New analytic models of "traversable" wormholes

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Abstract

The analytic solution of the general relativity equations for spherically symmetric wormholes are given. We investigate the special case of a "traversable" wormhole i.e., one allowing the signal to pass through it. The energy-momentum tensor of wormhole matter is represented as a superposition of a spherically symmetric magnetic field and dust matter with negative matter density. The dynamics of the model are investigated. We discuss both the solution of the equation with a Λ-term and without it. Superposing enough dust matter, a magnetic field, and a Λ-term can produce a static solution, which turns out to be a spherical Multiverse model with an infinite number of wormholes connected spherical universes. Corresponding solution can be static and dynamic.

1 Introduction

A wormhole (WH) [1]-[19]) is a hypothetical object described by a nonsingular solution of the Einstein equations with two large (or infinite) space-time regions connected by a throat. The two large space-time regions can be located in one universe or belong to different universes in the Multiverse model (see [18]). In the last case, "traversable" WHs afford a unique opportunity to explore other universes.

In the present paper, we analytically study the dynamics of a spherical model of a non-equilibrium WH filled with matter. This matter consists of a magnetic field and dust with negative mass density. The obtained solution therefore generalizes the Tolman solution [20] for a model with a spherically symmetric electromagnetic field. As we see below, this generalization leads to essentially new and important solutions. We use the method of calculations of physical quantities in the frame comoving with dust.

As the initial model for a WH, we use a static model in which gravitational accelerations are everywhere identically zero. Hence, the effective masses of both WH mouths vanish, although the geometry of three-dimensional space is certainly non-Euclidean. Such a model is considered in [19] and [17], where all matter is represented by a gravitating scalar field. We change the scalar field into a superposition of an electro- magnetic field and dust matter with negative energy density, which turns out to be a methodologically important development and generalization of these models. This allows us to apply methods of Tolman's problem to calculate the model (see [20], [21], [22]); these methods were generalized and further developed by Shatskiy (see [23] and [24]).

To further generalize and develop this method, we introduce the cosmological Λ-term into the model. This allows obtaining a principally new solution (see Section 6) for a static spherical model of the Multiverse. This model includes a infinite number of spherical worlds connected by throats. To our knowledge, this is the first analytic model of this type.

In the obtained solution, the Multiverse can have its total energy density positive everywhere in space. In addition, this solution can be generalized by the same method to the case of a dynamic model shifted from equilibrium by an excess (or shortage) of dust or by the Λ-term.

Some methodological details see in [25]

2 Einstein equations

We use the Armendariz-Picon static spherically symmetric solution of the Einstein equations [17] for the description of the unperturb WH model[1]:

$$ds^2 = dt^2 - dR^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \quad r^2(R) = q^2 + R^2\,. \tag{1}$$

The minimum allowed radius of this WH is $r_0 = q$ (the radius of the throat[2]).

Usually, this solution is related to the energy-momentum tensor of a scalar field (see [19], [17]). In the present paper, we suggest another interpretation of this solution related to a different representation of the energy-momentum tensor. This new interpretation of solution (1) can correspond to the energy-momentum tensor represented as

$$T^n_k = \begin{pmatrix} -\frac{q^2}{8\pi r^4} & 0 & 0 & 0 \\ 0 & +\frac{q^2}{8\pi r^4} & 0 & 0 \\ 0 & 0 & -\frac{q^2}{8\pi r^4} & 0 \\ 0 & 0 & 0 & -\frac{q^2}{8\pi r^4} \end{pmatrix} = \begin{pmatrix} +\frac{q^2}{8\pi r^4} & 0 & 0 & 0 \\ 0 & +\frac{q^2}{8\pi r^4} & 0 & 0 \\ 0 & 0 & -\frac{q^2}{8\pi r^4} & 0 \\ 0 & 0 & 0 & -\frac{q^2}{8\pi r^4} \end{pmatrix} + \begin{pmatrix} -\frac{q^2}{4\pi r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

The first term in the right-hand side corresponds to the energy-momentum tensor of a static magnetic field with the effective magnetic charge q and the second term describes dust matter with the negative energy density

$$\varepsilon_d = -\frac{q^2}{4\pi r^4} \tag{3}$$

This possible representation of the tensor T^n_k is different from its equivalent representation as the energy-momentum tensor of a scalar field adopted in previous papers [17]-[28].

An excess (or shortage) of dust (relative to ε_d) is not in equilibrium and initiates motion. Because dust layers are relatively independent, it is possible to integrate the equations of motion for dust in a way similar to the solution of Tolman's problem [20]. In essence, it is the same as Tolman's problem in a centrally symmetric and static electric (or magnetic) field for uncharged dust.

[1] We use units where $c = 1$ and $G = 1$.

[2] Here and below, all values at the throat have a subscript "$_0$".

We seek the metric tensor of the solution in the form[3]:

$$ds^2 = dt^2 - e^{\lambda} dR^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{4}$$

where r^2 and e^{λ} are functions of both R and t. Below, we justify this choice of the metric.

