Spin-spin interaction in general relativity and induced geometries with nontrivial topology

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We consider the dynamics of a self-gravitating spinor field and a self-gravitating rotating perfect fluid. It is shown that both these matter distributions can induce a vortex field described by the curl 4-vector of a tetrad: $\omega^i = \frac{1}{2}\varepsilon^{iklm}e_{(a)k}e^{(a)}_{l;m}$, where $e^{(a)}_k$ are components of the tetrad. The energy-momentum tensor $T_{ik}(\omega)$ of this field has been found and shown to violate the strong and weak energy conditions which leads to possible formation of geometries with nontrivial topology like wormholes. The corresponding exact solutions to the equations of general relativity have been found. It is also shown that other vortex fields, e.g., the magnetic field, can also possess such properties.

As we have shown earlier [1, 2], a self-gravitating Dirac spinor field with the Lagrangian

$$L(\psi) = \frac{\hbar c}{2}\left[\nabla_i\bar{\psi}\gamma^i\psi - \bar{\psi}\gamma^i\nabla_i\psi - F(\bar{\psi}\psi)\right] \qquad (1)$$

can interact with the vortex component of the gravitational field, which results in the appearance of a more general Lagrangian:

$$L(\psi) = \frac{\hbar c}{2}\Big[\partial_i\bar{\psi}\gamma^i\psi - \bar{\psi}\gamma^i\nabla_i\psi$$
$$+ \omega^i\cdot(\bar{\psi}\gamma_i\gamma_5\psi) - F(\bar{\psi}\psi)\Big]. \qquad (2)$$

Here, ω^i is the curl 4-vector of a tetrad: $\omega^i = \frac{1}{2}\varepsilon^{iklm}e^{(a)}_k e_{(a)l;m}$, i.e., the 4-vector of the gravitational field vortex; $\nabla_\alpha\psi$ is the covariant derivative of the spinor function $\psi(x^k)$: $\nabla_k\psi = \partial_k\psi - \Gamma_k\psi$, where Γ_k are the matrix spinor connection coefficients; γ_k are the curved-space Dirac matrices defined by the fundamental relation between the space-time metric and spin,

$$\gamma_i\gamma_k + \gamma_k\gamma_i = 2g_{ik}\cdot I,$$

and the axial vector $\bar{\psi}\gamma_i\gamma_5\psi$ is proportional to the proper angular momentum (spin) of the spinor field $S_k(\psi) = \frac{\hbar c}{2}\bar{\psi}\gamma_k\gamma_5\psi$; the function $F(\bar{\psi}\psi)$ is the spinor field potential depending on the invariant $\bar{\psi}\psi$. In particular, for a massive spinor field we have $F(\bar{\psi}\psi) = 2m\bar{\psi}\psi$.

Variation of the total Lagrangian of the gravitational and spinor fields $L = -R/(2\varkappa) + L(\psi)$ with respect to ω^i leads to a relation between the gravitational field vortex and the spin density of the spinor field:

$$\omega^i = \varkappa\frac{\hbar c}{4}\bar{\psi}\gamma^i\gamma_5\psi. \qquad (3)$$

Taking into account this relation, the spinor field Lagrangian (2) takes the form

$$L(\psi) = \frac{\hbar c}{2}\Big[\partial_k\bar{\psi}\gamma^k\psi - \bar{\psi}\gamma^k\partial_k\psi$$
$$+ \frac{\varkappa\hbar c}{2}(\bar{\psi}\gamma^k\gamma_5\psi)(\bar{\psi}\gamma_k\gamma_5\psi) - F(\bar{\psi}\psi)\Big], \quad (4)$$

i.e., we have obtained the Lagrangian of a nonlinear spinor field with a quadratic pseudovector nonlinearity.

Interaction of such a nonlinear spinor field with gravity, even if the latter has no vortex component, e.g., in the case of spherical symmetry, leads to an interesting result. Spherically symmetric spinor field configurations have a radially polarized spin density vector

$$S_i(\psi) = \frac{\hbar c}{2}\bar{\psi}\gamma_i\gamma_5\psi = \delta^1_i\bar{\psi}\gamma_1\gamma_5\psi\frac{\hbar c}{2},$$

distributed like the lines of force of a point electric charge. Let us choose the metric of a static, spherically symmetric space-time in the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2). \qquad (5)$$

