Evidence of time evolution in quantum gravity

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It is argued that the problem of time is not a crucial issue inherent in the quantum evolution of the universe. On the minisuperspace model example with the massless scalar field, it is shown that at least four ways of the quantum evolution description give similar results explicitly. Which of these methods will be suitable for the case of a full quantum theory of gravity is discussed.

I. INTRODUCTION

Usually, some crucial theoretical problems are self-created in some sense, and then these issues were solved successfully during some period. An example could be the spin crisis problem, which had been stating about 30 years [1]. The problem of time [2–6] holds a relatively long time from [7] and it was related closely with the variety of the points of view to a gravity quantization. The root of this issue is the gauge invariance of the general relativity. Such invariance allows choosing the equivalent time parametrizations, and one may suspect that the time is an "illusion."

On the other hand, the astrophysical data demonstrate that the universe evolves in time. The modern trends in the interpretation of quantum mechanics (e.g., see [8])

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suggest that all the phenomena, including the universe itself, are generally quantum. Thus, the time evolution in the frameworks of quantum cosmology has to been explained.

Although eternity and time are two sides of one coin, all the experimental observations are performed in the time. Thus, one needs to introduce time into a theory, in any case, to confront theory with the observations. However, sometimes, it could be useful to think in terms of eternity for the development of theoretical concepts *sub specie aeternitatis*.

The complexity of the full system of the equations for gravity does not prevent to consider this problem on example of the so-called minisuperspace models [9], which are extremely simple but have the Hamiltonian constraint like that in the general case.

Here we show that the problem of time does not prevent doing the concrete calculations of the time-dependent mean values, which could be, in principle, compared with the experimental observations.

II. CLASSICAL PICTURE

As it is well-known, there is no problem with defining time in the classical theory because it implies that if an observer has some particular clock, she can choose a gauge corresponding to this clock.

Let us consider action for gravity and a real massless scalar field ϕ:

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}\, d^4x + \frac{1}{2}\int \partial_\mu\phi\, g^{\mu\nu}\partial_\nu\phi\sqrt{-g}\, d^4x, \tag{1}$$

where R is a scalar curvature.

We restrict the consideration by the uniform, isotropic and flat universe

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(N^2 d\eta^2 - d^2\boldsymbol{r}), \tag{2}$$

where a scale factor a and a lapse function N depend on a conformal time η only.

Under these conditions, the action (1) becomes

$$S = \frac{1}{2}\int \frac{1}{N}\left(-M_p^2 a'^2 + a^2\phi'^2\right) d\eta, \tag{3}$$

where the reduced Planck mass $M_p = \sqrt{\frac{3}{4\pi G}}$ is used[2], which will be set to unity in the further consideration for simplicity.

The action (3) in the generalized form looks as

$$S = \int \left(-p_a a' + \pi_\phi \phi' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right)\right) d\eta, \tag{4}$$

which turns to (3) after variation on π_ϕ and p_a. The explicit expression for the Hamiltonian follows from (4):

$$H = N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right), \tag{5}$$

which is also the Hamiltonian constraint

$$\Phi_1 = -\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2} = 0, \tag{6}$$

due to $\frac{\delta S}{\delta N} = 0$.

Time evolution of an arbitrary quantity is expressed through the Poisson brackets

$$\frac{dA}{d\eta} = \frac{\partial A}{\partial \eta} + \{H, A\}, \tag{7}$$

defined as

$$\{A, B\} = \frac{\partial A}{\partial \pi_\phi}\frac{\partial B}{\partial \phi} - \frac{\partial A}{\partial \phi}\frac{\partial B}{\partial \pi_\phi} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial a} + \frac{\partial A}{\partial a}\frac{\partial A}{\partial p_a}. \tag{8}$$

The full system of the equations of motion has the form:

$$\begin{gathered} \pi_\phi' = -\frac{\partial H}{\partial \phi} = 0, \quad \Longrightarrow \quad \pi_\phi = k = const, \\ \phi' = \frac{\partial H}{\partial \pi_\phi} = \frac{k}{a^2}, \quad a' = -\frac{\partial H}{\partial p_a} = p_a, \quad p_a' = \frac{\partial H}{\partial a} = -\frac{k^2}{a^3}. \end{gathered} \tag{9}$$

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[2] The scale factor a in (3) becomes dimensional because it corresponds, in fact, to $aV^{3/2}$, where V is the volume of spatial integration in (1).

The solution of the equations of motion is

$$a = \sqrt{2|\pi_\phi|\eta}, \qquad\qquad \phi = \frac{\pi_\phi}{2|\pi_\phi|}\ln\eta + const. \tag{10}$$

According to Eq. (10), a gauge fixing condition

$$\Phi_2 = a - \sqrt{2|\pi_\phi|\eta} = 0, \tag{11}$$

which conserves in time, can be introduced in addition to the constraint Φ_1.