We consider the problem in the presence of a Λ-term. The Einstein equations corresponding to metric (4) can be written as[4]:

$$8\pi T_t^t = 8\pi\varepsilon + q^2/r^4 + \Lambda = -e^{-\lambda}\left(2rr_{,RR} + r_{,R}^2 - rr_{,R}\lambda_{,R}\right)/r^2 + \left(rr_{,t}\lambda_{,t} + r_{,t}^2\right)/r^2 + 1/r^2\,, \tag{5}$$

$$8\pi T_R^R = q^2/r^4 + \Lambda = \left(2rr_{,tt} + r_{,t}^2\right)/r^2 - e^{-\lambda} r_{,R}^2/r^2 + 1/r^2\,, \tag{6}$$

$$8\pi T_t^R = 0 = e^{-\lambda}\left(2r_{,Rt} - r_{,R}\lambda_{,t}\right)/r\,, \tag{7}$$

$$8\pi T_\theta^\theta = 8\pi T_\varphi^\varphi = -q^2/r^4 + \Lambda = -\left\{e^{-\lambda}\left[2r_{,RR} - r_{,R}\lambda_{,R}\right] - 2r_{,tt} - r\lambda_{,tt} - r\lambda_{,t}^2/2 - r_{,t}\lambda_{,t}\right\}/(2r)\,. \tag{8}$$

Thus, the energy-momentum tensor includes three types of matter:
a centrally symmetric magnetic field,
a cosmological Λ-term, and
dust matter with the density[5] ε.

3 Solution and the Initial Conditions

Equation (7) can be integrated with respect to time:

$$e^{-\lambda} r_{,R}^2 = F_1(R)\,. \tag{9}$$

At the throat, the condition $r_{,R} = 0$ must be satisfied, and therefore

$$F_1(0) = 0\,. \tag{10}$$

By substituting (9) in Eqn (6), we obtain

$$\frac{q^2}{r^2} + \Lambda r^2 = \left(rr_{,t}^2\right)_{,t}/r_{,t} - F_1 + 1\,. \tag{11}$$

Integrating this equation with respect to time yields

$$\frac{q^2}{r} - \Lambda r^3/3 + rr_{,t}^2 + (1 - F_1)r = F_2(R)\,. \tag{12}$$

The functions $F_1(R)$ and $F_2(R)$ determine the initial conditions for the velocity and dust density distributions.

With $\lambda_{,t}$ and $\lambda_{,tt}$ expressed from Eqn (7), Eqn (8) can be rewritten as

$$\left[q^2/r^2 + \Lambda r^2 + e^{-\lambda} r_{,R}^2 - r_{,t}^2 - 2rr_{,tt}\right]_{,R} = 0\,. \tag{13}$$

[3]Because dust is pressureless, we can choose the synchronous commoving frame (with the time metric component equal to 1).

[4]See, e.g., [31] for the derivation (problem 5 in §100).

[5]In the absence of the L term, solution (4) coincides with (1) only for the density ε equal to ε_d [see . (3)].

When Eqn (6) is satisfied, integrating Eqn (13) yields an identity. **This identity is the consequence of setting the time component of metric (4) to unity, which justifies this choice of the metric.**

It is worth noting from the methodological standpoint that this form of the metric for dust is obvious because dust is pressureless and hence the comoving frame is simultaneously a synchronous one. But this is not so obvious when a spherically symmetric electromagnetic field is added, and the obtained identity is a consequence of the absence of interaction between uncharged dust and the electromagnetic field. Moreover, the Lorentz transformation along the magnetic field lines does not change this field, which justifies the use of the dust comoving reference frame. For motion in this frame (along the field), the electromagnetic field remains invariant. If the metric could not be represented in form (4), further calculations would be impossible. It is this non-obvious fact that allowed us to successfully develop our method and to obtain all important results.

Multiplying Eqn (5) by $r^2 r_{,R}$ and expressing $\lambda_{,t}$ from Eqn (7), we obtain

$$8\pi\varepsilon r^2 r_{,R} + \left[\Lambda r^3/3 - q^2/r - r(1-F_1) - r r_{,t}^2\right]_{,R} = 0\,. \tag{14}$$

Integrating this with respect to R from 0 to R with account for (10) yields

$$\int\limits_0^R 8\pi\varepsilon r^2 r_{,R}\, dR - q^2/r + \Lambda r^3/3 - r(1-F_1) - r r_{,t}^2 = -q^2/r_{_0} + \Lambda r_{_0}^3/3 - r_{_0} - r_{_0} r_{_{0,t}}^2\,. \tag{15}$$

Here, we took into account that the function $r(R=0) = r_{_0}(t)$ can be time-dependent.

Expressing $r r_{,t}^2$ from (12) and substituting it in (15), we obtain

$$\int\limits_0^R 8\pi\varepsilon r^2 r_{,R}\, dR = F_2(R) - q^2/r_{_0} - r_{_0} + \Lambda r_{_0}^3/3 - r_{_0} r_{_{0,t}}^2\,. \tag{16}$$

The derivative of this expression with respect to R yields one more integral of motion (which expresses mass conservation in the comoving volume):

$$8\pi\varepsilon r^2 r_{,R} = \frac{dF_2}{dR}\,. \tag{17}$$

The Bianchi identities[6]:

$$T^n_{k;n} = \frac{1}{\sqrt{-g}}\,\frac{\partial(T^n_k\sqrt{-g})}{\partial x^n} - \frac{T^{nl}}{2}\,\frac{\partial g_{nl}}{\partial x^k} = 0 \tag{18}$$

are contained in the Einstein equations, their components $k = t, \theta, \varphi$ vanish identically, and the component $k = R$ yields the result in (17), which is an analogue of the integral for the pure-dust solution (see [31], §103) in the theory of the evolution of a dust cloud. The explicit form of the function $e^\lambda(R,t) = r_{,R}^2/F_1$ and an implicit form of the function $r^2(R,t)$ are obtained in Section 4.

Below, we use the index "i" to denote all quantities at the initial instant $t = 0$.

The velocity distribution is set to zero initially:

$$r_{,t}^2\Big|_{(t=0)} = 0 \tag{19}$$

[6]Here, g is the determinant of the metric tensor.