The components of the energy-momentum tensor of the nonlinear spinor field (4) in the absence of the potential $F(\bar{\psi}\psi)$ take the form

$$T_{ik}(\psi)$$
$$= \frac{\hbar c}{4}\left[\nabla_i\bar{\psi}\gamma_k\psi + \nabla_k\bar{\psi}\gamma_i\psi - \bar{\psi}\gamma_i\nabla_k\psi - \bar{\psi}\gamma_k\nabla_i\psi\right]$$
$$- \frac{\hbar c}{2}\frac{\varkappa\hbar c}{2}(\bar{\psi}\gamma^s\gamma_s\psi)(\bar{\psi}\gamma_s\gamma_5\psi)g_{ik}. \qquad (6)$$

Solving the set of Einstein-spinor equations due to the Lagrangian (4) with (6),

$$R_{ik} - \frac{1}{2} R g_{ik} = \varkappa T_{ik}(\psi),$$

$$\gamma^k \nabla_k \psi - \frac{\varkappa hc}{2} (\bar\psi \gamma^k \gamma_5 \psi) \gamma_k \gamma_5 \psi = 0 \qquad (7)$$

in a space-time with the metric (5), we find the function $\psi(r)$ and the metric coefficients $e^{\lambda(r)}$ and $e^{\nu(r)}$ [3, 4]. In particular, for the coefficients e^ν and $e^{\nu(r)}$ we obtain the expressions $e^\nu = 1$ and $e^\lambda = r^2/(r^2 - a^2)$ ($a = \text{const}$), and to keep the signature unchanged we must put $r^2 - a^2 > 0$. Then, after the transformation $r^2 - a^2 = x^2$ ($-\infty < x < +\infty$), the metric (5) is obtained in the form

$$ds^2 = dt^2 - dx^2 - (x^2 + a^2)(d\theta^2 + \sin^2\theta \, d\varphi^2). \quad (8)$$

It is the metric of a wormhole space-time (the so-called Ellis wormhole), connecting two asymptotically flat spaces. The constant a determines the wormhole throat radius. In this case, $a = l_0 s_0 \sqrt{\varkappa/\varepsilon_0}$, where $l_0 = \sqrt{\varkappa hc}$ is the Planck length while s_0 and ε_0 are the values of the spin flux density and energy density of the spinor field at $x = 0$. It can be seen that the throat radius of a wormhole formed by a polarized self-gravitating nonlinear spinor field is of the order of Planck's length. This is connected with the fact that the coefficient of nonlinearity $\varkappa hc/2 = l_0^2$ is of the order of Planck's length squared. If, however, this coefficient of nonlinearity can be much greater, the wormhole throat will also be much wider.

The metric (8) was discussed in detail by Ellis [5] and is a special case of metrics discussed in [6] in the context of scalar-tensor theories of gravity.

A simple example of a space-time with a stationary vortex gravitational field is the space-time with the cylindrically symmetric metric

$$ds^2 = D(x) \, dt^2 - A(x) \, dx^2 - B(x) \, d\alpha^2$$
$$- A(x) \, dz^2 - 2E(x) \, d\alpha \, dt. \quad (9)$$

The geometric properties of its spatial section are determined by the 3-dimensional spatial line element

$$dl^2 = A \, dx^2 + \frac{BD + E^2}{D} \, d\alpha^2 + A \, dz^2, \qquad (10)$$

while the intensity of the stationary gravitational vortex $\omega = (\omega_k \omega^k)^{1/2}$ is determined by the expression

$$\omega = \frac{E'D - D'E}{2DA^{1/2}(E^2 + BD)^{1/2}}, \qquad (11)$$

where the prime denotes d/dx. In the spatial metric (10), the coefficient $R(x) \equiv (BD + E^2)/D$ of the angular coordinate squared determines the length of the coordinate circle $x = x_0$, $z = z_0$. From the vacuum Einstein equations $R_{ik} = 0$ for the metric (9) one can obtain an equation for the coefficient $R(x)$:

$$A^{-1}\left[\frac{R''}{R} + \frac{R'}{2R}\left(\frac{D'}{D} - \frac{R'}{R}\right)\right] = \frac{4\omega^2}{c^2}. \qquad (12)$$

The right-hand side in Eq. (12), proportional to ω^2, is positive-definite. Therefore, at the point where $R' = 0$, one obtains $R'' > 0$ (a minimum), and this is a necessary condition for the existence of a wormhole. Thus a vortex gravitational field is able to induce wormhole formation. The definitions and general conditions for the existence of static cylindrical wormholes have been considered in [7].