One can see that there is an explicit time evolution under some particular gauge fixing. Moreover, for this simple example, the system could be reduced to a single degree of freedom [10, 11].

Let us take π_ϕ and ϕ as the physical variables, then a and p_a have to be excluded by the constraints (6),(11). Substituting p_a, a' and a into (3) results

$$S = \int \left(\pi_\phi\phi' - H_{phys}(\phi, \pi_\phi, \eta)\right) d\eta, \tag{12}$$

where

$$H_{phys}(\phi, \pi_\phi, \eta) = p_a a' = \frac{|\pi_\phi|}{2\,\eta}. \tag{13}$$

III. QUANTUM PICTURES WITH TIME

A. Schrödinger equation with physical Hamiltonian

The most simple and straightforward way to the description of the quantum evolution is based on the Schröodinger equation [10, 11]

$$i\partial_\eta\Psi = \hat{H}_{phys}\Psi \tag{14}$$

with the physical Hamiltonian (13). In the momentum representation, the operators become

$$\hat{\pi}_\phi = k, \qquad \hat{\phi} = i\frac{\partial}{\partial k}. \tag{15}$$

The solution of Eq. (14) is

$$\Psi(k, \eta) = C(k)|2k\eta|^{-i|k|/2}e^{i|k|/2}, \tag{16}$$

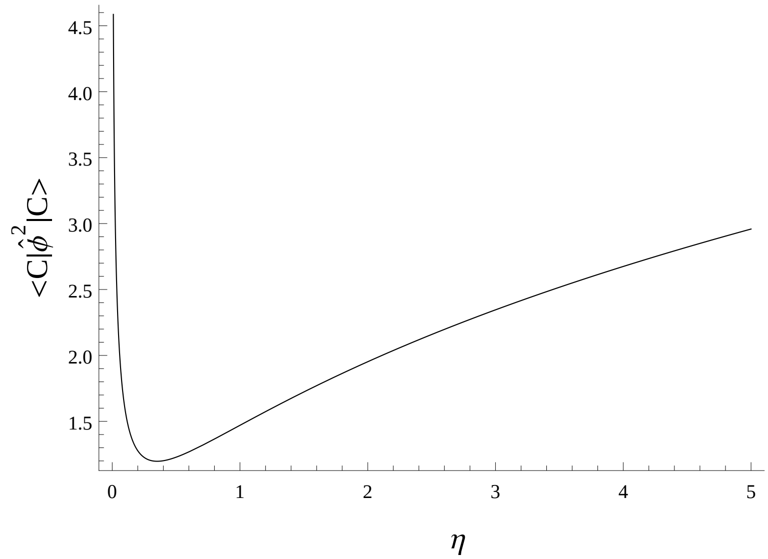

FIG. 1. The mean value of the square of the scalar field over the wave packet (18).

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where $C(k)$ is a momentum wave packet. An arbitrary operator $\hat{A}$ build from $\hat{\phi} = i\frac{\partial}{\partial k}$ and $a = \sqrt{2|k|\eta}$ is, in fact, the function of η, k, $i\frac{\partial}{\partial k}$. Using the wave function (16) allows calculating the mean value

$$< C|\hat{A}|C >= \int \Psi^*(k,\eta)\hat{A}\,\Psi(k,\eta)dk. \tag{17}$$

Since the base wave functions $\psi_k = |2k\eta|^{-i|k|/2}e^{i|k|/2}$ contain the module of k, a singularity may arises at $k = 0$ if $\hat{A}$ contains the degrees of the differential operator $\frac{\partial}{\partial k}$. This may violate hermicity. To avoid this, the wave packet $C(k)$ has to turn to zero at $k = 0$. For instance, it could be taken in the Gaussian form multiplied by k^2

$$C(k) = \frac{4\sigma^5}{3\sqrt{\pi}}k^2 \exp\left(-\frac{k^2}{2\sigma^2}\right). \tag{18}$$

Let us come to the calculation of the concrete mean values taking $\sigma = 1$ for simplicity. Mean values of the operators $\hat{\phi}^2$ and a are

$$< C|a|C >= \frac{4}{3}\sqrt{\frac{2}{\pi}}\sqrt{\eta}\int_{-\infty}^{\infty} e^{-k^2} k^{9/2} dk = \frac{4}{3}\sqrt{\frac{2}{\pi}}\Gamma(11/4)\sqrt{\eta}, \tag{19}$$

$$\begin{aligned} < C|\hat{\phi}^2|C >= \frac{1}{3\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-k^2}\Big(-4k^6 + k^4\left(20 + \ln^2 2 + \ln(\eta\,|k|)\ln(4\eta\,|k|)\right) - \\ 8k^2 + 2i\,|k|^3\left(-2k^2\ln(2\eta\,|k|) + 4\ln(\eta\,|k|) + 4\ln 2 + 1\right)\Big)dk = \\ \frac{1}{12}\ln\eta(3\ln\eta - 3\gamma + 8) + \frac{\pi^2}{32} + \frac{\gamma^2}{16} - \frac{\gamma}{3} + \frac{4}{3}, \end{aligned} \tag{20}$$

where Γ is the Gamma function, and γ is the Euler constant. One can note that the imaginary part in (20) disappears after integration on k due to hermicity of $\hat{\phi}$. As is shown in Fig. 1, the mean-square value of ϕ is infinite at $\eta = 0$, then it decreases and begins to increase finally.