A nonzero initial velocity is considered in Appendix A. We note that in all cases, the choice of initial conditions is not arbitrary and must satisfy constraints imposed by (5) and (7). These equations do not contain second time derivatives and, as is well known (see [27]), under the condition that $t =$ const at the initial cross section, they are automatically satisfied for all values t by virtue of the other Einstein equations.

We choose the initial coordinate scale of R in (4) such that

$$r_i = \sqrt{q^2 + R^2}\,, \qquad \varepsilon_{di} \equiv \varepsilon_d(r_i) = -\frac{q^2}{4\pi r_i^4}\,. \tag{20}$$

where $r_i(R)$ coincides with (1) and Eqn (10) is automatically satisfied. We set

$$s(r_i) \equiv \frac{1}{r_i}\int\limits_0^R 8\pi(\varepsilon_i - \varepsilon_{di}) r_i^2 r_{i,R}\, dR = \frac{1}{r_i}\int\limits_q^{r_i} 8\pi\varepsilon_i r_i^2\, dr_i + 2\frac{q}{r_i}\left(1 - \frac{q}{r_i}\right) \tag{21}$$

The quantity s has the meaning of the doubled Newtonian potential of dust matter excessive relative to ε_d [see (20)]. The relation of this potential to the functions F_1 and F_2 follows from (15) and (16) at the initial instant:

$$F_1 = 1 - s - \frac{q^2}{r_i^2} - \frac{\Lambda}{3}\left(r_i^2 - q^3/r_i\right)\,, \tag{22}$$

$$F_2 = r_i\left(s + 2\frac{q^2}{r_i^2} - \frac{\Lambda q^3}{3r_i}\right)\,. \tag{23}$$

Equation (9) implies that $F_1 \geq 0$. According to (22), this imposes constraints on the distribution $s(R)$.

From Eqn (6), we can find $r_{,tt}$ at the initial instant:

$$\text{at } t = 0\ : \quad r_{,tt} = -\frac{s}{2r_i} + \frac{\Lambda r_i}{3} + \frac{\Lambda q^3}{6r_i^2} \tag{24}$$

Equation (24) shows that for $\Lambda = 0$, the dynamics are absent in the region were $s = 0$ (at least up to the instant of intersection of the dust layers, when the model considered becomes inapplicable).

4 Solution without the Λ-term

We consider the case $\Lambda = 0$ in more detail.

For $\Lambda = 0$, it follows from Eqn (24) that **the WH throat size remains constant** $[r_0(t) = q]$ **in the whole region of the allowed values of R and t for the sought solution, until matter starts flowing through the throat. In addition, expression (24) implies that the dynamics of matter in a WH depend only on the internal layers of matter (with smaller values of r) and are independent of the external layers. Therefore, matter on one side of the throat does not influence matter on the opposite side (unless there is matter flowing through the throat or layer intersection occurs).**

It can be verified directly from (24) that at the initial instant, the acceleration $r_{,tt}$ is negative for $s > 0$. Hence, the excessive mass of dust is to collapse.