A solution to the vacuum Einstein equations $R_{ik} = 0$ for the metric under consideration describes a wormhole space-time:

$$A(x) = \frac{c}{b\omega_0 (x^2/b^2 + 1)^2};$$

$$D(x) = \frac{\exp\left[\arcsin(x^2/b^2 + 1)^{-1}\right]}{x^2/b^2 + 1};$$

$$R(x) = \frac{BD + E^2}{D}$$
$$= (x^2 + b^2) \exp\left[\arcsin(x^2/b^2 + 1)^{-1}\right];$$

$$\omega = \frac{\omega_0}{AD^{1/2}}; \qquad (\omega_0 = \text{const}, \; b = \text{const}), \qquad (13)$$

where $-\infty < x < \infty$. It is seen here that the "circular" metric coefficient $R(x)$, determining the coordinate circumference, nowhere turns to zero, and at the point $x = 0$ (throat) has a minimum: $R(x)_{\min} = b^2 \cdot \exp(\pi/2) \neq 0$, so that the throat radius of the obtained wormhole is $a = b \cdot \exp(\pi/4)$. Here b is an integration constant. Thus a free vortex gravitational field can form cylindrical wormholes, space-time tunnels connecting different regions of space-time.

Wormholes can be also induced by vortex fields other than spinor and gravitational ones, for example, an azimuthal magnetic field H_α. (A solution to the Einstein-Maxwell equations for different directions of electric and magnetic fields was obtained earlier in [8].)

In a static, cylindrically symmetric space-time with an azimuthal magnetic field n$H_\alpha = F_{13}$, described by the metric

$$ds^2 = D(x) \, dt^2 - A(x) \, dx^2$$

$$- R^2(x)\, d\alpha^2 - A(x)\, dz^2, \qquad (14)$$

from the set of Einstein-Maxwell equations we obtain the following equation for the circular metric coefficient $R(x)$:

$$A^{-1}\left[\frac{R''}{R} + \frac{R'}{2R}\left(\frac{D'}{D} - \frac{R'}{R}\right)\right] = 2\varkappa H_\alpha^2. \qquad (15)$$

Here, just as in the case of a vortex gravitational field, the right-hand side is positive-definite, and hence R'' is positive where $R' = 0$, which, as pointed out above, is a necessary condition for wormhole existence. The solution to the Einstein-Maxwell equations indeed describes a wormhole geometry:

$$A(x) = \frac{\varkappa I_z^2}{8\pi k^2}\cosh^2(kx)\cdot e^{5kx},$$

$$D(x) = \frac{\varkappa I_z^2}{8\pi k^2}\cosh^2(kx)\cdot e^{4kx}, \qquad (16)$$

while the function $R(x)$, determining the length of a coordinate circle $x = x_0$, $z = z_0$, is obtained in the form

$$R(x) = \frac{I_z}{k}\sqrt{\frac{\varkappa}{8\pi}}\cosh(kx)\cdot e^{kx/2} \qquad (17)$$

(where $-\infty < x < \infty$). Here k is an integration constant, I_z is the linear axial electric current density which is a source of the azimuthal magnetic field. From (17) it is seen that $R(x)$ nowhere turns to zero, and $R(x) \to \infty$ as $x \to +\infty$ and as $x \to -\infty$, i.e., there is a wormhole connecting two remote regions of space. The wormhole throat radius is proportional to the axial electric current density I_z.

The above results show that self-gravitating vortex fields (the spinor, gravitational and electromagnetic ones) can form wormholes, so that, for such a purpose, it is unnecessary to invoke phantom matter or scalar fields with negative kinetic energy, the more so as nobody knows how to get them. Meanwhile, as is known, a vortex azimuthal magnetic field is induced by a linear electric current, and a vortex gravitational field, as we have shown, is induced by a polarized spin of a spinor field.

In what follows, we will show that, in addition, a rotating continuous medium, e.g., a perfect fluid can also be a source of a vortex gravitational field. To this end, we will consider the gravitational interaction of a perfect fluid rotating with an angular velocity $\omega(x^k)$ in a space-time with the metric

(9), where, as we have shown above, there exists a stationary vortex gravitational field. We will show that a rotating self-gravitating perfect fluid can be a source of the metric (9). Consider the Einstein equations with the energy-momentum tensor of a perfect fluid

$$R_{ik} - \frac{1}{2}Rg_{ik} = \varkappa\left[U_iU_k(p + \varepsilon) - pg_{ik}\right] \qquad (18)$$

in a space-time with the metric (9). Let us use the comoving reference frame, in which the 4-velocity of the rotating fluid has the form $U^i = (1/\sqrt{D}, 0, 0, 0)$. In this case, the 4-vector U^i is a timelike monad vector, determining a rotating reference frame, and the angular velocity of the fluid is simultaneously the rotation angular velocity of the world-line congruence of the monad. In this problem setting, using the monad formalism [9], from the Einstein equations (18) in the metric (9) one obtains the following equation for the metric coefficient $R(x) = \frac{E^2 + BD}{D}$:

$$A^{-1}\left[\frac{R''}{R} + \frac{R'}{2R}\left(\frac{D'}{D} - \frac{R'}{R}\right)\right] = \varkappa(p - \varepsilon) + \frac{4\omega^2}{c^2}. \qquad (19)$$

It follows from this equation that if

$$\frac{4\omega^2}{c^2} > \varkappa(p - \varepsilon), \qquad (20)$$

then $R'' > 0$ at a point where $R' = 0$, which is a necessary condition for wormhole existence. Therefore the inequality (20) is a necessary condition for wormhole formation by a rotating perfect fluid.