Another more complicated example is

$$\begin{aligned} < C|\hat{\phi}^2\, a + a\,\hat{\phi}^2|C >= \frac{1}{3072}\Big(16\ln\eta\big(\ln\eta(4\ln\eta(3\ln\eta - 6\gamma + 16) + 9\pi^2 \\ +6\gamma(3\gamma - 16) + 384) - 9\gamma\pi^2 + 24\pi^2 - 6\gamma(64 + (\gamma - 8)\gamma) + 800\big) \\ +224\zeta(3)(-6\ln\eta + 3\gamma - 8) + 21\pi^4 + 12\gamma(3\gamma - 16)\pi^2 + 768\pi^2 \\ +4\gamma\big(\gamma(384 + \gamma(3\gamma - 32)) - 1600\big) + 16640\Big), \end{aligned} \tag{21}$$

where $\zeta(x)$ is the zeta-function.

B. Time evolution from WDW equation

The problem of time began from the discussion of the Wheeler-DeWitt (WDW) equation [7, 10, 12–14], which is a workhorse of the quantum cosmology and a "mathematical implementation of eternity." It is often stated that the WDW equation does not contain time explicitly. Indeed, it is true. Then, it is usually stated that the WDW equation forbids time evolution. Certainly, it is wrong if one considers a full quantum picture, including gauge fixing and evaluation of the mean values of the operators. The point is that the WDW equation has to be supplemented by the scalar product.

Let us to introduce the variable $\alpha = \ln a$ and perform the canonical quantization

$$[\hat{p}_\alpha, \alpha] = i, \qquad [\hat{\pi}_\phi, \phi] = -i \tag{22}$$

of the constraint $\Phi_1 = 0$. This results in the WDW equation

$$\left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2}\right)\Psi(\alpha,\phi) = 0 \tag{23}$$

of the Klein-Gordon type.

Scalar products for the Klein-Gordon equation are discussed in [15], where the "current" and "density" products were proposed. Here we will use only scalar product of the "current" type:

$$< \Psi|\Psi >= i\int\left(\Psi^*(\alpha,\phi)\frac{\partial}{\partial\alpha}\Psi(\alpha,\phi) - \Psi(\alpha,\phi)\frac{\partial}{\partial\alpha}\Psi^*(\alpha,\phi)\right)\Bigg|_{\alpha=\alpha_0} d\phi, \tag{24}$$

including the hyperplane $\alpha = \alpha_0$. For a mean value of some operator, the following formula has been introduced [15]

$$< \Psi|\hat{A}|\Psi >= i\int\left(\Psi^*\hat{D}^{1/4}\hat{A}\,\hat{D}^{-1/4}\frac{\partial\Psi}{\partial\alpha} - \left(\frac{\partial\Psi^*}{\partial\alpha}\right)\hat{D}^{-1/4}\hat{A}\,\hat{D}^{1/4}\,\Psi\right)\Bigg|_{\alpha=\alpha_0} d\phi, \tag{25}$$

where the operator $\hat{D} = \sqrt{-\frac{\partial^2}{\partial\phi^2}}$. In the momentum representation $\hat{\pi}_\phi = k$, $\hat{\phi} = i\frac{\partial}{\partial k}$, the WDW equation (23) looks as

$$\left(\frac{\partial^2}{\partial\alpha^2} + k^2\right)\psi(\alpha,k) = 0, \tag{26}$$

and, as a result of $\hat{D} = |k|$, the scalar product (25) takes the form:

$$< \Psi|\hat{A}|\Psi >= i\int C^*(k)e^{i|k|\alpha}\hat{A}e^{-i|k|\alpha}C(k)\Big|_{\alpha=\alpha_0} dk, \tag{27}$$

where

$$\Psi(\alpha,\phi) = \int e^{ik\phi}\psi(\alpha,k)dk = \int\frac{e^{ik\phi - i|k|\alpha}}{\sqrt{2|k|}}C(k)dk \tag{28}$$

is taken. To introduce the time evolution into this picture, one has to choose a time-dependent integration plane in (27) instead of $\alpha = \alpha_0$ by writing $\alpha = \frac{1}{2}\ln(2|k|\eta)$ according to (11), i.e., to $\Phi_2 = 0$.