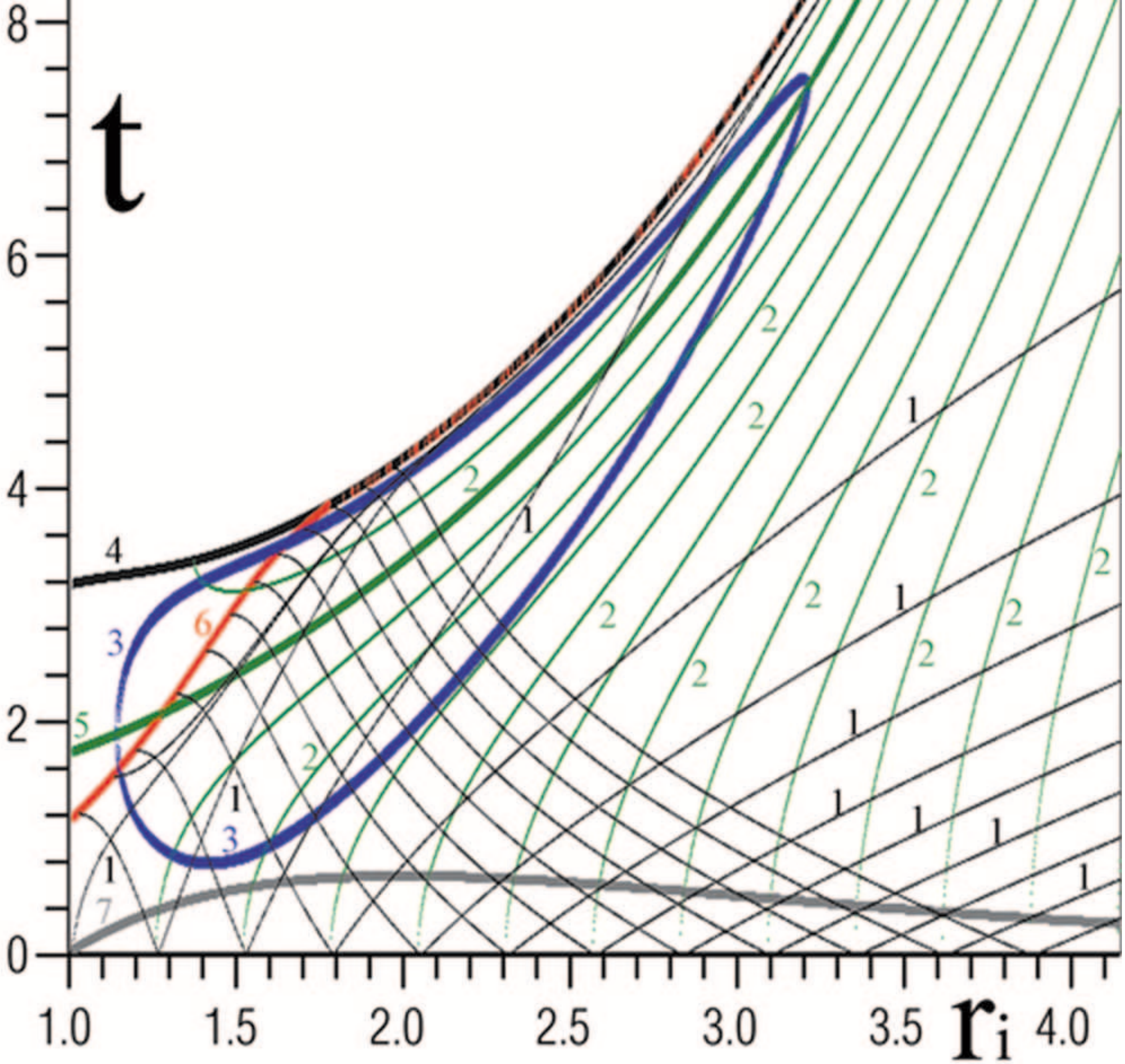

Figure 1: Diagrams $t(r_i/q)$: 1 - for light geodesics; 2 - for $r =$ const; 3 - for the apparent horizon and the inner horizon; 4 - for the limiting stopping time; 5 - for the throat $r = q$; 6 - for the intersection times of adjacent dust layers; 7 - the dependence $s(r_i/q) = 0.7(r_i/q - 1) \cdot \exp(2 - r_i/q)$ — the scale for s (vertical axis) coincides with the time scale t.

We introduce the definition of the apparent horizon.

The criterion of the absence of a apparent horizon during the collapse is given by

$$V < 1 \tag{25}$$

for the dust velocity [see also [23],[29],[30],[24] and (12)], where the dust velocity V is determined as

$$V^2 \equiv r^2_{,t} e^{\lambda}/r^2_{,R} = r^2_{,t}/F_1 = \frac{(s + 2q^2/r_i^2) r_i/r - s - q^2/r_i^2 - q^2/r^2}{1 - s - q^2/r_i^2} \tag{26}$$

For $q = 0$, this result coincides with the solution for collapsing dust (see [23]).

To avoid a $0/0$ ambiguity for the function V^2 at the throat at the initial instant ($F_1[r_i \to r_0] \to 0$), it is necessary to ensure a more rapid convergence to zero of the function $r^2_{,t}(t = 0, r_i \to r_0)$ than the function $F_1(r_i \to r_0)$ (see Appendix A).

