H}_\rho\widetilde{\psi}(\rho,T) + \frac{3\kappa^2}{4\Delta T}\rho\sum_{m\neq n}\frac{c_m^*}{c_n^*}\int_T^{T+\Delta T} dT' e^{i(E_mT-E_nT')}[\widehat{\mathcal{D}}^2]_{mn}\widetilde{\psi}(\rho,T')\,. \tag{50}$$

Although the obtained equation has the structure of a differential-integral equation, when $[\mathcal{D}^2]_{mn}$ has only diagonal non-vanishing components (the connection is zero), the Schrödinger equation with Hamiltonian $\widehat{H}_\rho$ is recovered for $\widetilde{\psi}$.

The first term of the RHS of (50) is inherited by conditioned state independently on the coupling between the clock and the geometry and corresponds to the mechanism discussed by Page and Wootters. On the other hand, the second term depends on the coupling through the connection in the geometric phase term. Notice that vanishing of this term does not necessarily require the coupling to be absent and one may obtain for $\widetilde{\psi}$ the Schrödinger evolution generated by $\widehat{H}_\rho$ also when the coupling is strong. On the other hand, recovery of semi-classical time (WKB time) itself requires that a coupling between matter and geometry exists.

In the semi-classical expansion of $\Psi(\rho,\phi)$ solving the WD equation, the geometrodynamical variable ρ and matter variable ϕ are treated respectively as "heavy" and "fast" degrees of freedom of the system. The evolution of the heavy degrees of freedom is described by the classical Einstein-Hamilton-Jacobi equation (32) for the chosen metric, while the matter field describes quantum perturbations of the system. For a cyclic evolution of ρ, the various components of (43) become contributions to the so-called Berry phase acquired by the system, which has been discussed both in the context of non-relativistic quantum mechanics, where it was first introduced, as well as in quantum cosmology [17–21]. In the present case, though, the relevance of the phase (43) originates from the fact that it determines the coupling of different components of the expansion (22) and the time evolution law of the geometrodynamical state.

When one takes into account the necessary coupling of gravity with "environmental" degrees of freedom [22], the fast energy eigen-modes in the expansion (22) decohere from each other and one can neglect the off-diagonal elements of (26), in which consists the so called "Born-Oppenheimer approximation". Since the coherence between different components of the clock decays, the clock cannot be used any longer to track time as the quantum superposition is destroyed and the time-dependent relative phases between distinct energy eigen-states is lost. This is reflected in the decoupling of the different indexes of ψ_n in (24) and the diagonalization of (40). In this limit, we obtain a effective Schrödinger equation for each component of $\widetilde{\psi}$

$$\widehat{H}_\rho\psi_n = -E_n\psi_n \tag{51}$$

which is equivalent to the equation defining G_n in the timeless picture (28). For a given n-branch with $E_n := E$, the solution is $\widetilde{\psi} \approx \exp\left[iET + iS(E,\rho)\right]$ and the classical trajectory is recovered by the condition

$$\text{const.} = \partial_E\left(ET+S\right) = T + \partial_E S(E,\rho)\,, \tag{52}$$

from which the kinematic expression for $\rho = \rho(T)$ can be obtained. For the present case, it is possible to check that we obtain the classical equation for ρ with pure cosmological constant with $N=1$ slice and clock time T corresponds to cosmic time.

IV. SOLUTION OF THE WD EQUATION WITH CLOCK

We will consider different clock models and show the behavior of the corresponding conditioned state $\widetilde{\psi}$.

A. Clock with a single eigen-state

To start, let us consider the total wave function with only the m-th component

$$\Psi(\rho,\phi) = \psi_m(\rho)\chi_m(\rho,\phi)\,. \tag{53}$$

For this state the connection vanishes $A = A_{mm} = 0$ and the wave function of the universe conditioned by the clock is simply

$$\widetilde{\psi}(T,\rho) = e^{i\mu(m+1/2)T} G_m(\rho)\,. \tag{54}$$

Thus the wave function has the trivial time dependence given by a overall phase factor, which is not measureable. The disappearance of clock time for the case of a single energy eigen-state reflects the fact that a working clock needs a superposition of at least two energy eigen-states to track the time evolution.

B. Clock with two eigen-states

Let us the consider the minimal case of a working clock, made up of the only two eigen-states 0 and 1 ($c_0 = c_1 = 1/\sqrt{2}$). The timeless wave function will be

$$\Psi(\rho,\phi) = \frac{1}{\sqrt{2}}\left(\psi_0(\rho)\chi_0(\phi|\rho) + \psi_1(\rho)\chi_1(\phi|\rho)\right)\,. \tag{55}$$

Notice that when looking for the solutions ψ_n (27) satisfying (24) we have assumed a expansion including all the eigen-states. This allows the use of the completeness relation to grant the vanishing of the extra ρ^{-2}-order term $\sum_k(\partial_\rho\chi_m,\chi_k)(\chi_k,\partial_\rho\chi_n) - (\partial_\rho\chi_m,\partial_\rho\chi_n)$. We may consider the present case of a finite number of energy eigen-states as an approximation where all other coefficients c_m are negligibly small and can be dropped from the equation. Also in this case, the connection A_{nm} ($n,m=0,1$) vanishes and $[e^{i\int d\rho A}]_{nm} = \mathbb{1}_{nm}$. The conditional state becomes

$$\widetilde{\psi}(T,\rho) = \frac{e^{i\mu T/2}}{\sqrt{2}}\left(G_0(\rho) + e^{i\mu T}G_1(\rho)\right)\,, \tag{56}$$

where the relative phase makes the time dependence observable.