However, if the operator $\hat{A}(\alpha, k, i\frac{\partial}{\partial\alpha}, i\frac{\partial}{\partial k})$ contains differentiations $\frac{\partial}{\partial k}$ or $\frac{\partial}{\partial\alpha}$, hermicity could be lost. To prevent this, let us rewrite (25), (27) in the form of

$$< \psi|\hat{A}|\psi >= \int \psi^*(\alpha, k)\Big(|k|^{1/4}\hat{A}|k|^{-1/4}\delta(\alpha - \frac{1}{2}\ln(2|k|\eta))\hat{p}_\alpha + \hat{p}_\alpha\delta(\alpha - \frac{1}{2}\ln(2|k|\eta))|k|^{-1/4}\hat{A}|k|^{1/4}\Big)\psi(\alpha, k)d\alpha dk, \qquad (29)$$

where $p_\alpha = i\frac{\partial}{\partial\alpha}$ and hermicity of $\hat{A}$ relative α, k variables are implied. In this case, no problem with hermicity arises if one takes the functions $\psi(\alpha, k)$ tending to zero at $\alpha \to \pm\infty$ to provide the throwing over the differential operators $\partial/\partial\alpha$ by the integration by parts. The functions $\psi(\alpha, k) = \frac{e^{-i|k|\alpha}}{\sqrt{2|k|}}C(k)$ do not poses such a property, thus, we shall take the functions

$$\psi(\alpha, k) = \frac{e^{-i|k|\alpha - \alpha^2/\Delta}}{\sqrt{2|k|}}C(k) \qquad (30)$$

in the intermediate calculations and, then, after integration over α, tend Δ to infinity. Performing the concrete calculations with the above wave packet (18), we again obtain the same values for (19) and (20). As for the mean value (21) of subsection III A, we cannot compare it using this picture because the particular operator ordering $a\hat{\phi}^2 + \hat{\phi}^2 a$ has been used in (21), but here the operators $a = \exp\alpha$ and $\hat{\phi}$ commute formally implying an existence of some intrinsic automatic ordering.

C. An evolution from the WDW using the Grassman variables

Another version with the anticommutative variables could be proposed in the form

$$< \psi|A|\psi >= \int \psi^*(\alpha, k)\exp\Big(i\lambda\big(\alpha - \frac{1}{2}\ln(2|k|\eta)\big) + \bar{\theta}\theta\hat{p}_\alpha + \frac{1}{2}\bar{\chi}\chi\left(|k|^{-i/2}\hat{A}\,|k|^{i/2} + |k|^{i/2}\hat{A}\,|k|^{-i/2}\right)\Big)\psi(\alpha, k)d\lambda d\alpha dk d\theta d\bar{\theta}d\chi d\bar{\chi}, \qquad (31)$$

where the anticommutating Grassman variables $\theta_i = (\theta, \chi)$, $\bar{\theta}_i = (\bar{\theta}, \bar{\chi})$ have the following properties: $\theta_i\theta_j + \theta_j\theta_i = 0$, $\int d\theta_i = 0$, $\int \theta_i d\theta_i = 1$, $(\bar{\theta}_i)^* = \theta_i$, $(\bar{\theta}_i\theta_j)^* = \bar{\theta}_j\theta_i$.

Again, for a reasons of hermicity, we take the functions (30) and then tend Δ to infinity. For the practical calculations, it is convenient to separate the expression in the exponent of Eq. (31) into two parts $M = i\lambda\left(\alpha - \frac{1}{2}\ln\left(2|k|\eta\right)\right)$, and $N = \bar{\theta}\theta\hat{p}_\alpha + \frac{1}{2}\bar{\chi}\chi\left(|k|^{-i/2}\hat{A}|k|^{i/2} + |k|^{i/2}\hat{A}|k|^{-i/2}\right)$ for using the formula [16]

$$\exp\left(\hat{M} + \hat{N}\right) = \left(1 + \sum_{m=1}^{\infty}\frac{\hat{X}_m}{m!}\right)\exp\hat{M}, \tag{32}$$

where $\hat{X}_m$ is set recurscively as $\hat{X}_1 = N$ and $\hat{X}_k = \hat{N}\hat{X}_{m-1} + [\hat{M}, \hat{X}_{m-1}]$. It is sufficient to take only three terms in a sun of Eq. (32) because $\hat{N}$ contains the Grassman variables.

D. Quasi-Heisenberg picture

Another approach to consider the time evolution is to take classical equations of motion and then quantize them, i.e., write "hat" under every quantity [17–20]. The operator equations of motion take the form:

$$\hat{\phi}'' + 2\hat{\alpha}'\hat{\phi}' = 0, \qquad \hat{\alpha}'' + \hat{\alpha}'^2 + \hat{\phi}'^2 = 0. \tag{33}$$