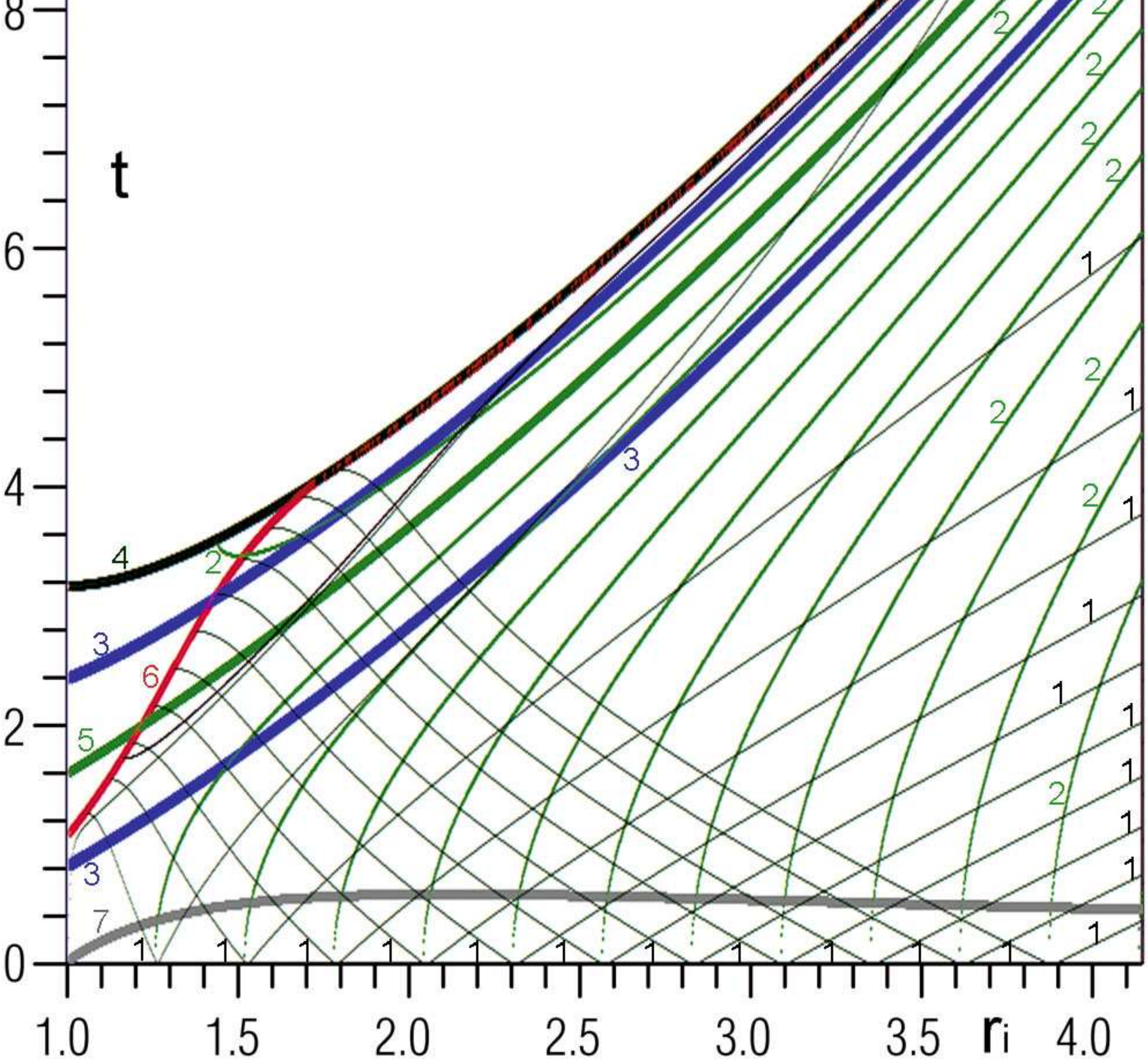

Figure 2: Diagrams $t(r_i/q)$: 1 - for light geodesics; 2 - for $r=$ const; 3 - for the apparent horizon and the inner horizon; 4 - for the limiting stopping time; 5 - for the throat $r=q$; 6 - for the intersection times of adjacent dust layers; 7 - the dependence $s(r_i/q)=(1-q/r_i)(2+0.5(1-q/r_i))q/r_i$ — the scale for s (vertical axis) coincides with the time scale t.

The law of motion of a dust layer can be found from (12):

$$t=\int\limits_{r_i}^{r}\frac{-r\,dr}{\sqrt{(F_1-1)r^2+F_2r-q^2}}=\int\limits_{r_i}^{r}\frac{-r\,dr}{r_i\sqrt{(1-r/r_i)[(s+q^2/r_i^2)r/r_i-q^2/r_i^2]}} \qquad (27)$$

This quadrature can be elementarily integrated:

$$t=r_i\frac{\sqrt{(1-r/r_i)[(s+q^2/r_i^2)r/r_i-q^2/r_i^2]}}{s+q^2/r_i^2}+r_i\frac{s+2q^2/r_i^2}{2(s+q^2/r_i^2)^{3/2}}\cdot\arccos\left[1-\frac{2(s+q^2/r_i^2)(1-r/r_i)}{s}\right] \qquad (28)$$

or in another form,

$$t = r_i \left[\frac{\sqrt{(1-r/r_i)[(s+q^2/r_i^2)r/r_i - q^2/r_i^2]}}{s+q^2/r_i^2} + \frac{s+2q^2/r_i^2}{(s+q^2/r_i^2)^{3/2}} \cdot \mathrm{arctg}\sqrt{\frac{(s+q^2/r_i^2)(1-r/r_i)}{(s+q^2/r_i^2)r/r_i - q^2/r_i^2}} \right] \tag{29}$$

Hence, the motion of dust layers depends on the excessive potential s and the initial distance from the WH throat: r_i/q.

However, this model is physically correct only until a possible intersection of dust layers. This is because after the intersection, an incoming energy flux appears in the matter-comoving frame (from the intersected matter layers), which has not been taken into account in the energy-momentum tensor. The intersection of adjacent matter layers corresponds to an infinite energy density. This corresponds to the condition $r_{,R} = 0$ (or $\frac{dr}{dr_i} = 0$) [see (20)]. Differentiation of (29) with respect to r_i allows explicitly determining the function $\frac{dr}{dr_i}(r, s, q)$. Differentiation of the left-hand side of (29) yields zero. We let a prime denote the derivative with respect to r_i, express all distances in the units of units r_i, and omit intermediate bulky calculations; the final expression for this function is given by

$$r' = 1 + \frac{(1-r)\left[s(s'-2q^2)(2q^2 - r\{s+q^2\}) - q^2(s+2q^2)(2s'+s)\right]}{2rs(s+q^2)^2} +$$
$$+ \frac{[2s(s+q^2) + (2q^2-s')(s+4q^2)]\sqrt{(1-r)[(s+q^2)r-q^2]}}{2r(s+q^2)^{5/2}} \cdot \mathrm{arctg}\sqrt{\frac{(s+q^2)(1-r)}{(s+q^2)r-q^2}} \tag{30}$$

It is impossible to use this formula to determine the instant of intersection of nonadjacent layers because the intersection with nonadjacent layers does not lead to any irregularities for dust. Clearly, the intersection of nonadjacent layers occurs after the intersection of adjacent ones and cannot propagate with a superluminal velocity. Therefore, after constructing diagrams $t(r_i)$ for light cones and the curve corresponding to the intersection of adjacent layers ($r' = 0$), we can determine regions that are definitely free from layer intersection (see the Figure 1 or 2).

Using formula (26), we can reach several important conclusions.

1. The formation of a horizon is possible only for a sufficiently large parameter s:

$$s > 2\frac{q}{r_i}\left(1 - \frac{q}{r_i}\right) \tag{31}$$

After the appearance of a horizon, the WH becomes "non-traversable". However, dust layers can already start intersecting before the horizon is reached.

2. Values of the function $t(r, r_i)$ are bounded by the maximum time t_{stop} (from the beginning of motion until the stop). In (28), this time corresponds to the argument of arccos being minus unity (or to the zero value of the square root expression in the same formula):

$$t_{stop} = \frac{\pi r_i (s + 2q^2/r_i^2)}{2(s+q^2/r_i^2)^{3/2}} \tag{32}$$

The time t_{stop} corresponds to the deviation $\Delta r \approx s r_i^3/q^2$ from the initial position. **For small deviations (corresponding to small values of s), harmonic oscillations with the period $T = 2t_{stop} \approx 2\pi r_i^2/q$ occur** (see Appendix B).