As discussed in the previous section, the simple time dependence of $\widetilde{\psi}(T,\rho)$ follows from the diagonality of (40). For the choice of matter Hamiltonian (16), diagonality is granted for $m \neq n \pm 2$. The simplest example in which this condition is not satisfied and $\widetilde{\psi}(T,\rho)$ takes a more complex time evolution is the case of eigen-states $n,m = 0,2$

$$\Psi = \frac{1}{\sqrt{2}}\left(\psi_0(\rho)\chi_0(\phi|\rho) + \psi_2(\rho)\chi_2(\phi|\rho)\right)\,. \tag{57}$$

The integral of the connection becomes

$$\int_{\rho_0}^{\rho} d\rho' A(\rho') = \frac{i}{\sqrt{8}}\ln(\rho/\rho_0)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{58}$$

and

$$B = \begin{pmatrix} \cos\left(\frac{\ln(\rho/\rho_0)}{\sqrt{8}}\right) & \sin\left(\frac{\ln(\rho/\rho_0)}{\sqrt{8}}\right) \\ -\sin\left(\frac{\ln(\rho/\rho_0)}{\sqrt{8}}\right) & \cos\left(\frac{\ln(\rho/\rho_0)}{\sqrt{8}}\right) \end{pmatrix}, \tag{59}$$

which is independent on the path ordering since the dependence on ρ appears in the connection integral as a overall multiplicative factor. The wave function of the universe conditioned by the clock (taking $b_n = 1/\sqrt{2}$) becomes

$$\widetilde{\psi}(T,\rho) = \frac{e^{i\mu T/2}}{2}\left[\left(B_{00}(\rho) + e^{2i\mu T}B_{20}(\rho)\right)G_0(\rho) + \left(B_{02}(\rho) + e^{2i\mu T}B_{22}(\rho)\right)G_2(\rho)\right]\,. \tag{60}$$

For the chosen clock Hamiltonian, the general time evolution (50) is simply

$$i\partial_T\widetilde{\psi}(\rho,T) = \widehat{H}_\rho\widetilde{\psi}(\rho,T) + \underbrace{\frac{3\kappa^2}{8}\left(\partial_\rho - \frac{1}{2\rho}\right)\sum_{m\neq n} c_m^* e^{iE_mT}\alpha_{mn}\psi_n(\rho)}_{X(\rho,T)}\,. \tag{61}$$

While the state (56) follows exactly the Schrödinger equation with Hamiltonian $\widehat{H}_\rho$, for the time evolution of the state (60) the second term X of (61) will not vanish generally.