One needs to find the commutation relations of the operators $\hat{\alpha}(\eta)$, $\hat{\phi}(\eta)$. The problem was solved by Dirac, who has introduced the "Dirac brackets" for the system with constraints postulating that commutator relations of the operators have to be analogous to the Dirac brackets. However, it is not always possible to find an operator realization of this commutator relations. The quasi-Heisenberg picture suggests to find an operator realization only at the initial moment and then allow operators to evolve according to the equations of motion. The initial conditions for operators could be taken in the form

$$\hat{\alpha}(0) = \alpha_0, \quad \hat{\alpha}'(0) = e^{-2\alpha_0}|k|, \quad \hat{\phi}(0) = i\frac{\partial}{\partial k}, \quad \hat{\phi}'(0) = e^{-2\alpha_0}k. \tag{34}$$

The solution of the operator equations of motion (33) with the initial conditions (34) is

$$\alpha(\eta) = \alpha_0 + \frac{1}{2}\ln\left(1 + 2|k|\eta\, e^{-2\alpha_0}\right), \quad \hat{\phi}(\eta) = i\frac{\partial}{\partial k} + \frac{k}{2|k|}\ln\left(1 + 2|k|\eta\, e^{-2\alpha_0}\right). \tag{35}$$

To built the Hilbert space, in which these quasi-Heisenberg operators act, one may use the WDW equation (23) and the scalar product (27) but at the end of evaluating the value of α_0 should be set to minus infinity, i.e., $\alpha_0 \to -\infty$, which corresponds to $a \to 0$ at $\eta = 0$. Explicit calculation gives the same mean values as (19), (20) and (21).

E. Evolution using the unconstraint Schrödinger equation in the extended space

It is believed [21–23] that the Grassman variables allow writing the Lagrangian without constraints at all. Here one has two possibilities: to set a gauge imposing an additional condition as a function of p_a, a, π_ϕ, ϕ such as (11). It is a canonical gauge setting. Another alternative is to impose that condition as a function of N (non-canonical gauge).

1. *Canonical gauge*

The discussion could be started in terms of continual integral which gives a transition amplitude from *in* to *out* states [21, 24, 25]:

$$< out|in >= Z = \int e^{i\int\left(\pi_\phi \phi' - H_{phys}(\phi,\pi_\phi)\right)d\eta} D\pi_\phi \mathcal{D}\phi, \tag{36}$$

where H_{phys} is given by (13). This functional can be rewritten in the form

$$Z = \int e^{i\int\left(\pi_\phi\phi' - p_a a' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right)\right)d\eta} \Pi_\eta p_a \Pi_\eta \delta(a - \sqrt{2\eta|\pi_\phi|}) \mathcal{D}p_a \mathcal{D}a \mathcal{D}\pi_\phi \mathcal{D}\phi \mathcal{D}N, \tag{37}$$

where [25] $p_a = \{\Phi_1, \Phi_2\}$ is the Faddeev-Popov determinant. Equivalence of (36) and (37) could be checked by transition to a new integration variable $\tilde{a} = a - \sqrt{2\eta|\pi_\phi|}$, and integrating on $\tilde{a}$, N, p_a in (37) gradually.

Using the Grassman anticommutative variables in Eq. (37) leads to the form containing the unconstraint Lagrangian in the exponent

$$Z = \int e^{i\int\left(\pi_\phi\phi' - p_a a' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) - \lambda\left(a - \sqrt{2\eta|\pi_\phi|}\right) - \bar{\theta}\theta p_a\right)d\eta}\mathcal{D}p_a\mathcal{D}a\mathcal{D}\pi_\phi\mathcal{D}\phi\mathcal{D}N\mathcal{D}\lambda\mathcal{D}\theta\mathcal{D}\bar{\theta}. \tag{38}$$

Eq. (38) allows writing the Hamiltonian

$$H = N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) + \lambda\left(a - \sqrt{2\eta|\pi_\phi|}\right) + \bar{\theta}\theta p_a, \tag{39}$$

which, after canonical quantization, could be used to describe evolution as in both Schrödinger and Heisenberg pictures.

2. *Non-canonical gauge*

Let us remind, how the Faddeev-Popov determinant appears in non-canonical gauge. The action (3) is invariant relatively the infinitesimal gauge transformations:

$$\tilde{a} = a + \delta a = a + \varepsilon\, a', \tag{40}$$

$$\tilde{\phi} = \phi + \delta\phi = \phi + \varepsilon\, \phi', \tag{41}$$

$$\tilde{N} = N + \delta N = N + (N\varepsilon)', \tag{42}$$

where ε is an infinitesimal function of time. When one sets a non-canonical gauge condition in the form $F(N) = 0$, the functional integral including a gauge fixing multiplier with the Dirac δ-function becomes [21]

$$Z = \int e^{i\int\left(\pi_\phi\phi' - p_a a' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right)\right)d\eta}\Pi_\eta\frac{\delta F}{\delta\varepsilon}\Pi_\eta\delta(F)\mathcal{D}p_a\mathcal{D}a\mathcal{D}\pi_\phi\mathcal{D}\phi\mathcal{D}N, \tag{43}$$