3. The existence of the second (smaller) root of the equation $V^2(r) = 1$ implies the appearance of the second (inner) horizon in the system (see the Figures 1 and 2).

4. For parameter $s = 2q(1 - q/r_i)/r_i$ (or $\varepsilon = 0$) the scalar $\mathbf{g^{ik}\, r_{;i} r_{;k} = (1 - q/r)^2}$ (scalar equal to the scalar of the extremal Reisner-Nordstrem solution). Therefore: **if the energy density tends to zero — the solution (29) tends to the extremal Reisner-Nordstrem solution of black hole (in the comoving, free-falling frame).**

5 Solution for $s = 0$ with the Λ-term

We consider the case where $s = 0$. Eqn (6) with (9) and (22) then becomes

$$2rr_{,tt} = \Lambda r^2 + q^2\left(\frac{1}{r^2} - \frac{1}{r_i^2}\right) - \frac{\Lambda}{3}\left(r_i^2 - q^3/r_i\right) - r_{,t}^2\,, \tag{33}$$

and Eqn (12) takes the form

$$r_{,t}^2 = (r - r_i)\left[-\frac{q^2}{r^2 r_i^2}(r - r_i) + \frac{\Lambda}{3}\left(r + r_i + \frac{q^3}{r r_i}\right)\right]. \tag{34}$$

These two equations imply that at $s = 0$, matter starts expanding from the rest state (see 19): inflation due to the Λ-term). **If the dimensionless parameter**

$$a \equiv \Lambda q^2 \tag{35}$$

exceeds some critical value a_{cr}, the inflation continues unlimitedly in the entire volume ($q \le r_i \le r < \infty$) **until an outer horizon forms, $V^2(r) = 1$.** The value a_{cr} is found from the condition of the maximum of the expression for the parameter a, which is obtained by equating the second factor (in square brackets) in Eqn (34) to zero:

$$a_{cr} = MAX\left[\frac{3\tilde{q}^4(1 - \tilde{r})\tilde{r}^2}{1 + \tilde{r} + \tilde{r}^2\tilde{q}^3}\right] \approx 0.22\,, \quad \tilde{r} \equiv r_i/r\,, \quad \tilde{q} \equiv q/r_i\,. \tag{36}$$

For $a < a_{cr}$, a region appears where the inflation stops. This region starts emerging at the throat and extends toward larger values of r_i as the parameter a decreases.

As $r/r_i \to \infty$, the inflation occurs exponentially: $r \propto \exp(t\sqrt{\Lambda/3})$.

The function r' can be expressed in quadratures (see Appendix C):

$$r' = 1 + \sqrt{r - r_i}\,\exp\left(-\int_{r_i}^{r} P_2(x)\,dx\right)\cdot\int_{r_i}^{r}\frac{Q_2(r) - P_2(r)}{\sqrt{r - r_i}}\exp\left(\int_{r_i}^{r} P_2(x)\,dx\right)dr\,,$$

$$P_2 = \frac{\frac{r_i^2}{2r^2} - \frac{r_i^2(r-r_i)}{r^3} - \frac{a}{2}\left(1 - \frac{q^3}{r^2 r_i}\right)}{-\frac{r_i^2(r-r_i)}{r^2} + a\left(r + r_i + \frac{q^3}{r r_i}\right)}\,,\quad Q_2 - P_2 = \frac{a + (r + r_i)\left(\frac{r_i(r-r_i)}{r^3} - \frac{aq^3}{2r^2 r_i^2}\right)}{-\frac{r_i^2(r-r_i)}{r^2} + a\left(r + r_i + \frac{q^3}{r r_i}\right)}\,. \tag{37}$$

Using this equation, we can find regions where the inflation occurs without dust layer intersections.

6 Model of the Multiverse

Dust with the positive s initially accelerates toward the center (without the Λ-term). The Λ-term without excessive dust leads to the original inflationary solution [see (24)]. At the WH throat, the potential $s = 0$, and the Λ-term at the throat provides nonzero matter acceleration. This contribution cannot be compensated at or near the throat [see (6)], where $r_{,t} = r_{,tt} = 0$ and $r = q$. Hence, there is no static solution for a WH with the throat radius $r_{_0} = q$ and the Λ-term.

Nevertheless, a static solution exists for a WH with the Λ-term and $r_{_0} \neq q$. This solution can be easily derived from Eqns (5), (6) and (8) (with $r_{,t} = 0$ and $r_{,tt} = 0$).

The metric of a static WH is determined by expression (1), taking into account that $r^2(R) \neq q^2 + R^2$.

From (6), we obtain $r^2_{,R} = 1 - q^2/r^2 - \Lambda r^2$ and easily find the expression for the throat radius $r_{_0}$,

$$\Lambda r_{_0}^4 - r_{_0}^2 + q^2 = 0 \quad \Rightarrow \quad r_{_0}^2 = \frac{1 - \sqrt{1 - 4\Lambda q^2}}{2\Lambda} \to q^2(1 + \Lambda q^2) \qquad (\text{at } \Lambda q^2 \to 0)\,, \tag{38}$$

and the dependence $r(R)$:

$$r^2(R) = \frac{1 - \sqrt{1 - 4\Lambda q^2} \cdot \cos(2\sqrt{\Lambda}R)}{2\Lambda} \tag{39}$$

The distribution of ε for this solution is determined[7] from (5):

$$\varepsilon = \frac{\Lambda}{4\pi} - \frac{q^2}{4\pi r^4} \tag{40}$$

Thus, for the static solution with the Λ-term, the throat radius and dust density are larger than without the Λ-term.