In the case of the maximally stiff equation of state $p = \varepsilon$, the condition (20) manifestly holds. The corresponding exact solution to the Einstein equations (18) for a self-gravitating rotating perfect fluid, with the equation of state $p = \varepsilon$, has the following form:

$$A = D = 1; \qquad \omega = \omega_0 = \text{const};$$

$$p = \varepsilon = \frac{\omega_0^2}{\varkappa c^2} = \text{const}; \qquad R(x) = b^2\cosh\frac{\omega_0 x}{c};$$

$$-\infty < x < \infty, \qquad b = \text{const}. \qquad (21)$$

The solution (21) shows that a self-gravitating fluid with the maximally stiff equation of state rotates like a rigid body and forms a wormhole since the metric coefficient $R(x) = (BD + E^2)/D$ of the angular part in the effective spatial metric is everywhere positive, and $R(x) \to \infty$ as $x \to \pm\infty$.

That a vortex gravitational field can induce wormhole formation can be explained by the fact hat it has a certain energy-momentum tensor $T_{ik}(\omega)$ whose all components are proportional to ω^2, and the components corresponding to pressure, $p_i(\omega)$, are negative [1], and it violates both the strong energy condition ($\varepsilon(\omega) + p_1 + p_2 + p_3(\omega) > 0$) and the weak one ($\varepsilon + (p_1 + p_2 + p_3)/3 > 0$), i.e., this tensor has "phantom" properties. It has the structure characteristic of a perfect fluid with an anisotropic negative pressure:

$$T_{ik}(\omega) = [p(\omega) + \varepsilon(\omega)]U_i U_k$$
$$- (p_1 - p)\chi_i \chi_k - p g_{ik}, \qquad (22)$$

where χ_i is the anisotropy vector directed along the rotation axis and satisfying the conditions $\chi_i U^i = 0$, $\chi_i \chi^i = -1$.

The tensor $T_{ik}(\omega)$ obeys the local conservation law: $T_k^i(\omega)_{;i} = 0$. The solution for the vortex intensity $\omega = \omega_0/(AD^{1/2})$ in Eqs. (13) is just an integral of this equation. In the case of a stationary vortex gravitational field in a space-time described by the metric (9), the energy-momentum tensor of this field $T_{ik}(\omega)$ has the following components:

$$T_k^i(\omega) = \frac{\omega^2}{\varkappa c^2} \cdot \mathrm{diag}\,(1,\ 1,\ 1,\ 3). \qquad (23)$$

It is seen that the negative pressure $p_z = -3\omega^2/(\varkappa c^2)$ along the rotation axis is three times as large as the radial and transversal pressures, $p_r = p_\alpha = -\omega^2/(\varkappa c^2)$, and the sum $\varepsilon(\omega) + \frac{1}{3}(p_r + p_\alpha + p_z) = -3\omega^2/(2\varkappa c^2) < 0$, i.e., the weak energy condition is violated, thus leading to possible wormhole existence. Besides, since the axial negative pressure is three times as large as the other components, at gravitational collapse of very massive rotating astrophysical objects (the most massive stars, galactic nuclei), in which process the matter density extremely grows along with rotation velocity, forming a vortex gravitational field like (23), such an object will stretch along its rotation axis. As a result, a stable, rapidly rotating astrophysical object can form, having a maximally stiff equation of state and stretched along its rotation axis, which can be a wormhole described by the above solution (21) for a stationary rotating perfect fluid configuration.

We can conclude that there can be a fourth final state of evolution of astrophysical objects (stars of various masses and galactic nuclei), in addition to three known states — a white dwarf, a neutron star (pulsar), and a black hole. Namely, a very massive rotating object can possibly form a wormhole with intense rotation and an equation of state close to the limiting one.

Thus we have shown that vortex fields (spinor, gravitational and magnetic) can form wormholes. A source of a vortex gravitational field can be a spinor field with polarized spin or a rapidly rotating continuous medium. This leads to one more possible final state of astrophysical object evolution, a wormhole.

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