where again the Faddev-Popov determinant $\Delta_{FP} = \frac{\delta F}{\delta\varepsilon}$ has been introduced [21]. In the particular case $F = N - 1$, it follows from (42) that the determinant is $\Delta_{FP} = \frac{\delta N}{\delta\varepsilon} = N' + N\frac{\partial}{\partial\eta}$. Using the Grassman variables raises the determinant into exponent

$$\begin{aligned} Z = i\int e^{i\int\left(\pi_\phi\phi' - p_a a' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) - \lambda(N-1) - N'\bar{\theta}\theta - N\bar{\theta}\theta'\right)d\eta}&\mathcal{D}p_a\mathcal{D}a\mathcal{D}\pi_\phi\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}N\mathcal{D}\theta\mathcal{D}\bar{\theta} \\ = \int e^{i\int\left(\pi_\phi\phi' - p_a a' - \left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) - \bar{\theta}\theta'\right)d\eta}&\mathcal{D}p_a\mathcal{D}a\mathcal{D}\pi_\phi D\phi\mathcal{D}\theta\mathcal{D}\bar{\theta}. \end{aligned} \tag{44}$$

An expression in the exponent of Eq. (44) could be treated as Lagrangian, but it cannot be put into the generalized Hamiltonian form, because velocity θ' cannot be expressed through a momentum. In this relation, an interesting trick has been suggested [22]: to take the gauge condition $N' = 0$, instead of $N = 1$. For this new gauge, it follows from (42) that

$$\delta F = \delta N' = (N\varepsilon)'', \tag{45}$$

and

$$Z = \int e^{i\int\left(\pi_\phi\phi' - p_a a' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) - \lambda N' - \bar{\theta}(N\theta)''\right)d\eta} \mathcal{D}p_a\mathcal{D}a\mathcal{D}\pi_\phi\mathcal{D}\phi\mathcal{D}N\mathcal{D}\lambda\mathcal{D}\theta\mathcal{D}\bar{\theta}. \tag{46}$$

The unconstraint Lagrangian is written from Eq. (46) as

$$L = \pi_\phi\phi' - p_a a' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) - \lambda N' + \bar{\theta}'(N\theta)', \tag{47}$$

For the momentums of the Grassman variables and N, one has from (47)

$$\pi_\theta = -\frac{\partial L}{\partial \theta'} = N\bar{\theta}', \quad \pi_{\bar{\theta}} = \frac{\partial L}{\partial \bar{\theta}'} = N'\theta + N\theta', \quad p_N = \frac{\partial L}{\partial N'} = -\lambda + \bar{\theta}'\theta, \tag{48}$$

where, as usual, left derivative over Grassman $\frac{\partial}{\partial\theta}\left(\theta f(\bar{\theta})\right) = f(\bar{\theta})$ is implied. The Lagrangian (47), rewritten in terms of momentums acquires the form

$$L = \pi_\phi\phi' - p_a a' + p_N\, N' + \bar{\theta}'\pi_{\bar{\theta}} + \pi_\theta\theta' - N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) - \frac{1}{N}\pi_\theta\pi_{\bar{\theta}}. \tag{49}$$

This means that the corresponding Hamiltonian is

$$H = N\left(-\frac{1}{2}p_a^2 + \frac{\pi_\phi^2}{2a^2}\right) + \frac{1}{N}\pi_\theta\pi_{\bar{\theta}}. \tag{50}$$

Thus, two Hamiltonians (39), (50), which drive unconstraint dynamics, have been obtained. The first one is time-dependent and contains the Grassman variables as parameters. The second one is time-independent and implies the time dynamics of the Grassman variables [22]. Further, we will consider only the Hamiltonian (50), because this timeless Hamiltonian seems more perspective in the general gravity quantization. Opposite to commutation relation (22), the anticommutation relation have to be introduced for the Grassman variables

$$\{\pi_\theta, \theta\} = -i, \qquad \{\pi_{\bar{\theta}}, \bar{\theta}\} = -i. \tag{51}$$

In the particular representation $\alpha = \ln a$, $\hat{p}_\alpha = i\frac{\partial}{\partial\alpha}$, $\hat{\phi} = i\frac{\partial}{\partial k}$, $\hat{\pi}_\phi = k$, $\hat{\pi}_\theta = -i\frac{\partial}{\partial\theta}$, $\hat{\pi}_{\bar{\theta}} = -i\frac{\partial}{\partial\bar{\theta}}$, the Schrödinger equation looks as

$$i\frac{\partial}{\partial\eta}\psi = \left(\frac{N}{2}e^{-2\alpha}\left(\frac{\partial^2}{\partial\alpha^2} + k^2\right) - \frac{1}{N}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}\right)\psi, \tag{52}$$

where the operator ordering in the form of the two-dimensional Laplacian has been used.