The total energy density for the static solution with the Λ-term is

$$T_t^t = \frac{3\Lambda}{8\pi} - \frac{q^2}{8\pi r^4} \tag{41}$$

The condition for the value T_t^t to be nonnegative everywhere in space (for $r \geq r_{_0}$) has the form

$$T_t^t \geq 0 \quad \text{at } a = \Lambda q^2 \geq \frac{3}{16} \tag{42}$$

Therefore, in the presence of the Λ-term, solutions for wormholes with a positive total energy density can be found.

From (39), we can obtain the maximum allowed radius of the static metric with the Λ-term:

$$r^2_{max} = \frac{1 + \sqrt{1 - 4\Lambda q^2}}{2\Lambda} \to \frac{1}{\Lambda} \qquad (\text{at } \Lambda q^2 \to 0)\,. \tag{43}$$

Beyond this radius, the Universe starts contracting again until a new throat occurs.

Solution (39) describes a static Multiverse with an infinite number of throats. When there is no charge ($q = 0$)**, this solution becomes a cosmological solution without wormholes, corresponding to a closed isotropic universe** (see §112, [31]).

When there is an excess (or shortage) of the dust part of the energy density or the Λ-term relative to (40), the obtained solution for the Multiverse becomes dynamical. Its analytic study is complicated by the need to solve a fourth-order algebraic equation and to calculate a quadrature similar to (37) for the instant of dust layer intersection.

[7] Here, the unknown value $r_{,RR} = q^2/r^3 - \Lambda r$ can be obtained from expression (8) or by directly differentiating Eqn (39).

7 Conclusion

In this paper, we have generalized and developed the method suggested earlier in papers [20], [22] and applied it to new problems in modern cosmology. We found and analyzed analytic solutions of the general relativity equations describing the dynamics of a traversable wormhole. The results obtained are important for the analysis of the general properties of traversable wormholes. We also obtained a solution describing a spherically symmetric model of the Multiverse. We have not analyzed the properties of geodesics that describe the motion of particles and other matter (energy and information) through a wormhole and its vicinity. These problems, which are mostly important for the analysis of the possible observational appearance of such objects, are considered elsewhere (see, e.g., [11]).

To conclude, we remark on the term "Multiverse". This term is used in physics and cosmology in two different senses.

First, this term assumes the possibility of the parallel co- existence of many or even an infinite number of different worlds, possibly emerged from a quantum vacuum (in some sense, in different places at different times). In this paper, this term is used exactly in this sense.

Second, this term is sometimes used to denote a sample of different realities in the Everett interpretation of quantum mechanics. The usage of one term for different notions is sometimes confusing. In our opinion, different terms should be used for these concepts. We propose keeping the term "Multiverse" for cosmology, and refer to the set of Everett worlds (following the proposal by M.B. Mensky) as the "Alterverse", keeping in mind different classical alternatives of the Everett world.

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A Initial conditions with a nonzero velocity

At the WH throat, the expression for V^2 in (26) involves a 0/0 ambiguity. To resolve this ambiguity, we consider the model with a nonzero initial velocity of matter. We introduce the notation

$$1-q/r_i \equiv \alpha\,, \quad r_i/r-1 \equiv \beta\,, \quad r^2_{,t}\Big|_{(t=0)} \equiv \gamma\,. \tag{44}$$

and assume that $\Lambda = 0$. The expressions for the functions F_1 and F_2 with $\gamma \neq 0$ [see (15) and (12)] are given by

$$F_1 = 1 - s - q^2/r_i^2 + \gamma - \gamma_0 q/r_i\,, \quad F_2 = r_i\left[s + 2q^2/r_i^2 + \gamma_0 q/r_i\right]\,. \tag{45}$$

With (12) and a nonzero initial velocity, the expression for V^2 in (26) takes the form

$$V^2 = \frac{-s - q^2/r_i^2 + \gamma - \gamma_0 q/r_i + (s + 2q^2/r_i^2 + \gamma_0 q/r_i)r_i/r - q^2/r^2}{1 - s - q^2/r_i^2 + \gamma - \gamma_0 q/r_i} \tag{46}$$

In the vicinity of the WH throat close to the initial instant, the functions α and β take small values. From (21), in the linear order in α, we have

$$s \approx \alpha\kappa\,, \quad \kappa = 8\pi r_i^2(\varepsilon_i - \varepsilon_d) \approx 8\pi\varepsilon_0 q^2 + 2 < 2\,.$$

We consider Eqn (46) in the linear order in α and β. After all transformations, we finally obtain

$$V^2 \approx \frac{\gamma + \gamma_0\beta}{\gamma - \gamma_0(1-\alpha) + \alpha(2-\kappa)} \rightarrow \frac{\gamma_0(1+\beta)}{\alpha(\gamma_0 + 2 - \kappa)} \text{ at } \gamma \rightarrow \gamma_0\,. \tag{47}$$

It follows that the condition for V^2 to be nonsingular at the throat is that $\gamma_0 = 0$. Then

$$V^2 \approx \frac{\gamma}{\gamma + \alpha(2-\kappa)} \tag{48}$$

Expression (48) for V^2 is regular and has no ambiguities if the function γ tends to zero at the throat faster than the function α. In this case, the rate of change of the radius r vanishes at the throat (as must be the case by the definition of the throat). In the linear order in β, the function V^2 is time-independent near the throat.

B Study of intersections of adjacent layers

We consider the model without the Λ-term for $s > 0$. In this case, the excessive matter collapses, i.e., $r \leq r_i$.