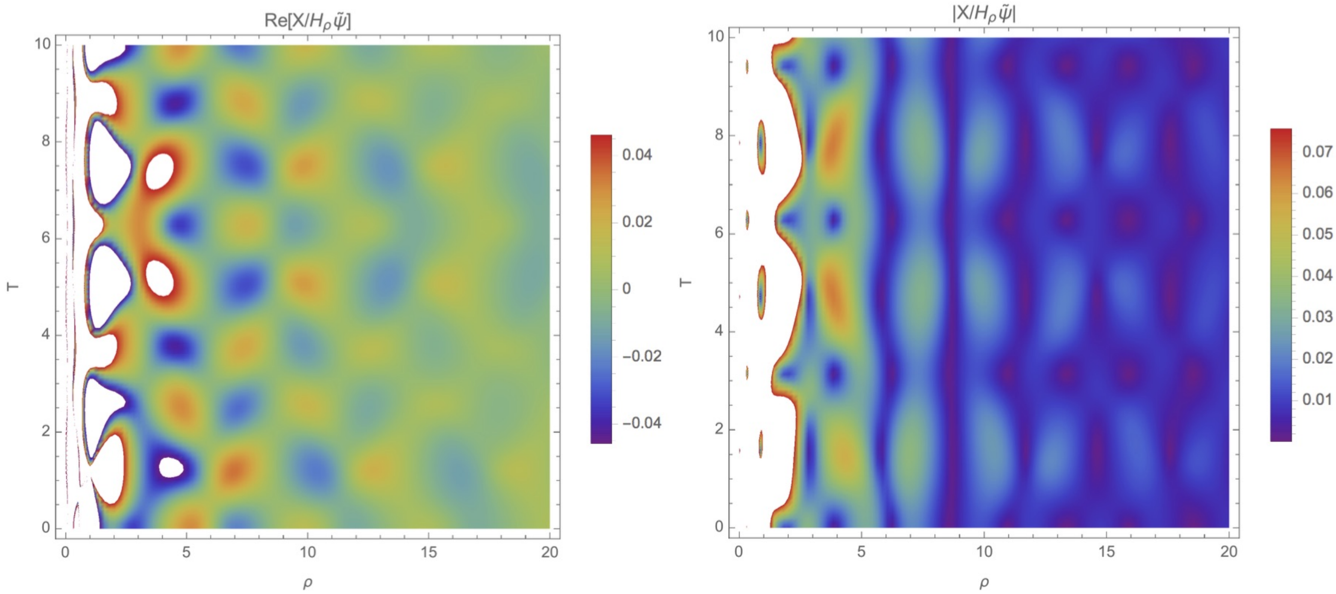

FIG. 1. Plot of $\mathrm{Re}[X/H_\rho\widetilde{\psi}]$ (left) and $|X/H_\rho\widetilde{\psi}|$ (right) for unitary values of the physical constants $\lambda = \mu = \kappa = \rho_0 = 1$ when the $\{0,2\}$-clock is employed.

The behavior of the ratio $X/H_\rho\widetilde{\psi}$ for the $\{0,2\}$-clock is shown in Fig.1. The values of $X(\rho,T)$ and $H_\rho\widetilde{\psi}$ are undetermined at $\rho = 0\,\forall T$ due to the "rotation" $B(\ln(\rho/\rho_0)/\sqrt{8})$, whose argument diverges at $\rho = 0$. In this regime, (61) diverges from the Schrödinger evolution. On the other hand, for

$$\rho \gg \lambda^{-\frac{1}{2}} = \ell_P^2\ell_H \tag{62}$$

where $\ell_H^{-1} := 2\sqrt{\Lambda/3}$ is the de Sitter horizon scale and $\ell_P = \kappa$ is the Planck length, the ratio becomes negligibly small. In this limit, $G_m(\rho)$ can be approximated as

$$G_m(\rho) \approx g_m e^{i\left(\sqrt{\lambda}\rho + \frac{\beta_m}{\sqrt{\lambda}}\log\left(\sqrt{\lambda}\rho\right)\right)}\,, \quad g_m = \frac{1}{4\sqrt{\lambda}}\frac{(2i)^{1+i\frac{\beta_m}{\sqrt{\lambda}}}}{\Gamma\left(1+i\frac{\beta_m}{\sqrt{\lambda}}\right)}\,. \tag{63}$$

On the other hand, $\partial_\rho B$ can be neglected in this limit and, neglecting the other contributions of order ρ^{-1}, $X(\rho,T)$ will be approximated as a oscillatory term of order κ^2

$$X(\rho,T) \approx i\frac{3\kappa^2}{8}\sqrt{\frac{\lambda}{2}}\left(\left(B_{22} - e^{i2\mu T}B_{02}\right)e^{i\frac{\beta_2}{\sqrt{\lambda}}\log\left(\sqrt{\lambda}\rho\right)}g_2 + \left(B_{20} - e^{i2\mu T}B_{00}\right)e^{i\frac{\beta_0}{\sqrt{\lambda}}\log\left(\sqrt{\lambda}\rho\right)}g_0\right)e^{i\left(\frac{\mu}{2}T + \sqrt{\lambda}\rho\right)}\,, \tag{64}$$

which can be neglected compared to the growing contribution of the geometrodynamical Hamiltonian. Therefore, in the limit (62) the Schrödinger equation is effectively recovered using this clock.

V. CONCLUSIONS

We have discussed the time dependence of the quantum state of the geometry obtained by conditioning the timeless solution of the Wheeler-DeWitt equation with a predefined clock state. The resulting time dependence is generally not trivial and the coupling between the clock and the geometry affects the quantum state through a geometric phase which couples different components of the expansion of the timeless state in the basis of the energy eigen-states of the clock. We have derived a evolution law for the geometry with respect to clock time which holds also when the coupling between the clock and the geometry cannot be neglected. A standard Scrödinger-type evolution generated by the geometrodynamical Hamiltonian is always recovered when the off-diagonal elements of the Berry connection, which determines the geometric phase, vanish. In the semi-classical limit when environment-induced decoherence is taken into account, the off-diagonal elements of the connection can effectively be neglected as well and the different components of the clock decohere from each other. The disappearance of quantum superposition between different energy eigen-states of the clock makes it impossible to use it to track time: in this limit observers will rely on the classical time parameter, which emerges in the WKB approximation of the total wave function. We have shown the explicit result for minimal working clocks made up of two harmonic oscillators. For a clock made by a superposition of the ground state and the first energy level (or indeed any two states which are not separated by two energy levels) the Schrödinger evolution is retrieved exactly. Using a superposition of the ground state and the second energy level harmonic oscillator the evolution of the quantum wave function of the geometry presents non-trivial deviation from the Schrödinger-type evolution generated by the pure geometrodynamical Hamiltonian. The Schrödinger equation is approximately recovered also for this clock for physical volume much larger than the scale $\ell_P^2 \ell_H$.

Although we have considered the simple case of a de Sitter mini-superspace, the extension to other geometries and different matter Hamiltonians for the clock may proceed along the same lines.

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