It should be supplemented by the scalar product

$$<\psi|\psi>= \int \psi^*(\eta, N, k, \alpha, \bar{\theta}, \theta)\psi(\eta, N, k, \alpha, \bar{\theta}, \theta)e^{2\alpha}d\alpha dk dN d\theta d\bar{\theta}, \tag{53}$$

where the measure $e^{2\alpha}$ arises due to hermicity requirement [25, 26]. This measure is a consequence of a minisuperspace metric if the Hamiltonian is written in the form $H = \frac{N}{2}g^{ij}p_i p_j + \frac{1}{N}\pi_\theta\pi_{\bar{\theta}}$ with $p_i \equiv \{\alpha, \phi\}$, $g^{ij} = \text{diag}\{-e^{-2\alpha}, e^{-2\alpha}\}$. Thus, the measure takes the form $\sqrt{|\det g_{ij}|} = e^{2\alpha}$ [26].

One of the particular formal solutions of the equation (52) could be written as

$$\psi(\eta, N, k, \alpha, \bar{\theta}, \theta) = (\bar{\theta} + \theta)\psi_1(\eta, N, k, \alpha) + i(\bar{\theta} - \theta)\psi_2(\eta, N, k, \alpha), \tag{54}$$

where the functions ψ_1 and ψ_2 satisfy the equation

$$i\frac{\partial}{\partial\eta}\psi_{1,2} = \hat{H}_0\psi_{1,2}, \tag{55}$$

where $\hat{H}_0 = \frac{N}{2}e^{-2\alpha}\left(\frac{\partial^2}{\partial\alpha^2} + k^2\right)$. Then, the scalar product reduces to

$$<\psi|\psi>= 2i\int (\psi_2^*\psi_1 - \psi_1^*\psi_2)\, e^{2\alpha}d\alpha dk dN. \tag{56}$$

To obtain the mean values close to that given by the previous methods, where Klein-Gordon scalar product is used, let us take the functions ψ_1, ψ_2 in the form

$$\psi_2 = e^{-i\hat{H}_0\eta}\psi_0(\alpha, k), \qquad \psi_1 = e^{-i\hat{H}_0\eta}\frac{\partial}{\partial\alpha}\psi_0(\alpha, k), \tag{57}$$

TABLE I. Comparison of the mean values calculated by the different methods. Capital letters denote the section of a method. A plus implies that the values obtained by the different methods coincide. Crosses of two types in a circle mean that the values obtained at least by two different methods coincide.

Method	A	B	C	D	E
a	$+$	$+$	$+$	$+$	
a^2	$+$	$+$	$+$	$+$	$+$
$\hat{\phi}^2$	$+$	$+$	$+$	$+$	
$\hat{\phi}^4$	$\oplus$	$\otimes$	$\otimes$	$\oplus$	
$\hat{\phi}^6$	$\oplus$			$\oplus$	
$a\hat{\phi}^2 + \hat{\phi}^2 a$	$\oplus$			$\oplus$	

where $\psi_0(\alpha, k)$ is given by (30). As one can see, at the limit $\Delta \to \infty$, the state (57) comes to the space of the WDW solution (see e.g. [27]), and the time evolution disappears. However, if this limit is taken after the calculating of the mean values, then the explicit time evolution could be caught. Let us consider the mean value of $a^2 = e^{2\alpha}$ for the wave packet (18). For the variable N, we will consider a very narrow packet around the value $N = 1$, i.e., simply set $N = 1$ and abandon integration over N. Remaining integrations give for the normalizing multiplier

$$< \psi|\psi >= 2i \int \left(\psi_2^* \psi_1 - \psi_1^* \psi_2\right) e^{2\alpha} d\alpha dk = \frac{3\pi e^{\Delta/2}\sqrt{\Delta}}{2\sqrt{2}}. \tag{58}$$

Then the mean value of a^2 becomes

$$< a^2 >= \frac{< \psi|e^{2\alpha}|\psi >}{< \psi|\psi >} = e^{3\Delta/2} + \frac{8(2\Delta+1)\eta}{3\sqrt{\pi}\Delta} + \frac{3e^{-\Delta/2}\eta^2}{\Delta}. \tag{59}$$

As one can see, three terms appear in Eq. (59). The first term is divergent at $\Delta \to \infty$, i.e., when one proceeds to the space of the WDW solutions, the evolution disappears, in a sense that this constant term dominates in (59). However if one omits this constant term (not dependent on time) and then proceeds to the limit $\Delta \to \infty$, then the value $< a^2 >= \frac{16\eta}{3\sqrt{\pi}}$ is the same as in the previous methods