Lemma. There always exists a nonzero time interval $[0, t]$ inside which no intersections of adjacent layers occur.

Equation (30) implies that this is obvious for a nonzero potential s. The only point at which this statement must be proved is the point corresponding to $s = 0$.

We study the asymptotic regime $s \rightarrow 0$ in more detail:

$$s << q^2/r_i^2 \leq 1 \tag{49}$$

From the allowed range of the radius (until the stopping point) $r \geq q^2/(sr_i + q^2/r_i)$, we deduce

$$1 - \frac{r}{r_i} \leq \frac{s}{s + q^2/r_i^2} \leq \frac{s}{q^2/r_i^2} << 1 \tag{50}$$

Because the last term in (30) contains no s in the denominator, it can be neglected compared to the previous term. This can be proved in more detail by considering the there exhaustive options for the ratio of the numerator and denominator in the square root expression of the arctan:

1. $(s + q^2/r_i^2)(1 - r/r_i) << (s + q^2/r_i^2)r/r_i - q^2/r_i^2$
2. $(s + q^2/r_i^2)(1 - r/r_i) \sim (s + q^2/r_i^2)r/r_i - q^2/r_i^2$
3. $(s + q^2/r_i^2)(1 - r/r_i) >> (s + q^2/r_i^2)r/r_i - q^2/r_i^2$

Keeping the leading terms, we obtain the asymptotic form of Eqn (30):

$$r' \rightarrow 1 + \frac{(1 - r/r_i)[-q^2 \cdot 2q^2 \cdot 2r_i s']}{2sq^4} = 1 - 2r_i s'(1 - r/r_i)/s \tag{51}$$

In a similar way, the first term in the right-hand side of expression (28) can be neglected (compared to the second term), and hence asymptotically with respect to time, we have

$$\cos\left(\frac{tq}{r_i^2}\right) \to 1 - \frac{2q^2}{sr_i^2}(1 - r/r_i) \quad \Longrightarrow \quad r \to r_i + \frac{sr_i^3}{2q^2}\left[\cos\left(\frac{tq}{r_i^2}\right) - 1\right] \tag{52}$$

Comparing Eqns (51) and (52), we obtain the sought asymptotic form as $s \to 0$:

$$r' \to 1 - \frac{2r_i^3 s'}{q^2} \sin^2\left(\frac{tq}{2r_i^2}\right). \tag{53}$$

This proves the lemma.
In addition, Eqn (52) implies harmonic dynamics (oscillations) for $s \to 0$ and $2r_i^3 s'/q^2 < 1$.

C Quadrature for r' in the solution with the Λ-term

We differentiate $r_{,t}^2$ with respect to r_i:

$$\frac{dr_{,t}^2}{dr_i} = 2r_{,t}r'_{,t} = 2r_{,t}^2\frac{dr'}{dr} \quad \Rightarrow \quad \frac{dr'}{dr} = \frac{1}{2r_{,t}^2} \cdot \frac{dr_{,t}^2}{dr_i} \tag{54}$$

Substituting Eqn (34) in (54) and differentiating it with respect to r_i, we obtain the following equation for the function $y(r) \equiv r'$:

$$\frac{dy}{dr} + P(r)y = Q(r)\,, \tag{55}$$

Here, the functions $P(r) \equiv P_1(r) + P_2(r)$ and $Q(r) \equiv Q_1(r) + Q_2(r)$ are given by (for convenience, r_i is set to unity below):

$$P_1 = \frac{-1}{2(r-1)}\,, \quad P_2 = \frac{1/(2r^2) - (r-1)/r^3 - (a/2)(1 - q^3/r^2)}{(1-r)/r^2 + a(1 + r + q^3/r)}\,,$$
$$Q_1 = \frac{-1}{2(r-1)}\,, \quad Q_2 = \frac{1/(2r^2) + (r-1)/r^2 + (a/2)(1 - q^3/r)}{(1-r)/r^2 + a(1 + r + q^3/r)}\,. \tag{56}$$

Equation (55) has a standard solution,

$$y(r) = \exp\left[-\int_{r_1}^{r} P(x)\,dx\right] \cdot \left\{\int_{r_2}^{r} Q(r) \cdot \exp\left[+\int_{r_1}^{r} P(x)\,dx\right] dr + C_1\right\} \tag{57}$$

The constants C_1, r_1 and r_2 are determined by the initial conditions. The exponential can be rewritten as (the first term with P_1 is integrated):

$$\exp\left[\int_{r_1}^{r} P(x)\,dx\right] = \frac{C_2}{\sqrt{r-1}} \cdot \exp\left[\int_{1}^{r} P_2(x)\,dx\right] \tag{58}$$

After that, the part of integral (57) corresponding to Q_1 can also be integrated (by parts), with the result

$$y(r) = \sqrt{r-1} \cdot \exp\left[-\int_1^r P_2(x)\,dx\right] \cdot \left\{\frac{1}{\sqrt{r-1}} \cdot \exp\left[+\int_1^r P_2(x)\,dx\right]\Bigg|_{r_2}^{r} - \right.$$
$$\left. -\int_{r_2}^{r} \frac{P_2(r)}{\sqrt{r-1}} \cdot \exp\left[+\int_1^r P_2(x)\,dx\right] dr + \int_{r_2}^{r} \frac{Q_2(r)}{\sqrt{r-1}} \cdot \exp\left[+\int_1^r P_2(x)\,dx\right] dr + C_1\right\} \tag{59}$$

We can then redefine the lower integration limit and determine the constant C_1 from the initial conditions. The final expression for the function $y(r) = r'$ is given in (37).

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