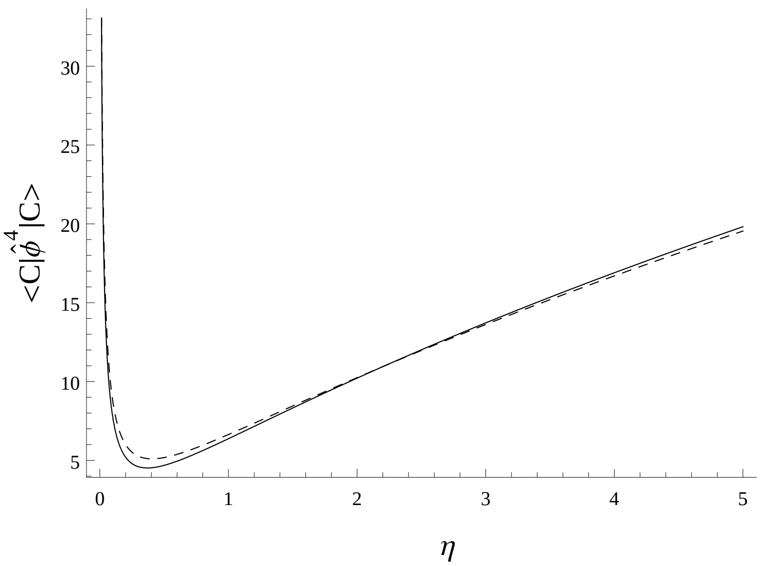

FIG. 2. The mean value of $\hat{\phi}^4$ over the wave packet (18) for the methods A,D- solid line and methods B,C- dashed line.

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A,B,C,D. In the general case, for instance, under evaluation a^4, the other diverging terms depending on the time appear. That prevents extracting the time evolution when one proceeds from the extended space to the space of WDW solutions. However, one could believe that some good regularization method has to exist.

IV. DISCUSSION AND POSSIBLE APPLICATION OF THE ABOVE APPROACHES TO THE GENERAL CASE OF GRAVITY QUANTIZATION

The results of the calculation of the mean values are presented in Table I. The mean value of $< C|a^2|C >$ turns out to be the same for all the methods considered. For the method III E, we are not able to calculate the mean values of the other operators for two reasons: because we use the most primitive way of calculation by expanding the exponent $e^{-i\hat{H}_0\eta}$ in Eq. (57) over the degrees of η, and use the primitive regularization under transition from extended spase [22, 23] to the space of WDW equation solutions.

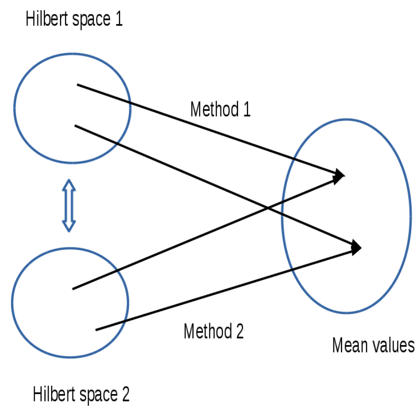

FIG. 3. The illustration that different methods could have different Hilbert spaces for producing the same set of the mean values for the arbitrarily given operators. Still, there should be correspondence between the state $C(k)$ of the Hilbert space 1 and the state $\tilde{C}(k)$ of the Hilbert space 2 producing the same mean values.

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The methods A,B,C,D produce the same value of the operators a, ϕ^2 as it is shown in Table I. For the mean value of $\hat{\phi}^4$ some difference emerges shown in Fig. 2. It is not the uncertainty of numerical calculations because they are fully analytical and have been performed using Mathematica. However, let us emphasize that it does not mean that the different methods are nonequivalent. Generally, as is illustrated in Fig. 3 different methods should not have the same Hilbert space when producing the same values of the different operators. Only the correspondence between these spaces should exist, i.e., these spaces have to be connected by some transformation.

In light of quantum gravity, one could guess that the method of subsection III A is not likely to be applicable to the building of the general quantum gravity theory. Simply, it is not possible to resolve the Hamiltonian and the momentum constraints to exclude the extra degrees of freedom.

Most of the considered methods use the time-dependent gauge condition. It seems the restrictive case for the general gravity if to demand conservation of the gauge condition in time. In fact, it is equivalent to the preliminary solution of the equations of motion for gravity. An exception is the quasi-Heisenberg picture III D, which demands to set a gauge condition only at the initial moment of time. Thus, it seems the most perspective picture. The unconstrained Schrödinger equation of subsection III E also seems attractive [14], but needs the invention of some regularization scheme when one proceeds from the extended space to the space of WDW equation solutions. One could hope that quantum computing will be applied [28–30] for a description of the quantum universe evolution in the future.

V. CONCLUSION

As one can see, the description of quantum evolution is very straightforward and unambiguous but teems with different details such as choosing a scalar product and operator ordering which are typical for quantization of the systems with constraints [31]. It is shown that if one wants to discuss the quantum evolution of the universe, there are no serious obstacles to this. Namely, the "problem of time" does not exist as a real problem.

Let us summarize the methods producing an explicit time evolution: time-dependent physical Hamiltonian with the excluded extra degrees of freedom, WDW equation with the time-dependent integration plane in the scalar product, quasi-Heisenberg picture quantizing the equations of motion, and unconstraint Hamiltonian with the Grassman variables. Since the WDW equation tells nothing about the time evolution without determining the scalar product, this equation alone is only halfway to a full picture.

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